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Research article

Innovative soliton solutions for a (2+1)-dimensional generalized KdV equation using two effective approaches

Ibrahim Alraddadi[1], Faisal Alsharif[2], Sandeep Malik[3], Hijaz Ahmad[1,4,5], Taha Radwan[6,*] and Karim K. Ahmed[7]

1 Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia

2 Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia

3 Department of Mathematics, Akal University, Talwandi Sabo, Bathinda, Punjab, 151302, India

4 Operational Research Center in Healthcare, Near East University, Nicosia/TRNC, 99138 Mersin 10, Turkey

5 Department of Technical Sciences, Western Caspian University, Baku, 1001, Azerbaijan

6 Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 51452, Saudi Arabia

7 Department of Mathematics, Faculty of Engineering, German International University (GIU), New Administrative Capital, Cairo, Egypt

* **Correspondence:** Email: t.radwan@qu.edu.sa.

Abstract: In this paper, we analyze and provide innovative soliton solutions for a (2+1)-dimensional generalized Korteweg-de Vries (gKdV) problem. We obtain phase shifts and dispersion relations by using the generalized Arnous technique and the Riccati equation approach, thus allowing different soliton solutions to be developed. Several precise solutions with special structural properties, including kink and solitary soliton solutions, are included in our study. This detailed examination demonstrates the complex behavior of the model and its capability to explain a large scale of nonlinear wave occurrences in many physical settings. Thus, in scientific domains such as fluid mechanics, plasma physics, and wave propagation in media ranging from ocean surfaces to optical fibers, our results are crucial to comprehend the principles behind the production and propagation of many complicated phenomena. Finally, we provide 2D and 3D graphs for various solutions that have been obtained using Maple.

Keywords: mathematical model; analytic solutions; soliton solutions; generalized KdV equation; generalized Arnous method; Riccati equation method

Mathematics Subject Classification: 35B35, 35C07, 35C08, 35C09

1. Introduction

A family of self-reinforcing wave packets known as soliton solutions, which maintain the balance between dispersive and nonlinear effects, are widely used in the physical and engineering disciplines [1–4]. N-soliton solutions have been extensively studied in various nonlinear partial differential (NLPD) systems, which may have numerous uses. The communications sector makes extensive use of optical soliton dynamics. In the absence of online activity, the modern world is still, as if life has come to an end. Consequently, a detailed analysis of the soliton's scientific dynamics is essential. For some wave equations that are part of the linear partial differential equations, the D'Alembert solutions represent a significant foundational rule. The well-known wave equation's D'Alembert solutions may be viewed as a thorough investigation. Some NLPD systems provide the D'Alembert traveling wave solution with a multitude of distinct functions [5, 6].

On the other hand, there are many interesting applications in science and technology to study and model evolution equations and systems, both integrable and non-integrable. They provide a solid basis to understand the intricate dynamics of various nonlinear natural systems. Over time, nonlinear evolution equations (NLEEs) have developed into essential tools to model and study a wide range of nonlinear natural and laboratory systems. Partial differential equations (PDEs) are frequently used to represent these equations, thereby encompassing the fundamental concepts of nonlinear wave propagation, including soliton interactions, rogue waves, and various complicated dynamical phenomena [7–10]. The previously mentioned types of such equations may be applied to mimic a broad range of stochastic dynamics in complex systems, either naturally occurring or artificially constructed. For instance, the family of nonlinear Schrödinger equation (NLSE) and the integrable / non-integrable group of the (non) planar Korteweg-de Vries (KdV) type equations were used to comprehend the dynamical behavior of the waves' propagation in shallow water, plasma waves, and magneto-hydrodynamic waves [11–14]. For instance, to investigate wave dynamics, Li and Yu [15] presented several families of non-autonomous soliton solutions with varying amplitude surfaces for the cubic-quintic Gross–Pitaevskii equation. Ahmed et al. [16] examined the dynamical behaviors of analytical solutions for the generalized Bogoyavlensky-Konopelchenko equation. A general approach was proposed by Li et al. [17] to control soliton waves, with potential applications in optical media.

Initially focused on waves in shallow water, the study of solitary waves has expanded to encompass a range of physical domains. These particular, nonlinear dispersive waves have generated significant research interests in several disciplines, foremost fluid dynamics, optical fibers, plasma physics, and other relevant scientific domains [18–22]. The improved $\tan\left(\frac{\phi}{2}\right)$−expansion strategy and the new extended generalized Kudryashov approach are among the most widely accepted methods for generating correct solutions to nonlinear partial differential equations (NLPDEs) [23, 24]. Solitary wave solutions may be extracted from various nonlinear situations using these simple procedures. These methods have the benefit of being applicable to problems with large balancing numbers.

In summary, bound solitons that consist of varying numbers of solitons constitute the major focus of the soliton molecule study. This paper establishes several soliton solutions for a (2+1)-dimensional gKdV problem [25–27].

This work is organized as follows: An explanation of the model and its uses are provided in Section 2. A summary of the two suggested alternatives is provided in Section 3, together with the mathematical background. Section 4 provides several precise soliton solutions as well as additional wave solutions for the model presented in (2.1) under a variety of assumptions. The discovered answers are graphically shown in both 2D and 3D forms in Section 5. A discussion about the novelty and innovations of the work is presented in Section 6. And finally, the paper's conclusion and the study viewpoint are presented in Section 7.

2. Description of the model and its applications

Water waves are among the most important natural phenomena. The acclaimed and well-known KdV equation represents weak, nonlinear water wave propagation in long, shallow, narrow channels. The mathematical foundation for the development of ideas about solitons and nonlinear dispersive waves came from observations of water waves. Equations such as the KdV equation have been immensely important because scholars, such as Korteweg and de Vries, utilized them to examine shallow-water waves in canals and oceans and contributed the theoretical foundation for the idea of solitary waves. Furthermore, KdV-type equations are now widely used models with a wide range of applications [28].

This research represents a novel search for the exact solutions for the (2+1)-dimensional gKdV problem, which can be read as follows [28]:

$$6\delta_1 u_x u_{xx} + \delta_1 u_{xxxx} + 3\delta_2 (u_x u_t)_x + \delta_2 u_{xxxt} + \delta_3 u_{xx} + \delta_4 u_{yy} + \delta_5 u_{xt} = 0, \tag{2.1}$$

where δ_i, $(i = 1, 2, 3, 4, 5)$ are nonzero constants. A class of gKdV equation may be obtained for one object if $\delta_4 = 0$, while the Kadomtsev-Petviashvili equation can be obtained for another if $\delta_2 = \delta_3 = 0$. In particular, the Hirota-Satsuma equation [29] associated with $\delta_1 = 0$ and the traditional KdV equation [30] related with $\delta_2 = \delta_3 = 0$ are obtained. Five nonzero constants δ_i, (where $i = 1, 2, 3, 4, 5$), in Eq (2.1) affect the (2+1)-dimensional generalized KdV equation's solutions. These constants have a major impact on the behavior of the solutions by affecting a number of distinct factors, including dispersion, the interaction between different wave components, and the intensity of nonlinear effects. Different kinds of precise solutions, such as solitons, periodic waves, or other kinds of localized structures, can be obtained by varying these parameters. Each constant has a different effect; some influence the solution's spatial or temporal evolution, while others regulate the nonlinearity and dispersion balance, which are essential for higher-dimensional wave propagation phenomena. In [28], Ma et al. obtained N-soliton solutions for the (2+1)-dimensional gKdV equation (2.1). In [31], the authors identified various types of exact solutions for the same equation using the Hirota direct method. Additionally, Sebogodi and Khalique [32] applied the Lie symmetry analysis, Kudryashov, and the simplest equation methods to derive multiple exact solutions of the considered equation. Moreover, they provided conservation laws. Furthermore, Raza et al. [33] employed the improved $\tan(\phi/2)$-expansion and extended the generalized Kudryashov methods to obtain soliton solutions for Eq (2.1). The present study aims to examine Eq (2.1) using two efficient approaches, which are the generalized Arnous technique and the Riccati equation approach.

3. The applied approaches in brief

In this section, we present the salient features of the applied approaches that will be used in this piece of work. First, consider the following NLPDE:

$$\mathcal{F}(\Upsilon, \Upsilon_x, \Upsilon_t, \Upsilon_y, \Upsilon_{xx}, \Upsilon_{xt}, \Upsilon_{tt}, \dots) = 0, \tag{3.1}$$

where $\mathcal{F}$ is a polynomial in the unknown function $\Upsilon = \Upsilon(x, y, t)$ and its corresponding various partial derivatives. To convert Eq (3.1) into an ordinary differential equation (ODE), we will assume the following wave transformation:

$$\Upsilon(x, y, t) = \mathcal{U}(\rho), \quad \rho = x + \lambda_1 y - \lambda_2 t, \tag{3.2}$$

where λ_1 and λ_2 are arbitrary constants. By inserting Eq (3.2) into Eq (3.1), it produces a nonlinear ODE with the following form:

$$P(\mathcal{U}, \mathcal{U}', \mathcal{U}'', \mathcal{U}''', \dots) = 0. \tag{3.3}$$

It is possible to pursue the exact traveling wave solutions of the equations by using the following basic steps of the Riccati equation approach and the generalized Arnous method.

3.1. Generalized Arnous method

The generalized Arnous technique provides a number of useful benefits for decomposing complicated nonlinear systems when used to solve the (2+1)-dimensional generalized KdV equation. This technique makes it easier to derive analytical solutions that are both accurate and approximative, which is essential to comprehend how waves behave in multidimensional domains. Its ability to be applied to equations in higher dimensions makes it possible to more effectively analyze phenomena such as wave propagation and solitons. Additionally, the technique minimizes the processing effort, which makes it a useful tool to resolve complex issues in nonlinear optics, fluid dynamics, and other domains where KdV-type equations are applied. The fundamental ideas of the generalized Arnous approach are as follows [34, 35]:

Algorithm 3.1.1. Assume that the solution of Eq (3.3) can be expressed as follows

$$\mathcal{U}(\rho) = \alpha_0 + \sum_{j=0}^{\mathbb{N}} \frac{\alpha_j + \beta_j (\Phi'(\rho))^j}{(\Phi(\rho))^j}, \tag{3.4}$$

where the constants $\alpha_0, \alpha_j, \beta_j (j = 1, 2, \dots, \mathbb{N})$ will be evaluated later. Additionally, the function $\Phi(\rho)$ obeys the following constraint:

$$[\Phi'(\rho)]^2 = \left[\Phi(\rho)^2 - \chi\right] \ln(\kappa)^2, \tag{3.5}$$

with

$$\Phi^{(n)}(\rho) = \begin{cases} \Phi(\rho) \ln(\kappa)^n, & n : \text{even}, \\ \Phi'(\rho) \ln(\kappa)^{n-1}, & n : \text{odd}, \end{cases} \quad n \geq 2, \tag{3.6}$$

where $\kappa > 0$, $\kappa \neq 1$, and χ is an arbitrary constant. The Eq (3.5) has a solution of the following form:

$$\Phi(\rho) = k \ln(\kappa) \kappa^{\rho} + \frac{\chi}{4k \ln(\kappa) \kappa^{\rho}}, \tag{3.7}$$

where k is an arbitrary parameter.
Algorithm 3.1.2. By applying the homogeneous balancing principle on Eq (3.3), the positive integer $\mathbb{N}$ in Eq (3.4) can be estimated.
Algorithm 3.1.3. Add (3.4) to (3.3) together with (3.5) and its derivatives. This substitution gives us a polynomial of the form $\frac{1}{\Phi(\rho)}\left(\frac{\Phi'(\rho)}{\Phi(\rho)}\right)$. By collecting all terms of the same powers and equating them to zero in this polynomial, one may now extract the unknown parameters $\alpha_0,\ \alpha_j,\ \beta_j\ (j = 1, 2, \ldots, \mathbb{N})$. This produces an over-determined system of algebraic equations. We obtain the exact solutions for Eq (3.1) after solving the derived system.

3.2. Riccati equation approach

A strong and organized technique to solve the (2+1)-dimensional generalized KdV equation is offered by the Riccati equation methodology, which is particularly useful for locating precise solutions to nonlinear PDEs. One of its main features is its capacity to simplify complicated, nonlinear equations into more understandable forms. This makes it easier to acquire solutions for soliton, periodic, and other waves. By changing certain parameters in the Riccati equation, this flexible method may be used to obtain a wide range of precise answers. Additionally, it offers an easy-to-use and straightforward framework to study higher-dimensional wave propagation in nonlinear optics, fluid dynamics, and plasma physics. Below is an overview of the Riccati equation approach as stated by [36, 37].
Algorithm 3.2.1. Equation (3.3) satisfies the following solution:

$$\mathcal{U}(\rho) = \sum_{i=0}^{\mathbb{N}} A_i R(\rho)^i, \tag{3.8}$$

where $A_0,\ A_1, ..., A_{\mathbb{N}}$ are unknown constants which will be estimated. To obtain the positive integer $\mathbb{N}$, we have to apply the homogeneous balancing principle on Eq (3.3). The used function $R(\rho)$ should satisfy the following Riccati equation:

$$R'(\rho) = B_0 + B_1 R(\rho) + B_2 R(\rho)^2, \tag{3.9}$$

where B_0, B_1, and B_2 are real-valued constants. Then, the solutions of Eq (3.9) can be expressed as follows:

$$\begin{aligned}
R(\rho) &= -\frac{B_1}{2B_2} - \frac{\sqrt{\mu}}{2B_2}\tanh\left(\frac{\sqrt{\mu}}{2}\rho + \rho_0\right), \quad \mu > 0,\\
R(\rho) &= -\frac{B_1}{2B_2} - \frac{\sqrt{\mu}}{2B_2}\coth\left(\frac{\sqrt{\mu}}{2}\rho + \rho_0\right), \quad \mu > 0,\\
R(\rho) &= -\frac{B_1}{2B_2} + \frac{\sqrt{-\mu}}{2B_2}\tan\left(\frac{\sqrt{-\mu}}{2}\rho + \rho_0\right), \quad \mu < 0,\\
R(\rho) &= -\frac{B_1}{2B_2} - \frac{\sqrt{-\mu}}{2B_2}\cot\left(\frac{\sqrt{-\mu}}{2}\rho + \rho_0\right), \quad \mu < 0,\\
R(\rho) &= -\frac{B_1}{2B_2} - \frac{1}{B_2\rho + \rho_0}, \quad \mu = 0,
\end{aligned} \tag{3.10}$$

where $\mu = B_1^2 - 4B_0B_2$.

Algorithm 3.2.2. Equations (3.8) and (3.9) can be inserted into Eq (3.3) to obtain the strategic equations. We obtain significant outcomes by resolving these strategic equations. Additionally, the solutions to Eq (3.1) may be obtained with the aid of Eq (3.10).

4. Implementing two efficient approaches on Eq (2.1)

This section applies the two efficient approaches on Eq (2.1) to present the newly created soliton solutions for this proposed model.

Let us assume the travelling wave solution of Eq (2.1) has the following form:

$$u(x, y, t) = U(\rho), \quad \rho = x + \lambda_1 y - \lambda_2 t. \tag{4.1}$$

By inserting Eq (4.1) in Eq (2.1), we can acquire the following non-linear ODE:

$$\left((-6\delta_2\lambda_2 + 6\delta_1) U' + \delta_4\lambda_1^2 - \delta_5\lambda_2 + \delta_3\right) U'' - (\delta_2\lambda_2 - \delta_1) U'''' = 0. \tag{4.2}$$

Integrating Eq (4.2) once w.r.t. ρ gives the following:

$$(-\delta_2\lambda_2 + \delta_1) U''' + 3(-\delta_2\lambda_2 + \delta_1)(U')^2 + \left(\delta_4\lambda_1^2 - \delta_5\lambda_2 + \delta_3\right) U' = 0, \tag{4.3}$$

thereby treating the integral constant as 0. Now, balancing $(U')^2$ with U''', we obtain that $\mathbb{N} = 1$.

4.1. Analytical solutions using the generalized Arnous approach

The generalized Arnous (GA) technique is used in this portion to solve the gKdV equation. According to the GA approach, the solution to Eq (4.3) is as follows:

$$U(\rho) = \alpha_0 + \sum_{j=0}^{1} \frac{\alpha_j + \beta_j (\Phi'(\rho))^j}{(\Phi(\rho))^j} = \alpha_0 + \frac{\alpha_1 + \beta_1 \Phi'(\rho)}{\Phi(\rho)}, \tag{4.4}$$

where α_0, α_1, and β_1 are arbitrary constants that, when present, ensure that either $\alpha_1 \neq 0$ or $\beta_1 \neq 0$ simultaneously. An expression in terms of $\frac{1}{\Phi(\rho)}\left(\frac{\Phi'(\rho)}{\Phi(\rho)}\right)$ is obtained. Equations (3.5) and (3.6) are substituted, together with Eq (4.4) into Eq (4.3). The result is an algebraic system of equations that, as the following example demonstrates, combine terms of the same power and equalize them to zero:

$$\begin{aligned}
0 &= 6\chi(\beta_1 - 1)(\delta_2\lambda_2 - \delta_1)\ln(\kappa)^2 \alpha_1, \\
0 &= \left((\delta_2\lambda_2 - \delta_1)\ln(\kappa)^2 - \delta_4\lambda_1^2 + \delta_5\lambda_2 - \delta_3\right)\alpha_1, \\
0 &= 3\left(\chi^2(\beta_1 - 2)\beta_1(\delta_2\lambda_2 - \delta_1)\ln(\kappa)^2 - \chi\alpha_1^2(\delta_2\lambda_2 - \delta_1)\right)\ln(\kappa)^2, \\
0 &= 4\ln(\kappa)^4\beta_1(\delta_2\lambda_2 - \delta_1)\chi - \left(3(\delta_2\lambda_2 - \delta_1)\alpha_1^2 + \chi\beta_1\left(-\delta_4\lambda_1^2 + \delta_5\lambda_2 - \delta_3\right)\right)\ln(\kappa)^2.
\end{aligned} \tag{4.5}$$

Utilizing the Maple software to resolve the system (4.5), we are able to acquire the following sets of solutions:

(4.1.1) $\alpha_1 = 0,\ \beta_1 = 2,\ \lambda_2 = \dfrac{4\ln(\kappa)^2\delta_1 + \delta_4\lambda_1^2 + \delta_3}{4\ln(\kappa)^2\delta_2 + \delta_5}.$

(4.1.2) $\alpha_1 = \sqrt{-\chi}\,\ln(\kappa)\,,\ \beta_1 = 1,\ \lambda_2 = \dfrac{\ln(\kappa)^2\,\delta_1 + \delta_4\lambda_1^2 + \delta_3}{\ln(\kappa)^2\,\delta_2 + \delta_5}.$

Through the solution set (4.1.1), we can express the solutions of Eq (4.3) as follows:

$$U_{4.1.1}(\rho) = \alpha_0 + \frac{4\left(\frac{4\mu\ln(\kappa)^2\kappa^{2\rho}}{\kappa^{\rho}} - \frac{4\mu^2\ln(\kappa)^2\kappa^{2\rho}+\chi}{2\mu\,\kappa^{\rho}}\right)\mu\ln(\kappa)\,\kappa^{\rho}}{4\mu^2\ln(\kappa)^2\,\kappa^{2\rho}+\chi}. \tag{4.6}$$

Thus, based on Eqs (3.2) and (4.6), the soliton solution of Eq (2.1) may be expressed as follows:

$$u_{4.1.1}(x,y,t) = \frac{8\left(\ln(\kappa)+\frac{\alpha_0}{2}\right)\ln(\kappa)^2\,\mu^2\kappa^{-\frac{2t\left(4\ln(\kappa)^2\delta_1+\delta_4\lambda_1^2+\delta_3\right)}{4\ln(\kappa)^2\delta_2+\delta_5}+2\lambda_1 y+2x} - 2\chi\left(\ln(\kappa)-\frac{\alpha_0}{2}\right)}{4\mu^2\ln(\kappa)^2\,\kappa^{-\frac{2t\left(4\ln(\kappa)^2\delta_1+\delta_4\lambda_1^2+\delta_3\right)}{4\ln(\kappa)^2\delta_2+\delta_5}+2\lambda_1 y+2x}+\chi}. \tag{4.7}$$

By taking $\chi = \pm 4\mu^2$, the above solution can be reduced to kink and singular soliton solutions, respectively, as

$$u^{(1)}_{4.1.1}(x,y,t) = \alpha_0 - 2\tanh\left(x+\lambda_1 y - \frac{\left(4\delta_1+\delta_4\lambda_1^2+\delta_3\right)}{4\delta_2+\delta_5}t\right), \tag{4.8}$$

and

$$u^{(2)}_{4.1.1}(x,y,t) = \alpha_0 - 2\coth\left(x+\lambda_1 y - \frac{\left(4\delta_1+\delta_4\lambda_1^2+\delta_3\right)}{4\delta_2+\delta_5}t\right). \tag{4.9}$$

Through the solution set (4.1.2), we can express the solutions of Eq (4.3) as follows:

$$U_{4.1.2}(\rho) = \alpha_0 + \frac{4\left(\frac{2\mu\ln(\kappa)^2\kappa^{2\rho}}{\kappa^{\rho}} - \frac{4\mu^2\ln(\kappa)^2\kappa^{2\rho}+\chi}{4\mu\,\kappa^{\rho}} + \sqrt{-\chi}\,\ln(\kappa)\right)\mu\ln(\kappa)\,\kappa^{\rho}}{4\mu^2\ln(\kappa)^2\,\kappa^{2\rho}+\chi}. \tag{4.10}$$

Thus, based on Eqs (3.2) and (4.10), the soliton solution of Eq (2.1) may be expressed as follows:

$$u_{4.1.2}(x,y,t) = \frac{4\ln(\kappa)^2\,\mu^2\left(\ln(\kappa)+\alpha_0\right)\kappa^{2\rho} + 4\ln(\kappa)^2\,\kappa^{\rho}\sqrt{-\chi}\,\mu - \chi\left(\ln(\kappa)-\alpha_0\right)}{4\mu^2\ln(\kappa)^2\,\kappa^{2\rho}+\chi}, \tag{4.11}$$

where $\rho = -\frac{t\left(\ln(\kappa)^2\delta_1+\delta_4\lambda_1^2+\delta_3\right)}{\ln(\kappa)^2\delta_2+\delta_5} + \lambda_1 y + x$. By considering $\chi = -4\mu^2$, the above solution can be reduced to the following singular soliton solutions:

$$u^{(1)}_{4.1.2}(x,y,t) = \alpha_0 - \coth\left(x+\lambda_1 y - \frac{\left(\delta_1+\delta_4\lambda_1^2+\delta_3\right)}{\delta_2+\delta_5}t\right). \tag{4.12}$$

4.2. Analytical solutions using the Riccati equation approach

The (2+1)-dimensional gKdV equation in Eq (2.1) is solved in this subsection using the Riccati equation technique. According to the Riccati equation method, the solution to Eq (4.3) can be expressed as follows:

$$U(\rho) = \sum_{i=0}^{1} A_i R(\rho)^i = A_0 + A_1\cdot R(\rho)\,, \tag{4.13}$$

where A_0 and A_1 are the indefinite constants to be examined such that $A_1 \neq 0$.

We obtain the following system by substituting Eq (4.13) and Eq (3.9) into Eq (4.3) and then setting the coefficients of the same power to zero:

$$\begin{aligned}
&0 = (-\delta_2\lambda_2 + \delta_1) A_1 \left(2B_0^2 B_2 + B_0 B_1^2\right) + 3(-\delta_2\lambda_2 + \delta_1) A_1^2 B_0^2 + \left(\delta_4\lambda_1^2 - \delta_5\lambda_2 + \delta_3\right) A_1 B_0,\\
&0 = (-\delta_2\lambda_2 + \delta_1) A_1 \left(8B_0 B_1 B_2 + B_1^3\right) + 6(-\delta_2\lambda_2 + \delta_1) A_1^2 B_0 B_1 + \left(\delta_4\lambda_1^2 - \delta_5\lambda_2 + \delta_3\right) A_1 B_1,\\
&0 = (-\delta_2\lambda_2 + \delta_1) A_1 \left(4B_0 B_2^2 + 5B_1^2 B_2 + 2\left(2B_0 B_2 + B_1^2\right) B_2\right)\\
&\quad + 3(-\delta_2\lambda_2 + \delta_1) A_1^2 \left(2B_0 B_2 + B_1^2\right) + \left(\delta_4\lambda_1^2 - \delta_5\lambda_2 + \delta_3\right) A_1 B_2,\\
&0 = 12(-\delta_2\lambda_2 + \delta_1) A_1 B_1 B_2^2 + 6(-\delta_2\lambda_2 + \delta_1) A_1^2 B_1 B_2,\\
&0 = 6(-\delta_2\lambda_2 + \delta_1) A_1 B_2^3 + 3(-\delta_2\lambda_2 + \delta_1) A_1^2 B_2^2.
\end{aligned} \tag{4.14}$$

The following can be discovered by resolving the algebraic system of equations described above:

$$A_1 = -2B_2,\ \lambda_2 = \frac{-4B_0B_2\delta_1 + B_1^2\delta_1 + \delta_4\lambda_1^2 + \delta_3}{-4B_0B_2\delta_2 + B_1^2\delta_2 + \delta_5}.$$

According to Eq (3.10), one can find the following possible solutions:

(i) A kink soliton solution such that $-4B_0B_2 + B_1^2 > 0$:

$$\begin{aligned}
u_{4.2,1}(x,y,t) =& A_0 + B_1 - \sqrt{-4B_0B_2 + B_1^2}\\
&\times \tanh\left(\frac{\left(\left(-4B_0B_2\delta_1 + B_1^2\delta_1 + \delta_4\lambda_1^2 + \delta_3\right)t - (\lambda_1 y + x)\left(-4B_0B_2\delta_2 + B_1^2\delta_2 + \delta_5\right)\right)\sqrt{-4B_0B_2 + B_1^2}}{-8B_0B_2\delta_2 + 2B_1^2\delta_2 + 2\delta_5}\right).
\end{aligned} \tag{4.15}$$

(ii) A singular soliton solution such that $-4B_0B_2 + B_1^2 > 0$:

$$\begin{aligned}
u_{4.2,2}(x,y,t) =& A_0 + B_1 - \sqrt{-4B_0B_2 + B_1^2}\\
&\times \coth\left(\frac{\left(\left(-4B_0B_2\delta_1 + B_1^2\delta_1 + \delta_4\lambda_1^2 + \delta_3\right)t - (\lambda_1 y + x)\left(-4B_0B_2\delta_2 + B_1^2\delta_2 + \delta_5\right)\right)\sqrt{-4B_0B_2 + B_1^2}}{-8B_0B_2\delta_2 + 2B_1^2\delta_2 + 2\delta_5}\right).
\end{aligned} \tag{4.16}$$

(iii) A singular periodic solution such that $4B_0B_2 - B_1^2 > 0$:

$$\begin{aligned}
u_{4.2,3}(x,y,t) =& A_0 + B_1 + \sqrt{4B_0B_2 - B_1^2}\\
&\times \tan\left(\frac{\left(\left(-4B_0B_2\delta_1 + B_1^2\delta_1 + \delta_4\lambda_1^2 + \delta_3\right)t - (\lambda_1 y + x)\left(-4B_0B_2\delta_2 + B_1^2\delta_2 + \delta_5\right)\right)\sqrt{4B_0B_2 - B_1^2}}{-8B_0B_2\delta_2 + 2B_1^2\delta_2 + 2\delta_5}\right).
\end{aligned} \tag{4.17}$$

(iv) A singular periodic solution such that $4B_0B_2 - B_1^2 > 0$:

$$\begin{aligned}
u_{4.2,4}(x,y,t) =& A_0 + B_1 - \sqrt{4B_0B_2 - B_1^2}\\
&\times \cot\left(\frac{\left(\left(-4B_0B_2\delta_1 + B_1^2\delta_1 + \delta_4\lambda_1^2 + \delta_3\right)t - (\lambda_1 y + x)\left(-4B_0B_2\delta_2 + B_1^2\delta_2 + \delta_5\right)\right)\sqrt{4B_0B_2 - B_1^2}}{-8B_0B_2\delta_2 + 2B_1^2\delta_2 + 2\delta_5}\right).
\end{aligned} \tag{4.18}$$

(v) A rational wave solution such that $B_2 \neq 0$:

$$u_{4.2,5}(x,y,t) = A_0 - 2B_2\left(-\frac{B_1}{2B_2} - \frac{1}{\left(-\frac{t\left(\delta_4\lambda_1^2+\delta_3\right)}{\delta_5} + \lambda_1 y + x\right)B_2}\right). \tag{4.19}$$

5. Graphical simulations of the extracted solutions

This investigation represents a novel study for the (2+1)-dimensional gKdV equation (2.1), which can play an essential role in various physical phenomena. By implementing the generalized Arnous method on the considered model, we have obtained two soliton solutions as shown in (4.7) and (4.11). We have derived many other soliton solutions as shown in (4.15)–(4.19) by using the Riccati equation approach. In this part of the manuscript, we shall see the dynamical behavior of some obtained solutions by demonstrating 3D plots (see Figures 1–5). To show the effect of time on the extracted solutions, 2D representations with varying t levels are also presented (see Figures 1–5).

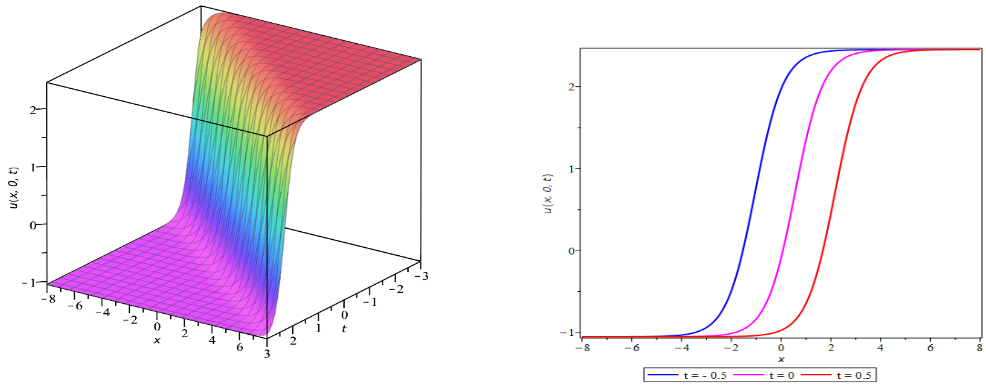

Figure 1. Graphical representation of solution (4.7) with $\alpha_0 = 0.7, \chi = 1.3, \mu = 0.4, \lambda_1 = 3, \kappa = 2.4, \delta_j = 1 (j = 1, 2, \ldots, 5)$ and $y = 0$.

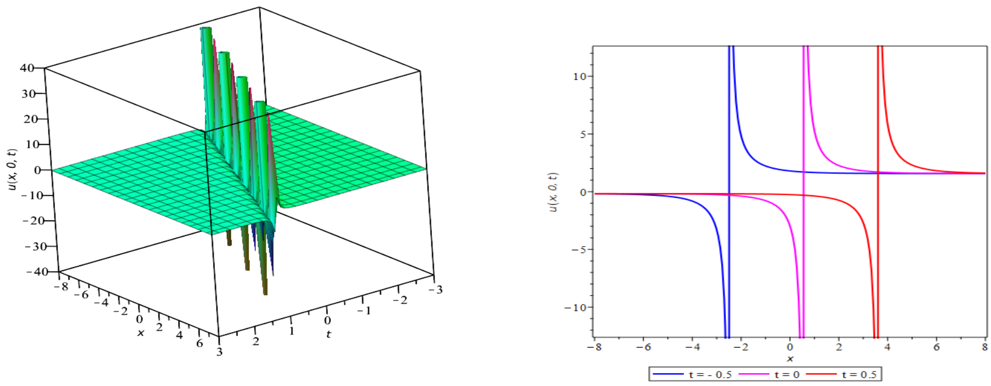

Figure 2. Graphical representation of solution (4.11) with $\alpha_0 = 0.7, \chi = -1.3, \mu = 0.4, \lambda_1 = 3, \kappa = 2.4, \delta_j = 1 (j = 1, 2, \ldots, 5)$ and $y = 0$.

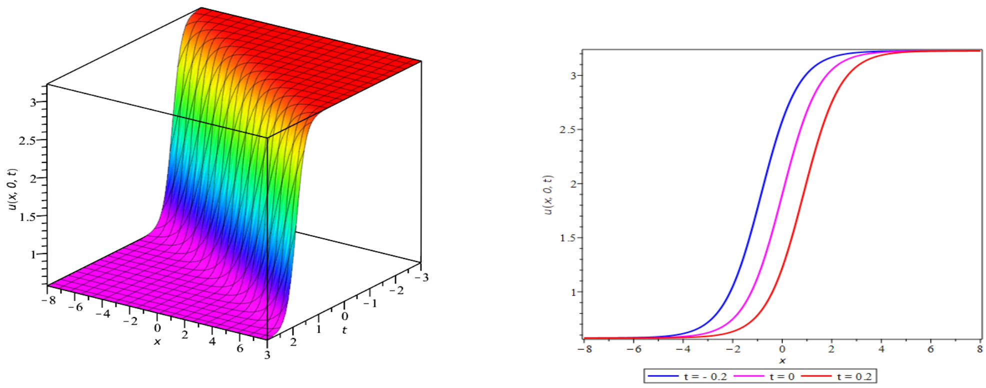

Figure 3. Graphical representation of solution (4.15) with $A_0 = 0.5, B_0 = 0.5, B_1 = 1.4, B_2 = 0.1, \lambda_1 = 3, \delta_j = 1 (j = 1, 2, \ldots, 5)$ and $y = 0$.

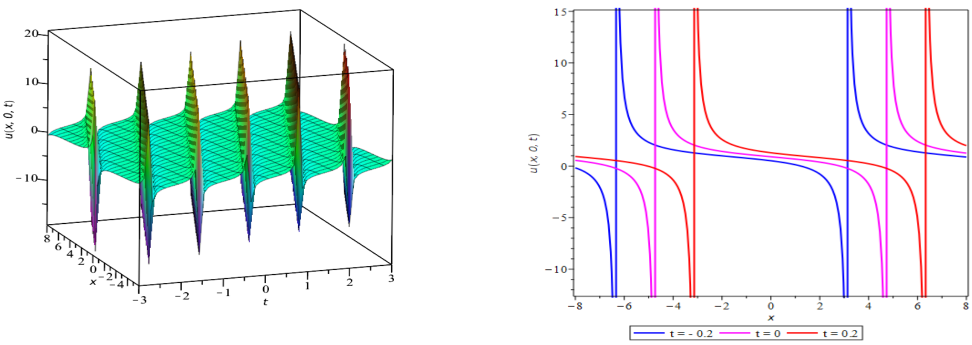

Figure 4. Graphical representation of solution (4.17) with $A_0 = 0.5, B_0 = 1.5, B_1 = 0.4, B_2 = 0.1, \lambda_1 = 3, \delta_3 = -4.08, \delta_j = 1 (j = 1, 2, 4, 5)$ and $y = 0$.

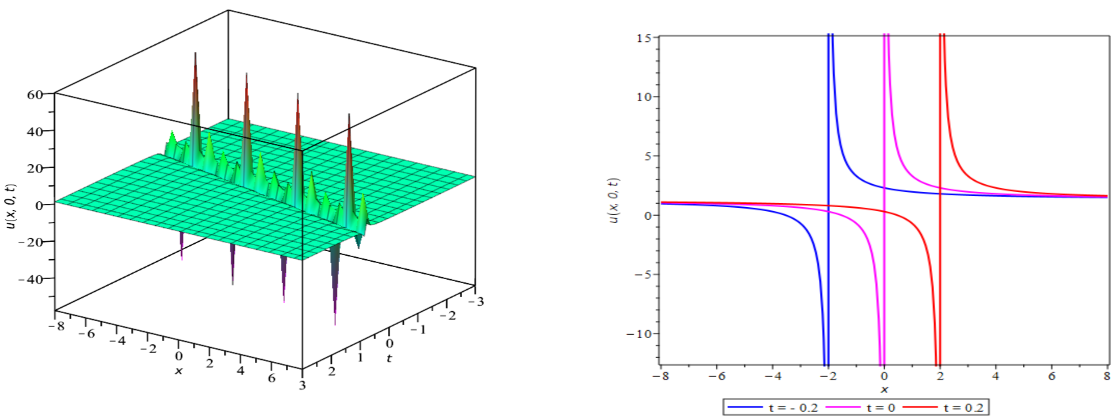

Figure 5. Graphical representation of solution (4.19) with $A_0 = 0.5, B_0 = 0.1, B_1 = 0.8, B_2 = 1.6, \lambda_1 = 3, \delta_j = 1 (j = 1, 2, \ldots, 5)$ and $y = 0$.

6. Novelty and innovations of the work

The study of solutions to the (2+1)-dimensional gKdV using the generalized Arnous technique and the Riccati equation approach provides several benefits and applications, along with a deeper understanding of the complex dynamics of the equation.

6.1. Potential benefits

The combination of the generalized Arnous technique and the Riccati equation approach allows for a more streamlined process to find the exact solutions to the (2+1)-dimensional gKdV equation. By simplifying these solutions, researchers can more easily interpret the underlying dynamics, which can be otherwise complex to analyze in higher-dimensional, nonlinear systems. These methods provide exact or nearly exact solutions, which offer a higher precision than the approximate or numerical solutions. Such exact solutions are valuable to develop benchmarks to validate numerical simulations and other analytical approximations in related studies.

6.2. More understanding of the obtained solutions

The implemented approaches lead to a more comprehensive understanding of the (2+1)-dimensional gKdV equation by enabling the discovery of various types of exact solutions, such as solitary waves, periodic waves, and even rogue wave forms. By analyzing these solutions, researchers can examine how different initial conditions or system parameters affect wave behaviors, stability, and the energy distribution. This insight contributes to a broader theoretical knowledge and provides guidance for practical implementations in physics, engineering, and beyond.

6.3. Practical applications of the solutions

The applications of exact solutions to the (2+1)-dimensional gKdV equation, which were obtained through the implemented approaches, span several advanced fields. In fluid dynamics, these solutions enhance the understanding of wave behaviors such as dispersion and breaking in shallow water, which is crucial for oceanography and environmental engineering. In plasma physics, they describe soliton propagation and wave stability, thus aiding the development of stable plasma containment methods which are important for nuclear fusion and space science. Additionally, in nonlinear optics, these solutions predict the behavior of optical pulses and solitons, thus supporting the design of efficient optical waveguides and telecommunications devices. Collectively, these applications leverage the precision of exact solutions to accurately model wave phenomena in various physical contexts.

7. Conclusions

In conclusion, this study introduced new soliton solutions for a (2+1)-dimensional gKdV problem using the generalized Arnous technique and the Riccati equation approach. The research yieled some soliton solutions, including kink and singular solitary solitons. These solutions highlight the complex behavior of the gKdV model and its relevance in explaining nonlinear wave phenomena across diverse fields, such as fluid mechanics, physics of plasmas, and wave propagation in different media. When we compared our results with those previously reported in the literature, we found that we obtained numerous soliton solutions that had not been discovered earlier. Moreover, the proposed model in Eq (2.1) had not been investigated before using the approaches implemented in this study. Additionally, the study included 2D and 3D visualizations using Maple, thus providing a deeper understanding of the solutions' structures and dynamics. The solutions and methods presented herein are crucial for ongoing research and practical applications in various scientific and engineering fields, thus providing a foundational understanding of the mechanisms governing the production and propagation of complex wave phenomena.

Author contributions

Ibrahim Alraddadi: Formal analysis, Software; Faisal Alsharif: Validation, Methodology; Sandeep Malik: Investigation, Writing–review & editing; Hijaz Ahmad: Software, Writing–review & editing; Taha Radwan: Formal analysis, Writing–review & editing; Karim K. Ahmed: Resources, Writing–review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. R. Hirota, *The direct method in soliton theory*, Cambridge University Press, 2004. http://doi.org/10.1017/CBO9780511543043
2. M. Jia, S. Y. Lou, Searching for missing D'alembert waves in nonlinear system: Nizhnik-Novikov-Veselov equation, *Chaos Soliton. Fract.*, **140** (2020), 110135. http://doi.org/10.1016/j.chaos.2020.110135
3. H. Qawaqneh, A. Altalbe, A. Bekir, K. U. Tariq, Investigation of soliton solutions to the truncated M-fractional (3+1)-dimensional Gross-Pitaevskii equation with periodic potential, *AIMS Mathematics*, **9** (2024), 23410–23433. http://doi.org/10.3934/math.20241138
4. A. M. G. Ahmed, A. Adjiri, S. Manukure, Soliton solutions and a bi-Hamiltonian structure of the fifth-order nonlocal reverse-spacetime Sasa-Satsuma-type hierarchy via the Riemann-Hilbert approach, *AIMS Mathematics*, **9** (2024), 23234–23267. http://doi.org/10.3934/math.20241130

5. B. Ren, Dynamics of a D'Alembert wave and a soliton molecule for an extended BLMP equation, *Commun. Theor. Phys.*, **73** (2021), 035003. http://doi.org/10.1088/1572-9494/abda17
6. B. Ren, J. Lin, D'Alembert wave and soliton molecule of the modified Nizhnik–Novikov–Veselov equation, *Eur. Phys. J. Plus*, **136** (2021), 123. http://doi.org/10.1140/epjp/s13360-021-01099-3
7. K. K. Ahmed, N. M. Badra, H. M. Ahmed, W. B. Rabie, Soliton solutions of generalized Kundu-Eckhaus equation with an extra-dispersion via improved modified extended tanh-function technique, *Opt. Quant. Electron.*, **55** (2023), 299. http://doi.org/10.1007/s11082-023-04599-x
8. A. S. Khalifa, W. B. Rabie, N. M. Badra, H. M. Ahmed, M. Mirzazadeh, M. S. Hashemi, et al., Discovering novel optical solitons of two CNLSEs with coherent and incoherent nonlinear coupling in birefringent optical fibers, *Opt. Quant. Electron.*, **56** (2024), 1340. https://doi.org/10.1007/s11082-024-07237-2
9. K. K. Ahmed, N. M. Badra, H. M. Ahmed, W. B. Rabie, Unveiling optical solitons and other solutions for fourth-order (2+1)-dimensional nonlinear Schrödinger equation by modified extended direct algebraic method, *J. Opt.*, (2024), 1–13. http://doi.org/10.1007/s12596-024-01690-8
10. C. Bhan, R. Karwasra, S. Malik, S. Kumar, A. H. Arnous, N. A. Shah, et al., Bifurcation, chaotic behavior, and soliton solutions to the KP-BBM equation through new Kudryashov and generalized Arnous methods, *AIMS Mathematics*, **9** (2024), 8749–8767. http://doi.org/10.3934/math.2024424
11. S. U. Rehman, J. Ahmad, Dispersive multiple lump solutions and soliton's interaction to the nonlinear dynamical model and its stability analysis, *Eur. Phys. J. D*, **76** (2022), 14. http://doi.org/10.1140/epjd/s10053-022-00351-4
12. M. Alquran, New interesting optical solutions to the quadratic-cubic Schrödinger equation by using the Kudryashov-expansion method and the updated rational sine-cosine functions, *Opt. Quant. Electron.*, **54** (2022), 666. http://doi.org/10.1007/s11082-022-04070-3
13. W. B. Rabie, K. K. Ahmed, N. M. Badra, H. M. Ahmed, M. Mirzazadeh, M. Eslami, New solitons and other exact wave solutions for coupled system of perturbed highly dispersive CGLE in birefringent fibers with polynomial nonlinearity law, *Opt. Quant. Electron.*, **56** (2024), 875. http://doi.org/10.1007/s11082-024-06644-9
14. A. S. Khalifa, N. M. Badra, H. M. Ahmed, W. B. Rabie, Retrieval of optical solitons in fiber Bragg gratings for high-order coupled system with arbitrary refractive index, *Optik*, **287** (2023), 171116. http://doi.org/10.1016/j.ijleo.2023.171116
15. L. Li, F. Yu, The fourth-order dispersion effect on the soliton waves and soliton stabilities for the cubic-quintic Gross-Pitaevskii equation, *Chaos Soliton. Fract.*, **179** (2024), 114377. http://doi.org/10.1016/j.chaos.2023.114377
16. K. K. Ahmed, H. H. Hussein, H. M. Ahmed, W. B. Rabie, W. Alexan, Analysis of the dynamical behaviors for the generalized Bogoyavlvensky-Konopelchenko equation and its analytical solutions occurring in mathematical physics, *Ain Shams Eng. J.*, **15** (2024), 103000. http://doi.org/10.1016/j.asej.2024.103000
17. L. Li, F. Yu, Q. Qin, Interaction and manipulation for non-autonomous bright soliton solution of the coupled derivative nonlinear Schrödinger equations with Riemann–Hilbert method, *Appl. Math. Lett.*, **149** (2024), 108924. https://doi.org/10.1016/j.aml.2023.108924

18. K. K. Ahmed, N. M. Badra, H. M. Ahmed, W. B. Rabie, M. Mirzazadeh, M. Eslami, et al., Investigation of solitons in magneto-optic waveguides with Kudryashov's law nonlinear refractive index for coupled system of generalized nonlinear Schrödinger's equations using modified extended mapping method, *Nonlinear. Anal.-Model.*, **29** (2024), 205–223. http://doi.org/10.15388/namc.2024.29.34070

19. A. S. Khalifa, H. M. Ahmed, N. M. Badra, W. B. Rabie, Exploring solitons in optical twin-core couplers with Kerr law of nonlinear refractive index using the modified extended direct algebraic method, *Opt. Quant. Electron.*, **56** (2024), 1060. http://doi.org/10.1007/s11082-024-06882-x

20. K. S. Nisar, O. A. Ilhan, S. T. Abdulazeez, J. Manafian, S. A. Mohammed, M. S. Osman, Novel multiple soliton solutions for some nonlinear PDEs via multiple Exp-function method, *Results Phys.*, **21** (2021), 103769. http://doi.org/10.1016/j.rinp.2020.103769

21. K. K. Ahmed, H. M. Ahmed, N. M. Badra, W. B. Rabie, Optical solitons retrieval for an extension of novel dual-mode of a dispersive non-linear Schrödinger equation, *Optik*, **307** (2024), 171835. http://doi.org/10.1016/j.ijleo.2024.171835

22. R. Alhami, M. Alquran, Extracted different types of optical lumps and breathers to the new generalized stochastic potential-KdV equation via using the Cole-Hopf transformation and Hirota bilinear method, *Opt. Quant. Electron.*, **54** (2022), 553. http://doi.org/10.1007/s11082-022-03984-2

23. A. R. Seadawy, A. Yasmeen, N. Raza, S. Althobaiti, Novel solitary waves for fractional (2+1)-dimensional Heisenberg ferromagnetic model via new extended generalized Kudryashov method, *Phys. Scr.*, **96** (2021), 125240. http://doi.org/10.1088/1402-4896/ac30a4

24. Y. S. Özkan, E. Yaşar, On the exact solutions of nonlinear evolution equations by the improved $\tan\left(\frac{\phi}{2}\right)$−expansion method, *Pramana-J. Phys.*, **94** (2020), 37. http://doi.org/10.1007/s12043-019-1883-3

25. W. X. Ma, $N-$soliton solutions and the Hirota conditions in (1+1)-dimensions, *Int. J. Nonlin. Sci. Num.*, **23** (2022), 123–133. http://doi.org/10.1515/ijnsns-2020-0214

26. B. P. Sebogodi, C. M. Khalique, Travelling wave solutions and conservation laws of the (2+1)-dimensional new generalized Korteweg–de Vries equation, *Partial Differ. Equ. Appl. Math.*, **11** (2024), 100815. http://doi.org/10.1016/j.padiff.2024.100815

27. N. Raza, A. Batool, M. Inc, New hyperbolic and rational form solutions of (2+1)-dimensional generalized Korteweg-de Vries model, *J. Ocean Eng. Sci.*, 2022, In Press. http://doi.org/10.1016/j.joes.2022.04.021

28. H. Ma, S. Yue, A. Deng, D'Alembert wave, the Hirota conditions and soliton molecule of a new generalized KdV equation, *J. Geom. Phys.*, **172** (2022), 104413. https://doi.org/10.1016/j.geomphys.2021.104413

29. R. Hirota, J. Satsuma, $N-$soliton solutions of model equations for shallow water waves, *J. Phys. Soc. Jpn.*, **40** (1976), 611–612. https://doi.org/10.1143/JPSJ.40.611

30. D. G. Crighton, Applications of KdV, In: *KdV '95*, Dordrecht, Springer, 1995. https://doi.org/10.1007/978-94-011-0017-5_2

31. N. Jannat, M. Kaplan, N. Raza, Abundant soliton-type solutions to the new generalized KdV equation via aut--Bäcklund transformations and extended transformed rational function technique, *Opt. Quant. Electron.*, **54** (2022), 466. https://doi.org/10.1007/s11082-022-03862-x

32. B. P. Sebogodi, C. M. Khalique, Travelling wave solutions and conservation laws of the (2+1)-dimensional new generalized Korteweg–de Vries equation, *Partial Differ. Equ. Appl. Math.*, **11** (2024), 100815. https://doi.org/10.1016/j.padiff.2024.100815

33. N. Raza, A. Batool, M. Inc, New hyperbolic and rational form solutions of (2+1)-dimensional generalized Korteweg-de Vries model, *J. Ocean Eng. Sci.*, 2022, In Press. https://doi.org/10.1016/j.joes.2022.04.021

34. S. Malik, S. Kumar, Pure-cubic optical soliton perturbation with full nonlinearity by a new generalized approach, *Optik*, **258** (2022), 168865. https://doi.org/10.1016/j.ijleo.2022.168865

35. S. Malik, S. Kumar, A. Biswas, Y. Yıldırım, L. Moraru, S. Moldovanu, et al., Cubic-quartic optical solitons in fiber Bragg gratings with dispersive reflectivity having parabolic law of nonlinear refractive index by Lie symmetry, *Symmetry*, **14** (2022), 2370. https://doi.org/10.3390/sym14112370

36. S. Malik, H. Almusawa, S. Kumar, A. M. Wazwaz, M. S. Osman, A (2+1)-dimensional Kadomtsev-Petviashvili equation with competing dispersion effect: Painlevé analysis, dynamical behavior and invariant solutions, *Results Phys.*, **23** (2021), 104043. https://doi.org/10.1016/j.rinp.2021.104043

37. Y. Yıldırım, A. Biswas, M. Ekici, H. Triki, O. Gonzalez-Gaxiola, A. K. Alzahrani, et al., Optical solitons in birefringent fibers for Radhakrishnan-Kundu-Lakshmanan equation with five prolific integration norms, *Optik*, **208** (2020), 164550. https://doi.org/10.1016/j.ijleo.2020.164550

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