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Research article

Global well-posedness and optimal decay rates for the n-D incompressible Boussinesq equations with fractional dissipation and thermal diffusion

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Abstract: In this paper, n-dimensional incompressible Boussinesq equations with fractional dissipation and thermal diffusion are investigated. Firstly, by applying frequency decomposition, we find that $\|(u,\theta)\|_{L^2(\mathbb{R}^n)} \to 0$, as $t \to \infty$. Secondly, by using energy methods, we can show that if the initial data is sufficiently small in $H^s(\mathbb{R}^n)$ with $s=1+\frac{n}{2}-2\alpha\,(0<\alpha<1)$, the global solutions are derived. Furthermore, under the assumption that the initial data $(u_0,\ \theta_0)$ belongs to L^p(where $1 \le p < 2$), using a more advanced frequency decomposition method, we establish optimal decay estimates for the solutions and their higher-order derivatives. Meanwhile, the uniqueness of the system can be obtained. In the case $\alpha = 0$, we obtained the regularity and decay estimate of the damped Boussinesq equation in Besov space.

Keywords: fractional Boussinesq equations; fractional dissipation; thermal diffusion; optimal decay; global well-posedness
Mathematics Subject Classification: 35A05, 35Q35, 76D03

1. Introduction

In this paper, we consider the generalized incompressible Boussinesq equations in $\mathbb{R}^n$:

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nu(-\Delta)^{\alpha}U + \nabla P = \Theta e_n, \\ \partial_t\Theta + U\cdot\nabla\Theta + \kappa(-\Delta)^{\beta}\Theta = 0, \\ \nabla\cdot U = 0, \\ U(x,0) = U_0(x), \Theta(x,0) = \Theta_0(x), \end{cases} \tag{1.1}$$

where α and β are nonnegative real parameters, $\nu \ge 0$ is the fluid kinematic viscosity coefficient, and $\kappa \ge 0$ is the thermal diffusion. The unknowns are the fluid velocity field $U = U(x,t)$ with $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, the temperature $\Theta = \Theta(x,t)$ can be understood in a physical context as a thermal variable when the $\kappa > 0$, or as a density variable when $\kappa = 0$, and the fluid pressure $P(x,t)$. We denote the vector $e_n = (0, 0, \cdots, 1)^t$. The term Θe_n in the momentum equation represents the influence of buoyancy on the motion of the fluid.

The fractional Laplacian operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is characterized by the Fourier transform, specifically,

$$(-\Delta)^\alpha \widehat{f} = \widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

As follows is the definition of the Fourier transform:

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

For the higher dimensional case $n \geq 3$, Ye [42] showed that Eq (1.1) admits a unique global classical solution as long as $\alpha \geq \frac{1}{2} + \frac{n}{4}, \beta > 0$. Moreover, reader's can also find many other results related to the higher dimensional cases in [18, 35] and references cited therein.

In the following, we focus on the 2D case. According to the value range of α and β, it can be classified into three distinct categories space (see [14]): The supercritical case when $\alpha + \beta < 1$, the critical case when $\alpha + \beta = 1$, and the subcritical case when $\alpha + \beta > 1$.

The Boussinesq system is usually referred to as (1.1) for $\alpha = \beta = 1$. The Boussinesq system is widely used in the fields of atmospheric science and oceanic turbulence, where rotation and stratification are crucial factors (see, e.g., [24]).The work of Tao and Wu [29] was able to establish the stability and the enhanced dissipation phenomenon for the linearized 2D Boussinesq equations with only vertical dissipation, Shang and Xu in [26] examined the stability and the decay of the corresponding linearized systems of 3D Boussinesq equations with horizontal viscosity and horizontal thermal diffusion. The stabilizing effect of the temperature on the buoyancy-driven fluids and the stability of the hydrostatic equilibrium were discovered for several partially dissipated 2D Boussinesq systems in [20, 25]. Important progress has been made on the stability and large-time behavior in [6, 19, 28, 31]. Therefore, the Boussinesq system has been extensively studied in the past few years; see [2, 3, 5, 7, 8, 22, 43] and references therein.

In the subcritical case, Xu in [34] investigated the subcritical cases where $\alpha + \beta > 2$ with $\alpha \geq 1$, and established the existence, uniqueness, and regularity of 2D fractional Boussinesq equations. Miao and Xue in [23] proved global well-posedness results for rough initial data when $\frac{6-\sqrt{6}}{4} < \alpha < 1$, $1 - \alpha < \beta < \min\left\{\frac{7+2\sqrt{6}\alpha-2}{5}, \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, 2 - 2\alpha\right\}$. Constantin and Vicol in [4] established the global regularity, when $\beta > \frac{2}{2+\alpha}$. Yang, Jiu and Wu in [36] verified the global well-posedness when $\beta > 1 - \frac{\alpha}{2}, \beta \geq \frac{2+\alpha}{3}$, $\beta > \frac{10-5\alpha}{10-4\alpha}$. Ye and Xu in [37], Ye and Xu in [38], Ye, Xu, and Xue in [39], Ye in [40], Wu, Xu, and Ye in [32], Zhou, Li, Shang, Wu, and Yuan in [44], and other literatures have made significant advancements.

The critical case is generally more challenging than the subcritical case. Hmidi, Keraani, and Rousset in [10, 11] studied two specific critical cases where $\alpha + \beta = 1$ with $\alpha = 0$ and $\beta = 1$, or with $\alpha = 1$ and $\beta = 0$, and they established the global regularity for the 2D Boussinesq equations with fractional diffusion in both cases. Jiu, Miao, Wu, and Zhang in [14]; verified the global regularity for the general case with $\alpha + \beta = 1$ and $0.9132 \approx \alpha_0 < \alpha < 1$. Stefanov and Wu in [27] advanced the previous work of [14]; they proved the global existence and smoothness of the classical solution. The conditions are as follows: $\frac{\sqrt{1777}-23}{24} = 0.798103. < \alpha < 1$, $\beta > 0$ and $\alpha + \beta = 1$. The range of α to

$0.7692 \approx \frac{10}{13} < \alpha < 1$ is further extended in [33] by Wu, Xu, Xue, and Ye. Some other developments are in [9, 12, 13, 41].

For the supercritical case, Jiu, Wu, and Yang in [15] consider the 2D incompressible Boussinesq equations with fractional dissipation; they verified the eventual regularity for $\alpha + \beta < 1$ and $0.9132 \approx \frac{23-\sqrt{145}}{12} < \alpha < 1$. Wu, Xu, Xue, and Ye in [33] proved the final regularity of the Leray-Hopf-type weak solution of the Boussinesq equation with supercritical dissipative and $0.7692 \approx \frac{10}{13} < \alpha < 1$. Some global regularity conclusions for supercritical equations can be obtained in [21].

We assume

$$u^{(0)} = (0,0,0),\ \theta^{(0)} = x_n,\ p^{(0)} = \frac{1}{2}x_n^2. \tag{1.2}$$

Then the perturbation (u, θ, p) with

$$u = U - u^{(0)},\ \theta = \Theta - \theta^{(0)},\ p = P - p^{(0)}. \tag{1.3}$$

Then, (u, θ, p) satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^{\alpha} u + \nabla p = \theta e_n, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^{\beta} \theta = -u_n, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x). \end{cases} \tag{1.4}$$

Our first result can be stated as follows.

Theorem 1.1. *Consider (1.4) with $\nu > 0$ and $\kappa > 0$. Assume the initial data $(u_0, \theta_0) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $n \geq 2$. Then (1.4) has a global weak solution (u, θ) that satisfies the following property:*

$$\lim_{t \to \infty} (\|u(t)\|_{L^2(\mathbb{R}^n)} + \|\theta(t)\|_{L^2(\mathbb{R}^n)}) = 0. \tag{1.5}$$

In the paper we focus on the global well-posedness and optimal decay of solutions of (1.4) with fractional dissipations. For the sake of simplicity, we set $\nu = \kappa = 1$ in (1.4). We shall consider the case with $0 \leq \alpha = \beta < 1$. In particular, we investigate the following Cauchy problem:

$$\begin{cases} \partial_t u + u \cdot \nabla u + (-\Delta)^{\alpha} u + \nabla p = \theta e_n, \\ \partial_t \theta + u \cdot \nabla \theta + (-\Delta)^{\alpha} \theta = -u_n, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x), \end{cases} \tag{1.6}$$

where $0 \leq \alpha < 1$. When $\alpha = 0$, (1.6) is reduced into the n-dimensional incompressible Boussinesq system with damping

$$\begin{cases} \partial_t u + u \cdot \nabla u + u + \nabla p = \theta e_n, \\ \partial_t \theta + u \cdot \nabla \theta + \theta = -u_n, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x). \end{cases} \tag{1.7}$$

The following is our second result.

Theorem 1.2. *Let $0 < \alpha < 1$ and $s \geq 1 + \frac{n}{2} - 2\alpha$. Assume that the initial data $(u_0, \theta_0) \in H^s(\mathbb{R}^n)$, $\nabla \cdot u_0 = 0$ and $n \geq 2$. Then there exists a constant $\varepsilon > 0$ such that, if*

$$\|u_0\|_{H^s(\mathbb{R}^n)} + \|\theta_0\|_{H^s(\mathbb{R}^n)} \leq \varepsilon, \tag{1.8}$$

then (1.6) has a global solution

$$(u,\theta) \in L^{\infty}([0,\infty); H^{s}(\mathbb{R}^n)) \cap L^{2}([0,\infty); H^{s+\alpha}(\mathbb{R}^n)). \tag{1.9}$$

In addition, for any $t > 0$,

$$\|u(t)\|^2_{H^s(\mathbb{R}^n)} + \|\theta(t)\|^2_{H^s(\mathbb{R}^n)} + \int_0^t \|\Lambda^{\alpha} u(\tau)\|^2_{H^s(\mathbb{R}^n)} + \|\Lambda^{\alpha}\theta(\tau)\|^2_{H^s(\mathbb{R}^n)} d\tau \le C\varepsilon^2. \tag{1.10}$$

Moreover, for any $0 \le m < s$,

$$\lim_{t\to\infty} (1+t)^{\frac{m}{2\alpha}} (\|\Lambda^m u(t)\|_{L^2(\mathbb{R}^n)} + \|\Lambda^m \theta(t)\|_{L^2(\mathbb{R}^n)}) = 0. \tag{1.11}$$

Furthermore, suppose that $(u_0, \theta_0) \in L^p(\mathbb{R}^n)$ *with* $1 \le p < 2$. *Then*

$$\|\Lambda^m u(t)\|^2_{L^2(\mathbb{R}^n)} + \|\Lambda^m \theta(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{m}{\alpha} - \frac{n}{\alpha}(\frac{1}{p} - \frac{1}{2})}. \tag{1.12}$$

When $s \ge 1 + \frac{n}{2} - \alpha$, *the solution derived above is unique.*

Our third result is focused on the regularity properties of the damped Boussinesq equations (1.7) with $\alpha = 0$.

Theorem 1.3. *Suppose the initial data* $(u_0, \theta_0) \in B^s_{2,1}(\mathbb{R}^n)$, $s \ge \frac{n}{2} + 1$ *with* $n \ge 2$ *and* $\nabla \cdot u_0 = 0$. *Then there exists a constant* $\epsilon > 0$ *such that, if*

$$\|u_0\|_{B^s_{2,1}} + \|\theta_0\|_{B^s_{2,1}} \le \epsilon, \tag{1.13}$$

then (1.7) has a unique global solution

$$(u,\theta) \in L^{\infty}([0,\infty); B^s_{2,1}) \cap L^2([0,\infty); B^s_{2,1}). \tag{1.14}$$

In addition, for any $t > 0$,

$$\|u(t)\|^2_{B^s_{2,1}} + \|\theta(t)\|^2_{B^s_{2,1}} + \int_0^t \|u(\tau)\|^2_{B^s_{2,1}} + \|\theta(\tau)\|^2_{B^s_{2,1}} d\tau \le C\epsilon^2. \tag{1.15}$$

Furthermore, for any $0 \le m < s$,

$$\|\Lambda^m u(t)\|_{L^2(\mathbb{R}^n)} + \|\Lambda^m \theta(t)\|_{L^2(\mathbb{R}^n)} \le C\epsilon^{\frac{m}{s}} e^{\frac{-s+m}{-s} t}. \tag{1.16}$$

2. Preliminary

2.1. Littlewood-Paley decomposition and Besov spaces

The purpose of this section is to introduce some basic knowledge of the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces, and some useful properties (for more details, please see [1]).

Let B and C represent the ball $\left\{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\right\}$ and the annulus $\left\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\}$, respectively. There exist radial functions χ and φ, both of which take values in the interval $[0,1]$, and which belong to $C_0^\infty(B)$ and $C_0^\infty(C)$, respectively, such that

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1,$$

$$\forall \xi \in \mathbb{R}^n, \chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1.$$

Let $u \in \mathcal{S}'(\mathbb{R})$ with $\mathcal{S}'$ being the set of tempered distributions. The nonhomogeneous dyadic blocks Δ_j are characterized by the following definition:

$$\Delta_j u = 0, \text{ if } \; j \le -2, \textit{ and } \; \Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^n} \widetilde{h}(y)u(x-y)dy,$$

$$\Delta_j u = \varphi(2^{-j}D)u := 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x-y)dy, \text{ if } \; j > 0$$

where $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$. Now we will define the low-frequency cut-off operator $\mathcal{S}_j$

$$\mathcal{S}_j u := \sum_{j' \le j-1} \Delta_{j'} u.$$

Lemma 2.1. (Bernstein inequality) *Let B denote a ball and C denote an annulus. There exists a constant C such that, for any nonnegative integer k, any couple $(p,q) \in [1,\infty]$ with $p \le q$, and any function $f \in L^p(\mathbb{R}^n)$, we have*

$$Supp\ \widehat{f} \subset \lambda B \Rightarrow \|D^k f\|_{L^q} \overset{def}{=} \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^q} \le C^{k+1}\lambda^{k+n(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$

$$Supp\ \widehat{f} \subset \lambda C \Rightarrow C^{-k-1}\lambda^k\|f\|_{L^p} \le \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \le C^{k+1}\lambda^k\|f\|_{L^p}.$$

Next we recall the definitions of the nonhomogeneous Besov spaces.

Definition 1. *Assume $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, the nonhomogeneous Besov space $B^s_{p,r}$ consists of all tempered distributions f such that*

$$\|f\|_{B^s_{p,r}} := \left\|(2^{js}\|\Delta_j f\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} < \infty.$$

When $p = r = 2$, we have $B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, where

$$\|f\|_{H^s(\mathbb{R}^n)} \triangleq \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Lemma 2.2. *Assume $s \in \mathbb{R}$ and $1 \le p, r \le \infty, f \in \mathcal{S}'$. Consequently, the following properties are valid: (1) Embedding: For $1 \le p, \bar{p} \le \infty$ and $1 \le r, \bar{r} \le \infty$, there holds*

$$B^s_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{\bar{p}})}_{\bar{p},\bar{r}}(\mathbb{R}^n),$$

$$B^s_{p,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n),$$
$$L^p(\mathbb{R}^n) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^n).$$

(2) Interpolation: For any $s_1 < s_2$ *and* $0 < \theta < 1$*, one has*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|f\|^\theta_{B^{s_1}_{p,r}} \|f\|^{1-\theta}_{B^{s_2}_{p,r}},$$
$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{s_2 - s_1}\Big(\frac{1}{\theta} + \frac{1}{1-\theta}\Big)\|f\|^\theta_{B^{s_1}_{p,\infty}} \|f\|^{1-\theta}_{B^{s_2}_{p,\infty}}.$$

Lemma 2.3. *Let* $\sigma \in \mathbb{R}$*,* $1 \le r \le \infty$*, and* $1 \le p \le p_1 \le \infty$*. Let* v *be a vector field over* $\mathbb{R}$*. Assume that*

$$\sigma > -n\ min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\} \quad or \quad \sigma > -1 - n\ min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\}\ \ if\ \nabla\cdot v = 0.$$

Define $R_j \triangleq [v\cdot\nabla, \Delta_j]f$*. There exists a constant* C*, depending continuously on* p*,* p_1*,* σ *and* d*, such that*

$$\|(2^{js}\|R_j\|_{L^p})_j\|_{\ell^r} \le C\|\nabla v\|_{B^{\frac{n}{p}}_{p_1,\infty}\cap L^\infty}\|f\|_{B^\sigma_{p,r}},\ if\ \sigma < 1 + \frac{n}{p_1}.$$

Further, if $\sigma > 0$ *(or* $\sigma > -1$*, if* $\nabla\cdot v = 0$*) and* $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$ *, then*

$$\|(2^{js}\|R_j\|_{L^p})_j\|_{\ell^r} \le C\left(\|\nabla v\|_{L^\infty}\|f\|_{B^\sigma_{p,r}} + \|\nabla f\|_{L^{p_2}}\|\nabla v\|_{B^{\sigma-1}_{p_1,r}}\right),$$

where

$$[v\cdot\nabla, \Delta_j]f = v\cdot\nabla(\Delta_j f) - \Delta_j(v\cdot\nabla f).$$

2.2. Interpolation and embedding in Sobolev spaces

Lemma 2.4. ([1]) *Let* $s \in \mathbb{R}$ *and* $\dot{H}^s(\mathbb{R}^n)$ *be the homogeneous Sobolev space with the definition*

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi < \infty.$$

If $k \in [0, \frac{n}{2})$*, then the space* $\dot{H}^k(\mathbb{R}^n)$ *is continuously embedded in* $L^{\frac{2n}{n-2k}}(\mathbb{R}^n)$*, namely*

$$\|u\|_{L^{\frac{2n}{n-2k}}} \le C\|\Lambda^k u\|_{L^2}.$$

Lemma 2.5. ([16, 17]). *Let* $s > 0$*,* $1 < p < \infty$*,* $f \in W^{1,p_1} \cap W^{s,p_2}$*,* $g \in L^{p_2} \cap W^{s,p_1}$ *, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C\left(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}\right),$$
$$\|\Lambda^s(fg)\|_{L^p} \le C\left(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}\right)$$

for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$*.*

Lemma 2.6. *Let* $s_0 < s < s_1$*. Then* $\dot{H}^{s_0} \cap \dot{H}^{s_1}$ *is included in* $\dot{H}^s$*, and*

$$\|\Lambda^s u\|_{L^2} \le \|\Lambda^{s_0}u\|^\theta_{L^2}\|\Lambda^{s_1}u\|^{1-\theta}_{L^2}$$

where $\theta \in [0, 1]$ *and* s*,* s_0*,* s_1 *satisfy*

$$s = \theta s_0 + (1-\theta)s_1.$$

Lemma 2.7. *If* $f \in L^p(\mathbb{R}^n)$*,* $1 \le p \le 2$*, and* $\frac{1}{p} + \frac{1}{\bar{p}} = 1$*, then* $\widehat{f} \in L^{\bar{p}}(\mathbb{R}^n)$ *and it satisfies*

$$\|\widehat{f}\|_{L^{\bar{p}}} \le C\|f\|_{L^p}.$$

3. Proof of the main results

Firstly, dotting (1.4) with (u, θ), we have

$$\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + 2\nu\|\Lambda^\alpha u\|_{L^2}^2 + 2\kappa\|\Lambda^\beta \theta\|_{L^2}^2 = 0, \tag{3.1}$$

integrating it in time yields

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{3.2}$$

Furthermore, by applying the Fourier transformation to (1.4), we conclude that

$$\partial_t \widehat{u}(\xi, t) + \nu|\xi|^{2\alpha}\widehat{u}(\xi, t) = \widehat{\theta}(\xi, t)e_n - \widehat{\nabla p}(\xi, t) - (\widehat{u \cdot \nabla)u}(\xi, t) \triangleq L_1(\xi, t), \tag{3.3}$$

$$\partial_t \widehat{\theta}(\xi, t) + \kappa|\xi|^{2\beta}\widehat{\theta}(\xi, t) = \widehat{-u_n}(\xi, t) - (\widehat{u \cdot \nabla)\theta}(\xi, t) \triangleq L_2(\xi, t). \tag{3.4}$$

By calculating the above equations, we obtain

$$\widehat{u}(\xi, t) = \widehat{u_0}(\xi)e^{-\nu|\xi|^{2\alpha}t} + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}L_1(\xi, t)d\tau, \tag{3.5}$$

$$\widehat{\theta}(\xi, t) = \widehat{\theta_0}(\xi)e^{-\kappa|\xi|^{2\beta}t} + \int_0^t e^{-\kappa|\xi|^{2\beta}(t-\tau)}L_2(\xi, t)d\tau. \tag{3.6}$$

Due to the simple fact that $\nabla \cdot u = 0$, we have

$$|(\widehat{u \cdot \nabla)u}(\xi, t)| = |\nabla \cdot \widehat{(u \otimes u)}(\xi, t)| \le |\xi|\|u \otimes u\|_{L^1} \le |\xi|\|u\|_{L^2}^2. \tag{3.7}$$

Similarly,

$$|(\widehat{u \cdot \nabla)\theta}(\xi, t)| \le |\xi|\|u\|_{L^2}\|\theta\|_{L^2}. \tag{3.8}$$

Applying the divergence operator to the velocity equation in (1.4), we obtained

$$\begin{aligned}\nabla p &= (-\Delta)^{-1}\nabla\nabla \cdot (\theta e_n - u \cdot \nabla u)\\ &= (-\Delta)^{-1}\nabla\nabla \cdot (\theta e_n) - (-\Delta)^{-1}\nabla\nabla \cdot \nabla \cdot (u \otimes u).\end{aligned} \tag{3.9}$$

By (3.7), we obtain

$$|\widehat{\nabla p}(\xi, t)| \le |\widehat{\theta}| + |\xi|\|u \otimes u\|_{L^1} \le |\widehat{\theta}| + |\xi|\|u\|_{L^2}^2. \tag{3.10}$$

Invoking these estimates, it leads to

$$\begin{aligned}|\widehat{u}(\xi, t)| &\le |\widehat{u_0}(\xi)|e^{-\nu|\xi|^{2\alpha}t} + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}|L_1(\xi, t)|d\tau,\\ &\le |\widehat{u_0}(\xi)| + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}C\left(|\xi|\|u(\tau)\|_{L^2}^2 + |\widehat{\theta}(\xi, \tau)|\right)d\tau\\ &\le |\widehat{u_0}(\xi)| + C|\xi|\int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}\left(\|u(\tau)\|_{L^2}^2 + \|\theta(\tau)\|_{L^2}^2\right)d\tau + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}|\widehat{\theta}|d\tau\\ &\le |\widehat{u_0}(\xi)| + C|\xi|(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2)\int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}d\tau + C\int_0^t |\widehat{\theta}|d\tau\end{aligned}$$

$$\leq |\widehat{u_0}(\xi)| + Ct|\xi| + C\int_0^t |\widehat{\theta}| d\tau. \tag{3.11}$$

The other term can be similarly bounded,

$$|\widehat{\theta}(\xi,t)| \leq |\widehat{\theta_0}(\xi)| + Ct|\xi| + C\int_0^t |\widehat{u}| d\tau. \tag{3.12}$$

By employing the Plancherel theorem in conjunction with the Fourier splitting technique, we obtain

$$\begin{aligned} \|\Lambda^\alpha u\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\widehat{u}(\xi,t)|^2 d\xi \\ &\geq \int_{|\xi|\geq r_1} |\xi|^{2\alpha} |\widehat{u}(\xi,t)|^2 d\xi \\ &\geq r_1^{2\alpha} \int_{|\xi|\geq r_1} |\widehat{u}(\xi,t)|^2 d\xi \\ &= r_1^{2\alpha} \left(\|u\|_{L^2}^2 - \int_{|\xi|\leq r_1} |\widehat{u}|^2 d\xi \right) \end{aligned} \tag{3.13}$$

and

$$\|\Lambda^\alpha \theta\|_{L^2}^2 \geq r_2^{2\beta} \left(\|\theta\|_{L^2}^2 - \int_{|\xi|\leq r_2} |\widehat{\theta}|^2 d\xi \right). \tag{3.14}$$

By using Fubini's theorem, the integral terms on the right-hand side of (3.13) and (3.14) can be dominated

$$\begin{aligned} \int_{|\xi|\leq r_1} |\widehat{u}|^2 d\xi &\leq \int_{|\xi|\leq r_1} \left(|\widehat{u_0}(\xi)| + Ct|\xi| + C\int_0^t |\widehat{\theta}| d\tau \right)^2 d\xi \\ &\leq \int_{|\xi|\leq r_1} |\widehat{u_0}(\xi)|^2 d\xi + Cr_1^{n+2}(1+t)^2 + C\int_{|\xi|\leq r_1} \left(\int_0^t |\widehat{\theta}| \cdot 1 d\tau \right)^2 d\xi \\ &\leq \int_{|\xi|\leq r_1} |\widehat{u_0}(\xi)|^2 d\xi + Cr_1^{n+2}(1+t)^2 + Ct\int_{|\xi|\leq r_1} \int_0^t |\widehat{\theta}|^2 d\tau d\xi \\ &\leq \int_{|\xi|\leq r_1} |\widehat{u_0}(\xi)|^2 d\xi + Cr_1^{n+2}(1+t)^2 + Ct\int_0^t \int_{|\xi|\leq r_1} |\widehat{\theta}|^2 d\xi d\tau \\ &\leq \int_{|\xi|\leq r_1} |\widehat{u_0}(\xi)|^2 d\xi + Cr_1^{n+2}(1+t)^2 + Ct\int_0^t \int_{\mathbb{R}^n} |\widehat{\theta}|^2 d\xi d\tau \\ &\leq \int_{|\xi|\leq r_1} |\widehat{u_0}(\xi)|^2 d\xi + Cr_1^{n+2}(1+t)^2 + Ct\int_0^t \|\theta\|_{L^2}^2 d\tau. \end{aligned} \tag{3.15}$$

Similarly,

$$\int_{|\xi|\leq r_2} |\widehat{\theta}|^2 d\xi \leq \int_{|\xi|\leq r_2} |\widehat{\theta_0}(\xi)|^2 d\xi + Cr_2^{n+2}(1+t)^2 + Ct\int_0^t \|u\|_{L^2}^2 d\tau. \tag{3.16}$$

Substituting (3.13)–(3.16) into (3.9), it yields

$$
\begin{aligned}
&\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + 2\nu r_1^{2\alpha}\|u\|_{L^2}^2 + 2\kappa r_2^{2\beta}\|\theta\|_{L^2}^2 \\
&\le 2\nu r_1^{2\alpha}\left(\int_{|\xi|\le r_1} |\widehat{u}(\xi)|^2 d\xi\right) + 2\kappa r_2^{2\beta}\left(\int_{|\xi|\le r_2} |\widehat{\theta}(\xi)|^2 d\xi\right) \\
&\le 2\nu r_1^{2\alpha}\int_{|\xi|\le r_1} |\widehat{u_0}(\xi)|^2 d\xi + 2\kappa r_2^{2\beta}\int_{|\xi|\le r_2} |\widehat{\theta_0}(\xi)|^2 d\xi + C\left(r_1^{2\alpha+n+2} + r_2^{2\alpha+n+2}\right)(1+t)^2 \\
&\quad + Ctr_1^{2\alpha}\int_0^t \|\theta\|_{L^2}^2 d\tau + Ctr_2^{2\beta}\int_0^t \|u\|_{L^2}^2 d\tau \\
&\le 2\nu r_1^{2\alpha}\int_{|\xi|\le r_1} |\widehat{u_0}(\xi)|^2 d\xi + 2\kappa r_2^{2\beta}\int_{|\xi|\le r_2} |\widehat{\theta_0}(\xi)|^2 d\xi + C\left(r_1^{2\alpha+n+2} + r_2^{2\alpha+n+2}\right)(1+t)^2 \\
&\quad + C\left(r_1^{2\alpha} + r_2^{2\beta}\right)(1+t)^2 \\
&\le 2\nu r_1^{2\alpha}\int_{|\xi|\le r_1} |\widehat{u_0}(\xi)|^2 d\xi + 2\kappa r_2^{2\beta}\int_{|\xi|\le r_2} |\widehat{\theta_0}(\xi)|^2 d\xi + C\left(r_1^{2\alpha+n+2} + r_2^{2\alpha+n+2}\right)(1+t)^2. \qquad (3.17)
\end{aligned}
$$

Let

$$
r_1 = \left(\frac{k}{2\nu(1+t)\ln(1+t)}\right)^{\frac{1}{2\alpha}}, \quad r_2 = \left(\frac{k}{2\kappa(1+t)\ln(1+t)}\right)^{\frac{1}{2\beta}} \qquad (3.18)
$$

for some $k > 1 + \frac{n+2}{\min\{2\alpha,2\beta\}}$. Then, we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \frac{k}{(1+t)\ln(1+t)}\left(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) \\
&\le \frac{k}{(1+t)\ln(1+t)}\left(\int_{|\xi|\le\max\{r_1,r_2\}} |\widehat{u_0}(\xi)|^2 + |\widehat{\theta_0}(\xi)|^2 d\xi\right) \\
&\quad + C\left((\frac{k}{2\nu(1+t)\ln(1+t)})^{1+\frac{n+2}{2\alpha}} + (\frac{k}{2\kappa(1+t)\ln(1+t)})^{1+\frac{n+2}{2\beta}}\right)(1+t)^2. \qquad (3.19)
\end{aligned}
$$

Multiplying (3.19) by $\ln^k(1+t)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\ln^k(1+t)(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2)\right) \\
&\le \frac{k\ln^{k-1}(1+t)}{1+t}\left(\int_{|\xi|\le\max\{r_1,r_2\}} |\widehat{u_0}(\xi)|^2 + |\widehat{\theta_0}(\xi)|^2 d\xi\right) \\
&\quad + C\ln^k(1+t)\left((\frac{k}{2\nu(1+t)\ln(1+t)})^{1+\frac{n+2}{2\alpha}} + (\frac{k}{2\kappa(1+t)\ln(1+t)})^{1+\frac{n+2}{2\beta}}\right)(1+t)^2 \\
&\triangleq \mathrm{J}_1(t) + \mathrm{J}_2(t) + \mathrm{J}_3(t). \qquad (3.20)
\end{aligned}
$$

Integrating (3.20) in $[0,t]$, we obtain

$$
\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \le \ln^{-k}(1+t)\left(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2\right) + \ln^{-k}(1+t)\int_0^t \mathrm{J}_1(\tau)d\tau
$$

$$+\ln^{-k}(1+t)\int_0^t \mathrm{J}_2(\tau)d\tau + \ln^{-k}(1+t)\int_0^t \mathrm{J}_3(\tau)d\tau. \tag{3.21}$$

Naturally,

$$\lim_{t\to\infty} \ln^{-k}(1+t)\left(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2\right) = 0. \tag{3.22}$$

For the second term on the right-hand side of (3.21), we have

$$\int_0^t \mathrm{J}_1(\tau)d\tau = \left(\int_{|\xi|\le \max\{r_1,r_2\}} |\widehat{u_0}(\xi)|^2 + |\widehat{\theta_0}(\xi)|^2 d\xi\right)(\ln^k(1+t)-1). \tag{3.23}$$

When $t\to\infty$, it is easy to show that r_1 and r_2 will converge towards zero. By using Theorem 1.1, we have

$$\lim_{t\to\infty}\int_{|\xi|\le \max\{r_1,r_2\}} \left(|\widehat{u_0}(\xi)|^2 + |\widehat{\theta_0}(\xi)|^2\right)d\xi = 0. \tag{3.24}$$

This indicates that

$$\lim_{t\to\infty} \ln^{-k}(1+t)\int_0^t \mathrm{J}_1(\tau)d\tau = 0. \tag{3.25}$$

For the third term on the right-hand side of (3.21),

$$\begin{aligned}
\int_0^t \mathrm{J}_2(\tau)d\tau &= C\int_0^t \ln^k(1+\tau)(1+\tau)^2\left(\frac{k}{2\nu(1+\tau)\ln(1+\tau)}\right)^{1+\frac{n+2}{2\alpha}} d\tau\\
&\le C\int_0^t \frac{\ln(1+\tau)^{k-\frac{n+2}{2\alpha}-1}}{(1+\tau)^{\frac{n+2}{2\alpha}-1}} d\tau\\
&\le C\int_0^t \frac{\ln(1+\tau)^{k-\frac{n+2}{2\alpha}-1}}{1+\tau} d\tau\\
&= C(\ln(1+\tau)^{k-\frac{n+2}{2\alpha}}-1),
\end{aligned} \tag{3.26}$$

then

$$\lim_{t\to\infty} \ln^{-k}(1+t)\int_0^t \mathrm{J}_2(\tau)d\tau = 0. \tag{3.27}$$

Similar to J_2, one gets

$$\lim_{t\to\infty} \ln^{-k}(1+t)\int_0^t \mathrm{J}_3(\tau)d\tau = 0. \tag{3.28}$$

Inserting (3.22), (3.25), (3.27), and (3.28) into (3.21), we finally obtain

$$\lim_{t\to\infty} \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 = 0. \tag{3.29}$$

As a result, Theorem 1.1 is proved.

3.1. Proof of Theorem 1.2

Step 1. A prior estimation and existence. By using the basic energy estimate of (1.6),

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\Lambda^\alpha \theta(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{3.30}$$

Applying Λ^s to both sides of (1.6) and then dotting the results with $(\Lambda^s u, \Lambda^s\theta)$, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2) + (\|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+\alpha}\theta\|_{L^2}^2) = I_1 + I_2, \tag{3.31}$$

where we have used the following facts:

$$\begin{aligned}
&\int_{\mathbb{R}^n}(u\cdot\nabla\Lambda^s u)\cdot\Lambda^s u dx = 0, \ \int_{\mathbb{R}^n}(u\cdot\nabla\Lambda^s\theta)\cdot\Lambda^s\theta dx = 0,\\
&\int \Lambda^s\theta e_n\cdot\Lambda^s u dx - \int \Lambda^s u_n\cdot\Lambda^s\theta dx = 0,
\end{aligned} \tag{3.32}$$

and

$$I_1 = -\int_{\mathbb{R}^n}[\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u]\cdot\Lambda^s u dx, \tag{3.33}$$

$$I_2 = -\int_{\mathbb{R}^n}[\Lambda^s(u\cdot\nabla\theta) - u\cdot\nabla\Lambda^s\theta]\cdot\Lambda^s\theta dx. \tag{3.34}$$

Using Hölder's inequality, Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}
I_1 &\le \|\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\lesssim \|\Lambda^s u\|_{L^{\frac{2n}{n-2\alpha}}}\|\nabla u\|_{L^{\frac{n}{\alpha}}}\|\Lambda^s u\|_{L^2} + \|\nabla\Lambda^{s-1}u\|_{L^{\frac{2n}{n-2\alpha}}}\|\nabla u\|_{L^{\frac{n}{\alpha}}}\|\Lambda^s u\|_{L^2}\\
&\lesssim \|\Lambda^{s+\alpha}u\|_{L^2}^2\|\Lambda^s u\|_{L^2}\\
&\lesssim \|\Lambda^\alpha u\|_{H^s}^2\|u\|_{H^s}
\end{aligned} \tag{3.35}$$

with $s = 1 + \frac{n}{2} - 2\alpha$.
Similarly,

$$I_2 \lesssim (\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\alpha\theta\|_{H^s}^2)\|\theta\|_{H^s}. \tag{3.36}$$

Inserting (3.35) and (3.36) in (3.32) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2) + (\|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+\alpha}\theta\|_{L^2}^2)\\
&\quad\lesssim (\|u\|_{H^s} + \|\theta\|_{H^s})(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\alpha\theta\|_{H^s}^2).
\end{aligned} \tag{3.37}$$

Integrating (3.37) over $[0, t]$ and combining it with (3.30), we have

$$\begin{aligned}
&\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + 2\int_0^t(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\alpha\theta\|_{H^s}^2)d\tau\\
&\quad\lesssim (\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2) + 2\int_0^t(\|u\|_{H^s} + \|\theta\|_{H^s})(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\alpha\theta\|_{H^s}^2)d\tau.
\end{aligned} \tag{3.38}$$

We set

$$E(t) = \sup_{0\le\tau\le t}\Big(\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t (\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\alpha\theta\|_{H^s}^2)d\tau\Big). \tag{3.39}$$

Consequently,

$$E(t) \le C_0E(0) + C_1E^{\frac{3}{2}}(t). \tag{3.40}$$

By using the bootstrapping argument ([30]), if

$$E(0) = \|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2 < \varepsilon^2 \tag{3.41}$$

for sufficiently small $\varepsilon > 0$, we have

$$\|u(t)\|_{H^s(\mathbb{R}^n)}^2 + \|\theta(t)\|_{H^s(\mathbb{R}^n)}^2 + \int_0^t (\|\Lambda^\alpha u\|_{H^s(\mathbb{R}^n)}^2 + \|\Lambda^\alpha\theta\|_{H^s(\mathbb{R}^n)}^2)d\tau \le C\varepsilon^2. \tag{3.42}$$

Step 2. We demonstrate (1.10) in this step. Comparable to the derivation of (3.37), for $0 < m \le s$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2) + (\|\Lambda^{m+\alpha}u\|_{L^2}^2 + \|\Lambda^{m+\alpha}\theta\|_{L^2}^2)\\ &\quad\lesssim (\|u\|_{H^s} + \|\theta\|_{H^s})(\|\Lambda^{m+\alpha}u\|_{L^2}^2 + \|\Lambda^{m+\alpha}\theta\|_{L^2}^2).\end{aligned} \tag{3.43}$$

Then from (3.43)

$$\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2) + (\|\Lambda^{m+\alpha}u\|_{L^2}^2 + \|\Lambda^{m+\alpha}\theta\|_{L^2}^2) \le 0. \tag{3.44}$$

By Lemma 2.6, we have

$$\|\Lambda^m u\|_{L^2} \le C\|u\|_{L^2}^{\frac{\alpha}{m+\alpha}}\|\Lambda^{m+\alpha}u\|_{L^2}^{\frac{m}{m+\alpha}}. \tag{3.45}$$

Similarly,

$$\|\Lambda^m \theta\|_{L^2} \le C\|\theta\|_{L^2}^{\frac{\alpha}{m+\alpha}}\|\Lambda^{m+\alpha}\theta\|_{L^2}^{\frac{m}{m+\alpha}}. \tag{3.46}$$

Substituting these two bounds into (3.47), we derive

$$\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2) + C\left(\frac{\|\Lambda^m u\|_{L^2}^{\frac{2(m+\alpha)}{m}}}{\|u\|_{L^2}^{\frac{2\alpha}{m}}} + \frac{\|\Lambda^m \theta\|_{L^2}^{\frac{2(m+\alpha)}{m}}}{\|\theta\|_{L^2}^{\frac{2\alpha}{m}}}\right) \le 0, \tag{3.47}$$

which implies

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2) + \frac{C}{\max\left\{\|u\|_{L^2}^{\frac{2\alpha}{m}}, \|\theta\|_{L^2}^{\frac{2\alpha}{m}}\right\}}\left(\|\Lambda^m u\|_{L^2}^{\frac{2(m+\alpha)}{m}} + \|\Lambda^m \theta\|_{L^2}^{\frac{2(m+\alpha)}{m}}\right)\\ &\le 0.\end{aligned} \tag{3.48}$$

Using the Cauchy inequality

$$(f+g)^k \le 2^{k-1}(f^k + g^k),\ \ k \ge 1, f, g \ge 0, \tag{3.49}$$

therefore,

$$\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) + \frac{C}{\max\left\{\|u\|_{L^2}^{\frac{2\alpha}{m}}, \|\theta\|_{L^2}^{\frac{2\alpha}{m}}\right\}}\left(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2\right)^{\frac{m+\alpha}{m}}$$
$$\leq 0. \tag{3.50}$$

Integrating (3.50) over $[0, t]$, we obtain

$$\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2 \leq C\left(1 + \int_0^t \frac{1}{\max\left\{\|u(\tau)\|_{L^2}^{\frac{2\alpha}{m}}, \|\theta(\tau)\|_{L^2}^{\frac{2\alpha}{m}}\right\}} d\tau\right)^{-\frac{m}{\alpha}}. \tag{3.51}$$

Multiplying (3.51) by $(1+t)^{\frac{m}{\alpha}}$, we obtain

$$(1+t)^{\frac{m}{\alpha}}\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2 \leq C\left(\frac{1+t}{1 + \int_0^t \frac{1}{\max\left\{\|u(\tau)\|_{L^2}^{\frac{2\alpha}{m}}, \|\theta(\tau)\|_{L^2}^{\frac{2\alpha}{m}}\right\}} d\tau}\right)^{\frac{m}{\alpha}}. \tag{3.52}$$

We found that using the L'Hopital rule, yields

$$\lim_{t\to\infty}\left(\frac{1+t}{1 + \int_0^t \frac{1}{\max\left\{\|u(\tau)\|_{L^2}^{\frac{2\alpha}{m}}, \|\theta\tau\|_{L^2}^{\frac{2\alpha}{m}}\right\}} d\tau}\right) = \lim_{t\to\infty} \max\left\{\|u(t)\|_{L^2}^{\frac{2\alpha}{m}}, \|\theta(t)\|_{L^2}^{\frac{2\alpha}{m}}\right\}. \tag{3.53}$$

Finally, we have

$$\lim_{t\to\infty}(1+t)^{\frac{m}{2\alpha}}(\|\Lambda^m u\|_{L^2} + \|\Lambda^m \theta\|_{L^2}) = 0. \tag{3.54}$$

Step 3. We demonstrate (1.11) in this step. Similar to (3.13) and (3.14), we obtain

$$\begin{aligned}\|\Lambda^{m+\alpha} u\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2\alpha+2m}|\widehat{u}(\xi)|^2 d\xi \\ &\geq \int_{|\xi|\geq r} |\xi|^{2\alpha+2m}|\widehat{u}(\xi)|^2 d\xi \\ &= r^{2\alpha}\left(\|\Lambda^m u\|_{L^2}^2 - \int_{|\xi|\leq r} |\xi|^{2m}|\widehat{u}(\xi)|^2 d\xi\right)\end{aligned} \tag{3.55}$$

and

$$\|\Lambda^{m+\alpha}\theta\|_{L^2}^2 \geq r^{2\alpha}\left(\|\Lambda^m \theta\|_{L^2}^2 - \int_{|\xi|\leq r} |\xi|^{2m}|\widehat{\theta}(\xi)|^2 d\xi\right). \tag{3.56}$$

Inserting these two bounds into (3.44), it yields

$$\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) + r^{2\alpha}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2)$$
$$\le r^{2\alpha}\left(\int_{|\xi|\le r} |\xi|^{2m}|\widehat{u}(\xi)|^2 d\xi + \int_{|\xi|\le r} |\xi|^{2m}|\widehat{\theta}(\xi)|^2 d\xi\right). \tag{3.57}$$

Utilizing $\nabla \cdot u = 0$, due to the property of the Fourier transform, which constitutes a bounded linear operator from L^1 into L^∞, one can derive the following result:

$$\begin{aligned}|\Lambda^m \widehat{(u\cdot\nabla)}u(\xi,t)| &= |\nabla \cdot \Lambda^m \widehat{(u \otimes u)}(\xi,t)| \\ &\le |\xi|\|\Lambda^m (u\otimes u)\|_{L^1} \\ &\le |\xi|\|u\|_{L^2}\|\Lambda^m u\|_{L^2}.\end{aligned} \tag{3.58}$$

Similarly,

$$|\Lambda^m \widehat{(u\cdot\nabla)}\theta(\xi,t)| \le |\xi|\|u\|_{L^2}\|\Lambda^m \theta\|_{L^2} + |\xi|\|\theta\|_{L^2}\|\Lambda^m u\|_{L^2}, \tag{3.59}$$

$$|\widehat{\Lambda^m \nabla} P(\xi,t)| \le |\xi|\|u\|_{L^2}\|\Lambda^m u\|_{L^2} + |\widehat{\Lambda^m}\theta(\xi,t)|. \tag{3.60}$$

Hence,

$$\begin{aligned}|\widehat{\Lambda^m}u(\xi,t)| &\le |\widehat{\Lambda^m u_0}(\xi)|e^{-\nu|\xi|^{2\alpha}t} + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}\, \||\xi|^m L_1(\xi,\tau)|\, d\tau \\ &\le |\widehat{\Lambda^m u_0}(\xi)| + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)} C|\xi|\|u\|_{L^2}\|\Lambda^m u\|_{L^2} d\tau \\ &\quad + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)} |\widehat{\Lambda^m}\theta(\xi,t)| d\tau \\ &\le |\widehat{\Lambda^m u_0}(\xi)| + C|\xi| \int_0^t (\|u(\tau)\|_{L^2} + \|\theta(\tau)\|_{L^2})(\|\Lambda^m u(\tau)\|_{L^2} + \|\Lambda^m \theta(\tau)\|_{L^2}) d\tau \\ &\quad + \int_0^t |\widehat{\Lambda^m}\theta(\xi,t)| d\tau \\ &\le |\widehat{\Lambda^m u_0}(\xi)| + C|\xi| \int_0^t (\|u_0\|_{L^2} + \|\theta_0\|_{L^2})(\|\Lambda^m u(\tau)\|_{L^2} + \|\Lambda^m \theta(\tau)\|_{L^2}) d\tau \\ &\quad + \int_0^t |\widehat{\Lambda^m}\theta(\xi,t)| d\tau \\ &\le |\widehat{\Lambda^m u_0}(\xi)| + C|\xi| \int_0^t (\|\Lambda^m u(\tau)\|_{L^2} + \|\Lambda^m \theta(\tau)\|_{L^2}) d\tau \\ &\quad + C\int_0^t |\widehat{\Lambda^m}\theta(\xi,t)| d\tau\end{aligned} \tag{3.61}$$

and

$$|\widehat{\Lambda^m}\theta(\xi,t)| \le |\widehat{\Lambda^m \theta_0}(\xi)| + C|\xi| \int_0^t (\|\Lambda^m u(\tau)\|_{L^2} + \|\Lambda^m \theta(\tau)\|_{L^2}) d\tau$$

$$+ C\int_0^t |\widehat{\Lambda^m u}(\xi,t)|d\tau. \tag{3.62}$$

The right-hand side of (3.57) can be bounded in the following manner:

$$\begin{aligned}
&\int_{|\xi|\le r}\left(|\xi|^{2m}|\widehat{u}(\xi)|^2+|\xi|^{2m}|\widehat{\theta}(\xi)|^2\right)d\xi\\
&\quad\le \int_{|\xi|\le r}|\widehat{\Lambda^m u_0}(\xi)|^2+|\widehat{\Lambda^m\theta_0}(\xi)|^2d\xi\\
&\quad\quad + C\int_{|\xi|\le r}|\xi|^2\left(\int_0^t(\|\Lambda^m u\|_{L^2}+\|\Lambda^m\theta\|_{L^2})d\tau\right)^2d\xi\\
&\quad\quad + Ct\int_{|\xi|\le r}\int_0^t(|\xi|^{2m}|\widehat{u}(\xi,\tau)|^2+|\xi|^{2m}|\widehat{\theta}(\xi,\tau)|^2)d\tau d\xi\\
&\quad\le \int_{|\xi|\le r}|\widehat{\Lambda^m u_0}(\xi)|^2+|\widehat{\Lambda^m\theta_0}(\xi)|^2d\xi\\
&\quad\quad + C\int_{|\xi|\le r}|\xi|^2\left[\left(\int_0^t d\tau\right)^{\frac12}\left(\int_0^t(\|\Lambda^m u\|_{L^2}+\|\Lambda^m\theta\|_{L^2})^2d\tau\right)^{\frac12}\right]^2d\xi\\
&\quad\quad + Ct\int_0^t\int_{|\xi|\le r}(|\xi|^{2m}|\widehat{u}(\xi,\tau)|^2+|\xi|^{2m}|\widehat{\theta}(\xi,\tau)|^2)d\xi d\tau\\
&\quad\le \int_{|\xi|\le r}|\widehat{\Lambda^m u_0}(\xi)|^2+|\widehat{\Lambda^m\theta_0}(\xi)|^2d\xi + Cr^{n+2}t\int_0^t(\|\Lambda^m u\|_{L^2}+\|\Lambda^m\theta\|_{L^2})^2d\tau\\
&\quad\quad + Ct\int_0^t(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m\theta\|_{L^2}^2)d\tau.
\end{aligned} \tag{3.63}$$

By the following inequality

$$\begin{aligned}
\int_{|\xi|\le r}|\widehat{u_0}(\xi)|^2d\xi &\le \left(\int_{|\xi|\le r}|\widehat{u_0}(\xi)|^{\frac{p}{p-1}}d\xi\right)^{\frac{2p-2}{p}}\left(\int_{|\xi|\le r}d\xi\right)^{\frac{2-p}{2}}\\
&\le Cr^{n(\frac{2}{p}-1)}\|u_0\|_{L^p}^2,
\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
\int_{|\xi|\le r}|\widehat{\theta_0}(\xi)|^2d\xi &\le \left(\int_{|\xi|\le r}|\widehat{\theta_0}(\xi)|^{\frac{p}{p-1}}d\xi\right)^{\frac{2p-2}{p}}\left(\int_{|\xi|\le r}d\xi\right)^{\frac{2-p}{2}}\\
&\le Cr^{n(\frac{2}{p}-1)}\|\theta_0\|_{L^p}^2.
\end{aligned} \tag{3.65}$$

Thus,

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m\theta\|_{L^2}^2)+r^{2\alpha}(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m\theta\|_{L^2}^2)\\
&\quad\le r^{2\alpha}\left(\int_{|\xi|\le r}|\widehat{\Lambda^m u_0}(\xi)|^2+|\widehat{\Lambda^m\theta_0}(\xi)|^2d\xi\right)
\end{aligned}$$

$$
\begin{aligned}
&+ Cr^{2\alpha+2+n} t \int_0^t (\|\Lambda^m u\|_{L^2} + \|\Lambda^m \theta\|_{L^2})^2 d\tau \\
&+ Cr^{2\alpha} t \int_0^t (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) d\tau \\
\leq\ & r^{2m+2\alpha+n(\frac{2}{p}-1)} (\|u_0\|_{L^p}^2 + \|\theta_0\|_{L^p}^2) \\
&+ Cr^{2\alpha+2+n}(1+t) \int_0^t (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) d\tau. \qquad (3.66)
\end{aligned}
$$

Now we set $r = (\frac{k}{1+t})^{\frac{1}{2\alpha}}$ with $k > \frac{n}{\alpha}(\frac{1}{p} - \frac{1}{2}) + \frac{m}{\alpha}$, and insert it into (3.63), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(1+t)^k (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) \\
&\leq C(1+t)^{k-\frac{m}{\alpha}-1-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} + C(1+t)^{k-\frac{2+n}{2\alpha}} \int_0^t (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) d\tau. \qquad (3.67)
\end{aligned}
$$

Then integrating (3.67) from 0 to t, one has

$$
\begin{aligned}
&(\|\Lambda^m u(t)\|_{L^2}^2 + \|\Lambda^m \theta(t)\|_{L^2}^2) \\
&\leq C(1+t)^{-k} (\|\Lambda^m u_0\|_{L^2}^2 + \|\Lambda^m \theta_0\|_{L^2}^2) + C(1+t)^{-k} \int_0^t (1+\tau)^{k-\frac{m}{\alpha}-1-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} d\tau \\
&\quad + C(1+t)^{-k} \int_0^t (1+\tau)^{k-\frac{2+n}{2\alpha}} \int_0^\tau (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) dz d\tau \\
&\leq C(1+t)^{-k} (\|\Lambda^m u_0\|_{L^2}^2 + \|\Lambda^m \theta_0\|_{L^2}^2) + C(1+t)^{-\frac{m}{\alpha}-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} \\
&\quad + C(1+t)^{1-\frac{2+n}{2\alpha}} \int_0^t (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) d\tau. \qquad (3.68)
\end{aligned}
$$

By the following basic inequality

$$
(1+t)^{\frac{m}{\alpha}+\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} \int_0^t (1+\tau)^{-\frac{m}{\alpha}-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} d\tau \leq \begin{cases} (1+t)\ln(1+t), & \frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2}) + \frac{m}{\alpha} = 1, \\ C(1+t), & \frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2}) + \frac{m}{\alpha} \neq 1, \end{cases}
$$

we have

$$
\begin{aligned}
&(1+t)^{\frac{m}{\alpha}+\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} (\|\Lambda^m u(t)\|_{L^2}^2 + \|\Lambda^m \theta(t)\|_{L^2}^2) \\
&\leq C(1+t)^{\frac{m}{\alpha}+\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})-k} (\|\Lambda^m u_0\|_{L^2}^2 + \|\Lambda^m \theta_0\|_{L^2}^2) + C + C(1+t)^{\frac{m}{\alpha}+\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})+1-\frac{2+n}{2\alpha}} \\
&\quad \times \int_0^t C(1+t)^{-\frac{m}{\alpha}-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} d\tau \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{m}{\alpha}+\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} (\|\Lambda^m u(\tau)\|_{L^2}^2 + \|\Lambda^m \theta(\tau)\|_{L^2}^2) \\
&\leq C + C(1+t)^{2-\frac{2+n}{2\alpha}} \max\{1, \ln(1+t)\} \\
&\quad \times \sup_{0\leq\tau\leq t} \Big((1+\tau)^{\frac{m}{\alpha}+\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})} (\|\Lambda^m u(\tau)\|_{L^2}^2 + \|\Lambda^m \theta(\tau)\|_{L^2}^2) \Big). \qquad (3.69)
\end{aligned}
$$

Since $0 < \alpha < 1 < \frac{2+n}{4}$, we obtain,

$$
(\|\Lambda^m u(t)\|_{L^2}^2 + \|\Lambda^m \theta(t)\|_{L^2}^2) \leq C(1+t)^{-\frac{m}{\alpha}-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{2})}. \qquad (3.70)
$$

Step 4. Uniqueness. Assume that

$$(U,\Theta),(u,\theta)\in L^\infty([0,T];H^s)\cap L^2([0,T];H^{s+\alpha}). \tag{3.71}$$

are two solutions of (1.6). Consider the difference $(\widetilde{u},\widetilde{\theta})$ with

$$\widetilde{u}=u-U,\ \widetilde{\theta}=\theta-\Theta. \tag{3.72}$$

Then it follows that

$$\begin{cases} \partial_t\widetilde{u}+u\cdot\nabla\widetilde{u}+\widetilde{u}\cdot\nabla U+(-\Delta)^\alpha\widetilde{u}+\nabla\widetilde{p}=\widetilde{\theta}e_n,\\ \partial_t\widetilde{\theta}+u\cdot\nabla\widetilde{\theta}+\widetilde{u}\cdot\nabla\Theta+(-\Delta)^\alpha\widetilde{\theta}=-\widetilde{u}_n,\\ \nabla\cdot\widetilde{u}=0,\\ \widetilde{u}(x,0)=0,\widetilde{\theta}(x,0)=0, \end{cases} \tag{3.73}$$

where $\widetilde{p}=p-P$. Dotting (3.73) with $(\widetilde{u},\widetilde{\theta})$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)+(\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+\|\Lambda^\alpha\widetilde{\theta}\|_{L^2}^2)\\ &\quad=-\int_{\mathbb{R}^n}\widetilde{u}\cdot\nabla U\cdot\widetilde{u}dx--\int_{\mathbb{R}^n}\widetilde{u}\cdot\nabla\Theta\cdot\widetilde{\theta}dx\\ &\quad=K_1+K_2. \end{aligned} \tag{3.74}$$

By Young's inequality and Lemma 2.4,

$$\begin{aligned} |K_1|=\left|\int_{\mathbb{R}^n}\widetilde{u}\cdot\nabla U\widetilde{u}dx\right| &\le \|\widetilde{u}\|_{L^2}\|\widetilde{u}\|_{L^{\frac{2n}{n-2\alpha}}}\|\nabla U\|_{L^{\frac{n}{\alpha}}}\\ &\le \|\widetilde{u}\|_{L^2}\|\Lambda^\alpha\widetilde{u}\|_{L^{\frac{2n}{n-2\alpha}}}\|\Lambda^{\frac{n}{2}+1-\alpha}U\|_{L^2}\\ &\le \frac{1}{4}\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+C\|\widetilde{u}\|_{L^2}^2\|\Lambda^s U\|_{L^2}^2. \end{aligned} \tag{3.75}$$

Similarly,

$$|K_2|=\left|\int_{\mathbb{R}^n}\widetilde{u}\cdot\nabla\Theta\widetilde{\theta}dx\right|\le\frac{1}{4}\|\Lambda^\alpha\widetilde{\theta}\|_{L^2}^2+C\|\widetilde{u}\|_{L^2}^2\|\Lambda^s\Theta\|_{L^2}^2. \tag{3.76}$$

Invoking the estimates in (3.75) and (3.76), it infers that

$$\begin{aligned} &\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)+(\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+\|\Lambda^\alpha\widetilde{\theta}\|_{L^2}^2)\\ &\quad\le C\|\widetilde{u}\|_{L^2}^2(\|\Lambda^s U\|_{L^2}^2+\|\Lambda^s\Theta\|_{L^2}^2)\\ &\quad\le (\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)(\|\Lambda^s U\|_{L^2}^2+\|\Lambda^s\Theta\|_{L^2}^2). \end{aligned} \tag{3.77}$$

The application of Gronwall's inequality consequently establishes the required uniqueness. Then this concludes the proof of Theorem 1.2.

3.2. Proof of Theorem 1.3

Applying the inhomogeneous blocks Δ_j operator to Eq (1.7) yields

$$\begin{cases} \partial_t \Delta_j u + \Delta_j(u \cdot \nabla u) + \Delta_j u + \nabla \Delta_j p = \Delta_j \theta e_n, \\ \partial_t \Delta_j \theta + \Delta_j(u \cdot \nabla \theta) + \Delta_j \theta = -\Delta_j u_n. \end{cases} \tag{3.78}$$

Taking the inner product of (3.78) with $\Delta_j u$ and $\Delta_j \theta$ respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2) + (\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2) \\ &= -\left\langle \Delta_j(u \cdot \nabla u), \Delta_j u \right\rangle - \left\langle \Delta_j(u \cdot \nabla \theta), \Delta_j \theta \right\rangle \\ &= Z_1 + Z_2, \end{aligned} \tag{3.79}$$

where

$$Z_1 = -\int_{\mathbb{R}^n} \left[\Delta_j(u \cdot \nabla u) - u \cdot \Delta_j \nabla u\right] \cdot \Delta_j u dx, \tag{3.80}$$

$$Z_2 = -\int_{\mathbb{R}^n} \left[\Delta_j(u \cdot \nabla \theta) - u \cdot \Delta_j \nabla \theta\right] \cdot \Delta_j \theta dx. \tag{3.81}$$

By Lemma 2.2, it yields

$$\|\nabla f\|_{L^\infty} \lesssim \|\nabla f\|_{B^{\frac{n}{2}}_{2,1}} \lesssim \|\nabla f\|_{B^{s-1}_{2,1}} \leq \|\nabla f\|_{B^s_{2,1}}, \tag{3.82}$$

where $s \geq \frac{n}{2} + 1$. By Lemma 2.3 and (3.82),

$$\begin{aligned} |Z_1| &\leq \left| -\int_{\mathbb{R}^n} \left[\Delta_j(u \cdot \nabla u) - u \cdot \Delta_j \nabla u\right] \cdot \Delta_j u dx \right| \\ &\leq \|\Delta_j u\|_{L^2} \| \left[u \cdot \nabla, \Delta_j\right] u \|_{L^2} \\ &\leq C c_{j,1} 2^{-js} \|u\|_{B^s_{2,1}}^2 \|\Delta_i u\|_{L^2}. \end{aligned} \tag{3.83}$$

$$\begin{aligned} |Z_2| &\leq \left| -\int_{\mathbb{R}^n} \left[\Delta_j(u \cdot \nabla \theta) - u \cdot \Delta_j \nabla \theta\right] \cdot \Delta_j \theta dx \right| \\ &\leq \|\Delta_j \theta\|_{L^2} \| \left[u \cdot \nabla, \Delta_j\right] \theta \|_{L^2} \\ &\leq C c_{j,1} 2^{-js} \|u\|_{B^s_{2,1}} \|\theta\|_{B^s_{2,1}} \|\Delta_i \theta\|_{L^2}. \end{aligned} \tag{3.84}$$

Here $\|c_{j,1}\|_{l^1} = 1$, where l^1 stands for

$$\|c_{j,1}\|_{l^1} = \sum_{j \in \mathbb{Z}} |c_{j,1}|. \tag{3.85}$$

Using these estimates, it leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2) + (\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2) \\ &\lesssim c_{j,1} 2^{-js} (\|\Delta_j u\|_{L^2} + \|\Delta_j \theta\|_{L^2})(\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2) \\ &\lesssim c_{j,1} 2^{-js} (\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2)^{\frac{1}{2}} (\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2) \end{aligned}$$

$$\lesssim \frac{1}{2}(\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2) + Cc_{j,1}2^{-js}(\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2)^2. \tag{3.86}$$

According to the Cauchy inequality

$$\sqrt{2}(f+g) \geq \sqrt{2}(f^2+g^2)^{\frac{1}{2}} \geq f+g (f, g \geq 0) \tag{3.87}$$

and Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}(\|\Delta_j u\|_{L^2} + \|\Delta_j \theta\|_{L^2}) \\
&\leq \frac{d}{dt}(\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2)^{\frac{1}{2}} + (\|\Delta_j u\|_{L^2}^2 + \|\Delta_j \theta\|_{L^2}^2)^{\frac{1}{2}} \\
&\lesssim c_{j,1}2^{-js}(\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2).
\end{aligned} \tag{3.88}$$

Integrating (3.88) in time from 0 to t yields

$$\begin{aligned}
&\sqrt{2}(\|\Delta_j u(t)\|_{L^2} + \|\Delta_j \theta(t)\|_{L^2}) + \sqrt{2}\int_0^t (\|\Delta_j u\|_{L^2} + \|\Delta_j \theta\|_{L^2}) d\tau \\
&\leq 2(\|\Delta_j u_0\|_{L^2} + \|\Delta_j \theta_0\|_{L^2}) + C\int_0^t c_{j,1}2^{-js}(\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2) d\tau.
\end{aligned} \tag{3.89}$$

Multiplying it by 2^{2js} and subsequently performing the summation with respect to j results in the following transformation

$$\begin{aligned}
&\sqrt{2}(\|u\|_{B^s_{2,1}} + \|\theta\|_{B^s_{2,1}}) + \sqrt{2}\int_0^t (\|\Delta_j u\|_{B^s_{2,1}} + \|\Delta_j \theta\|_{B^s_{2,1}}) d\tau \\
&\leq 2(\|u_0\|_{B^s_{2,1}} + \|\theta_0\|_{B^s_{2,1}}) + C\int_0^t c_{j,1}2^{-js}(\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2) d\tau.
\end{aligned} \tag{3.90}$$

Set

$$E(t) = \sup_{0\leq\tau\leq t}(\|u\|_{B^s_{2,1}} + \|\theta\|_{B^s_{2,1}}) + \int_0^t (\|u\|_{B^s_{2,1}} + \|\theta\|_{B^s_{2,1}}) d\tau. \tag{3.91}$$

Consequently, (3.90) implies that

$$E(t) \leq C_0 E(0) + C_1 E^2(t). \tag{3.92}$$

According to the bootstrapping argument, if

$$E(0) = \|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2 < \varepsilon^2 \tag{3.93}$$

for sufficiently small $\varepsilon > 0$, then

$$\|u\|_{B^s_{2,1}}^2 + \|\theta\|_{B^s_{2,1}}^2 + \int_0^t (\|u(\tau)\|_{B^s_{2,1}}^2 + \|\theta(\tau)\|_{B^s_{2,1}}^2) d\tau \leq C\varepsilon^2 \tag{3.94}$$

for any $t > 0$.

Multiplying (1.7) with e^t, it yields that

$$\begin{cases} e^t\partial_t u + u\cdot\nabla(e^t u) + e^t u + e^t\nabla p = e^t\theta e_n, \\ e^t\partial_t\theta + u\cdot\nabla(e^t\theta) + e^t\theta = -e^t u_n, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x). \end{cases} \tag{3.95}$$

Dotting (3.95) with $(e^t u, e^t\theta)$, it yields that

$$\frac{1}{2}\frac{d}{dt}(\|e^t u\|_{L^2}^2 + \|e^t\theta\|_{L^2}^2) = 0. \tag{3.96}$$

Integrating it in $[0,t]$, we have

$$e^{2t}(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2) = \|e^t u\|_{L^2}^2 + \|e^t\theta\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{3.97}$$

Therefore,

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \le Ce^{-t}. \tag{3.98}$$

By utilizing the interpolation inequality, we have

$$\begin{aligned} \|\Lambda^m u(t)\|_{L^2} + \|\Lambda^m\theta(t)\|_{L^2} &\le C\|\Lambda^s(u,\theta)\|_{L^2}^{\frac{m}{s}}\|(u,\theta)\|_{L^2}^{\frac{s-m}{s}} \\ &\le C\|(u,\theta)\|_{B_{2,1}}^{\frac{m}{s}}\|(u,\theta)\|_{L^2}^{\frac{s-m}{s}} \\ &\le C\varepsilon^{\frac{m}{s}}e^{\frac{-s+m}{-s}t}. \end{aligned} \tag{3.99}$$

Hence, Theorem 1.3 has been proved.

4. Conclusions

In summary, our study provides a comprehensive analysis of the n-dimensional incompressible Boussinesq equations with fractional dissipation and thermal diffusion, including conditions for global existence, convergence, decay, uniqueness, and regularity of solutions, depending on the size and nature of the initial data and the presence of fractional dissipation.

Author contribution

Xinli Wang developed the concept with her supervisor, designed the manuscript and provided key information; Haiyang Yu and Tianfeng Wu helped revise the manuscript and provided the intellectual support. All of authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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