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Research article

Temporal Hölder continuity of the parabolic Anderson model driven by a class of time-independent Gaussian fields with rough initial conditions

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Abstract: In this paper, we considered the parabolic Anderson model with a class of time-independent generalized Gaussian fields on $\mathbb{R}^d$, which included fractional white noise, Bessel field, massive free field, and other nonstationary Gaussian fields. Under the rough initial conditions, we constructed the Feynman-Kac formula as a solution in the Stratonovich integral by Brownian bridge, and then proved the Hölder continuity of the solution with respect to the time variable. As a comparison, we also studied the Hölder continuity under the regular initial conditions that $u_0 \equiv C$ and $u_0 \in C^\kappa(\mathbb{R}^d)$ with $\kappa \in (0, 1]$.

Keywords: parabolic Anderson model; Feynman-Kac formula; generalized Gaussian field; Brownian bridge; Hölder continuity; measure-valued initial data
Mathematics Subject Classification: 60F99, 60G60, 60H15

1. Introduction

In this paper, we study the following stochastic heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\triangle u(t,x) + \theta V(x)u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1.1}$$

which is also called parabolic Anderson model. Here, parameter $\theta > 0$ and V is a centered generalized Gaussian field which is defined by the Gaussian family $\{\langle V, \varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ with mean zero and covariance

$$\mathbb{E}[\langle V, \phi\rangle\langle V, \psi\rangle] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\psi(y)k(x,y)dxdy, \qquad \forall\, \phi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{1.2}$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space, and $k(x, y)$ is a symmetric positive definite kernel function. We assume that there exists a constant $C > 0$ such that for almost everywhere $(x, y) \in \mathbb{R}^{2d}$,

$$|k(x,y)| \le C(\gamma_h(x-y) + 1). \tag{1.3}$$

Here, γ_h is a nonnegative and nonnegative definite function which satisfies that $\gamma_h(x) \in L^1_{loc}(\mathbb{R}^d)$, and there exists a $\alpha \in (0, 2 \wedge d)$ such that $\gamma_h(rx) = r^{-\alpha}\gamma_h(x)$ for all $r > 0$.

There exist many Gaussian fields satisfying (1.3). For example, the stationary case includes Bessel field [1], Gaussian field with Riesz potential covariance [2], and fractional white noise [3] (Hurst parameters $H_i \in (1/2, 1)$ for $1 \leq i \leq d$), while the nonstationary case partly includes 2-d massive free field [4] and log-correlated Gaussian field [5]. In these fields, the covariances of Gaussian field with Riesz potential covariance and fractional white noise are homogeneous themselves, and γ_h in (1.3) can be taken as them. The covariance of Bessel field is represented as the Bessel function $G_b(x)$, which is not homogeneous but satisfies the asymptotic behaviours for when $x \to 0$,

$$G_b(x) \sim \begin{cases} \frac{\Gamma(\frac{d-b}{2})}{2^b\pi^{b/2}}|x|^{b-d}, & \text{if} \quad 0 < b < d, \\ \frac{1}{2^{d-1}\pi^{d/2}} \ln\frac{1}{|x|}, & \text{if} \quad b = d, \\ \frac{\Gamma(\frac{b-d}{2})}{2^b\pi^{b/2}}, & \text{if} \quad b > d, \end{cases} \tag{1.4}$$

when $|x| \to \infty$, $G_b(x) \sim (2^{\frac{d+b-1}{2}}\pi^{\frac{d-1}{2}}\Gamma(\frac{b}{2}))^{-1}|x|^{\frac{b-1-d}{2}}e^{-|x|}$. The covariances of 2-$d$ massive free field and log-correlated Gaussian field satisfy that when $x \to y$, $k(x, y) \sim \ln\frac{1}{|x-y|}$, which are bounded away from the diagonal region $\{x = y\}$. It can be observed that the covariance of Bessel field ($0 < b < d$) is asymptotically homogeneous, where it requires that $b > d-2$ such that (1.3) is satisfied when $\alpha = d-b$; the covariances of Bessel field ($b \geq d$), 2-d massive free field, and log-correlated Gaussian field are bounded or asymptotically logarithmic, satisfying (1.3) for all α sufficiently closed to 0. In addition, we can construct a series of nonstationary fields satisfying (1.3) by setting $V(x) = g(x)\tilde{V}(x)$ for the nontrivial, bounded, and measurable function g and stationary Gaussian field $\tilde{V}$ satisfying (1.3).

At present, the rough initial conditions are getting more and more attention in the field of stochastic partial differential equations. Bertini and Giacomin [6] focused on the initial conditions with growing tails in stochastic Burgers and Kardar-Parisi-Zhang (abbr. KPZ) equations. Amir, Corwin, and Quastel [7] utilized the Dirac δ initial condition (or narrow wedge initial conditions) to study the distribution of stochastic heat (or KPZ) equations. Until the publishing of [8], Chen and Dalang first introduced and studied the rough initial conditions for the nonlinear stochastic heat equation, which are quite extensive, including Dirac δ measure, non-tempered measure with exponentially growing tails, etc.

For (1.1), we consider the rough initial condition: the initial value u_0 is a Borel measure on $\mathbb{R}^d$ owing a Jordan decomposition $u_0 = u_0^+ - u_0^-$. Let $|u_0| := u_0^+ + u_0^-$ be the variation measure of u_0. We assume that for $t > 0$ and $x \in \mathbb{R}^d$,

$$p_t * |u_0|(x) := \int_{\mathbb{R}^d} p_t(x-y)|u_0|(dy) < \infty, \tag{1.5}$$

where "$*$" represents the convolution and $p_t(x) := (2\pi t)^{-d/2}\exp\{-|x|^2/(2t)\}$ is the usual heat kernel function. It is worth noting that due to the temporal continuity of $p_t(x)$ on $(0, \infty)$, condition (1.5) implies that for $0 < \delta < T$ and $x \in \mathbb{R}^d$,

$$\sup_{t\in[\delta,T]} p_t * |u_0|(x) < \infty. \tag{1.6}$$

There have been many results for the Hölder continuity of the stochastic heat equation in the Itô-Skorokhod integral and rough initial conditions, such as [9–13]. In the earlier literatures [8, 14], Chen and Dalang studied the continuity for the nonlinear stochastic heat and fractional heat equations with rough initial conditions in the Itô-Skorokhod integral, including the parabolic Anderson model. In Chen and Huang [15], the time-space Hölder continuity was established for nonlinear stochastic heat equations driven by time-white and space-colored Gaussian fields, with rough initial conditions concerning Itô-Skorokhod integral. However, the published papers about Hölder continuity in the Stratonovich sense are not as rich as in the Itô-Skorokhod sense due to the technical complexity. When initial value $u_0 \equiv 1$, Hu, Huang, Nualart and Tindel [16] proved the time-space Hölder continuity for the stochastic heat equation driven by time-space stationary Gaussian fields in the Stratonovich integral. For the similar model, under the rough initial condition, Lyu [17] obtained the spatial Hölder continuity in the case of time-space stationary Gaussian fields, which are homogeneous on space. Later, Lyu and Li [18] proved the time-space Hölder continuity for time-independent log-correlated Gaussian field and initial value $u_0 \equiv 1$. As far as we know, there are very few results for temporal Hölder continuity in the case of nonstationary Gaussian field and rough initial condition.

In this paper, under the conditions (1.3) and (1.5), we tend to prove the temporal Hölder continuity for the Feynman-Kac formula of (1.1) in the Stratonovich integral. According to [[17], Lemma 3.1], the Feynman-Kac formula is a mild solution to (1.1) in the Stratonovich integral. As mentioned in [16], the path-wise solution in the Young integral can be viewed as a version of the Feynman-Kac formula in the Stratonovich integral. Thus, to obtain the Hölder continuity in the Stratonovich sense, we only need to prove the Hölder continuity in the Young sense. However, the strategy is usually unsuccessful for the rough initial condition.

According to (5.13) in [16], when the initial value u_0 belongs to the weighted Besov-Hölder space $\mathcal{B}^{\kappa,e_\lambda}_{\infty,\infty}(\mathbb{R}^d)$ ($\kappa \in (0,1)$), it was obtained as the temporal Hölder continuity of solution in the sense of the norm of $\mathcal{B}^{\kappa_u,w_t}_{\infty,\infty}(\mathbb{R}^d)$ ($\kappa_u \in (\kappa, 1)$). Because the weighted Besov space $\mathcal{B}^{\kappa_u,w_t}_{\infty,\infty}$ coincides with the weighted Hölder space $C^{\kappa_u}(\mathbb{R}^d; w_t)$, we can directly obtain the temporal Hölder continuity in the point-wise sense. Unluckily, if u_0 is a measure, it usually does not belong to $\mathcal{B}^{\kappa,e_\lambda}_{\infty,\infty}(\mathbb{R}^d)$ ($\kappa \in (0,1)$), such as Dirac $\delta_0 \in \mathcal{B}^{-d(1-1/q),e_\lambda}_{q,\infty}$ ($q \in [1,\infty]$) but $\notin \mathcal{B}^{\kappa,e_\lambda}_{\infty,\infty}(\mathbb{R}^d)$ ($\kappa \in (0,1)$). When u_0 belongs to the Besov space on torus $\mathcal{B}^{-\kappa}_{q,\infty}$ ($\kappa \in [0, 1/2)$), by reference to [19,20], the temporal Hölder continuity of solution was obtained in the sense of the norm of $\mathcal{B}^{\kappa_u}_{q,\infty}(\mathbb{T}^d)$ ($\kappa_u \in (\kappa, 1)$), but q cannot arrive at infinity in solution space $\mathcal{B}^{\kappa_u}_{q,\infty}(\mathbb{T}^d)$. This leads to that we still have no way to prove the temporal Hölder continuity in the point-wise sense.

Instead of the above method, we directly prove the Hölder continuity for the Feynman-Kac formula by the Kolmogorov continuity theorem. It has been known that under the rough initial condition, the previous Feynman-Kac formula based on Brownian motion is not well-defined any more. Hence, we will use the Feynman-Kac formula based on Brownian bridge. In the earlier work [21], Chen, Hu, and Nualart proved the Feynman-Kac formula for the nonlinear stochastic heat equation on $\mathbb{R}$ in the Itô-Skorokhod integral with time-space white noise and rough initial conditions. Hu, Nualart, and Song [3] (also see [16]) obtained the Feynman-Kac formula for the stochastic heat equation driven by time-space Gaussian fields with function-valued initial data in the Itô-Skorokhod and Stratonovich integral. After it, Huang, Lê, and Nualart [22] obtained the Feynman-Kac moment representation based on Brownian bridge for the stochastic heat equation in the Itô-Skorokhod integral, driven by time-white Gaussian fields with rough initial conditions. Inspired by it, Lyu [17] proved the Feynman-Kac formula for the stochastic heat equation in the Stratonovich integral, with time-space Gaussian fields and rough initial condition. Similarly, this paper also obtained the Feynman-Kac formula based on Brownian bridge $u_\theta(t, x)$ defined in (2.1) in the case of nonstationary Gaussian field and rough initial condition, but the

Feynman-Kac moment representation of $u_\theta(t,x)$ that we get in (2.6) is different from the representation in [17].

Different from Brownian motion and stationary Gaussian field, the computations of Hölder continuity are complex in the case of Brownian bridge and nonstationary Gaussian field. To overcome the difficulty, on the one hand, we construct a novel decomposition of Brownian bridge in Lemma 2.5; on the other hand, because the technique of Fourier transform cannot be directly applied to estimate positive definite kernel $k(x,y)$, we will use the estimates of the heat kernel in Lemma 2.1.

We state the temporal Hölder continuity of the Feynman-Kac formula $u_\theta(t,x)$ in (2.1) as follows.

Theorem 1.1. *Assume that conditions* (1.3) *and* (1.5) *hold. Set* $0<\delta<1\le T$ *and* $\beta\in(0,1-\alpha/2)$, *where* α *is taken from* (1.3). *Then, there exists some constant* $C>0$ *such that for all* $\theta>0$, $t,s\in[\delta,T]$, $x\in\mathbb{R}^d$, *and integer* $n\ge 1$,

$$\mathbb{E}|u_\theta(t,x)-u_\theta(s,x)|^n \le C^n e^{C\theta^2 n^2 T^2}\exp\left\{C\theta^{\frac{4}{2-\alpha}}T^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\right\}((2n-1)!!)^{1/2}T^{(\beta d/2+1)n} \cdot \delta^{-(d/2+1)\beta n}\Big(\sup_{r\in[\delta,T/(1-\beta)]} p_r * |u_0|(x)\Big)^n |t-s|^{\beta n/2}. \tag{1.7}$$

Moreover, there exists a temporal $\frac{\beta}{2}$*-Hölder continuous modification of* $u_\theta(t,x)$ *on* $(0,\infty)$.

As an extension of temporal Hölder continuity in [[16], Theorem 4.12], where the Gaussian fields are stationary and initial value $u_0\equiv 1$, Theorem 1.1 contains the case of nonstationary Gaussian fields and initial value of measure. However, patient readers may observe from Theorem 1.1 that when initial value $u_0\equiv 1$, on the one hand, the order of Hölder continuity is not optimal, where $\beta/2<1/2$; on the other hand, the Hölder continuity of the solution is limited on open interval $(0,\infty)$ excluding the zero point. For this reason, we intend to make some technical explanations as follows:

(1) Because the measure-valued initial data u_0 is considered, we choose to use the Feynman-Kac formula based on Brownian bridge (2.1). In the estimates of the Hölder continuity, (2.1) leads to the need to utilize the continuity of bridge $B_{0,t}$ with respect to t; see Proposition 5.3. Here, remark that the continuity of $\int_0^s V(B_{0,t}^{x,z}(r))dr$ at the $t=s$ point is necessary for our estimates. If we consider the Feynman-Kac formula based on Brownian motion with function-valued initial data, then the continuity of the term can be bypassed. So, when $u_0\equiv 1$, the order of Hölder continuity is low in Theorem 1.1.
(2) Under condition (1.5), the proof of Hölder continuity can only depend on the regularity of heat kernel $p_t(x)$ rather than of u_0. However, in the step of estimates of the heat kernel, the terms t and s with negative power are produced; see Lemma 2.1. For the Feynman-Kac formula (2.1), in the computations of (2.37)–(2.39), we have no way to get rid of the term $(t^{-d/2-1}+s^{-d/2-1})^{\beta n}$ produced in estimates of the heat kernel. Moreover, we obtain an additional term $\delta^{-(d/2+1)\beta n}$ in (1.7) relative to the estimates of moment in Proposition 4.1, which implies that δ cannot tend to 0. Thus, when $u_0\equiv 1$, the coefficient in the right side of (1.7) is not exact, such that the Hölder continuity cannot be proved at zero point.

In order to compensate the defect of Theorem 1.1 in the case of function-valued initial data, we specifically show the following result in which the Hölder continuity is extended to the zero point.

Theorem 1.2. *Under condition* (1.3), *the following results hold:*

(i) When initial value u_0 is a κ-Hölder continuous function in $C^\kappa(\mathbb{R}^d)$ with $\kappa \in (0, 1]$, for $\rho \in (0, \kappa)$, $\theta > 0$, and $x \in \mathbb{R}^d$, there exists a modification of $u_\theta(t, x)$, which is $\frac{\rho}{2}$-Hölder continuous on $[0, \infty)$.

(ii) When initial value u_0 is a constant, that is, $u_0 \equiv C$, for $\nu \in (0, 1 - \alpha/4)$, $\theta > 0$, and $x \in \mathbb{R}^d$, there exists a modification of $u_\theta(t, x)$, which is ν-Hölder continuous on $[0, \infty)$.

The order of Hölder continuity in Theorem 1.2 (i) coincides with it in [8, 15], though their settings are different from ours, where they considered the Itô-Skorokhod integral and time-white Gaussian fields which are colored in space. The order in Theorem 1.2 (ii) is the same as it is in [[16], Theorem 4.12].

Next, we make some comments on the results in Theorems 1.1 and 1.2 as follows:

(a) With respect to the special fields, including Bessel field ($b \geq d$), 2-d massive free field, and log-correlated Gaussian field, the orders of Hölder continuity are sufficiently closed to $1/2$ and 1 in Theorem 1.1 and Theorem 1.2 (ii), respectively, because these fields always satisfy (1.3) for all small α. For the special fields, we can prove a more precise modulus of continuity in Theorem 1.1 than (1.7), which is similar to [[18], Proposition 2.4]. However, the more precise modulus does not impact on the order of Hölder continuity. For the homogeneous or asymptotically homogeneous Gaussian fields, like Gaussian field with Riesz potential covariance, fractional white noise, and Bessel field ($0 < b < d$), the orders of Hölder continuity in Theorems 1.1 and 1.2 are optimal within our framework if we take α equal to the (asymptotically) homogeneous degree of these fields in condition (1.3).

(b) For Theorem 1.2, the Hölder continuity in (i) is limited by the Hölder continuity of u_0. Though the initial condition in (ii) is a special case in (i), the order in (ii) is obviously higher than it is in (i), i.e., $\nu > 1/2 > \rho/2$. It is found that, different from (ii), the Hölder continuity in (i) is only determined by the regularity of u_0 by comparing (2.41) and (2.47) in the proof of Theorem 1.2.

(c) Notice that the order in Theorem 1.2 (i) is not necessarily higher than Theorem 1.1, because the Hölder continuity at zero point is considered in Theorem 1.2 (i). To sum up Theorem 1.1 and Theorem 1.2 (i), the order of Hölder continuity on $(0, \infty)$ is $(\beta \vee \rho)/2$ when $u_0 \in C^\kappa(\mathbb{R}^d)$. On the other hand, it is found that the order in Theorem 1.1 is lower than it is in Theorem 1.2 (ii), i.e., $\beta/2 < 1/2 < \nu$. Obviously, the initial condition is very special in Theorem 1.2 (ii).

Methodology: In the sense of the Stratonovich integral, our method heavily depends on the Feynman-Kac formula based on the Brownian bridge (2.1) and Feynman-Kac formula based on Brownian motion (2.4), which produce the different Hölder continuities and modulus of continuity in Theorems 1.1 and 1.2. Meanwhile, our method can only be applied to the linear model. However, in the sense of the Itô-Skorokhod integral, the method in [8, 13–15] can cover the case of the nonlinear model, the advantages of which are that the estimates of Hölder continuity are stable for rough and regular initial conditions. In fact, the above settings of the integral are different, and our method mainly compensates the lack of result in the Stratonovich integral (or Young integral) rather than the Itô-Skorokhod integral.

Organisation: Section 2 is the preliminaries about Fourier transform, estimates of heat kernel, and Brownian bridge. In Section 3, we give the definitions of the Feynman-Kac formula, Feynman-Kac functional, and Feynman-Kac moment representation. In Section 4, we show the well-definiteness and moment estimates of the Feynman-Kac functional and Feynman-Kac formula. Section 5 is the proof of temporal Hölder continuity in Theorem 1.1.

2. Preliminaries

Notations: Write $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_+ := \{1,2,3,\cdots\}$. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be the probability space with expectation $\mathbb{E}$. Set $p\in[1,\infty]$, and denote the Lebesgue space on $(\Omega,\mathfrak{F},\mathbb{P})$ by $L^p(\Omega)$. For region $D\subseteq\mathbb{R}^d$, let $L^p(D)$ be Lebesgue space on D. Denote by $L^1_{loc}(\mathbb{R}^d)$ the space composed of locally integrable functions on $\mathbb{R}^d$. For $\kappa\in(0,1]$, $C^\kappa(\mathbb{R}^d)$ is the space composed of κ-Hölder continuous functions. $\mathcal{S}(\mathbb{R}^d)$ is Schwartz space on $\mathbb{R}^d$, and its dual space $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions. Let C be a universal nonnegative constant. $f\lesssim g$ represents that there is a constant $C>0$ not dependent on variables such that $f\le Cg$.

Fourier transform: The Fourier transform of a function $f\in\mathcal{S}(\mathbb{R}^d)$ is defined as

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)dx,$$

and the inverse Fourier transform is given by $\mathcal{F}^{-1}f(\xi) = (2\pi)^{-d}\mathcal{F}f(-\xi)$. The generalized Fourier transform of $f\in\mathcal{S}'(\mathbb{R}^d)$ is defined by the dual

$$\langle\mathcal{F}f,g\rangle = \langle f,\mathcal{F}g\rangle, \qquad \forall g\in\mathcal{S}(\mathbb{R}^d). \tag{2.1}$$

For nonnegative definite function γ_h in (1.3), according to the Bochner theorem (e.g., p.158, [23]), there exists a nonnegative and symmetric tempered measure μ_h such that $\gamma_h = \mathcal{F}\mu_h$. Noticing that $\gamma_h(x)$ is a function, it is found that

$$\gamma_h(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\mu_h(d\xi), \qquad \text{a.e.}, \tag{2.2}$$

by (2.1) and the Fubini theorem. Because γ_h satisfies that $\gamma_h(rx) = r^{-\alpha}\gamma_h(x)$ for all $r>0$, μ_h is homogeneous, that is, $\mu_h(d(r\xi)) = r^\alpha\mu_h(d\xi)$ for all $r>0$.

Estimates of heat kernel: We give the estimates of heat kernel used to prove the Hölder continuity. The results similar to (i) and (iii) in Lemma 2.1 have been proved in [[15], Lemma 3.1], but our proof is slightly different from [15] in details.

Lemma 2.1. *For the heat kernel $p_t(x) = (2\pi t)^{-d/2}\exp\{-|x|^2/(2t)\}$, the following results hold.*

(i) For all $x,y\in\mathbb{R}^d$ and $t>0$, it holds that

$$|p_t(x)-p_t(y)| \lesssim t^{-(d+1)/2}|x-y|. \tag{2.3}$$

(ii) For all $z_1,z_2,x,y\in\mathbb{R}^d$, and $t>0$, it holds that

$$|p_t(z_1+x)-p_t(z_1+y)-p_t(z_2+x)+p_t(z_2+y)| \lesssim t^{-d/2-1}|z_1-z_2||x-y|. \tag{2.4}$$

(iii) For all $x\in\mathbb{R}^d$ and $t,s>0$, it holds that

$$|p_t(x)-p_s(x)| \lesssim (t^{-d/2-1}+s^{-d/2-1})|t-s|. \tag{2.5}$$

Proof. (i) By $p_t = \mathcal{F}^{-1}e^{-\frac{t}{2}|\cdot|^2}$, the inequality $|e^{i\xi\cdot y}-e^{i\xi\cdot x}| \le |\xi||x-y|$, and the integral substitution, we have

$$|p_t(x)-p_t(y)| = (2\pi)^{-d}\left|\int_{\mathbb{R}^d}(e^{-i\xi\cdot x}-e^{-i\xi\cdot y})e^{-t|\xi|^2/2}d\xi\right|$$

$$\leq (2\pi)^{-d}\int_{\mathbb{R}^d}|\xi|e^{-t|\xi|^2/2}d\xi|x-y|$$
$$\lesssim t^{-(d+1)/2}|x-y|. \tag{2.6}$$

(ii) According to the arguments similar to (2.6), it holds that

$$\begin{aligned}&|p_t(z_1+x)-p_t(z_1+y)-p_t(z_2+x)+p_t(z_2+y)|\\ =&(2\pi)^{-d}\left|\int_{\mathbb{R}^d}(e^{-i\xi\cdot z_1}-e^{-i\xi\cdot z_2})(e^{-i\xi\cdot x}-e^{-i\xi\cdot y})e^{-t|\xi|^2/2}d\xi\right|\\ \leq&(2\pi)^{-d}\int_{\mathbb{R}^d}|\xi|^2e^{-t|\xi|^2/2}d\xi|z_1-z_2||x-y|\\ \lesssim&t^{-d/2-1}|z_1-z_2||x-y|.\end{aligned}$$

(iii) From $p_t=\mathcal{F}^{-1}e^{-\frac{t}{2}|\cdot|^2}$, $|e^{i\xi\cdot x}|=1$, and the inequality $|e^a-e^b|\leq|a-b|(e^a+e^b)$, it implies that

$$\begin{aligned}|p_t(x)-p_s(x)|&=(2\pi)^{-d}\left|\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\Big(\exp\{-\frac{t}{2}|\xi|^2\}-\exp\{-\frac{s}{2}|\xi|^2\}\Big)d\xi\right|\\ &\leq 2^{-d-1}\pi^{-d}\int_{\mathbb{R}^d}|\xi|^2\Big(\exp\{-\frac{t}{2}|\xi|^2\}+\exp\{-\frac{s}{2}|\xi|^2\}\Big)d\xi|t-s|\\ &=2^{-d-1}\pi^{-d}(t^{-d/2-1}+s^{-d/2-1})\int_{\mathbb{R}^d}|\xi|^2\exp\{-\frac{1}{2}|\xi|^2\}d\xi|t-s|\\ &\lesssim(t^{-d/2-1}+s^{-d/2-1})|t-s|,\end{aligned}$$

where the second to last step is due to the integral substitution.

So, we complete the proof. □

Lemma 2.2. *Under condition* (1.3)*, for $\beta>0$, there exist some $C>0$ dependent on α and β such that for all $t>0$,*

$$\int_{\mathbb{R}^d}(\gamma_h(y+x)+1)p_t^\beta(y)dy\leq Ct^{(1-\beta)d/2}(t^{-\alpha/2}+1). \tag{2.7}$$

Proof. By the spherical substitution, $\gamma_h\in L^1_{loc}(\mathbb{R}^d)$, and $\gamma_h(tx)=t^{-\alpha}\gamma_h(x)$ ($\alpha\in(0,2\wedge d)$), it gives that

$$\begin{aligned}\int_{\mathbb{R}^d}\gamma_h(y)p_1^\beta(y)dy&=(2\pi)^{-\beta d/2}\int_0^\infty r^{-\alpha+d-1}e^{-\beta r^2/2}dr\int_{\{|y|=1\}}\gamma_h(y)dS\\ &<\infty.\end{aligned} \tag{2.8}$$

By the facts that $\gamma_h=\mathcal{F}\mu_h$ and $\mathcal{F}p_t(\xi)=e^{-\frac{t|\xi|^2}{2}}$, $p_t^\beta(x)=(2\pi)^{(1-\beta)d/2}\beta^{-d/2}t^{(1-\beta)d/2}p_{t/\beta}(x)$, and $|e^{ia}|=1$,

$$\begin{aligned}\int_{\mathbb{R}^d}(\gamma_h(y+x)+1)p_t^\beta(y)dy&=(2\pi)^{(1-\beta)d/2}\beta^{-d/2}t^{(1-\beta)d/2}\Big(\int_{\mathbb{R}^d}\gamma_h(y+x)p_{t/\beta}(y)dy+1\Big)\\ &=(2\pi)^{(1-\beta)d/2}\beta^{-d/2}t^{(1-\beta)d/2}\Big(\int_{\mathbb{R}^d}e^{i\xi\cdot x}\exp\{-\frac{t}{2\beta}|\xi|^2\}\mu_h(d\xi)+1\Big)\\ &\leq(2\pi)^{(1-\beta)d/2}\beta^{-d/2}t^{(1-\beta)d/2}\Big(\int_{\mathbb{R}^d}\exp\{-\frac{t}{2\beta}|\xi|^2\}\mu_h(d\xi)+1\Big)\end{aligned}$$

$$= (2\pi)^{(1-\beta)d/2}\beta^{-d/2}t^{(1-\beta)d/2}\Big(\int_{\mathbb{R}^d}\gamma_h(y)p_{t/\beta}(y)dy + 1\Big). \tag{2.9}$$

Moreover, using the integral substitution, $\gamma_h(tx) = t^{-\alpha}\gamma_h(x)$, and (2.8), it gives that

$$\begin{aligned}\int_{\mathbb{R}^d}(\gamma_h(y+x)+1)p_t^\beta(y)dy &\le (2\pi)^{(1-\beta)d/2}\beta^{-d/2}t^{(1-\beta)d/2}\Big((t/\beta)^{-\alpha/2}\int_{\mathbb{R}^d}\gamma_h(y)p_1(y)dy + 1\Big)\\ &\le Ct^{(1-\beta)d/2}(t^{-\alpha/2}+1). \end{aligned}\tag{2.10}$$

Thus, (2.7) is proved. □

Brownian bridge: Let $B(s)$ or B_s be a d-dimensional standard Brownian motion on $\mathbb{R}_+$, which is independent of V. Set $B_s^x := B_s + x$ as a Brownian motion starting from point $x \in \mathbb{R}^d$. Moreover, for $t > 0$, the d-dimensional standard Brownian bridge is defined as

$$B_{0,t}(s) := B_s - \frac{s}{t}B_t, \quad \forall s \in [0,t]. \tag{2.11}$$

For $0 \le s \le t$ and $x, y \in \mathbb{R}^d$, write $a_{s,t}^{x,y} := \frac{t-s}{t}x + \frac{s}{t}y$. Based on the notations $B_{0,t}$ and $a_{s,t}^{x,y}$, the Brownian bridge from x to y is defined as

$$B_{0,t}^{x,y}(s) := B_{0,t}(s) + a_{s,t}^{x,y}, \quad \forall s \in [0,t]. \tag{2.12}$$

Write $B^0 = B$ and $B_{0,t}^{0,0} = B_{0,t}$ without ambiguity.

By the relation that $a_{s,t}^{x,y} = a_{t-s,t}^{y,x}$ and the computations of covariances, it can be directly checked that the following two elementary lemmas hold.

Lemma 2.3. *$\{B_{0,t}^{x,y}(s)\}_{s\in[0,t]}$ is identically distributed as $\{B_{0,t}^{y,x}(t-s)\}_{s\in[0,t]}$.*

Lemma 2.4. *$\{B_{0,t}^{x,y}(s)\}_{s\in[0,t]}$ is independent of $\{B^x(s)\}_{s\ge t}$.*

Based on Lemma 2.4, we obtain a decomposition of the Brownian bridge.

Lemma 2.5. *For $0 < t_2 < t_1$ and $0 \le r \le t_2$, let*

$$G_{t_2,t_1} := \frac{B(t_2)}{t_2} - \frac{B(t_1)}{t_1}. \tag{2.13}$$

Then,

$$B_{0,t_1}(r) = B_{0,t_2}(r) + rG_{t_2,t_1}, \tag{2.14}$$

where G_{t_2,t_1} is independent of $\{B_{0,t_2}(r)\}_{r\in[0,t_2]}$ and $G_{t_2,t_1} \sim N\Big(0, \frac{t_1-t_2}{t_2t_1}\Big)$.

Lemma 2.6. *Let F be a nonnegative measurable functional on $C([0,\lambda t])$, where $C([0,\lambda t])$ is the space composed of continuous functions on $[0,\lambda t]$ for $t > 0$ and $\lambda \in (0,1)$. Then,*

$$\mathbb{E}F(\{B_{0,t}(s)\}_{0\le s\le\lambda t}) \le (1-\lambda)^{-d/2}\mathbb{E}F(\{B(s)\}_{0\le s\le\lambda t}). \tag{2.15}$$

Proof. Using [[22], (2.8)] in the case of $x = y = 0$ and the nonnegativity of F, we obtain

$$\begin{aligned}\mathbb{E}F(\{B_{0,t}(s)\}_{0\le s\le\lambda t}) &= (1-\lambda)^{-d/2}\mathbb{E}\Big[F(\{B(s)\}_{0\le s\le\lambda t})\exp\Big\{-\frac{|B(\lambda t)|^2}{2(1-\lambda)t}\Big\}\Big]\\ &\le (1-\lambda)^{-d/2}\mathbb{E}F(\{B(s)\}_{0\le s\le\lambda t}).\end{aligned}\tag{2.16}$$

Thus, the proof is completed. □

3. Feynman-Kac representations

When u_0 is a measure satisfying (1.5), we consider the following Feynman-Kac formula:

$$u_\theta(t,x) := \int_{\mathbb{R}^d} \mathbb{E}_B \exp\Big\{\theta \int_0^t V(B_{0,t}^{x,y}(s))ds\Big\} p_t(y-x)u_0(dy), \tag{2.1}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Here, $B_{0,t}^{x,y}$ is the d-dimensional Brownian bridge from x to y, and the integral $\int_0^t V(B_{0,t}^{x,y}(s))ds$ is defined as a $L^2(\Omega)$-limit, that is,

$$\int_0^t V(B_{0,t}^{x,y}(s))ds := \lim_{\varepsilon\to 0} \int_0^t V_\varepsilon(B_{0,t}^{x,y}(s))ds, \qquad \forall (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^{2d}, \tag{2.2}$$

where we set $V_\varepsilon(x) := \langle V(\cdot), p_{2\varepsilon}(x-\cdot)\rangle$ with $p_{2\varepsilon}(x) = (4\pi\varepsilon)^{-d/2}e^{-|x|^2/(4\varepsilon)}$. To simplify it, we also use the notation

$$\hat{V}_{x,y}(t) = \int_0^t V(B_{0,t}^{x,y}(s))ds. \tag{2.3}$$

We will prove the well-definiteness of $\hat{V}_{x,y}(t)$ in Lemma 3.1. Based on it, if the Feynman-Kac formula $u_\theta(t,x)$ is a $L^1(\Omega)$-integrable stochastic process, we call $u_\theta(t,x)$ well-defined, which will be proved in Corollary 4.2.

When u_0 is a measurable function, (2.1) is rewritten as

$$u_\theta(t,x) := \mathbb{E}_B\Big[\exp\Big\{\theta\int_0^t V(B_s^x)ds\Big\}u_0(B_t^x)\Big], \tag{2.4}$$

where B_t^x is a d-dimensional Brownian motion at starting point $x \in \mathbb{R}^d$, and the integral $\int_0^t V(B_s^x)ds$ is similarly defined, like (2.2).

Let $\mathbb{E}_V$ be the expectation with respect to V, and $\mathbb{E}_B$ be the expectation with respect to B. Then, by the independence between V and B, $\mathbb{E}$ can be represented as $\mathbb{E}_B \otimes \mathbb{E}_V$. Conditioning on the Brownian motion, $\hat{V}_{x,y}(t)$ is a centered Gaussian process with conditional covariance

$$\mathbb{E}_V[\hat{V}_{x,y_1}(t)\hat{V}_{x,y_2}(t)] = \int_0^t\int_0^t k(B_{0,t}^{x,y_1}(s), B_{0,t}^{x,y_2}(r))dsdr, \qquad \forall y_1,y_2 \in \mathbb{R}^d. \tag{2.5}$$

Let $\{B_j; j=1,\cdots,n\}$ be a family of d-dimensional independent standard Brownian motions for positive integer n. Set $\{B_{j,0,t}^{x,y}(s) := B_j(s) - \frac{s}{t}B_j(t) + a_{s,t}^{x,y}, \forall s \in [0,t]; j=1,\cdots,n\}$ as a family of independent Brownian bridges from x to y. Then, based on (2.1) and (2.5), the n-order Feynman-Kac moment representation satisfies that

$$\mathbb{E}u_\theta^n(t,x) = \int_{\mathbb{R}^{dn}} \mathbb{E}\exp\Big\{\frac{\theta^2}{2}\sum_{j,k=1}^n \int_0^t\int_0^t k\Big(B_{j,0,t}^{x,y_j}(s), B_{k,0,t}^{x,y_k}(r)\Big)drds\Big\}\prod_{j=1}^n p_t(y_j-x)u_0(dy_1)\cdots u_0(dy_n). \tag{2.6}$$

Similar to (2.1), for $(t,s,x) \in (0,\infty)^2 \times \mathbb{R}^d$, we define the Feynman-Kac functional $\bar{u}_\theta(t,s,x)$ as

$$\bar{u}_\theta(t,s,x) := \int_{\mathbb{R}^d} \mathbb{E}_B \exp\Big\{\theta\hat{V}_{x,y}(t)\Big\}\, p_s(y-x)|u_0|(dy). \tag{2.7}$$

When $s = t$, we write $\bar{u}_\theta(t,x) := \bar{u}_\theta(t,t,x)$. Through (2.5) and (2.7), we can obtain the n-order moment representation

$$\mathbb{E}\bar{u}_\theta^n(t,s,x) = \int_{\mathbb{R}^{dn}} \mathbb{E}\exp\left\{\frac{\theta^2}{2}\sum_{j,k=1}^n \int_0^t\int_0^t k\Big(B_{j,0,t}^{x,y_j}(s), B_{k,0,t}^{x,y_k}(r)\Big)drds\right\}\prod_{j=1}^n p_s(y_j - x)|u_0|(dy_1)\cdots|u_0|(dy_n). \tag{2.8}$$

Lemma 3.1. *If condition* (1.3) *holds, then* $\hat{V}_{x,y}(t)$ *in* (2.2) *is well-defined.*

Proof. By the similar method to [[16], Proposition 4.2.] and [[17], Proposition 3.1.], we only need to show that for $T > 0$,

$$\sup_{\varepsilon>0}\sup_{(t,x,y)\in[0,T]\times\mathbb{R}^{2d}} \mathbb{E}\left|\int_0^t V_\varepsilon(B_{0,t}^{x,y}(s))ds\right|^2 < \infty. \tag{2.9}$$

In fact, using the inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ and the integral substitution, we obtain

$$\begin{aligned}
\mathbb{E}\left|\int_0^t V_\varepsilon(B_{0,t}^{x,y}(s))ds\right|^2 &\le 2\mathbb{E}\left|\int_0^{t/2} V_{\varepsilon_1}(B_{0,t}^{x,y}(s))ds\right|^2 + 2\mathbb{E}\left|\int_{t/2}^t V_{\varepsilon_1}(B_{0,t}^{x,y}(s))ds\right|^2\\
&\le 2\mathbb{E}\left|\int_0^{t/2} V_{\varepsilon_1}(B_{0,t}^{x,y}(s))ds\right|^2 + 2\mathbb{E}\left|\int_0^{t/2} V_{\varepsilon_1}(B_{0,t}^{x,y}(t-s))ds\right|^2\\
&\le 2\mathbb{E}\left|\int_0^{t/2} V_\varepsilon(B_{0,t}^{x,y}(s))ds\right|^2 + 2\mathbb{E}\left|\int_0^{t/2} V_{\varepsilon_1}(B_{0,t}^{y,x}(s))ds\right|^2,
\end{aligned} \tag{2.10}$$

where the last step is due to $\{B_{0,t}^{y,x}(s)\}_{s\in[0,t]} \overset{d}{=} \{B_{0,t}^{x,y}(t-s)\}_{s\in[0,t]}$. Notice that the above two terms are similar, and we only need to show the estimates of the first term. Recall that $a_{s,t}^{x,y} = \frac{t-s}{t}x + \frac{s}{t}y$. Then, by Lemma 2.6 for $\{B_{0,t}(s)\}_{s\in[0,t/2]}$ and the integral substitution, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^{t/2} V_\varepsilon(B_{0,t}^{x,y}(s))ds\right|^2\\
&\le 2^{d/2}\mathbb{E}_B\left[\mathbb{E}_V\left|\int_0^{t/2} V_{\varepsilon_1}(B(s)+a_{s,t}^{x,y})ds\right|^2\right]\\
&\le 2^{d/2}\mathbb{E}\int_0^{t/2}\int_0^{t/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x_1+B(s)+a_{s,t}^{x,y}, y_1+B(r)+a_{r,t}^{x,y})p_\varepsilon(x_1)p_\varepsilon(y_1)dx_1dy_1dsdr\\
&\lesssim 2^{d/2}\mathbb{E}\int_0^{t/2}\int_0^{t/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\gamma_h(x_1+B(s)+a_{s,t}^{x,y}-y_1-B(r)-a_{r,t}^{x,y})+1)p_\varepsilon(x_1)p_\varepsilon(y_1)dx_1dy_1dsdr\\
&\lesssim \int_0^{t/2}\int_0^{t/2}\int_{\mathbb{R}^d} \mathbb{E}e^{i\xi\cdot(B(s)-B(r))}e^{i\xi\cdot(a_{s,t}^{x,y}-a_{r,t}^{x,y})}e^{-\varepsilon|\xi|^2}\mu_h(d\xi)dsdr + t^2\\
&\lesssim \int_0^{t/2}\int_0^{t/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|s-r||\xi|^2}\mu_h(d\xi)dsdr + t^2\\
&\lesssim t^{2-\frac{\alpha}{2}} + t^2,
\end{aligned} \tag{2.11}$$

where the second to last step is due to (2.2), $\mathcal{F}p_{2\varepsilon}(\xi) = e^{-\varepsilon|\xi|^2}$, and $|e^{ia}| = 1$, and the last step is due to (2.8), $\gamma_h = \mathcal{F}\mu_h$, and $\mu_h(d(r\xi)) = r^\alpha\mu_h(d\xi)$ for all $r > 0$.

Finally, substituting (2.11) into (2.10), and by $\alpha < 2$, we can obtain that

$$\sup_{\varepsilon>0}\ \sup_{(t,x,y)\in[0,T]\times\mathbb{R}^{2d}} \mathbb{E}\left|\int_0^t V_\varepsilon(B_{0,t}^{x,y}(s))ds\right|^2 \lesssim T^{2-\frac{\alpha}{2}} + T^2, \tag{2.12}$$

which shows that (2.9) holds. □

4. Moment estimates of Feynman-Kac formulas

Lemma 4.1. *Under condition* (1.3)*, there exist some* $C > 0$ *dependent on* $k(x,y)$ *such that for all* $\theta, t > 0$ *and* $n \in \mathbb{N}_+$*,*

$$\mathbb{E}\exp\left\{\theta^2 \sum_{j,k=1}^n \int_0^t \int_0^t \gamma_h\Big(B_j(s) - B_k(r)\Big)drds\right\} \le C^n \exp\left\{C\theta^{\frac{4}{2-\alpha}} t^{\frac{4-\alpha}{2-\alpha}} n^{\frac{4-\alpha}{2-\alpha}}\right\}. \tag{2.1}$$

Proof. By (2.2), the Jensen inequality, and the independence of $\{B_j\}_{1\le j\le n}$, we have

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta^2 \sum_{j,k=1}^n \int_0^t \int_0^t \gamma_h(B_j(s) - B_k(r))dsdr\right\}\\
&= \mathbb{E}\exp\left\{\theta^2 n^2 \int_{\mathbb{R}^d} \left|\frac{1}{n}\sum_{j=1}^n \int_0^t e^{i\xi\cdot B_j(s)}ds\right|^2 \mu_h(d\xi)\right\}\\
&\le \mathbb{E}\exp\left\{\theta^2 n \sum_{j=1}^n \int_{\mathbb{R}^d} \left|\int_0^t e^{i\xi\cdot B_j(s)}ds\right|^2 \mu_h(d\xi)\right\}\\
&\le \left(\mathbb{E}\exp\left\{\theta^2 n \int_{\mathbb{R}^d} \left|\int_0^t e^{i\xi\cdot B(s)}ds\right|^2 \mu_h(d\xi)\right\}\right)^n.
\end{aligned} \tag{2.2}$$

By Brownian scaling $\{B(rs)\}_{s\in\mathbb{R}_+} \overset{d}{=} \{r^{1/2}B(s)\}_{s\in\mathbb{R}_+}$ and $\mu_h(d(r\xi)) = r^\alpha \mu_h(d\xi)$ for any $r > 0$ and the integral substitution, we find that for $r > 0$,

$$\int_{\mathbb{R}^d} \left|\int_0^t e^{i\xi\cdot B(s)}ds\right|^2 \mu_h(d\xi) \overset{d}{=} r^{\frac{\alpha}{2}-2} \int_{\mathbb{R}^d} \left|\int_0^{rt} e^{i\xi\cdot B(s)}ds\right|^2 \mu_h(d\xi). \tag{2.3}$$

Set process $A_t := \int_{\mathbb{R}^d} |\int_0^t e^{i\xi\cdot B(s)}ds|^2 \mu_h(d\xi)$. Then, taking $r = (\theta^2 nt)^{\frac{2}{2-\alpha}}$ and using (2.3), we obtain

$$\mathbb{E}\exp\left\{\theta^2 n A_t\right\} = \mathbb{E}\exp\left\{(rt)^{-1} A_{rt}\right\}. \tag{2.4}$$

Using the similar methods to [[17], (3.20)], we find that there exist some $C > 0$ such that for all $t > 0$,

$$\mathbb{E}\exp\left\{(rt)^{-1} A_{rt}\right\} \le Ce^{Crt}, \tag{2.5}$$

by Lemma 2.2 in [17]. At last, summing up (2.2), (2.4), and (2.5), and using $(\theta^2 nt)^{\frac{2}{2-\alpha}}$ instead of r, the proof of (2.5) can be completed. □

Proposition 4.1. *Under conditions* (1.3) *and* (1.5)*, there exist some* $C > 0$ *such that for all* $t, s, \theta > 0$, $x \in \mathbb{R}^d$*, and* $n \in \mathbb{N}_+$,

$$\mathbb{E}\bar{u}_\theta^n(t,s,x) \le C^n e^{C\theta^2 n^2 t^2} \exp\Big\{C\theta^{\frac{4}{2-\alpha}} t^{\frac{4-\alpha}{2-\alpha}} n^{\frac{4-\alpha}{2-\alpha}}\Big\}\big(p_s * |u_0|(x)\big)^n. \tag{2.6}$$

Proof. By (2.8) and (1.3), we obtain

$$\mathbb{E}\bar{u}_\theta^n(t,s,x) \le e^{C\theta^2 n^2 t^2} \int_{\mathbb{R}^{dn}} \mathbb{E}\exp\Big\{C\theta^2 \sum_{j,k=1}^n \int_0^t \int_0^t \gamma_h\Big(B_{j,0,t}^{x,y_j}(s) - B_{k,0,t}^{x,y_k}(r)\Big)drds\Big\}$$
$$\cdot \prod_{j=1}^n p_s(y_j - x)|u_0|(dy_1)\cdots|u_0|(dy_n). \tag{2.7}$$

By (2.2) and the inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$, we obtain

$$\sum_{j,k=1}^n \int_0^t \int_0^t \gamma_h\Big(B_{j,0,t}^{x,y_j}(s) - B_{k,0,t}^{x,y_k}(r)\Big)drds \le 2\int_{\mathbb{R}^d}\Big|\sum_{j=1}^n \int_0^{t/2} e^{i\xi\cdot B_{j,0,t}^{x,y_j}(s)}ds\Big|^2 \mu_h(d\xi)$$
$$+ 2\int_{\mathbb{R}^d}\Big|\sum_{j=1}^n \int_{t/2}^{t} e^{i\xi\cdot B_{j,0,t}^{x,y_j}(s)}ds\Big|^2 \mu_h(d\xi). \tag{2.8}$$

In addition, by the integral substitution and $\{B_{j,0,t}^{x,y}(s)\}_{s\in[0,t]} \overset{d}{=} \{B_{j,0,t}^{y,x}(t-s)\}_{s\in[0,t]}$, we have

$$\int_{\mathbb{R}^d}\Big|\sum_{j=1}^n \int_{t/2}^{t} e^{i\xi\cdot B_{j,0,t}^{x,y_j}(s)}ds\Big|^2 \mu_h(d\xi) \overset{d}{=} \int_{\mathbb{R}^d}\Big|\sum_{j=1}^n \int_0^{t/2} e^{i\xi\cdot B_{j,0,t}^{y_j,x}(s)}ds\Big|^2 \mu_h(d\xi). \tag{2.9}$$

Recall that $a_{s,t}^{x,y} = \frac{t-s}{t}x + \frac{s}{t}y$. To substitute (2.8) and (2.9) into (2.7), and by using (2.2) and the Cauchy-Schwartz inequality, we obtain

$$\mathbb{E}\bar{u}_\theta^n(t,s,x) \le e^{C\theta^2 n^2 t^2}\Big(\int_{\mathbb{R}^{dn}} \mathbb{E}\exp\Big\{C\theta^2 \sum_{j,k=1}^n \int_0^{t/2}\int_0^{t/2} \gamma_h\Big(B_{j,0,t}(s) - B_{k,0,t}(r) + a_{s,t}^{x,y_j} - a_{r,t}^{x,y_k}\Big)drds\Big\}$$
$$\cdot \prod_{j=1}^n p_s(y_j - x)|u_0|(y_j)dy_1\cdots dy_n\Big)^{1/2}$$
$$\cdot\Big(\int_{\mathbb{R}^{dn}} \mathbb{E}\exp\Big\{C\theta^2 \sum_{j,k=1}^n \int_0^{t/2}\int_0^{t/2} \gamma_h\Big(B_{j,0,t}(s) - B_{k,0,t}(r) + a_{s,t}^{y_j,x} - a_{r,t}^{y_k,x}\Big)drds\Big\}$$
$$\cdot \prod_{j=1}^n p_s(y_j - x)|u_0|(y_j)dy_1\cdots dy_n\Big)^{1/2}. \tag{2.10}$$

Let $a(s,r,t,x,y,z)$ be a measurable fucntion from $\mathbb{R}_+^3 \times \mathbb{R}^{3d}$ to $\mathbb{R}^d$. We claim that for all $t, \theta > 0$ and $x, y_1, \cdots, y_n \in \mathbb{R}^d$, it holds that

$$\mathbb{E}\exp\Big\{\theta^2 \sum_{j,k=1}^n \int_0^t\int_0^t \gamma_h(B_{j,0,t}(s) - B_{k,0,t}(r) + a(s,r,t,x,y_j,y_k))drds\Big\}$$

$$\leq \mathbb{E}\exp\Big\{\theta^2\sum_{j,k=1}^{n}\int_0^t\int_0^t\gamma_h(B_{j,0,t}(s)-B_{k,0,t}(r))dsdr\Big\}. \tag{2.11}$$

In fact, through the Taylor expansion, we only need to compare their m-order moments. Precisely, using (2.2), we find that for any positive integer m,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{j,k=1}^{n}\int_0^t\int_0^t\gamma_h\Big(B_{j,0,t}(s)-B_{k,0,t}(r)+a(s,r,t,x,y_j,y_k)\Big)dsdr\Big]^m\\
&=\int_{\mathbb{R}^{dm}}\int_{[0,t]^m}\int_{[0,t]^m}\sum_{j_l,\cdots,j_m=1}^{n}\sum_{k_l,\cdots,k_m=1}^{n}\mathbb{E}\prod_{l=1}^{m}e^{i\xi_l\cdot(B_{j_l,0,t}(s_l)-B_{k_l,0,t}(r_l))}\\
&\quad\cdot\prod_{l=1}^{m}e^{i\xi_l\cdot a(s,r,t,x,y_j,y_k)}ds_1\cdots ds_m dr_1\cdots dr_m\mu_h(d\xi_1)\cdots\mu_h(d\xi_m)\\
&\leq\mathbb{E}\Big[\sum_{j,k=1}^{n}\int_0^t\int_0^t\gamma_h(B_{0,t}(s)-B_{0,t}(r))dsdr\Big]^m.
\end{aligned}$$

Here in the last inequality, we have used $|e^{ia}|=1$, the nonnegativity of μ_h, and the fact that

$$\mathbb{E}\prod_{j=1}^{m}e^{i\xi_j\cdot(B_{j_l,0,t}(s_l)-B_{k_l,0,t}(r_l))}=\exp\Big\{-\frac{1}{2}Var\Big(\sum_{j=1}^{m}\xi_j\cdot(B_{j_l,0,t}(s_l)-B_{k_l,0,t}(r_l))\Big)\Big\}\geq 0.$$

Then, by (2.10), (2.11), and Lemma 2.6, we obtain

$$\begin{aligned}
\mathbb{E}\bar{u}_\theta^n(t,s,x)&\leq e^{C\theta^2n^2t^2}\mathbb{E}\exp\Big\{C\theta^2\sum_{j,k=1}^{n}\int_0^{t/2}\int_0^{t/2}\gamma_h\Big(B_{j,0,t}(s)-B_{k,0,t}(r)\Big)drds\Big\}(p_s*|u_0|(x))^n\\
&\leq 2^{d/2}e^{C\theta^2n^2t^2}\mathbb{E}\exp\Big\{C\theta^2\sum_{j,k=1}^{n}\int_0^{t/2}\int_0^{t/2}\gamma_h\Big(B_j(s)-B_k(r)\Big)drds\Big\}(p_s*|u_0|(x))^n\\
&\leq C^ne^{C\theta^2n^2t^2}\exp\Big\{C\theta^{\frac{4}{2-\alpha}}t^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\Big\}(p_s*|u_0|(x))^n,
\end{aligned} \tag{2.12}$$

where the last step is due to Lemma 4.1. Hence, we complete the proof of (2.6). □

Corollary 4.1. *Under conditions* (1.3) *and* (1.5)*, there exist some $C>0$ such that for all $t,\theta>0$, $x\in\mathbb{R}^d$, and $n\in\mathbb{N}_+$,*

$$\mathbb{E}|u_\theta(t,x)|^n\leq C^ne^{C\theta^2n^2t^2}\exp\Big\{C\theta^{\frac{4}{2-\alpha}}t^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\Big\}(p_t*|u_0|(x))^n. \tag{2.13}$$

Proof. By the Cauchy-Schwartz inequality and (2.6), it is readily checked that

$$\mathbb{E}|u_\theta(t,x)|^n\leq(\mathbb{E}u_\theta^{2n}(t,x))^{1/2}\leq(\mathbb{E}\bar{u}_\theta^{2n}(t,x))^{1/2}. \tag{2.14}$$

Recalling $\bar{u}_\theta(t,x):=\bar{u}_\theta(t,t,x)$, and by (2.14) and Proposition 4.1, we complete the proof of (2.13). □

By Proposition 4.1 and Corollary 4.1, we directly obtain the following result.

Corollary 4.2. *Under conditions* (1.3) *and* (1.5)*, for $t,s>0$, $x\in\mathbb{R}^d$, and $n\in\mathbb{N}_+$, $\bar{u}_\theta(t,s,x)$ and $u_\theta(t,x)$ are well-defined as the $L^n(\Omega)$-integrable stochastic processes.*

5. Hölder continuity on time variable

In this section, we will prove Theorem 1.1. Before it, the following results are required.

Proposition 5.1. *Under conditions* (1.3) *and* (1.5)*, for all* $t \geq s > 0$*,* $n \in \mathbb{N}_+$*, and* $x \in \mathbb{R}^d$*,*

$$\begin{aligned}
&\mathbb{E}\Big|\int_{\mathbb{R}^d} \mathbb{E}_B[\exp\{\hat{V}_{x,z}(t)\} - \exp\{\hat{V}_{x,z}(s)\}]p_t(z-x)u_0(dz)\Big|^n \\
&\leq 2^{n-1}\theta^n((2n-1)!!)^{1/2}\left\{(\mathbb{E}\bar{u}^n_{2\theta}(t,x))^{1/2} + (\mathbb{E}\bar{u}^n_{2\theta}(s,t,x))^{1/2}\right\} \\
&\quad \cdot \left(\int_{\mathbb{R}^d} \mathbb{E}\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^2 p_t(z-x)|u_0|(dz)\right)^{n/2}.
\end{aligned} \tag{2.1}$$

Proof. Using the inequalities $|e^a - e^b| \leq |a-b|(e^a + e^b)$, $(|a| + |b|)^n \leq 2^{n-1}(|a|^n + |b|^n)$ and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big|\int_{\mathbb{R}^d} \mathbb{E}_B[\exp\{\hat{V}_{x,z}(t)\} - \exp\{\hat{V}_{x,z}(s)\}]p_t(z-x)u_0(dz)\Big|^n \\
&\leq \theta^n\mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{E}_B\Big[\big(\exp\{\theta\hat{V}_{x,z}(t)\} + \exp\{\theta\hat{V}_{x,z}(s)\}\big)\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|\Big]p_t(z-x)|u_0|(dz)\right]^n \\
&\leq 2^{n-1}\theta^n\mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{E}_B\left[\exp\{\theta\hat{V}_{x,z}(t)\}\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|\right] p_t(z-x)|u_0|(dz)\right]^n \\
&\quad + 2^{n-1}\theta^n\mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{E}_B\left[\exp\{\theta\hat{V}_{x,z}(s)\}\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|\right] p_t(z-x)|u_0|(dz)\right]^n \\
&\leq 2^{n-1}\theta^n\mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} \mathbb{E}_B \exp\{2\theta\hat{V}_{x,z}(t)\}p_t(z-x)|u_0|(dz)\Big)^{1/2} \\
&\quad \cdot \Big(\int_{\mathbb{R}^d} \mathbb{E}_B\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^2 p_t(z-x)|u_0|(dz)\Big)^{1/2}\Big]^n \\
&\quad + 2^{n-1}\theta^n\mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} \mathbb{E}_B \exp\{2\theta\hat{V}_{x,z}(s)\}p_t(z-x)|u_0|(dz)\Big)^{1/2} \\
&\quad \cdot \Big(\int_{\mathbb{R}^d} \mathbb{E}_B\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^2 p_t(z-x)|u_0|(dz)\Big)^{1/2}\Big]^n \\
&\leq 2^{n-1}\theta^n\left\{(\mathbb{E}\bar{u}^n_{2\theta}(t,x))^{1/2} + (\mathbb{E}\bar{u}^n_{2\theta}(s,t,x))^{1/2}\right\} \\
&\quad \cdot \left\{\mathbb{E}_V\Big[\int_{\mathbb{R}^d} \mathbb{E}_B\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^2 p_t(z-x)|u_0|(dz)\Big]^n\right\}^{1/2}.
\end{aligned} \tag{2.2}$$

Using the Minkowsky integral inequality and (conditional) Gaussian variance property, we get

$$\begin{aligned}
&\left\{\mathbb{E}_V\Big[\int_{\mathbb{R}^d} \mathbb{E}_B\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^2 p_t(z-x)|u_0|(dz)\Big]^n\right\}^{1/2} \\
&\leq \left(\int_{\mathbb{R}^d} \mathbb{E}_B\Big[\mathbb{E}_V\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^{2n}\Big]^{\frac{1}{n}} p_t(z-x)|u_0|(dz)\right)^{n/2} \\
&\leq ((2n-1)!!)^{1/2}\left(\int_{\mathbb{R}^d} \mathbb{E}\Big|\hat{V}_{x,z}(t) - \hat{V}_{x,z}(s)\Big|^2 p_t(z-x)|u_0|(dz)\right)^{n/2}.
\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we can complete the proof of (2.1). □

Proposition 5.2. *Under condition* (1.3)*, there exists a* $C > 0$ *dependent on* α *such that for all* $x, z \in \mathbb{R}^d$*,* $T > 1$*, and* $0 \le s \le t \le T$*,*

$$\mathbb{E}\left|\int_s^t V(B_{0,t}^{x,z}(r))dr\right|^2 \le CT^{\alpha/2}|t-s|^{2-\alpha/2}. \tag{2.4}$$

Proof. Case I: $t/2 \le s \le t$. Recall that $a_{r,t}^{z,x} := \frac{t-r}{t}z + \frac{r}{t}x$ and $B_{0,t}^{z,x}(r) = B_{0,t}(r) + a_{r,t}^{z,x}$. Then, by the integral substitution, $\{B_{0,t}^{x,z}(s)\}_{s\in[0,t]} \overset{d}{=} \{B_{0,t}^{z,x}(t-s)\}_{s\in[0,t]}$, and Lemma 2.6, we get

$$\begin{aligned}
\mathbb{E}\left|\int_s^t V(B_{0,t}^{x,z}(r))dr\right|^2 &= \mathbb{E}\left|\int_0^{t-s} V(B_{0,t}(r) + a_{r,t}^{z,x})dr\right|^2 \\
&\le (\frac{t}{s})^{d/2}\mathbb{E}\left|\int_0^{t-s} V(B(r) + a_{r,t}^{z,x})dr\right|^2 \\
&\le 2^{d/2}\int_0^{t-s}\int_0^{t-s} \mathbb{E}k(B(r_1) + a_{r_1,t}^{z,x}, B(r_2) + a_{r_2,t}^{z,x})dr_1dr_2 \\
&\lesssim \int_0^{t-s}\int_0^{t-s} \mathbb{E}[\gamma_h(B(r_1) + a_{r_1,t}^{z,x} - B(r_2) - a_{r_2,t}^{z,x}) + 1]dr_1dr_2 \\
&\lesssim \int_0^{t-s}\int_0^{t-s}\int_{\mathbb{R}^d} (\gamma_h(y + a_{r_1,t}^{z,x} - a_{r_2,t}^{z,x}) + 1)p_{|r_1-r_2|}(y)dydr_1dr_2, \qquad (2.5)
\end{aligned}$$

where the second to last step is due to (1.3). By Lemma 2.2, we have

$$\int_{\mathbb{R}^d} (\gamma_h(y + a_{r_1,t}^{z,x} - a_{r_2,t}^{z,x}) + 1)p_{|r_1-r_2|}(y)dy \lesssim (|r_1 - r_2|^{-\alpha/2} + 1). \tag{2.6}$$

Substituting (2.6) into (2.5), it is obtained that

$$\begin{aligned}
\mathbb{E}\left|\int_s^t V(B_{0,t}^{x,z}(r))dr\right|^2 &\lesssim \int_0^{t-s}\int_0^{t-s} (|r_1 - r_2|^{-\alpha/2} + 1)dr_1dr_2 \\
&\lesssim \left((1-\alpha/2)^{-1}|t-s|^{2-\alpha/2} + |t-s|^2\right) \\
&\le CT^{\alpha/2}|t-s|^{2-\alpha/2}, \qquad (2.7)
\end{aligned}$$

by the relations that $s \le t \le T$, $T > 1$, and $\alpha \in (0, 2 \wedge d)$.

Case II: $0 \le s < t/2$. From the inequality $|a + b|^2 \le 2(|a|^2 + |b|^2)$, it gives that

$$\mathbb{E}\left|\int_s^t V(B_{0,t}^{x,z}(r))dr\right|^2 \le 2\mathbb{E}\left|\int_{t/2}^t V(B_{0,t}^{x,z}(r))dr\right|^2 + 2\mathbb{E}\left|\int_s^{t/2} V(B_{0,t}^{x,z}(r))dr\right|^2. \tag{2.8}$$

Using Lemma 2.6, (1.3), and the integral substitution, we have

$$\begin{aligned}
\mathbb{E}\left|\int_s^{t/2} V(B_{0,t}^{x,z}(r))dr\right|^2 &\le 2^{d/2}\mathbb{E}\left|\int_s^{t/2} V(B(r) + a_{r,t}^{z,x})dr\right|^2 \\
&\lesssim \int_s^{t/2}\int_s^{t/2} \mathbb{E}[\gamma_h(B(r_1) + a_{r_1,t}^{z,x} - B(r_2) - a_{r_2,t}^{z,x}) + 1]dr_1dr_2
\end{aligned}$$

$$\lesssim \int_0^{t/2-s}\int_0^{t/2-s}\int_{\mathbb{R}^d}(\gamma_h(y+a^{z,x}_{r_1+s,t}-a^{z,x}_{r_2+s,t})+1)p_{|r_1-r_2|}(y)dydr_1dr_2$$
$$\lesssim \int_0^{t/2-s}\int_0^{t/2-s}\int_{\mathbb{R}^d}(\gamma_h(y+a^{z,x}_{r_1+s,t}-a^{z,x}_{r_2+s,t})+1)p_{|r_1-r_2|}(y)dydr_1dr_2$$
$$\le CT^{\alpha/2}|t/2-s|^{2-\alpha/2}, \tag{2.9}$$

where we have used the computations similar to (2.7) in the last step.

To combine (2.8) with (2.7) and (2.5), it is found that

$$\mathbb{E}\left|\int_s^t V(B^{x,z}_{0,t}(r))dr\right|^2 \le CT^{\alpha/2}(|t/2|^{2-\alpha/2}+|t/2-s|^{2-\alpha/2}) \le CT^{\alpha/2}|t-s|^{2-\alpha/2}. \tag{2.10}$$

So, to sum up (2.7) and (2.10) in the above two cases, we can complete the proof. □

Proposition 5.3. *Under condition* (1.3)*, set* $\beta\in(0,1-\alpha/2)$*, and there exists* $C>0$ *dependent on* α *and* β *such that for all* $x,z\in\mathbb{R}^d$*,* $T>1$*, and* $0<s\le t\le T$*,*

$$\mathbb{E}\left|\int_0^s V(B^{x,z}_{0,t}(r))dr-\int_0^s V(B^{x,z}_{0,s}(r))dr\right|^2 \le CT^{\alpha/2+\beta}s^{2-\beta-\alpha/2}t^{-\beta}|t-s|^{\beta}(|x-z|^{2\beta}+1). \tag{2.11}$$

Proof. By $B^{x,z}_{0,t}(r)=B_{0,t}(r)+a^{x,z}_{r,t}$, Lemma 2.5, and the inequality $|a+b|^n\le 2^{n-1}(|a|^n+|b|^n)$, we have

$$\begin{aligned}
\Im &:= \mathbb{E}\left|\int_0^s V(B^{x,z}_{0,t}(r))dr-\int_0^s V(B^{x,z}_{0,s}(r))dr\right|^2\\
&= \mathbb{E}\left|\int_0^s V(B_{0,s}(r)+rG_{s,t}+a^{x,z}_{r,t})dr-\int_0^s V(B_{0,s}(r)+a^{x,z}_{r,s})dr\right|^2\\
&\le \int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^{s/2} V(B_{0,s}(r)+ry+a^{x,z}_{r,t})dr-\int_0^{s/2} V(B_{0,s}(r)+a^{x,z}_{r,s})dr\right|^2 p_{\frac{t-s}{st}}(y)dy\\
&\quad+\int_{\mathbb{R}^d}\mathbb{E}\left|\int_{s/2}^{s} V(B_{0,s}(r)+ry+a^{x,z}_{r,t})dr-\int_{s/2}^{s} V(B_{0,s}(r)+a^{x,z}_{r,s})dr\right|^2 p_{\frac{t-s}{st}}(y)dy\\
&\le \int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^{s/2} V(B_{0,s}(r)+ry+a^{x,z}_{r,t})dr-\int_0^{s/2} V(B_{0,s}(r)+a^{x,z}_{r,s})dr\right|^2 p_{\frac{t-s}{st}}(y)dy\\
&\quad+\int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^{s/2} V(B_{0,s}(r)+(s-r)y+a^{x,z}_{s-r,t})dr-\int_0^{s/2} V(B_{0,s}(r)+a^{x,z}_{s-r,s})dr\right|^2 p_{\frac{t-s}{st}}(y)dy\\
&\le 2^{d/2}\int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^{s/2} V(B(r)+ry+a^{x,z}_{r,t})dr-\int_0^{s/2} V(B(r)+a^{x,z}_{r,s})dr\right|^2 p_{\frac{t-s}{st}}(y)dy\\
&\quad+2^{d/2}\int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^{s/2} V(B(r)+(s-r)y+a^{x,z}_{s-r,t})dr-\int_0^{s/2} V(B(r)+a^{x,z}_{s-r,s})dr\right|^2 p_{\frac{t-s}{st}}(y)dy\\
&:= \Im_1+\Im_2,
\end{aligned} \tag{2.12}$$

where the second to last inequality is due to the integral substitution and $\{B_{0,t}(s)\}_{s\in[0,t]} \overset{d}{=} \{B_{0,t}(t-s)\}_{s\in[0,t]}$, and the last inequality is due to Lemma 2.6.

For $\Im_1$, using the symmetry of $k(x,y)$ and the integral substitution, it is obtained that

$$\Im_1 = 2^{d/2}\int_{\mathbb{R}^d}\int_0^{s/2}\int_0^{s/2}\mathbb{E}\Big[k(B_{r_1}+r_1y+a^{x,z}_{r_1,t}, B_{r_2}+r_2y+a^{x,z}_{r_2,t})$$

$$
\begin{aligned}
&- k(B_{r_1} + r_1 y + a^{x,z}_{r_1,t}, B_{r_2} + a^{x,z}_{r_2,s}) - k(B_{r_1} + a^{x,z}_{r_1,s}, B_{r_2} + r_2 y + a^{x,z}_{r_2,t}) \\
&+ k(B_{r_1} + a^{x,z}_{r_1,s}, B_{r_2} + a^{x,z}_{r_2,s})\Big] dr_1 dr_2 p_{\frac{t-s}{st}}(y) dy \\
= {}& 2^{d/2+1} \int_{\mathbb{R}^d} \int_0^{s/2} \int_0^{r_1} \eta^{r_1,r_2,y}_{s,t,x,z} dr_1 dr_2 p_{\frac{t-s}{st}}(y) dy,
\end{aligned} \tag{2.13}
$$

where we set

$$
\begin{aligned}
\eta^{r_1,r_2,y}_{s,t,x,z} :=& \mathbb{E}\Big[k(B_{r_1} + r_1 y + a^{x,z}_{r_1,t}, B_{r_2} + r_2 y + a^{x,z}_{r_2,t}) - k(B_{r_1} + r_1 y + a^{x,z}_{r_1,t}, B_{r_2} + a^{x,z}_{r_2,s}) \\
&- k(B_{r_1} + a^{x,z}_{r_1,s}, B_{r_2} + r_2 y + a^{x,z}_{r_2,t}) + k(B_{r_1} + a^{x,z}_{r_1,s}, B_{r_2} + a^{x,z}_{r_2,s})\Big].
\end{aligned} \tag{2.14}
$$

By $r_2 \le r_1$ and the independence of Brownian increments and the integral substitutions, we obtain

$$
\begin{aligned}
\eta^{r_1,r_2,y}_{s,t,x,z} =& \mathbb{E}\Big[k(B_{r_1} - B_{r_2} + B_{r_2} + r_1 y + a^{x,z}_{r_1,t}, B_{r_2} + r_2 y + a^{x,z}_{r_2,t}) \\
&- k(B_{r_1} - B_{r_2} + B_{r_2} + r_1 y + a^{x,z}_{r_1,t}, B_{r_2} + a^{x,z}_{r_2,s}) - k(B_{r_1} - B_{r_2} + B_{r_2} + a^{x,z}_{r_1,s}, B_{r_2} + r_2 y + a^{x,z}_{r_2,t}) \\
&+ k(B_{r_1} - B_{r_2} + B_{r_2} + a^{x,z}_{r_1,s}, B_{r_2} + a^{x,z}_{r_2,s})\Big] \\
=& \int\int_{\mathbb{R}^{2d}} k(\overline{x} + \overline{y}, \overline{y})\Big[p_{r_1-r_2}(\overline{x} + (r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\Big] \\
&\cdot \Big[p_{r_2}(\overline{y} - r_2 y - a^{x,z}_{r_2,t}) - p_{r_2}(\overline{y} - a^{x,z}_{r_2,s})\Big] d\overline{x} d\overline{y} \\
&+ \int\int_{\mathbb{R}^{2d}} k(\overline{x} + \overline{y}, \overline{y})\Big[p_{r_1-r_2}(\overline{x} + (r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s}) \\
&- p_{r_1-r_2}(\overline{x} - r_1 y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big] p_{r_2}(\overline{y} - a^{x,z}_{r_2,s}) d\overline{x} d\overline{y}.
\end{aligned} \tag{2.15}
$$

We write $b_{s,t} := (\frac{t-s}{st})^{1/2}$. To substitute (2.15) into (2.13), and by the absolute-value inequality and the integral substitutions about y, we get

$$
\begin{aligned}
\mathfrak{I}_1 \le& 2^{d/2+1} \int_{\mathbb{R}^d} \int_0^{s/2} \int_0^{r_1} \int\int_{\mathbb{R}^{2d}} |k(\overline{x} + \overline{y}, \overline{y})| \Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) \\
&- p_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\Big| \Big| p_{r_2}(\overline{y} - b_{s,t} r_2 y - a^{x,z}_{r_2,t}) - p_{r_2}(\overline{y} - a^{x,z}_{r_2,s})\Big| d\overline{x} d\overline{y} dr_1 dr_2 p_1(y) dy \\
&+ 2^{d/2+1} \int_{\mathbb{R}^d} \int_0^{s/2} \int_0^{r_1} \int\int_{\mathbb{R}^{2d}} |k(\overline{x} + \overline{y}, \overline{y})| \Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) \\
&- p_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s}) - p_{r_1-r_2}(\overline{x} - b_{s,t} r_1 y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big| \\
&\cdot p_{r_2}(\overline{y} - a^{x,z}_{r_2,s}) d\overline{x} d\overline{y} dr_1 dr_2 p_1(y) dy \\
=:& \mathfrak{I}_{11} + \mathfrak{I}_{12}.
\end{aligned} \tag{2.16}
$$

Notice that $\beta \in (0, 1)$. Thanks to (2.3) and (1.3), it holds that

$$
\begin{aligned}
\tilde{J}_{11} :=& \int\int_{\mathbb{R}^{2d}} |k(\overline{x} + \overline{y}, \overline{y})| \Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) \\
&- p_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\Big| \Big| p_{r_2}(\overline{y} - b_{s,t} r_2 y - a^{x,z}_{r_2,t}) - p_{r_2}(\overline{y} - a^{x,z}_{r_2,s})\Big| d\overline{x} d\overline{y} \\
\lesssim& \int\int_{\mathbb{R}^{2d}} (\gamma_h(\overline{x}) + 1) \Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\Big|^{1-\beta}
\end{aligned}
$$

$$\cdot \Big| p_{r_2}(\overline{y} - b_{s,t} r_2 y - a^{x,z}_{r_2,t}) - p_{r_2}(\overline{y} - a^{x,z}_{r_2,s}) \Big|^{1-\beta} d\overline{x} d\overline{y} (r_1 - r_2)^{-\beta(d+1)/2} r_2^{-\beta(d+1)/2}$$
$$\cdot | - b_{s,t} r_1 y - a^{x,z}_{r_1,t} + a^{x,z}_{r_1,s} |^{\beta} | - b_{s,t} r_2 y - a^{x,z}_{r_2,t} + a^{x,z}_{r_2,s} |^{\beta}. \tag{2.17}$$

On the one hand, by $b_{s,t} = (\frac{t-s}{st})^{1/2}$ and $a^{x,z}_{r,t} - a^{x,z}_{r,s} = \frac{(t-s)}{st}(x-z)r$, it is found that

$$| - b_{s,t} r_1 y - a^{x,z}_{r_1,t} + a^{x,z}_{r_1,s} |^{\beta} | - b_{s,t} r_2 y - a^{x,z}_{r_2,t} + a^{x,z}_{r_2,s} |^{\beta} = (\frac{t-s}{st})^{\beta} r_1^{\beta} r_2^{\beta} |y + b_{s,t}(x-z)|^{2\beta}. \tag{2.18}$$

On the other hand, notice the fact that $p_t^{\beta}(x) = (2\pi)^{(1-\beta)d/2} \beta^{-d/2} t^{(1-\beta)d/2} p_{t/\beta}(x)$. Then, by the inequality $|a+b|^{\beta} \le |a|^{\beta} + |b|^{\beta} (\beta \in [0,1])$ and Lemma 2.2, we have

$$\begin{aligned}
&\int\int_{\mathbb{R}^{2d}} (\gamma_h(\overline{x}) + 1) \Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1) y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s}) \Big|^{1-\beta} \\
&\quad \cdot \Big| p_{r_2}(\overline{y} - b_{s,t} r_2 y - a^{x,z}_{r_2,t}) - p_{r_2}(\overline{y} - a^{x,z}_{r_2,s}) \Big|^{1-\beta} d\overline{x} d\overline{y} \\
&\le \int_{\mathbb{R}^d} (\gamma_h(\overline{x}) + 1) \Big(p^{1-\beta}_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1) y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) + p^{1-\beta}_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s}) \Big) d\overline{x} \\
&\quad \cdot \int_{\mathbb{R}^d} \Big(p^{1-\beta}_{r_2}(\overline{y} - b_{s,t} r_2 y - a^{x,z}_{r_2,t}) + p^{1-\beta}_{r_2}(\overline{y} - a^{x,z}_{r_2,s}) \Big) d\overline{y} \\
&\le C(r_1 - r_2)^{\beta d/2}((r_1 - r_2)^{-\alpha/2} + 1) r_2^{\beta d/2} \int_{\mathbb{R}^d} \Big(p_{r_2/(1-\beta)}(\overline{y} - b_{s,t} r_2 y - a^{x,z}_{r_2,t}) + p_{r_2/(1-\beta)}(\overline{y} - a^{x,z}_{r_2,s}) \Big) d\overline{y} \\
&\le C(r_1 - r_2)^{\beta d/2}((r_1 - r_2)^{-\alpha/2} + 1) r_2^{\beta d/2},
\end{aligned} \tag{2.19}$$

where the last step is due to the integral substitutions about $\overline{y}$ and $\|p_t\|_{L^1(\mathbb{R}^d)} = 1$.

To substitute (2.18) and (2.19) into (2.17), we get

$$\tilde{J}_{11} \le C(\frac{t-s}{st})^{\beta} (r_1 - r_2)^{-\beta/2}((r_1 - r_2)^{-\alpha/2} + 1) r_1^{\beta} r_2^{\beta/2} |y + b_{s,t}(x-z)|^{2\beta}. \tag{2.20}$$

In addition, by the inequality $|a+b|^{2\beta} \le 2^{2\beta-1} \vee 1(|a|^{2\beta} + |b|^{2\beta})$ $(\beta \in (0,1))$,

$$\begin{aligned}
\int_{\mathbb{R}^d} |y + b_{s,t}(x-z)|^{2\beta} p_1(y) dy &\le C \int_{\mathbb{R}^d} (|y|^{2\beta} + (\frac{t-s}{st})^{\beta} |x-z|^{2\beta}) p_1(y) dy \\
&\le C((\frac{t-s}{st})^{\beta} |x-z|^{2\beta} + 1).
\end{aligned} \tag{2.21}$$

Noticing that $-\alpha/2 - \beta/2 > -1$ (i.e., $\beta < 1 - \alpha/2 < 2 - \alpha$), and by (2.20), (2.21), and the Fubini theorem,

$$\begin{aligned}
\mathfrak{I}_{11} &\le C(\frac{t-s}{st})^{\beta} \int_0^{s/2} \int_0^{r_1} (r_1 - r_2)^{-\beta/2}((r_1 - r_2)^{-\alpha/2} + 1) r_1^{\beta} r_2^{\beta/2} dr_1 dr_2 \int_{\mathbb{R}^d} |y + b_{s,t}(x-z)|^{2\beta} p_1(y) dy \\
&\le C s^{2-\alpha/2}(s^{\alpha/2} + 1) t^{-\beta}(t-s)^{\beta} \int_{\mathbb{R}^d} |y + b_{s,t}(x-z)|^{2\beta} p_1(y) dy \\
&\le C s^{2-\alpha/2}(s^{\alpha/2} + 1) t^{-\beta}(t-s)^{\beta}((\frac{t-s}{st})^{\beta} |x-z|^{2\beta} + 1).
\end{aligned} \tag{2.22}$$

For $\beta \in (0,1)$, by (1.3) and (2.4),

$$\tilde{J}_{12} := \int\int_{\mathbb{R}^{2d}} |k(\overline{x} + \overline{y}, \overline{y})| \Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1) y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + b_{s,t} r_2 y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})$$

$$
\begin{aligned}
&- p_{r_1-r_2}(\overline{x} - b_{s,t}r_1y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big| p_{r_2}(\overline{y} - a^{x,z}_{r_2,s})d\overline{x}d\overline{y}\\
&\lesssim \int\int_{\mathbb{R}^{2d}} (|\gamma_h(\overline{x})| + 1)\Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + b_{s,t}r_2y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\\
&- p_{r_1-r_2}(\overline{x} - b_{s,t}r_1y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big|^{1-\beta} p_{r_2}(\overline{y} - a^{x,z}_{r_2,s})d\overline{x}d\overline{y}\\
&(r_1 - r_2)^{-\beta d/2-\beta}|b_{s,t}r_1y + a^{x,z}_{r_1,t} - a^{x,z}_{r_1,s}|^{\beta}|b_{s,t}r_2y + a^{x,z}_{r_2,t} - a^{x,z}_{r_2,s}|^{\beta}. \qquad (2.23)
\end{aligned}
$$

Using the inequality $|a + b|^{\beta} \le |a|^{\beta} + |b|^{\beta}(\beta \in [0, 1])$ and Lemma 2.2,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (|\gamma_h(\overline{x})| + 1)\Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + b_{s,t}r_2y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\\
&- p_{r_1-r_2}(\overline{x} - b_{s,t}r_1y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big|^{1-\beta} d\overline{x}\\
&\le \int_{\mathbb{R}^d} (|\gamma_h(\overline{x})| + 1)\Big(p^{1-\beta}_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) + p^{1-\beta}_{r_1-r_2}(\overline{x} + b_{s,t}r_2y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\\
&+ p^{1-\beta}_{r_1-r_2}(\overline{x} - b_{s,t}r_1y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p^{1-\beta}_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big)d\overline{x}\\
&\le C(r_1 - r_2)^{\beta d/2}((r_1 - r_2)^{-\alpha/2} + 1). \qquad (2.24)
\end{aligned}
$$

Using the Fubini theorem for (2.23), and substituting (2.24) and (2.18) into (2.23),

$$
\begin{aligned}
\tilde{J}_{12} &\lesssim \int_{\mathbb{R}^d} (|\gamma_h(\overline{x})| + 1)\Big| p_{r_1-r_2}(\overline{x} + b_{s,t}(r_2 - r_1)y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,t}) - p_{r_1-r_2}(\overline{x} + b_{s,t}r_2y + a^{x,z}_{r_2,t} - a^{x,z}_{r_1,s})\\
&- p_{r_1-r_2}(\overline{x} - b_{s,t}r_1y + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,t}) + p_{r_1-r_2}(\overline{x} + a^{x,z}_{r_2,s} - a^{x,z}_{r_1,s})\Big|^{1-\beta} d\overline{x} \int_{\mathbb{R}^d} p_{r_2}(\overline{y} - a^{x,z}_{r_2,s})d\overline{y}\\
&(r_1 - r_2)^{-\beta d/2-\beta}|b_{s,t}r_1y + a^{x,z}_{r_1,t} - a^{x,z}_{r_1,s}|^{\beta}|b_{s,t}r_2y + a^{x,z}_{r_2,t} - a^{x,z}_{r_2,s}|^{\beta}\\
&\le C(\frac{t-s}{st})^{\beta}(r_1 - r_2)^{-\beta}((r_1 - r_2)^{-\alpha/2} + 1)r_1^{\beta}r_2^{\beta}|y + b_{s,t}(x - z)|^{2\beta}. \qquad (2.25)
\end{aligned}
$$

Recalling that $-\alpha/2 - \beta > -1$, and by (2.25), (2.21), and the similar computations to (2.22),

$$
\begin{aligned}
\mathfrak{I}_{12} &\le C(\frac{t-s}{st})^{\beta} \int_0^{s/2}\int_0^{r_1} (r_1 - r_2)^{-\beta}((r_1 - r_2)^{-\alpha/2} + 1)r_1^{\beta}r_2^{\beta}dr_1dr_2 \int_{\mathbb{R}^d} |y + b_{s,t}(x - z)|^{2\beta}p_1(y)dy\\
&\le Cs^{2-\alpha/2}(s^{\alpha/2} + 1)t^{-\beta}(t - s)^{\beta}((\frac{t-s}{st})^{\beta}|x - z|^{2\beta} + 1). \qquad (2.26)
\end{aligned}
$$

To substitute (2.22) and (2.26) into (2.16),

$$
\mathfrak{I}_1 \le Cs^{2-\alpha/2}(s^{\alpha/2} + 1)t^{-\beta}(t - s)^{\beta}((\frac{t-s}{st})^{\beta}|x - z|^{2\beta} + 1). \qquad (2.27)
$$

Notice that $\mathfrak{I}_2$ is similar to $\mathfrak{I}_1$. By $a^{x,z}_{s-r,t} - a^{x,z}_{s-r,s} = \frac{t-s}{st}(x - z)(r - s)$ and the similar computations to (2.27), we obtain

$$
\mathfrak{I}_2 \le Cs^{2-\alpha/2}(s^{\alpha/2} + 1)t^{-\beta}(t - s)^{\beta}((\frac{t-s}{st})^{\beta}|x - z|^{2\beta} + 1). \qquad (2.28)
$$

At last, substituting (2.27) and (2.28) into (2.12), and by the relations that $2-\beta-\alpha/2>0$ (because of $\beta\in(0,1-\alpha/2)$ and $\alpha\in(0,2\wedge d)$), $T>1$, and $s\le t\le T$,

$$\begin{aligned}\Im &\le Cs^{2-\alpha/2}(s^{\alpha/2}+1)t^{-\beta}(t-s)^{\beta}((\frac{t-s}{st})^{\beta}|x-z|^{2\beta}+1)\\ &\le CT^{\alpha/2+\beta}s^{2-\beta-\alpha/2}t^{-\beta}(t-s)^{\beta}(|x-z|^{2\beta}+1).\end{aligned}\tag{2.29}$$

So, we complete the proof. □

The proof of Theorem 1.1. Without loss of generality, we assume that $t\ge s$. Firstly, by (2.1), we have

$$\begin{aligned}u_\theta(t,x)-u_\theta(s,x) &= \int_{\mathbb{R}^d}\mathbb{E}_B[\exp\{\hat{V}_{x,z}(t)\}-\exp\{\hat{V}_{x,z}(s)\}]p_t(z-x)u_0(dz)\\ &\quad+\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}[p_t(z-x)-p_s(z-x)]u_0(dz).\end{aligned}\tag{2.30}$$

Then, by the inequality $|a+b|^n\le 2^{n-1}(|a|^n+|b|^n)$, we obtain

$$\begin{aligned}\mathbb{E}|u_\theta(t,x)-u_\theta(s,x)|^n &\le 2^{n-1}\mathbb{E}\left|\int_{\mathbb{R}^d}\mathbb{E}_B[\exp\{\hat{V}_{x,z}(t)\}-\exp\{\hat{V}_{x,z}(s)\}]p_t(z-x)u_0(dz)\right|^n\\ &\quad+2^{n-1}\mathbb{E}\left|\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}[p_t(z-x)-p_s(z-x)]u_0(dz)\right|^n\\ &:=\mathcal{I}_1+\mathcal{I}_2.\end{aligned}\tag{2.31}$$

In $\mathcal{I}_1$, by the elementary inequality $(a+b)^2\le 2(a^2+b^2)$, we find that for $x,z\in\mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}\left|\hat{V}_{x,z}(t)-\hat{V}_{x,z}(s)\right|^2 &= \mathbb{E}\left|\int_0^t V(B^{x,z}_{0,t}(r))dr-\int_0^s V(B^{x,z}_{0,s}(r))dr\right|^2\\ &\le 2\mathbb{E}\left|\int_s^t V(B^{x,z}_{0,t}(r))dr\right|^2+2\mathbb{E}\left|\int_0^s V(B^{x,z}_{0,t}(r))dr-\int_0^s V(B^{x,z}_{0,s}(r))dr\right|^2.\end{aligned}\tag{2.32}$$

Thanks to $\beta<1-\alpha/2$ and $\alpha>0$, it holds that $2-\alpha/2-2\beta>\alpha/2>0$. To combine (2.32) with Propositions 5.2 and 5.3, and by the relations that $T>1$ and $s\le t\le T$,

$$\begin{aligned}\mathbb{E}\left|\hat{V}_{x,z}(t)-\hat{V}_{x,z}(s)\right|^2 &\le CT^{\alpha/2}|t-s|^{2-\alpha/2}+CT^{\alpha/2+\beta}s^{2-\beta-\alpha/2}t^{-\beta}|t-s|^{\beta}(|x-z|^{2\beta}+1)\\ &\le CT^{2-\beta}|t-s|^{\beta}(|x-z|^{2\beta}+1).\end{aligned}\tag{2.33}$$

In addition, by the inequality $|a|^{2\beta}\le e^{|a|^2}(\beta\in(0,1))$, we find that

$$\int_{\mathbb{R}^d}(|x-z|^{2\beta}+1)p_t(z-x)|u_0|(dz)\lesssim t^{\beta}p_{t/(1-\beta)}*|u_0|(x)+p_t*|u_0|(x).\tag{2.34}$$

Hence, by (2.33), (2.34), and $T>1$, we find that

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}\left|\hat{V}_{x,z}(t)-\hat{V}_{x,z}(s)\right|^2 p_t(z-x)|u_0|(dz) &\le CT^{2-\beta}|t-s|^{\beta}\int_{\mathbb{R}^d}(|x-z|^{2\beta}+1)p_t(z-x)|u_0|(dz)\\ &\le CT^2\sup_{r\in[\delta,T/(1-\beta)]}p_r*|u_0|(x)|t-s|^{\beta},\end{aligned}\tag{2.35}$$

where the last step is due to $\delta \le s \le t$, too.

Using Proposition 5.1 and (2.35), we obtain

$$\mathcal{I}_1 \le C^n\theta^n((2n-1)!!)^{1/2}T^n\left\{\left(\mathbb{E}\bar{u}^n_{2\theta}(t,x)\right)^{1/2} + \left(\mathbb{E}\bar{u}^n_{2\theta}(s,t,x)\right)^{1/2}\right\} \cdot \Big(\sup_{r\in[\delta,T/(1-\beta)]} p_r * |u_0|(x)\Big)^{n/2}|t-s|^{\beta n/2}. \tag{2.36}$$

Second, from (2.5), we find that for $\beta \in (0,1)$,

$$\begin{aligned}\mathcal{I}_2 &\le C^n\mathbb{E}\Big[\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}|p_t(z-x)-p_s(z-x)||u_0|(dz)\Big]^n\\ &\le C^n(t^{-d/2-1}+s^{-d/2-1})^{\beta n}|t-s|^{\beta n}\\ &\quad\cdot\mathbb{E}\Big[\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}|p_t(z-x)-p_s(z-x)|^{1-\beta}|u_0|(dz)\Big]^n.\end{aligned} \tag{2.37}$$

Then, by the inequalities that $|a+b|^\beta \le |a|^\beta + |b|^\beta (\beta \in [0,1])$ and $|a+b|^n \le 2^{n-1}(|a|^n+|b|^n)$, and $p_t^{1-\beta}(x) = (2\pi)^{\beta d/2}(1-\beta)^{-d/2}t^{\beta d/2}p_{t/(1-\beta)}(x)$,

$$\begin{aligned}&\mathbb{E}\Big[\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}|p_t(z-x)-p_s(z-x)|^{1-\beta}|u_0|(dz)\Big]^n\\ &\le \mathbb{E}\Big[\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}\Big(p_t^{1-\beta}(z-x)+p_s^{1-\beta}(z-x)\Big)|u_0|(dz)\Big]^n\\ &\le C^n(t^{\beta d/2}+s^{\beta d/2})^n\mathbb{E}\Big[\int_{\mathbb{R}^d}\mathbb{E}_B\exp\{\hat{V}_{x,z}(s)\}(p_{\frac{t}{1-\beta}}(z-x)+p_{\frac{s}{1-\beta}}(z-x))|u_0|(dz)\Big]^n\\ &\le C^n(t^{\beta d/2}+s^{\beta d/2})^n\Big[\mathbb{E}\bar{u}^n_\theta(s,t/(1-\beta),x)+\mathbb{E}\bar{u}^n_\theta(s,s/(1-\beta),x)\Big],\end{aligned} \tag{2.38}$$

where we recall that $\bar{u}_\theta(t,s,x)$ is defined in (2.7).

To substitute (2.38) into (2.37), and by the relation $\delta \le s \le t \le T$,

$$\begin{aligned}\mathcal{I}_2 &\le C^n(t^{\beta d/2}+s^{\beta d/2})^n(t^{-d/2-1}+s^{-d/2-1})^{\beta n}\Big[\mathbb{E}\bar{u}^n_\theta(s,t/(1-\beta),x)+\mathbb{E}\bar{u}^n_\theta(s,s/(1-\beta),x)\Big]|t-s|^{\beta n}\\ &\le C^nT^{\beta dn/2}\delta^{-(d/2+1)\beta n}\Big[\mathbb{E}\bar{u}^n_\theta(s,t/(1-\beta),x)+\mathbb{E}\bar{u}^n_\theta(s,s/(1-\beta),x)\Big]|t-s|^{\beta n}.\end{aligned} \tag{2.39}$$

To combine (2.31) with (2.36) and (2.39),

$$\begin{aligned}\mathbb{E}|u_\theta(t,x)-u_\theta(s,x)|^n &\le C^n\theta^n((2n-1)!!)^{1/2}T^n\left\{\left(\mathbb{E}\bar{u}^n_{2\theta}(t,x)\right)^{1/2}+\left(\mathbb{E}\bar{u}^n_{2\theta}(s,t,x)\right)^{1/2}\right\}\\ &\quad\cdot\Big(\sup_{r\in[\delta,T/(1-\beta)]}p_r*|u_0|(x)\Big)^{n/2}|t-s|^{\beta n/2}\\ &\quad+C^nT^{\beta dn/2}\delta^{-(d/2+1)\beta n}\Big[\mathbb{E}\bar{u}^n_\theta(s,t/(1-\beta),x)+\mathbb{E}\bar{u}^n_\theta(s,s/(1-\beta),x)\Big]|t-s|^{\beta n}.\end{aligned}$$

Moreover, by Proposition 4.1 and the relations that $\delta < 1 \le T, \beta < 1$ and $\theta \le e^{\theta^2}$, we can obtain that for all $n \in \mathbb{N}_+$,

$$\begin{aligned}\mathbb{E}|u_\theta(t,x)-u_\theta(s,x)|^n &\le C^n\theta^ne^{C\theta^2n^2t^2}((2n-1)!!)^{1/2}T^n\exp\left\{C\theta^{\frac{4}{2-\alpha}}t^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\right\}\\ &\quad\cdot\Big(\sup_{r\in[\delta,T/(1-\beta)]}p_r*|u_0|(x)\Big)^n|t-s|^{\beta n/2}\end{aligned}$$

$$
\begin{aligned}
&+ C^n e^{C\theta^2 n^2 t^2} T^{\beta d n/2} \delta^{-(d/2+1)\beta n} \exp\left\{C\theta^{\frac{4}{2-\alpha}} t^{\frac{4-\alpha}{2-\alpha}} n^{\frac{4-\alpha}{2-\alpha}}\right\} \\
&\cdot \Big(\sup_{r\in[\delta,T/(1-\beta)]} p_r * |u_0|(x)\Big)^n |t-s|^{\beta n} \\
\le{}& C^n e^{C\theta^2 n^2 T^2} \exp\left\{C\theta^{\frac{4}{2-\alpha}} T^{\frac{4-\alpha}{2-\alpha}} n^{\frac{4-\alpha}{2-\alpha}}\right\} ((2n-1)!!)^{1/2} T^{(\beta d/2+1)n} \\
&\cdot \delta^{-(d/2+1)\beta n} \Big(\sup_{r\in[\delta,T/(1-\beta)]} p_r * |u_0|(x)\Big)^n |t-s|^{\beta n/2}. \qquad (2.40)
\end{aligned}
$$

At last, by (1.6), (2.40), and the classic Kolmogorov continuity theorem, we find that for all $\beta \in (0, 1-\alpha/2)$, there exists a temporal $\frac{\beta}{2}$-Hölder continuous modification of $u_\theta(t,x)$ on $[\delta, T]$. Because δ and T are any, the proof can be completed. □

The proof of Theorem 1.2. Assume that $T > 1$ and $0 \le s \le t \le T$. Let n be a positive integer.

(i) Through (2.4) and Lemma 2.4, it can be proved that

$$
\begin{aligned}
u_\theta(t,x) - u_\theta(s,x) ={}& \mathbb{E}_B\Big[\Big(\exp\Big\{\theta\int_0^t V(B_r^x)dr\Big\} - \exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}\Big)u_0(B_t^x)\Big] \\
&+ \mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_t^x)\Big] - \mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_s^x)\Big] \\
={}& \int_{\mathbb{R}^d}\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^t V(B_{0,t}^{x,z}(r))dr\Big\} - \exp\Big\{\theta\int_0^s V(B_{0,t}^{x,z}(r))dr\Big\}\Big]p_t(z-x)u_0(dz) \\
&+ \mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_t^x)\Big] - \mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_s^x)\Big]. \qquad (2.41)
\end{aligned}
$$

Next, by the similar computations to (2.31), we obtain

$$
\begin{aligned}
&\mathbb{E}|u_\theta(t,x) - u_\theta(s,x)|^n \\
\le{}& 2^{n-1}\mathbb{E}\Big|\int_{\mathbb{R}^d}\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^t V(B_{0,t}^{x,z}(r))dr\Big\} - \exp\Big\{\theta\int_0^s V(B_{0,t}^{x,z}(r))dr\Big\}\Big]p_t(z-x)u_0(dz)\Big|^n \\
&+ 2^{n-1}\mathbb{E}\Big|\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_t^x)\Big] - \mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_s^x)\Big]\Big|^n \\
:={}& \mathcal{D}_1 + \mathcal{D}_2. \qquad (2.42)
\end{aligned}
$$

For $\mathcal{D}_1$, using the method of proof similar to Proposition 5.1, it not difficult to check that

$$
\begin{aligned}
\mathcal{D}_1 \le{}& 2^{n-1}\theta^n((2n-1)!!)^{1/2}\left\{\left(\mathbb{E}\bar{u}_{2\theta}^n(t,x)\right)^{1/2} + \left(\mathbb{E}\bar{u}_{2\theta}^n(s,t,x)\right)^{1/2}\right\} \\
&\cdot\left(\int_{\mathbb{R}^d}\mathbb{E}\Big|\int_s^t V(B_{0,t}^{x,z}(r))dr\Big|^2 p_t(z-x)|u_0|(dz)\right)^{n/2}. \qquad (2.43)
\end{aligned}
$$

To associate the above (2.43) with Propositions 5.2 and 4.1,

$$
\begin{aligned}
\mathcal{D}_1 \le{}& C^n\theta^n((2n-1)!!)^{1/2}T^{\alpha n/4}\left\{\left(\mathbb{E}\bar{u}_{2\theta}^n(t,x)\right)^{1/2} + \left(\mathbb{E}\bar{u}_{2\theta}^n(s,t,x)\right)^{1/2}\right\}|t-s|^{(1-\alpha/4)n} \\
\le{}& C^n e^{C\theta^2 n^2 t^2}((2n-1)!!)^{1/2}T^{\alpha n/4}\exp\left\{C\theta^{\frac{4}{2-\alpha}} t^{\frac{4-\alpha}{2-\alpha}} n^{\frac{4-\alpha}{2-\alpha}}\right\}(p_t * |u_0|(x))^n|t-s|^{(1-\alpha/4)n}. \qquad (2.44)
\end{aligned}
$$

For $\mathcal{D}_2$, from the independence of Brownian increments and κ-Hölder continuity of u_0, it is found that

$$\begin{aligned}\mathcal{D}_2 &= 2^{n-1}\mathbb{E}\Big|\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_t-B_s+B_s^x)\Big]-\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}u_0(B_s^x)\Big]\Big|^n\\ &\le 2^{n-1}\mathbb{E}\Big|\int_{\mathbb{R}^d}\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}\Big|u_0((t-s)^{1/2}y+B_s^x)-u_0(B_s^x)\Big|\Big]p_1(y)dy\Big|^n\\ &\le C^n\mathbb{E}\Big[\mathbb{E}_B\exp\Big\{\theta\int_0^s V(B_r^x)dr\Big\}\Big]^n\Big(\int_{\mathbb{R}^d}|y|^\kappa p_1(y)dy\Big)^n(t-s)^{\kappa n/2}\\ &\le C^n e^{C\theta^2n^2t^2}\exp\Big\{C\theta^{\frac{4}{2-\alpha}}t^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\Big\}(t-s)^{\kappa n/2},\end{aligned}\tag{2.45}$$

where the last step is due to Proposition 4.1.

Notice that $0\le s\le t\le T$. To combine (2.42) with (2.44) and (2.45), it is found that for all $x\in\mathbb{R}^d$ and integer $n\ge 1$,

$$\begin{aligned}\mathbb{E}|u_\theta(t,x)-u_\theta(s,x)|^n \le\, & C^n e^{C\theta^2n^2T^2}\exp\Big\{C\theta^{\frac{4}{2-\alpha}}T^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\Big\}((2n-1)!!)^{1/2}T^{(1-\kappa/2)n}\\ &\cdot\Big(\sup_{r\in[0,T]}p_r*|u_0|(x)\Big)^n|t-s|^{\kappa n/2},\end{aligned}\tag{2.46}$$

where we have used the fact that $\kappa/2<1-\alpha/4$ for $\kappa\in(0,1]$ and $\alpha\in(0,2\wedge d)$.

So, by (2.46) and the Kolmogorov continuity theorem, we can prove the result.

(ii) By $u_0\equiv C$ and the method similar to (2.41), it is obtained that

$$u_\theta(t,x)-u_\theta(s,x)=C^n\int_{\mathbb{R}^d}\mathbb{E}_B\Big[\exp\Big\{\theta\int_0^t V(B_{0,t}^{x,z}(r))dr\Big\}-\exp\Big\{\theta\int_0^s V(B_{0,t}^{x,z}(r))dr\Big\}\Big]p_t(z-x)dz.\tag{2.47}$$

Moreover, using the computations similar to (2.44) and $0\le s\le t\le T$, we find that for all $x\in\mathbb{R}^d$ and integer $n\ge 1$,

$$\mathbb{E}|u_\theta(t,x)-u_\theta(s,x)|^n\le C^ne^{C\theta^2n^2T^2}((2n-1)!!)^{1/2}T^{\alpha n/4}\exp\Big\{C\theta^{\frac{4}{2-\alpha}}T^{\frac{4-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}}\Big\}|t-s|^{(1-\alpha/4)n}.\tag{2.48}$$

Lastly, through (2.48) and the Kolmogorov continuity theorem, we can complete the proof. □

6. Conclusions

This work mainly studies the temporal Hölder continuity for the Feynman-Kac formula of the parabolic Anderson model under the rough initial condition $p_t*|u_0|(x)<\infty$. As a comparison, we also consider the function-valued initial conditions $u_0\equiv C$ and $u_0\in C^\kappa(\mathbb{R}^d)$ with $\kappa\in(0,1]$. However, many function-valued initial data have not been considered in this paper, which will be a future work. Besides, our future work is also going to investigate the case of time-space generalized Gaussian field and rough initial condition.

Author contributions

Hui Sun: Dealt with conceptualization, supervision, formal analysis, writing-original draft, review, edition; Yangyang Lyu: Investigation, methodology, writing-original draft, edition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The author(s) declare(s) that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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