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Research article

Fast finite difference/Legendre spectral collocation approximations for a tempered time-fractional diffusion equation

Zunyuan Hu[1], **Can Li**[1,*] **and Shimin Guo**[2]

[1] Department of Applied Mathematics, Xi'an University of Technology, Xi'an, Shaanxi 710054, China

[2] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, China

* **Correspondence:** Email: mathlican@xaut.edu.cn.

Abstract: The present work is concerned with the efficient numerical schemes for a time-fractional diffusion equation with tempered memory kernel. The numerical schemes are established by using a $L1$ difference scheme for generalized Caputo fractional derivative in the temporal variable, and applying the Legendre spectral collocation method for the spatial variable. The sum-of-exponential technique developed in [Jiang et al., Commun. Comput. Phys., 21 (2017), 650–678] is used to discrete generalized fractional derivative with exponential kernel. The stability and convergence of the semi-discrete and fully discrete schemes are strictly proved. Some numerical examples are shown to illustrate the theoretical results and the efficiency of the present methods for two-dimensional problems.

Keywords: spectral collocation method; generalized memory kernel; sum-of-exponential; error estimates
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1. Introduction

Fractional differential equations have attracted extensive attention for their ability to accurately describe complex physical phenomena in both micro and macro perspectives [1–3]. The differential equations with fractional derivative in time variable can better describe genetic and memory characteristics of some physical processes. For different physical situations, different memory function arise from various fractional derivative operators, such as the power memory kernel, exponential memory kernel, and Mittag-Leffler memory kernel functions [4].

In this work, we consider the following initial boundary value problem:

$$\begin{cases} \partial_{0,t}^{\alpha,\lambda(t)} u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + f(\mathbf{x},t), \mathbf{x} \in \Omega, t \in (0,T], \\ u(\mathbf{x},t) = 0, \mathbf{x} \in \partial\Omega, t \in (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ denotes the Laplace operator, $\Omega = [-1,1]^2$ denotes the computational domain with boundary $\partial\Omega$, $\mathbf{x} = (x_1, x_2) \in \Omega$ stands for the space variables, and $\partial_{0,t}^{\alpha,\lambda(t)}$ denotes the generalized Caputo time-fractional derivative [5]

$$\partial_{0,t}^{\alpha,\lambda(t)} u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\lambda(t-s)}{(t-s)^\alpha} \frac{\partial u}{\partial s}(\mathbf{x},s) \mathrm{d}s, 0 < \alpha < 1. \tag{1.2}$$

In fractional derivative (1.2), we assume that the kernel function satisfies $\lambda(t) \in C^2[0,T]$, $\lambda(t) > 0$ and $\lambda'(t) \le 0$ for all $t \in [0,T]$. Particularly, if kernel takes $\lambda(t) = e^{-bt} (b > 0)$, fractional derivative (1.2) gives

$$\partial_{0,t}^{\alpha,e^{-bt}} u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-b(t-s)}}{(t-s)^\alpha} \frac{\partial u}{\partial s}(\mathbf{x},s) \mathrm{d}s, \tag{1.3}$$

provides the tempered fractional derivative and integral

$$\partial_{0,t}^{\alpha,e^{-bt}} u(\mathbf{x},t) = \frac{e^{-bt}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s}(u(\mathbf{x},s)e^{bs}) \mathrm{d}s - \frac{b}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-b(t-s)}}{(t-s)^\alpha} u(\mathbf{x},s) \mathrm{d}s. \tag{1.4}$$

Obviously, from a mathematical perspective, the Caputo-type fractional derivative (1.2) can be interpreted as a kind of generalized fractional derivative [4, 6, 7]. From Figure 1, we can see the influence of parameters on the solution. More application (in macroscopic and microscopic scales) of the model given in problem (1.1) with tempered fractional derivative, see references [3, 8, 9]. There are many research works on the discretization of fractional derivative (1.2). By splitting into two parts, Alikhanov first discussed the $L1$ formula of fractional derivative (1.2) in the work [5], and proposed a new difference scheme for the time-fractional diffusion equation with the fractional derivatives (1.2). Recently, in view of relation (1.4), Chen et al. [10] established two different $L1$ discretizations for tempered Caputo fractional derivative on the graded meshes, which developed in reference [11].

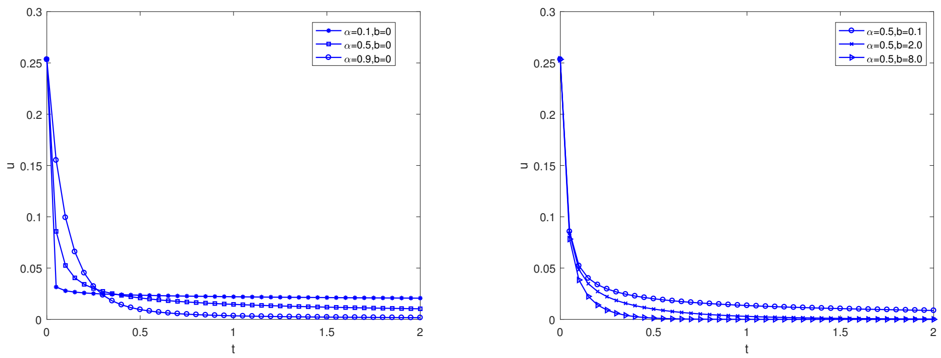

Figure 1. The evolution of solution of problem (1.1) with $\lambda(t) = e^{-bt}$. Here, we take zero source term (i.e., $f = 0$). The left figure illustrates behavior of solution with different α, the right figure illustrates behavior of solution with different b.

The nonlocality of the fractional derivative will make storage and computation very expensive if we adopt a lower-order method. For example, if we use the $L1$ scheme to calculate (1.1), the required computational and storage amounts are $O(NN_T)$ and $O(NN_T^2)$, respectively, where N is the number of spatial grid points and N_T stands for the number of grid points in time. When the time step is small enough, it means that the numerical calculation needs expensive storage costs and computational costs, so it is particularly important to develop fast or high-order numerical schemes. The sum-of-exponential (SOE) technique is one of the effective techniques to approximate the fractional derivatives with the power kernel function [12, 13]. In this topic, Jiang et al. [13] used the SOE method to approximate the kernel function $t^{-(\alpha+1)}(0 < \alpha < 1)$ and solved fractional diffusion equations on unbounded domains. Yan et al. [14] combined the $L2$-1_σ scheme with SOE approximation successfully solved an initial boundary value problem of the time fractional diffusion equation. Combining the SOE approximation, Xu et al. [15] proposed a fast $L2$ discretization of Caputo fractional derivatives, and they get the error estimates by rearranging the coefficients of the fast $L2$ difference approximation. With the help of the SOE technique, Gu et al. [16] proposed a fast difference algorithm to solve the generalized time-space fractional diffusion equation with fractional derivative (1.3). It seems that the authors of [16] fail to give any convergence analysis of the fast $L1$ difference scheme.

As a higher-order numerical method, the spectral method has been widely used for solving many kinds of differential equations [17, 18] including fractional differential equations [15]. Lin and Xu [19] solved the time-fractional diffusion equation based on the finite difference method in the time direction and the Legendre spectral method in space. Later, Li and Xu [20] improved their previous results and proposed a time-space spectral method for the time-fractional diffusion equation. Compared with one-dimensional problems, solving high-dimensional problems is more difficult due to the increase in storage and computational cost. Zeng et al. [21] proposed the Legendre Galerkin spectral method for two-dimensional Riesz space fractional nonlinear reaction-diffusion equations and obtained the optimal spatial error estimation. Guo et al. [22] used the Legendre spectral method to solve two-dimensional fractional nonlinear reaction-diffusion wave equations. Recently, combining the fast $L2$ discretization of Caputo fractional derivatives, Xu et al. [15] proposed a fast difference/spectral method for a time fractional equation in one dimension. In their numerical scheme, the convergence order in the temporal direction is $(3 - \alpha)$ and the convergence order in the spatial direction is $O(N^{1-m})$, where m is the regularity of the exact solution in the spatial direction. This work suggests that the fast difference approximation with high-order discretization schemes is an effective way to solve the time fractional differential equations. Following this line, in this paper, we try to develop a fast finite difference/Legendre spectral collocation method for problem (1.1) with smooth solution and discuss the error estimates for the present numerical schemes. First, the generalized fractional derivative is discretized by $L1$ discretization, and the time direction convergence order is analyzed. Secondly, by using SOE approximation of exponential kernel function, a fast $L1$ scheme with exponential memory kernel function is obtained. Therefore, it can be seen that the calculation and storage costs of the fast scheme are significantly reduced. With the similar technique given in [15], we will discuss the error estimates when the fast L1 method is used to solve problem (1.1).

The rest of this paper is organized as follows: In Section 2, we present time discretizations for problem (1.1) and discuss the stability and convergence of the present semi-discrete scheme. Combing the Legendre spectral collocation method for spatial variables, in Section 3, we discuss the finite difference/spectral collocation method for problem (1.1). The error estimates of the fully discrete scheme are also established in this section. For the generalized fractional derivative with exponential kernel, we develop the sum-of-exponentials technique developed in [13] to accelerate the calculation process. The error estimates of the related numerical schemes are discussed in Section 4. In Section 5, we present two numerical examples to verify the efficiency of our numerical algorithm and theoretical results. Finally, some conclusions are provided in Section 6.

2. Time semi-discrete scheme

First, we discretize the fractional derivative with a generalized memory kernel, assuming the solution $u(t) \in C^2[0,T]$. The computational domain $[0,T]$ is uniformly divided into N_T intervals with the uniform grids $t_k = k\tau$, $k = 0, 1, \ldots, N_T$. For $\lambda(t) \in C^2[0,T]$, the time fractional derivative with a generalized memory kernel is discretized as [5]

$$\begin{aligned}\partial_{0,t}^{\alpha,\lambda(t)} u(t)|_{t=t_{k+1}} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1}} \frac{\lambda(t_{k+1}-s)}{(t_{k+1}-s)^\alpha} u'(s)ds \\ &= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{l=0}^{k} (\lambda_{k-l+1/2} a_{k-l} + (\lambda_{k-l} - \lambda_{k-l+1}) b_{k-l}) u_{t,l} + R_1^k + R_2^k,\end{aligned} \tag{2.1}$$

where

$$\lambda_l = \lambda(t_l),\ u_{t,l} = \frac{u(t_{l+1}) - u(t_l)}{\tau},\ \Pi_{1,l} u(s) = u(t_{l+1}) \frac{s - t_l}{\tau} + u(t_l) \frac{t_{l+1} - s}{\tau},$$
$$a_l = (l+1)^{1-\alpha} - l^{1-\alpha},\ b_l = \frac{1}{2-\alpha} \left[(l+1)^{2-\alpha} - l^{2-\alpha}\right] - \frac{1}{2} \left[(l+1)^{1-\alpha} + l^{1-\alpha}\right],\ l \geq 0,$$

and the error

$$R_1^k = \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{l=0}^{k} u_{t,l} \int_0^1 \frac{\lambda_{k-l+1-z} - \lambda_{k-l+1/2} - (\lambda_{k-l} - \lambda_{k-l+1})(z - 1/2)}{(k-l+1-z)^\alpha} dz,$$
$$R_2^k = \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{k} \int_{t_l}^{t_{l+1}} \frac{\lambda(t_{k+1}-s)(u(s) - \Pi_{1,l} u(s))'}{(t_{k+1}-s)^\alpha} ds,$$

give [5]

$$|R_1^k| \leq \frac{t_{k+1}^{1-\alpha}}{4\Gamma(2-\alpha)} \tau^2 \max_{0\leq t\leq t_{k+1}} |u'(t)| \max_{0\leq t\leq t_{k+1}} |\lambda''(t)|,\ |R_2^k| \leq \tilde{c} \max_{0\leq t\leq t_{k+1}} |u''(t)| \lambda(0) \tau^{2-\alpha},$$

where $\tilde{c}$ depends on α.

We rewrite (2.1) as follows:

$$\partial_{0,t}^{\alpha,\lambda(t)} u(t)|_{t=t_{k+1}} = \alpha_0^{-1} \Big[\sum_{j=0}^{k-1} (c_{j+1} - c_j) u(t_{k-j}) + c_0 u(t_{k+1}) - c_k u(t_0) \Big] + r_\tau^{k+1},$$

where $\alpha_0 = \tau^\alpha \Gamma(2-\alpha)$, $c_l = (\lambda_{l+1/2} a_l + (\lambda_l - \lambda_{l+1}) b_l)(l \geq 0)$, and the truncation error gives $r_\tau^{k+1} = R_1^k + R_2^k$. For the sake of simplification, we define the fractional differential operator $L_t^{\alpha,\lambda(t)} u(\mathbf{x}, t_{k+1})$ by

$$L_t^{\alpha,\lambda(t)} u(\mathbf{x}, t_{k+1}) = \alpha_0^{-1} \Big[\sum_{j=0}^{k-1} (c_{j+1} - c_j) u(\mathbf{x}, t_{k-j}) + c_0 u(\mathbf{x}, t_{k+1}) - c_k u(\mathbf{x}, t_0) \Big]. \tag{2.2}$$

Using $L_t^{\alpha,\lambda(t)}u(\mathbf{x},t_{k+1})$ to approximate $\partial_{0,t}^{\alpha,\lambda(t)}u(\mathbf{x},t)$ in problem (1.1) at t_{k+1}, and denoting u^{k+1} is an approximation of $u(\mathbf{x},t_{k+1})$, $f^{k+1} = f(\mathbf{x},t_{k+1})$, we obtain the time semi-discrete scheme of problem (1.1), i.e.,

$$c_0u^1 - \alpha_0\Delta u^1 = c_0u^0 + \alpha_0 f^1, \tag{2.3}$$

for $k = 0$.

For $1 \le k \le N_T - 1$,

$$\sum_{j=0}^{k-1}(c_j - c_{j+1})u^{k-j} + c_ku^0 = c_0u^{k+1} - \alpha_0\Delta u^{k+1} - \alpha_0 f^{k+1}. \tag{2.4}$$

After a simple calculation, the truncation error of the semi-discretized schemes (2.6) and (2.7) give $r^{k+1} = \alpha_0 r_\tau^{k+1}$ and satisfies

$$|r^{k+1}| = \alpha_0|r_\tau^{k+1}| \le \tilde{c}\max_{0\le t\le t_{j+1}}|\partial_t^2u(\mathbf{x},t)|\lambda(0)\tau^2. \tag{2.5}$$

Considering the variational formulation of (1.1), we define some functional spaces endowed with standard norms and inner products that will be used hereafter [17, 18, 26]

$$H^m(\Omega) = \left\{u|u \in L^2(\Omega), D^\beta u \in L^2(\Omega) \text{ for } 0 \le |\beta| \le m\right\}, H_0^1(\Omega) = \left\{u|u \in H^1(\Omega), u|_{\partial\Omega} = 0\right\},$$

where $\beta = (\beta_1,\beta_2)$, $|\beta| = \beta_1 + \beta_2$, $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}$, $L^2(\Omega)$ is the space of measurable functions whose square is Lebesgue integrable in Ω, which equip with the inner products

$$(u,v) = \int_\Omega uv d\mathbf{x}, \quad (u,v)_1 = (u,v) + \alpha_0c_0^{-1}(\nabla u,\nabla v),$$

and the normH^1 is defined by

$$\|v\|_1 = (v,v)_1^{1/2}, \quad \|v\|_1 = \left(\|v\|^2 + \alpha_0c_0^{-1}\|\nabla v\|^2\right)^{1/2}.$$

The weak formulation of the (2.3) and (2.4) reads: Find $u^{k+1} \in H_0^1(\Omega)$, such that

$$(u^1,v) + \alpha_0c_0^{-1}(\nabla u^1,\nabla v) = (u^0,v) + \alpha_0c_0^{-1}(f^1,v), \tag{2.6}$$

for $k = 0$, and

$$(u^{k+1},v) + \alpha_0c_0^{-1}(\nabla u^{k+1},\nabla v) = c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})(u^{k-j},v) + c_0^{-1}c_k(u^0,v) + \alpha_0c_0^{-1}(f^{k+1},v), \tag{2.7}$$

for all $v \in H_0^1(\Omega)$ and $1 \le k \le N_T - 1$.

2.1. Stability analysis of the semi-discrete scheme

Lemma 1. *[5] If $\lambda(t) > 0, \lambda'(t) \le 0$, and $\lambda(t) \in C^2[0,T]$, the weighted coefficients a_l, b_l and c_l ($l = 0,1,\dots$) are given by (2.1), satisfy*

$$a_0 > a_1 > a_2 > \ldots > a_l > \frac{1-\alpha}{(l+1)^\alpha},\ b_0 > b_1 > \ldots b_l > 0,$$

$$c_0 > c_1 > \ldots > c_l > \frac{\alpha_0 \lambda(t_{l+1/2})}{\Gamma(1-\alpha) t_{l+1}^\alpha} > \frac{\alpha_0 \lambda(T)}{\Gamma(1-\alpha) T^\alpha}.$$

Theorem 1. *For all $\tau > 0$, the semi-discretized schemes* (2.6) *and* (2.7) *are unconditionally stable, i.e.,*

$$\|u^{k+1}\|_1 \leq \|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k+1} \|f^j\|, k = 0, 1, \ldots, N_T - 1. \tag{2.8}$$

Proof. For $k = 0$, substituting $v = u^1$ in (2.6) and using $\|u^1\| \leq \|u^1\|_1$, we have

$$\|u^1\|_1^2 \leq \|u^0\|\|u^1\| + \alpha_0 c_0^{-1} \|f^1\|\|u^1\| \leq \|u^0\|\|u^1\|_1 + \alpha_0 c_0^{-1} \max_{0\leq j\leq 1} \|f^j\|\|u^1\|_1. \tag{2.9}$$

From (2.9), we have $\|u^1\|_1 \leq \|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq 1} \|f^j\|$. For $k > 1$, we prove (2.8) by mathematical induction. Assume that there holds

$$\|u^j\|_1 \leq \|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq l\leq j} \|f^l\|, j = 1, 2, \ldots, k. \tag{2.10}$$

Now prove (2.8) using inequality (2.10). Substituting $v = u^{k+1}$ into (2.7), we obtain

$$\|u^{k+1}\|^2 + \alpha_0 c_0^{-1} \|\nabla u^{k+1}\|^2 = c_0^{-1} \sum_{j=0}^{k-1} (c_j - c_{j+1})(u^{k-j}, u^{k+1}) + \alpha_0 c_0^{-1} (f^{k+1}, u^{k+1}) + c_k c_0^{-1} (u^0, u^{k+1}), \tag{2.11}$$

which implies

$$\|u^{k+1}\|_1^2 \leq\ c_0^{-1} \sum_{j=0}^{k-1} (c_j - c_{j+1}) \|u^{k-j}\|\|u^{k+1}\| + \alpha_0 c_0^{-1} \|f^{k+1}\|\|u^{k+1}\| + c_k c_0^{-1} \|u^0\|\|u^{k+1}\|.$$

Using Lemma 1 and the inequality $\|u^{k+1}\| \leq \|u^{k+1}\|_1$, we have

$$\begin{aligned} \|u^{k+1}\|_1 \leq & c_0^{-1} \sum_{j=0}^{k-1} (c_j - c_{j+1})(\|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k-j} \|f^j\|) + c_k c_0^{-1} \|u^0\| + \alpha_0 c_0^{-1} \|f^{k+1}\| \\ = & c_0^{-1} \Big[\sum_{j=0}^{k-1} (c_j - c_{j+1})(\|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k-j} \|f^j\|) + c_k \|u^0\| + \alpha_0 \|f^{k+1}\| \Big]. \end{aligned}$$

Note that $\|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k-j} \|f^j\| \leq \|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k} \|f^j\| (j = 0, 1, \ldots, k-1)$. Therefore

$$\begin{aligned} \|u^{k+1}\|_1 \leq & c_0^{-1} \Big[\Big(\sum_{j=0}^{k-1} (c_j - c_{j+1}) + c_k \Big)(\|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k} \|f^j\|) \Big] + \alpha_0 c_0^{-1} \|f^{k+1}\| \\ \leq & \|u^0\| + \tilde{c}\alpha_0 c_0^{-1} \max_{0\leq j\leq k+1} \|f^j\|. \end{aligned}$$

Finally, we obtain (2.8). □

2.2. Convergence analysis of the semi-discrete scheme

Here we provide some lemmas to analyze coefficients c_l and a_l, which will help to study the convergence analysis of the solution. In the following sections, $\tilde{c}$ is used to represent a constant, which may be different under different situations. Here, we employ the technique used in reference [19] to do the error estimates of schemes (2.6) and (2.7).

Theorem 2. *Let $u(\boldsymbol{x}, t_k)$ be the exact solution of problem* (1.1)*, $\{u^k\}_{k=0}^{N_T}$ be the solution of schemes* (2.6) *and* (2.7)*, there holds*

$$\|u(\boldsymbol{x}, t_k) - u^k\|_1 \le \begin{cases} \tilde{c}T^\alpha \max\limits_{t\in(0,T]} \|\partial_t^2 u(\boldsymbol{x}, t)\|\tau^{2-\alpha}, 0 < \alpha < 1, \\ \tilde{c}T \max\limits_{t\in(0,T]} \|\partial_t^2 u(\boldsymbol{x}, t)\|\tau, \alpha \to 1, \end{cases} \tag{2.12}$$

where $\tilde{c}$ is independent of u, τ and T.

Proof. Firstly, we prove that the following inequality

$$\|u(\mathbf{x}, t_j) - u^j\|_1 \le \tilde{c}c_{j-1}^{-1} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2, j = 1, 2, \ldots, N_T. \tag{2.13}$$

Let $e^j = u(\mathbf{x}, t_j) - u^j$, for $j = 1$, we obtain

$$(e^1, v) + \alpha_0 c_0^{-1}(\nabla e^1, \nabla v) = (e^0, v) + (r^1, v). \tag{2.14}$$

If we assume the initial value is exact (i.e., $e^0 = 0$), let $v = e^1$ in (2.14), using (2.5), we obtain

$$\|e^1\|_1 \le \|r^1\| \le \tilde{c}c_0^{-1} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2.$$

Suppose (2.13) holds for all $j = 1, 2, \ldots, k$, we need to prove it also holds for $j = k + 1$. In view of

$$(e^{k+1}, v) + \alpha_0 c_0^{-1}(\nabla e^{k+1}, \nabla v) = c_0^{-1} \sum_{j=0}^{k-1}(c_j - c_{j+1})(e^{k-j}, v) + c_0^{-1}c_k(e^0, v) + (r^{k+1}, v). \tag{2.15}$$

Let $v = e^{k+1}$ in (2.15), we have

$$\|e^{k+1}\|^2 + \alpha_0 c_0^{-1}\|\nabla e^{k+1}\|^2 = c_0^{-1} \sum_{j=0}^{k-1}(c_j - c_{j+1})(e^{k-j}, e^{k+1}) + c_0^{-1}c_k(e^0, e^{k+1}) + (r^{k+1}, e^{k+1}). \tag{2.16}$$

We rewrite (2.16) as follows:

$$\|e^{k+1}\|_1^2 \le c_0^{-1}\Big[\sum_{j=0}^{k-1}(c_j - c_{j+1})\tilde{c}c_{k-j}^{-1} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2 + \tilde{c} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2\Big]\|e^{k+1}\|.$$

Using Lemma 1, we have $c_{k-j}^{-1} < c_k^{-1}$, it follows that

$$\begin{aligned}\|e^{k+1}\|_1 &\le c_0^{-1}\Big[\sum_{j=0}^{k-1}(c_j - c_{j+1})\tilde{c}c_k^{-1} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2 + \tilde{c} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2\Big] \\ &\le \tilde{c}c_k^{-1} \max_{t\in(0,T]} |\partial_t^2 u(\mathbf{x}, t)|\lambda(0)\tau^2,\end{aligned}$$

which implies (2.13). Lemma 1 suggests that

$$k^{-\alpha}c_{k-1}^{-1}\lambda(t_{k+1/2}) \leq k^{-\alpha}\frac{\Gamma(1-\alpha)t_k^{\alpha}}{\alpha_0\lambda(t_{k-1/2})}\lambda(t_{k+1/2}) \leq \frac{\Gamma(1-\alpha)\tau^{\alpha}}{\alpha_0}\frac{\lambda(t_{k+1/2})}{\lambda(t_{k-1/2})} \leq \frac{\Gamma(1-\alpha)\tau^{\alpha}}{\Gamma(2-\alpha)\tau^{\alpha}} = \frac{1}{1-\alpha}.$$

Hence, we have

$$\begin{aligned}\|u(\mathbf{x},t_k)-u^k\|_1 &\leq \tilde{c}c_{k-1}^{-1}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\lambda(0)\tau^2\\ &= \tilde{c}k^{-\alpha}\lambda(t_{k+1/2})c_{k-1}^{-1}\frac{1}{\lambda(t_{k+1/2})}k^{\alpha}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\lambda(0)\tau^2\\ &\leq \tilde{c}\frac{1}{1-\alpha}\frac{1}{\lambda(t_{k+1/2})}(k\tau)^{\alpha}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\lambda(0)\tau^{2-\alpha}\\ &\leq \tilde{c}T^{\alpha}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^{2-\alpha}.\end{aligned}$$

Next we consider the case $\alpha \to 1$, we firstly prove that

$$\left\|u(t_j)-u^j\right\|_1 \leq \tilde{c}\,j\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^2,\ j=1,2,\ldots,N_T. \tag{2.17}$$

Inequality (2.17) obviously holds for $j = 1$. Suppose (2.17) holds for $j = 1,2,\ldots,k$. Now we will prove that (2.17) still holds when $j = k+1$. In fact, with the similar method given in Theorem 1, we have

$$\begin{aligned}&\|u(t_{k+1})-u^{k+1}\|_1\\ &\leq c_0^{-1}(c_0-c_1)\|e^k\| + c_0^{-1}\sum_{j=1}^{k-1}(c_j-c_{j+1})\|e^{k-j}\| + c_0^{-1}c_k\|e^0\| + \|r^{k+1}\|\\ &\leq c_0^{-1}[(c_0-c_1)\tilde{c}k\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^2 + \sum_{j=1}^{k-1}(c_j-c_{j+1})(k-j)\tilde{c}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^2] + \tilde{c}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^2\\ &= c_0^{-1}\Big[(c_0-c_1)\frac{k}{k+1} + \sum_{j=1}^{k-1}(c_j-c_{j+1})\frac{k-j}{k+1} + \frac{c_0}{k+1}\Big](k+1)\tilde{c}\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^2.\end{aligned} \tag{2.18}$$

Simple calculations, we can check that

$$\begin{aligned}(c_0-c_1)\frac{k}{k+1} + \sum_{j=1}^{k-1}(c_j-c_{j+1})\frac{k-j}{k+1} + \frac{c_0}{k+1} &= (c_0-c_1) + \sum_{j=1}^{k-1}(c_j-c_{j+1}) - (c_0-c_1)\frac{1}{k+1}\\ &\quad - \sum_{j=1}^{k-1}(c_j-c_{j+1})\frac{j+1}{k+1} + \frac{c_0}{k+1}.\end{aligned} \tag{2.19}$$

With the help of (2.18), Eq (2.19) provides that

$$\begin{aligned}\|e^{k+1}\|_1 \leq c_0^{-1}\Big[&(c_0-c_1) + \sum_{j=1}^{k-1}(c_j-c_{j+1}) - (c_0-c_1)\frac{1}{k+1} - \sum_{j=1}^{k-1}(c_j-c_{j+1})\frac{j+1}{k+1}\\ &+ \frac{c_0}{k+1}\Big]\tilde{c}(k+1)\max_{t\in(0,T]}|\partial_t^2 u(\mathbf{x},t)|\tau^2.\end{aligned} \tag{2.20}$$

Since

$$(c_0-c_1)\frac{1}{k+1}+\sum_{j=1}^{k-1}(c_j-c_{j+1})\frac{j+1}{k+1}+c_k\geq\frac{1}{k+1}\Big[(c_0-c_1)+\sum_{j=1}^{k-1}(c_j-c_{j+1})+c_k\Big]=\frac{c_0}{k+1},$$

we have

$$-(c_0-c_1)\frac{1}{k+1}-\sum_{j=1}^{k-1}(c_j-c_{j+1})\frac{j+1}{k+1}+\frac{c_0}{k+1}\leq c_k. \tag{2.21}$$

Substituting (2.21) into (2.20), we obtain

$$\begin{aligned}\|e^{k+1}\|_1&\leq c_0^{-1}\Big[(c_0-c_1)+\sum_{j=1}^{k-1}(c_j-c_{j+1})+c_k\Big](k+1)\tilde{c}\max_{t\in(0,T]}|\partial_t^2u(\mathbf{x},t)|\tau^2\\&\leq\tilde{c}T\max_{t\in(0,T]}|\partial_t^2u(\mathbf{x},t)|\tau.\end{aligned}$$

We finally obtain our conclusion. □

3. The fully discrete scheme

Let $P_N(I_x)$ and $P_N(I_y)$ be the spaces of all polynomials of degree less than or equal to N defined on domains $I_x=(-1,1)$ and $I_y=(-1,1)$, respectively. Let $V_N(\Omega)=P_N(I_x)\bigotimes P_N(I_y)$, $\{\xi_j\}_{j=0}^N$,$\{\xi_i\}_{i=0}^N$ are Legendre-Gauss-Labatoo (LGL) points, and $\{\omega_j\}_{j=0}^N$ and $\{\omega_i\}_{i=0}^N$ are the weights, which satisfy the following quadrature formula [25, 26]

$$\int_{-1}^1\int_{-1}^1\varphi(x_1,x_2)dx_1x_2=\sum_{j_1=0}^N\sum_{j_2=0}^N\varphi(\xi_{j_1},\xi_{j_2})\omega_{j_1}\omega_{j_2},\forall\varphi\in V_{2N-1}(\Omega).$$

Define the inner product and norm as follows [18, 25, 26]

$$(\phi,\psi)_N=\sum_{i=0}^N\sum_{j=0}^N\phi(\xi_i,\xi_j)\psi(\xi_i,\xi_j)\omega_i\omega_j,\ \ \|\psi\|_{0,N}=(\phi,\phi)_N^{1/2}.$$

We will use the H_0^1-orthogonal projection operator π_N such that [24]

$$(\nabla(u-\pi_Nu),\nabla v)=0,\forall v\in V_N^0(\Omega), \tag{3.1}$$

for all $u\in H_0^1(\Omega)$. Here $V_N^0(\Omega)$ is defined as

$$V_N^0(\Omega)=\Big(P_N(I_x)\otimes P_N(I_y)\Big)\cap H_0^1(\Omega).$$

For this projection, there have estimates [25, 26]

$$\|u-\pi_Nu\|_l\leq\tilde{c}N^{l-m}\|u\|_m,u\in H^m(\Omega)\cap H_0^1(\Omega),m\geq1,l=0,1. \tag{3.2}$$

Lemma 2. *[25] For all $\varphi\in H^m(\Omega)(\ m\geq1)$, $v_N\in P_N(\Omega)$, there holds*

$$(\varphi,v_N)-(\varphi,v_N)_N\leq\tilde{c}N^{-m}\|\varphi\|_m\|v_N\|_{0,N}. \tag{3.3}$$

Finite difference/Legendre collocation approximation of problem (1.1) gives: Find $u_N^k \in V_N^0(\Omega)$,

$$(u_N^{k+1}, v_N)_N + \alpha_0 c_0^{-1}(\nabla u_N^{k+1}, \nabla v_N)_N = c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})(u_N^{k-j}, v_N)_N + c_k c_0^{-1}(u_N^0, v_N)_N + (f^{k+1}, v_N)_N, \quad (3.4)$$

for all $v_N \in V_N^0(\Omega)$. The discrete H^1-norm is defined by

$$\|\psi_N\|_{1,N} = \left((\psi_N, \psi_N)_N + \alpha_0 c_0^{-1}(\nabla\psi_N, \nabla\psi_N)_N\right)^{1/2}, \forall \psi_N \in V_N(\Omega).$$

To do the error estimate, we split the error as $u(t_k) - u_N^k = (u(t_k) - \pi_N u(t_k)) - (u_N^k - \pi_N u(t_k)) := \epsilon_N^k - e_N^k$. Without loss of generality, we only consider the homogeneous case of problem (1.1), i.e., $f = 0$.

Theorem 3. *Let $u(\boldsymbol{x}, t)$ be the exact solution of problem* (1.1)*, and u_N^k be the solution of fully discrete scheme* (3.4) *with the initial condition $u_N^0 = \pi_N u^0$. Suppose $\partial_t^2 u \in L^\infty((0, T]; H^m(\Omega))(m \geq 1)$, then*

$$\|u(\boldsymbol{x}, t_k) - u_N^k\|_{1,N} \leq \begin{cases} \frac{\tilde{c}T^\alpha}{1-\alpha}\frac{1}{\lambda(T)}\left(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \tau^{2-\alpha}\lambda(0)(N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)} + \|\partial_t^2 u\|_{L^\infty(L^2)})\right) \\ +\tilde{c}N^{1-m}\|u\|_{L^\infty(H^m)}, 0 < \alpha < 1, \\ \\ \tilde{c}\frac{T}{\lambda(T)}\left(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \tau\lambda(0)(N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)} + \|\partial_t^2 u\|_{L^\infty(L^2)})\right) \\ +\tilde{c}N^{1-m}\|u\|_{L^\infty(H^m)}, \alpha \to 1, \end{cases} \quad (3.5)$$

where $\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} = \sup_{t\in(0,T)}\|\partial_t^{\alpha,\lambda(t)}u(\boldsymbol{x}, t)\|_m$, $\|u\|_{L^\infty(H^m)} = \sup_{t\in(0,T)}\|u(\boldsymbol{x}, t)\|_m$, $\|\partial_t^2 u\|_{L^\infty(H^m)} = \sup_{t\in(0,T)}\|\partial_t^2 u(\boldsymbol{x}, t)\|_m$, *and* $\|\partial_t^2 u\|_{L^\infty(L^2)} = \sup_{t\in(0,T)}\|\partial_t^2 u(\boldsymbol{x}, t)\|$.

Proof. Subtracting $e_N^k = u_N^k - \pi_N u(t_k)$ on both sides of Eq (3.4), we obtain

$$\begin{aligned} &\left(e_N^{k+1}, v_N\right)_N + \alpha_0 c_0^{-1}\left(\nabla e_N^{k+1}, \nabla v_N\right)_N \\ &= c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})(u_N^{k-j}, v_N)_N + c_k c_0^{-1}(u_N^0, v_N)_N - (\pi_N u(t_{k+1}), v_N)_N - \alpha_0 c_0^{-1}\left(\nabla\pi_N u(t_{k+1}), \nabla v_N\right)_N. \end{aligned}$$

Rewriting the above formula, we obtain

$$\begin{aligned} &\left(e_N^{k+1}, v_N\right)_N + \alpha_0 c_0^{-1}\left(\nabla e_N^{k+1}, \nabla v_N\right)_N \\ &= c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})\left(e_N^{k-j}, v_N\right)_N + c_k c_0^{-1}\left(e_N^0, v_N\right)_N + \left(\varepsilon_1^{k+1}, v_N\right)_N + \left(\varepsilon_2^{k+1}, v_N\right)_N, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1^{k+1} = &(u(t_{k+1}) - \pi_N u(t_{k+1})) - c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})(u(t_{k-j}) - \pi_N u(t_{k-j})) \\ &- c_k c_0^{-1}(u(t_0) - \pi_N u(t_0)), \end{aligned}$$

and

$$\begin{aligned} \varepsilon_2^{k+1} = &-(u(t_{k+1}) - \pi_N u(t_{k+1})) + c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})u(t_{k-j}) + c_k c_0^{-1}u(t_0) \\ &- \pi_N u(t_{k+1}) + \alpha_0 c_0^{-1}\Delta\pi_N u(t_{k+1}). \end{aligned}$$

For term $\left(\varepsilon_1^{k+1}, v_N\right)_N$, we have

$$\begin{aligned}\left(\varepsilon_1^{k+1}, v_N\right)_N &= (I_d - \pi_N)\Big(u(t_{k+1}) - c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})u(t_{k-j}) - c_k c_0^{-1}u(t_0), v_N\Big)_N\\ &= \alpha_0 c_0^{-1}\Big((I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)}u(t_{k+1}) - r_\tau^{k+1}), v_N\Big)_N,\end{aligned}$$

where I_d is the identity operator. Using (3.3), we have

$$\begin{aligned}&\left|\left(\varepsilon_1^{k+1}, v_N\right)_N\right|\\ &\le \alpha_0 c_0^{-1}\left[\Big((I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)}u(t_{k+1}) - r_\tau^{k+1}), v_N\Big) + \tilde{c}N^{-1}\|(I_d - \pi_N)(\partial_t^{\alpha}u(t_{k+1}) - r_\tau^{k+1})\|_1\|v_N\|_{0,N}\right]\\ &\le \alpha_0 c_0^{-1}\left[\|(I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)}u(t_{k+1}) - r_\tau^{k+1})\|_{0,N} + \tilde{c}N^{-1}\|(I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)}u(t_{k+1}) - r_\tau^{k+1})\|_1\right]\|v_N\|_{0,N}.\end{aligned}$$

By using of (3.2), we obtain

$$\left|\left(\varepsilon_1^{k}, v_N\right)_N\right| \le \tilde{c}\alpha_0 c_0^{-1}\left(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \tau^{2-\alpha}\lambda(0)N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\right)\|v_N\|_{0,N}.$$

Note that

$$\left(\varepsilon_2^{k+1}, v_N\right)_N = -\Big(u(t_{k+1}) - c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})u(t_{k-j}) - c_k c_0^{-1}u(t_0), v_N\Big)_N - \alpha_0 c_0^{-1}\Big(\nabla\pi_N u(t_{k+1}), \nabla v_N\Big)_N,$$

the facts $(\nabla\pi_N u(t_k), \nabla v_N)_N = (\nabla\pi_N u(t_k), \nabla v_N)$ and $\left(\partial_t^{\alpha,\lambda(t)}u(t_k), v_N\right) = -(\nabla u(t_k), \nabla v_N)$, we obtain

$$\begin{aligned}\left|\left(\varepsilon_2^{k+1}, v_N\right)_N\right| \le{}& \alpha_0 c_0^{-1}\left(L_t^{\alpha,\lambda(t)}u(t_{k+1}), v_N\right) - \alpha_0 c_0^{-1}\left(L_t^{\alpha,\lambda(t)}u(t_{k+1}), v_N\right)_N\\ &+ \alpha_0 c_0^{-1}\left(\partial_t^{\alpha,\lambda(t)}u(t_{k+1}) - L_t^{\alpha,\lambda(t)}u(t_{k+1}), v_N\right).\end{aligned}$$

Furthermore, we obtain

$$\left|\left(\varepsilon_2^{k+1}, v_N\right)_N\right| \le \tilde{c}\alpha_0 c_0^{-1}\left[N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \lambda(0)\tau^{2-\alpha}\left(\|\partial_t^2 u\|_{L^\infty(L^2)} + N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\right)\right]\|v_N\|_{0,N}.$$

Combining the estimates of $\left(\varepsilon_1^{k}, v_N\right)_N$ and $\left(\varepsilon_2^{k}, v_N\right)_N$, we have

$$\begin{aligned}\|e_N^{k+1}\|_{1,N} \le{}& c_0^{-1}\sum_{j=0}^{k-1}(c_j - c_{j+1})\|e_N^{k-j}\|_{0,N} + c_k c_0^{-1}\|e_N^0\|_{0,N}\\ &+ \tilde{c}\alpha_0 c_0^{-1}\left[N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \lambda(0)\tau^{2-\alpha}\left(\|\partial_t^2 u\|_{L^\infty(L^2)} + N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\right)\right].\end{aligned} \tag{3.6}$$

Moreover, with the same technique used in Theorem 2, using inequality (3.6), we can prove that

$$\begin{aligned}\|e_N^k\|_{1,N} &\le c_{k-1}^{-1}\tilde{c}\alpha_0 c_0^{-1}\left[N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \lambda(0)\tau^{2-\alpha}\left(\|\partial_t^2 u\|_{L^\infty(L^2)} + N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\right)\right]\\ &\le \tilde{c}c_{k-1}^{-1}k^{-\alpha}\lambda(t_{k+1/2})\frac{\Gamma(2-\alpha)}{\lambda(t_{k+1/2})}k^\alpha\tau^\alpha(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \tau^{2-\alpha}\lambda(0)\|\partial_t^2 u\|_{L^\infty(L^2)}\\ &+ \tau^{2-\alpha}\lambda(0)N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)})\\ &\le \frac{\tilde{c}T^\alpha}{(1-\alpha)(\lambda(T))}\left[N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + \lambda(0)\tau^{2-\alpha}\left(\|\partial_t^2 u\|_{L^\infty(L^2)} + N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\right)\right].\end{aligned}$$

Hence, applying triangle inequality, we obtain

$$\begin{aligned}\|u(t_k)-u_N^k\|_{1,N} &\le \|u(t_k)-\pi_N u(t_k)\|_{1,N}+\|u_N^k-\pi_N u(t_k)\|_{1,N}\\ &\le \tilde{c}N^{1-m}\|u\|_{L^\infty(H^m)}+\frac{\tilde{c}T^\alpha}{(1-\alpha)\lambda(T)}\times\\ &\left[N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)}+\lambda(0)\tau^{2-\alpha}\left(\|\partial_t^2 u\|_{L^\infty(L^2)}+N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\right)\right].\end{aligned}$$

For $\alpha\to 1$, similar to the case discussed in Theorem 2, we obtain

$$\|e_N^k\|_{1,N}\le \tilde{c}\frac{T}{\lambda(T)}\Big(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)}+\tau\lambda(0)\|\partial_t^2 u\|_{L^\infty(L^2)}+\tau\lambda(0)N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\Big).$$

Applying triangle inequality once again, we have

$$\begin{aligned}\|u(t_k)-u_N^k\|_{1,N} &\le \|u(t_k)-\pi_N u(t_k)\|_{1,N}+\|u_N^k-\pi_N u(t_k)\|_{1,N},\\ &\le \tilde{c}N^{1-m}\|u\|_{L^\infty(H^m)}+\tilde{c}\frac{T}{\lambda(T)}\Big(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)}+\tau\lambda(0)\|\partial_t^2 u\|_{L^\infty(L^2)}\\ &+\tau\lambda(0)N^{-m}\|\partial_t^2 u\|_{L^\infty(H^m)}\Big),\end{aligned}$$

which implies the desired result. □

4. Fast time discretization

4.1. Construction of the fast difference operators

Applying the same technique given in reference [13], we use the sum-of-exponentials (SOE) approximation to approximate the power function $t^{-\alpha}(0<\alpha<1)$ in fractional derivative (2.1). And for the approach error, we have the following results:

Lemma 3. *[13] For a given absolute error ε, the kernel functions $\frac{1}{t^\alpha}(\alpha\in(0,1))$, there exist positive real numbers $\{\eta_i^\alpha\}_{i=1}^{N_{exp}}$ and $\{w_i^\alpha\}_{i=1}^{N_{exp}}$ such that*

$$\left|t^{-\alpha}-\sum_{i=1}^{N_{exp}}w_i^\alpha e^{-\eta_i^\alpha t}\right|\le\varepsilon, t\in[\tau,T],\tag{4.1}$$

where $N_{exp}=O\left(\log\frac{1}{\varepsilon}\left(\log\log\frac{1}{\varepsilon}+\log\frac{T}{\tau}\right)+\log\frac{1}{\tau}\left(\log\log\frac{1}{\varepsilon}+\log\frac{1}{\tau}\right)\right)$.

For the details of the proof of Lemma 3, see [13]. Following the same idea given in references [13, 16], we split the Caputo fractional derivative (1.3) into a local term $L(t_k)$ and a history term $H(t_k)$ in discretized fractional derivative (2.1) with kernel $\lambda(t)=e^{-bt}(b\ge 0)$, i.e.,

$$\begin{aligned}\partial_{0,t}^{\alpha,\lambda(t)}u(t)|_{t=t_k} &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k-1}}\frac{e^{b(t_k-s)}}{(t_k-s)^\alpha}u'(s)ds+\frac{1}{\Gamma(1-\alpha)}\int_{t_{k-1}}^{t_k}\frac{e^{b(t_k-s)}}{(t_k-s)^\alpha}u'(s)ds\\ &:= H(t_k)+L(t_k).\end{aligned}$$

For the local term, we discrete it as the follows

$$
\begin{aligned}
L(t_k) :\approx & \frac{u(t_k)-u(t_{k-1})}{\tau\Gamma(1-\alpha)}\int_{t_{k-1}}^{t_k}\frac{e^{-b(t_k-s)}}{(t_k-s)^{\alpha}}ds, \\
= & \frac{u(t_k)-u(t_{k-1})}{\tau\Gamma(2-\alpha)}\left(e^{-b\tau}\tau^{1-\alpha}+b\int_0^{\tau}e^{-b\theta}\theta^{1-\alpha}d\theta\right).
\end{aligned} \tag{4.2}
$$

The history term, using SOE approximation, we obtain

$$
\begin{aligned}
H(t_k) :=& \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k-1}}\frac{e^{-b(t_k-s)}}{(t_k-s)^{\alpha}}u'(s)ds \\
\approx & \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k-1}}\sum_{i=1}^{N_{exp}} w_i^{\alpha}e^{-\eta_i^{\alpha}(t_k-s)}e^{-b(t_k-s)}u'(s)ds \\
= & \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{N_{exp}}\int_0^{t_{k-1}} w_i^{\alpha}e^{-\widetilde{\eta}_i^{\alpha}(t_k-s)}u'(s)ds,
\end{aligned}
$$

where $\widetilde{\eta}_i^{\alpha} = \eta_i^{\alpha} + b$. For simplicity, we rewrite $H(t_k)$ as

$$
H(t_k) \approx \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{N_{exp}} w_i^{\alpha}U_{h,i}^{\alpha}(t_k),
$$

with

$$
U_{h,i}^{\alpha}(t_k) = e^{-\widetilde{\eta}_i^{\alpha}\tau}U_{h,i}^{\alpha}(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} e^{-\widetilde{\eta}_i^{\alpha}(t_k-s)}u'(s)ds.
$$

Using linear interpolation operators to approximate $u'(s)$, we have

$$
\int_{t_{k-2}}^{t_{k-1}} e^{-\widetilde{\eta}_i^{\alpha}(t_k-s)}u'(s)ds = \int_{t_{k-2}}^{t_{k-1}} e^{-\widetilde{\eta}_i^{\alpha}(t_k-s)}(\Pi_{1,k-1}u(s))'ds = \hat{a}_i u(t_{k-1}) + \hat{b}_i u(t_{k-2}) + O(\tau^{2-\alpha}),
$$

where $\hat{a}_i = \frac{1-e^{-(\eta_i^{\alpha}+b)}\tau}{\tau(\eta_i^{\alpha}+b)e^{(\eta_i^{\alpha}+b)\tau}}$, $\hat{b}_i = -\hat{a}_i$. Hence, history term has recurrence relation

$$
U_{h,i}^{\alpha}(t_k) = e^{-\widetilde{\eta}_i^{\alpha}\tau}U_{h,i}^{\alpha}(t_{k-1}) + \hat{a}_i u(t_{k-1}) + \hat{b}_i u(t_{k-2}).
$$

Collecting the above formulas, we obtain the fast finite difference operator of fractional derivative (1.3)

$$
\begin{aligned}
F_t^{\alpha,\lambda(t)}u^1 =& L_t^{\alpha,\lambda(t)}(t_1), k=1, \\
F_t^{\alpha,\lambda(t)}u^k =& \frac{u(t_k)-u(t_{k-1})}{\tau\Gamma(2-\alpha)}\left(e^{-b\tau}\tau^{1-\alpha}+b\int_0^{\tau}e^{-b\theta}\theta^{1-\alpha}d\theta\right) + \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{N_{exp}} w_i^{\alpha}U_{h,i}^{\alpha}(t_k), 2 \le k \le N_T,
\end{aligned} \tag{4.3}
$$

where $U_{h,i}^{\alpha}(t_1) = 0$.

4.2. The fast semi-discrete problem

If we use the difference operator (4.3) to approximate fractional derivatives (1.3) at time level t_k, we get the semi-discrete scheme

$$F_t^{\alpha,\lambda(t)}u^k - \Delta u^k = f(t_k), 1 \le k \le N_T.$$

The weak form in L^2-inner product gives: Find $u^k \in H_0^1(\Omega)(1 \le k \le N_T)$, such that

$$\left(F_t^{\alpha,\lambda(t)}u^k, v\right) + \left(\nabla u^k, \nabla v\right) = (f^k, v), \forall v \in H_0^1(\Omega). \quad (4.4)$$

For the sake of simplification, (2.2) can be rewritten as

$$L_t^{\alpha,\lambda(t)}u(t_k) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)c_0^{-1}}\Big[u(t_k) - \sum_{j=1}^{k}\rho_{k-j}^{k,\alpha}u(t_{k-j})\Big],$$

where $\rho_{k-j}^{k,\alpha} = \frac{c_{j-1}-c_j}{c_0}$ as $1 \le j \le k-1$, $\rho_{k-j}^{k,\alpha} = \frac{c_{k-1}}{c_0}$ as $j = k$. By comparing the two operators $F_t^{\alpha,\lambda(t)}$ and $L_t^{\alpha,\lambda(t)}$, we have

$$\begin{aligned}
&F_t^{\alpha,\lambda(t)}u(t_k) - L_t^{\alpha,\lambda(t)}u(t_k)\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\Big[(t_k-s)^{-\alpha} - \sum_{i=1}^{Nexp} w_i^\alpha e^{-\eta_i^\alpha(t_k-s)}\Big]e^{-b(t_k-s)}(\Pi_{1,j}u(s))' ds\\
&= \frac{1}{\alpha_0(1-\alpha)^{-1}}\sum_{j=1}^{k}[\widetilde{a}_{k-j}u(t_{j-1}) + \widetilde{b}_{k-j}u(t_j)],
\end{aligned}$$

where $\Pi_{1,j}u(s) = u(t_{j-1})\frac{t_j-s}{\tau} + u(t_j)\frac{s-t_{j-1}}{\tau}$, $s \in [t_{j-1}, t_j]$,

$$\widetilde{a}_{k-j} = -\tau^{\alpha-1}\int_{t_{j-1}}^{t_j}\Big[(t_k-s)^{-\alpha} - \sum_{i=1}^{Nexp} w_i^\alpha e^{-\eta_i^\alpha(t_k-s)}\Big]e^{-b(t_k-s)}ds,$$

and

$$\widetilde{b}_{k-j} = \tau^{\alpha-1}\int_{t_{j-1}}^{t_j}\Big[(t_k-s)^{-\alpha} - \sum_{i=1}^{Nexp} w_i^\alpha e^{-\eta_i^\alpha(t_k-s)}\Big]e^{-b(t_k-s)}ds.$$

Simple calculations, we can check that

$$\left|\widetilde{a}_{k-j}\right| = \left|\widetilde{b}_{k-j}\right| \le \tau^\alpha \varepsilon.$$

For the truncation error of the fast difference operator $F_t^{\alpha,\lambda(t)}$ gives

$$R_\tau^{k,\alpha} := \partial_t^{\alpha,\lambda(t)}u(t_k) - F_t^{\alpha,\lambda(t)}u(t_k), 2 \le k \le N_T.$$

Lemma 4. *Suppose that $u(t) \in C^2[0,T]$, for any $0 < \alpha < 1$, there holds*

$$|R_\tau^{k,\alpha}| \le \tilde{c}\tau^{2-\alpha}\max_{0\le t\le t_k}|u''(t)| + \tilde{c}\varepsilon\max_{0\le t\le t_k}|u(t)|, 2 \le k \le N_T.$$

Proof. By the triangle inequality

$$\begin{aligned}|R_\tau^{k,\alpha}| = |\partial_t^{\alpha,\lambda(t)}u(t_k) - F_t^\alpha u(t_k)| &\le |\partial_t^{\alpha,\lambda(t)}u(t_k) - L_t^\alpha u(t_k)| + |L_t^\alpha u(t_k) - F_t^\alpha u(t_k)|\\ &\le \tilde{c}\tau^{2-\alpha}\max_{0\le t\le t_k}|u''(t)| + \frac{1}{\alpha_0(1-\alpha)^{-1}}\sum_{j=1}^{k}[\widetilde{a}_{k-j}u(t_{j-1}) + \widetilde{b}_{k-j}u(t_j)]\\ &\le \tilde{c}\tau^{2-\alpha}\max_{0\le t\le t_k}|u''(t)| + \tilde{c}\varepsilon\max_{0\le t\le t_k}|u(t)|.\end{aligned}$$

□

Furthermore, we can rewrite the fast difference operator as

$$\begin{aligned}F_t^{\alpha,\lambda(t)}u^k &= \frac{1}{\alpha_0 c_0^{-1}}\Big[u^k - \sum_{j=1}^{k}\rho_{k-j}^{k,\alpha}u^{k-j}\Big] + \frac{1}{\alpha_0(1-\alpha)^{-1}}\sum_{j=1}^{k}\Big[\widetilde{a}_{k-j}u^{j-1} + \widetilde{b}_{k-j}u^j\Big]\\ &= \frac{1}{\alpha_0 c_0^{-1}}\Big[u^k - \sum_{j=1}^{k}\rho_{k-j}^{k,\alpha}u^{k-j}\Big] + \frac{1}{\alpha_0(1-\alpha)^{-1}}\sum_{j=0}^{k}\zeta_{k-j}^{k,\alpha}u^{k-j},\end{aligned}$$

where

$$\begin{aligned}&\zeta_0^{1,\alpha} = \widetilde{a}_0, \zeta_1^{1,\alpha} = \widetilde{b}_0;\\ &\zeta_0^{2,\alpha} = \widetilde{a}_1, \zeta_1^{2,\alpha} = \widetilde{a}_0 + \widetilde{b}_1, \zeta_2^{2,\alpha} = \widetilde{b}_0;\\ &\zeta_0^{3,\alpha} = \widetilde{a}_2, \zeta_1^{3,\alpha} = \widetilde{a}_1 + \widetilde{b}_2, \zeta_2^{3,\alpha} = \widetilde{a}_0 + \widetilde{b}_1, \zeta_3^{3,\alpha} = \widetilde{b}_0;\\ &\zeta_0^{k,\alpha} = \widetilde{a}_{k-1}, \zeta_{k-j}^{3,\alpha} = \widetilde{a}_{j-1} + \widetilde{b}_j, \zeta_{k-1}^{k,\alpha} = \widetilde{b}_0, k \ge 4, j = 2, 3, \ldots, k-1.\end{aligned}$$

The coefficients $\rho_{k-j}^{k,\alpha}$ have the following properties.

Lemma 5. *For $k \ge 1$, the coefficients $\rho_{k-j}^{k,\alpha}$ and c_0^{-1} satisfy*

(1) $\rho_{k-j}^{k,\alpha} > 0, j = 1, 2\ldots, k-1$; *(2)* $\sum\limits_{j=1}^{k}\rho_{k-j}^{k,\alpha} = 1, j = 1, 2\ldots, k-1$;

(3) $\rho_0^{k,\alpha} > \frac{2(1-\alpha)\lambda_{k-1/2}}{3k^\alpha}$; *(4)* $0 < \rho_{k-1}^{k,\alpha} < 1$; *(5)* $1 < c_0^{-1} < e^{\frac{b\tau}{2}}$.

Proof. See the Appendix A. □

To do the error estimates of the fast difference scheme, we follow the technique used in reference [24]. Introducing a parameter σ, then $u^k - \sum_{j=1}^{k}\rho_{k-j}^{k,\alpha}u^{k-j}$ can be rewritten as following

$$\begin{aligned}u^k - \sum_{j=1}^{k}\rho_{k-j}^{k,\alpha}u^{k-j} =&(u^k - \sigma u^{k-1}) - (\rho_{k-1}^{k,\alpha} - \sigma)(u^{k-1} - \sigma u^{k-2}) - \cdots\\ &- (\rho_2^{k,\alpha} + \sigma\rho_3^{k,\alpha} + \cdots + \sigma^{k-3}\rho_{k-1}^{k,\alpha} - \sigma^{k-2})(u^2 - \sigma u^1)\\ &- (\rho_1^{k,\alpha} + \sigma\rho_2^{k,\alpha} + \cdots + \sigma^{k-2}\rho_{k-1}^{k,\alpha} - \sigma^{k-1})(u^1 - \sigma u^0)\\ &- (\rho_0^{k,\alpha} + \sigma\rho_1^{k,\alpha} + \cdots + \sigma^{k-1}\rho_{k-1}^{k,\alpha} - \sigma^k)u^0\\ =& \bar{u}^k - \sum_{j=1}^{k}\bar{\rho}_{k-j}^{k,\alpha}\bar{u}^{k-j},\end{aligned}$$

where $\sigma = \frac{1}{2}\rho_{k-1}^{k,\alpha}$, σ^j means power of σ, and

$$\bar{u}^0 = u^0, \bar{u}^j = u^j - \sigma u^{j-1}, \bar{\rho}_{k-j}^{k,\alpha} = \sum_{l=1}^{j} \sigma^{j-l}\rho_{k-l}^{k,\alpha} - \sigma^j, j = 1, 2, \ldots, k.$$

Similarly, the sum-of-exponentials part can be rewritten as

$$\frac{1}{\alpha_0(1-\alpha)^{-1}} \sum_{j=0}^{k} \zeta_{k-j}^{k,\alpha} u^{k-j} = \frac{1}{\alpha_0(1-\alpha)^{-1}} \sum_{j=0}^{k} \bar{\zeta}_{k-j}^{k,\alpha} \bar{u}^{k-j},$$

where $\bar{\zeta}_{k-j}^{k,\alpha} = \sum_{l=0}^{j} \sigma^{j-l}\zeta_{k-l}^{k,\alpha}$, $j = 0, 1, \ldots, k$. The above statements show that the fast difference operator can be rewritten as

$$\bar{F}_t^{\alpha,\lambda(t)}\bar{u}^k = F_t^{\alpha,\lambda(t)}u^k = \frac{1}{\alpha_0 c_0^{-1}}[\bar{u}^k - \sum_{j=1}^{k}\bar{\rho}_{k-j}^{k,\alpha}\bar{u}^{k-j}] + \frac{1}{\alpha_0(1-\alpha)^{-1}} \sum_{j=1}^{k} \bar{\zeta}_{k-j}^{k,\alpha}\bar{u}^{k-j}.$$

Thus, the semi-discrete scheme (4.4) can be rewritten as

$$\left(\bar{F}_t^{\alpha,\lambda(t)}\bar{u}^k, v\right) + \left(\nabla u^k, \nabla v\right) = (f^k, v), \forall v \in H_0^1(\Omega). \tag{4.5}$$

For the weighted coefficients $\bar{\rho}_{k-j}^{k,\alpha}$ and $\bar{\zeta}_{k-j}^{k,\alpha}$, there holds

Lemma 6. *For $k \geq 2$, the coefficients $\bar{\rho}_{k-j}^{k,\alpha}$ and $\bar{\zeta}_{k-j}^{k,\alpha}$ satisfy*
(1) $\bar{\rho}_{k-j}^{k,\alpha} > 0$, $j = 1, 2 \ldots, k$; (2) $\frac{1}{\bar{\rho}_0^{k,\alpha}} < \frac{1}{\rho_0^{k,\alpha}} < \frac{3k^\alpha}{2(1-\alpha)\lambda_{k-1/2}}$;
(3) $\sum\limits_{j=1}^{k} \bar{\rho}_{k-j}^{k,\alpha} \leq 1 - \frac{2\sigma(1-\alpha)\lambda_{k-1/2}}{3(1-\sigma)k^\alpha}$; (4) $|\bar{\zeta}_{k-j}^{k,\alpha}| \leq 4\varepsilon\tau^\alpha$, $j = 1, 2 \ldots, k$; (5) $0 < \sigma < \frac{1}{2}$.

Proof. See the Appendix B. □

Theorem 4. *Assume that the permissible error ε satisfies $\varepsilon \leq \min\left\{\frac{1}{4}\sqrt{\frac{\Gamma(1-\alpha)\sigma\lambda(T)}{3\hat{c}(1-\sigma)T^\alpha N_T e^{b\tau/2}}}, \frac{\Gamma(1-\alpha)}{4\hat{c}}\right\}$, then the semi-discrete scheme* (4.4) *is stable, and its solution satisfies*

$$\|u^k\|_1 \leq (c_0^{-1} + \tilde{c}(\alpha_0 c_0)^{-1/2})(\|u^0\| + \max_{0\leq j\leq k}\|f^j\|), 1 \leq k \leq N_T. \tag{4.6}$$

Proof. For $k = 1$, taking $v = u^1$ in (4.4), there holds

$$(u^1, u^1) + \alpha_0(\nabla u^1, \nabla u^1) = (u^0, u^1) + (f^1, u^1).$$

Using Schwarz inequality, we arrive at $\|u^1\|_1^2 \leq \|u^1\|\|u^0\| + \|f^1\|\|u^1\|$, which produces $\|u^1\|_1 \leq \|u^0\| + \|f^1\|$. Using (5) of Lemma 5, we have

$$\|u^1\|_1 \leq c_0^{-1}(\|u^0\| + \|f^1\|) \leq c_0^{-1}(1 + \tilde{c}(\alpha_0 c_0)^{-1/2})(\|u^0\| + \max_{0\leq j\leq 1}\|f^j\|).$$

For $k \geq 2$, using mathematical induction on the index k, we will prove the following result:

$$\alpha_0 c_0^{-1}\|\nabla u^k\|^2 \leq c_0^{-2}(\|u^0\|^2 + \max_{0\leq j\leq k}\|f^j\|), 2 \leq k \leq N_T. \tag{4.7}$$

Firstly, substituting $v = 2\bar{u}^k$ into semi-discrete scheme (4.5), it yields

$$2(\bar{u}^k, \bar{u}^k) + 2\alpha_0 c_0^{-1}(\nabla u^k, \nabla \bar{u}^k) + 2c_0^{-1}(1-\alpha)\sum_{j=0}^{k} \bar{\zeta}_{k-j}^{k,\alpha}(\bar{u}^{k-j}, \bar{u}^k) = 2\sum_{j=1}^{k} \bar{\rho}_{k-j}^{k,\alpha}(\bar{u}^{k-j}, \bar{u}^k) + 2(f^k, \bar{u}^k).$$

Using the identity $2(\nabla u^k, \nabla \bar{u}^k) = \|\nabla u^k\|^2 + \|\nabla \bar{u}^k\|^2 - \sigma^2\|\nabla u^{k-1}\|^2$ and (3) of Lemma 6, we have

$$\begin{aligned} &\|\bar{u}^k\|^2 + \alpha_0 c_0^{-1}\|\nabla u^k\|^2 + \alpha_0 c_0^{-1}\|\nabla \bar{u}^k\|^2 \\ &\leq \sigma\|\bar{u}^{k-1}\|^2 + \alpha_0 c_0^{-1}\sigma\|\nabla u^{k-1}\|^2 + \sum_{j=2}^{k} \bar{\rho}_{k-j}^{k,\alpha}\|u^{k-j}\|^2 \\ &- 2c_0^{-1}(1-\alpha)\sum_{j=0}^{k} \bar{\zeta}_{k-j}^{k,\alpha}(\bar{u}^{k-j}, \bar{u}^k) + \|f^k\|^2 + \|\bar{u}^k\|^2. \end{aligned} \tag{4.8}$$

Furthermore, using Poincaré inequality [26]

$$\|\bar{u}^k\| \leq \hat{c}\|\nabla \bar{u}^k\|, \tag{4.9}$$

and (5) of Lemma 5, if $\varepsilon \leq \frac{\Gamma(1-\alpha)}{4\hat{c}}$, we obtain

$$\left| -2c_0^{-1}(1-\alpha)\bar{\zeta}_k^{k,\alpha}(\bar{u}^k, \bar{u}^k)\right| \leq 2\tau^\alpha \varepsilon c_0^{-1}(1-\alpha)\|\bar{u}^k\|^2 \leq \frac{\alpha_0 c_0^{-1}}{2}\|\nabla \bar{u}^k\|^2,$$

where we used the relation $\bar{\zeta}_k^{k,\alpha} = \zeta_k^{k,\alpha} = \tilde{b}_0 \leq \tau^\alpha \varepsilon$ in above derivation. Setting $\theta_k = 1 - \sum_{j=1}^{k} \bar{\rho}_{k-j}^{k,\alpha}$ and using Young inequality, we obtain

$$\left| -2c_0^{-1}(1-\alpha)\sum_{j=1}^{k} \bar{\zeta}_{k-j}^{k,\alpha}(\bar{u}^{k-j}, \bar{u}^k)\right| \leq \frac{\theta_k}{k}\sum_{j=1}^{k}\|\bar{u}^{k-j}\|^2 + \frac{kc_0^{-2}(1-\alpha)^2}{\theta_k}\sum_{j=1}^{k}(\bar{\zeta}_{k-j}^{k,\alpha})^2\|\bar{u}^k\|^2.$$

According to (3) and (4) of Lemma 6, we obtain $\theta_k = 1 - \sum_{j=1}^{k} \bar{\rho}_{k-j}^{k,\alpha} \geq \frac{2\sigma(1-\alpha)\lambda_{k-1/2}}{3(1-\sigma)k^\alpha}$. Moreover, with the help of Poincaré inequality, we obtain

$$\left| -2c_0^{-1}(1-\alpha)\sum_{j=1}^{k} \bar{\zeta}_{k-j}^{k,\alpha}(\bar{u}^{k-j}, \bar{u}^k)\right| \leq \frac{\theta_k}{k}\sum_{j=1}^{k}\|\bar{u}^{k-j}\|^2 + \frac{3\hat{c}c_0^{-2}(1-\alpha)^2(1-\sigma)k^{\alpha+1}}{2\sigma(1-\alpha)\lambda(T)}(4\tau^\alpha\varepsilon)^2\|\nabla \bar{u}^k\|^2.$$

If $\varepsilon \leq \frac{1}{4}\sqrt{\frac{\Gamma(1-\alpha)\sigma\lambda(T)}{3\hat{c}(1-\sigma)T^\alpha N_T e^{b\tau/2}}}$, there holds

$$\begin{aligned} \left| -2c_0^{-1}(1-\alpha)\sum_{j=1}^{k} \bar{\zeta}_{k-j}^{k,\alpha}(\bar{u}^{k-j}, \bar{u}^k)\right| &\leq \frac{\theta_k}{k}\sum_{j=1}^{k}\|\bar{u}^{k-j}\|^2 + \frac{3\hat{c}c_0^{-2}(1-\alpha)^2(1-\sigma)k^{\alpha+1}}{2\sigma(1-\alpha)\lambda(T)}(4\tau^\alpha\varepsilon)^2\|\nabla \bar{u}^k\|^2 \\ &\leq \frac{\theta_k}{k}\sum_{j=1}^{k}\|\bar{u}^{k-j}\|^2 + \frac{\alpha_0 c_0^{-1}}{2}\|\nabla \bar{u}^k\|^2. \end{aligned}$$

Rewriting (4.8) as follows:

$$\alpha_0 c_0^{-1}\|\nabla u^k\|^2 \le (\sigma + \frac{\theta_k}{k})\|u^{k-1}\|^2 + \alpha_0 c_0^{-1}\sigma\|\nabla u^{k-1}\|^2 + \sum_{j=2}^{k}(\bar{\rho}_{k-j}^{k,\alpha} + \frac{\theta_k}{k})\|\bar{u}^{k-j}\|^2 + \|f^k\|^2.$$

In view of relation $\sigma + \theta_2 + \bar{\rho}_0^{2,\alpha} = 1$, we obtain (4.7) for $k = 2$. Assuming that (4.7) is already correct for all $k = 2, 3, \ldots, n-1$, we can check that

$$\begin{aligned}\alpha_0 c_0^{-1}\|\nabla u^n\|^2 &\le c_0^{-2}(\sigma + \frac{\theta_n}{n} + \sum_{j=2}^{k}\bar{\rho}_{k-j}^{k,\alpha} + \frac{\theta_n}{n}(n-1))\|u^0\|^2 + \|f^n\|^2 \\ &= c_0^{-2}(\|u^0\|^2 + \|f^n\|^2) \\ &\le c_0^{-2}(\|u^0\| + \max_{0\le j\le n}\|f^j\|)^2, \forall\, n = 2, \ldots, N_T,\end{aligned}$$

which implies

$$\sqrt{\alpha_0 c_0^{-1}}\|\nabla u^n\| \le c_0^{-1}(\|u^0\| + \max_{0\le j\le n}\|f^j\|). \tag{4.10}$$

Furthermore, using (4.10) and Poincaré inequality (4.9), we obtain

$$\|u^n\|^2 \le \hat{c}\|\nabla u^n\|^2 \le \tilde{c}(\alpha_0 c_0)^{-1/2}(\|u^0\| + \max_{0\le j\le n}\|f^j\|). \tag{4.11}$$

Combining (4.10) and (4.11), we obtain the desired results. □

For the convergence of time semi-discrete scheme (4.4), we have

Theorem 5. *Let $u(\boldsymbol{x}, t)$ be the exact solution of problem* (1.1)*, and $\{u^k\}_{k=1}^{N_T}$ be the semi-discrete solution of scheme* (4.4) *with the initial $u^0 = u(0)$. Under the assumption given in Theorem 4 and $\partial_t^2 u(\boldsymbol{x}, t) \in L^\infty((0, T]; L^2(\Omega))$, there holds*

$$\left\|u(t_k) - u^k\right\|_1 \le \tilde{c}_{\alpha,T}\Big(\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(L^2)} + \varepsilon\|u\|_{L^\infty(L^2)}\Big), 2 \le k \le N_T, \tag{4.12}$$

where $\tilde{c}_{\alpha,T}$ depends only on α and T.

Proof. Let $e^k = u(t_k) - u^k$. Subtracting (4.4) from problem (1.1) at time level t_k, it follows that

$$\Big(F_t^{\alpha,\lambda(t)}e^k, v\Big) + \Big(\nabla e^k, \nabla v\Big) = \Big(R_\tau^{k,\alpha}, v\Big), \forall v \in H_0^1(\Omega).$$

With the similar technique in Theorem 4, we obtain

$$\|e^k\|^2 + \alpha_0 c_0^{-1}\|\nabla e^k\|^2 \le \tilde{c}_{\alpha,T}\max_{2\le i\le N_T}\|R_\tau^{k,\alpha}\|^2, k \ge 2.$$

Applying Lemma 4, we obtain (4.12). □

4.3. Fast difference/Legendre spectral collocation

We consider the spectral collocation method in space as follows: Find $u_N^k \in V_N^0(\Omega)$, such that

$$\left(F_t^{\alpha,\lambda(t)} u_N^k, v_N\right)_N + \left(\nabla u_N^k, \nabla v_N\right)_N = 0, \forall v_N \in V_N^0(\Omega). \tag{4.13}$$

Theorem 6. *Let $u(\boldsymbol{x}, t)$ be the exact solution of* (1.1)*, and u_N^k be the solution of scheme* (4.13) *with the initial condition $u_N^0 = \pi_N u^0$. Suppose $\partial_t^2 u(\boldsymbol{x}, t) \in L^\infty((0, T]; H^m(\Omega))(m \geq 1)$, for $k = 2, \ldots, N_T$, there holds*

$$\begin{aligned} \|u(t_k) - u_N^k\|_1 \leq \tilde{c}_{\alpha,T}\Big(&\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(L^2)} + N^{-m}\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(H^m)} \\ &+ N^{-m}\|\partial_t^{\alpha,\lambda(t)} u\|_{L^\infty(H^m)} + N^{1-m}\|u\|_{L^\infty(H^m)} \\ &+ \varepsilon\|u\|_{L^\infty(L^2)} + \varepsilon N^{-m}\|u\|_{L^\infty(H^m)}\Big). \end{aligned} \tag{4.14}$$

Proof. Let $e_N^k = u_N^k - \pi_N u(t_k)$, using (4.13), we have

$$\begin{aligned} &(F_t^{\alpha,\lambda(t)} e_N^k, v_N)_N + (\nabla e_N^k, \nabla v_N)_N \\ &= (F_t^{\alpha,\lambda(t)} u_N^k, v_N)_N + (\nabla u_N^k, \nabla v_N)_N - (F_t^{\alpha,\lambda(t)} \pi_N u(t_k), v_N)_N - (\nabla \pi_N u(t_k), \nabla v_N)_N \\ &= (F_t^{\alpha,\lambda(t)} u_N^k - F_t^{\alpha,\lambda(t)} \pi_N u(t_k), v_N)_N - (F_t^{\alpha,\lambda(t)} u(t_k), v_N)_N - (\nabla \pi_N u(t_k), \nabla v_N)_N \\ &:= \varepsilon_1^k(v_N) + \varepsilon_2^k(v_N), \forall v_N \in V_N^0(\Omega), \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \varepsilon_1^k(v_N) &= (F_t^{\alpha,\lambda(t)} u_N^k - F_t^{\alpha,\lambda(t)} \pi_N u(t_k), v_N)_N, \\ \varepsilon_2^k(v_N) &= -(F_t^{\alpha,\lambda(t)} u(t_k), v_N)_N - (\nabla \pi_N u(t_k), \nabla v_N)_N. \end{aligned} \tag{4.16}$$

Using equality $\bar{F}_t^{\alpha,\lambda(t)} \bar{u}^k = F_t^{\alpha,\lambda(t)} u^k$, scheme (4.15) can be rewritten as

$$(\bar{F}_t^{\alpha,\lambda(t)} \bar{e}_N^k, v_N)_N + (\nabla e_N^k, \nabla v_N)_N = \varepsilon_1^k(v_N) + \varepsilon_2^k(v_N), \forall v_N \in V_N^0(\Omega),$$

where $\bar{e}_N^k = e_N^k - \sigma e_N^{k-1}$. Moreover, using Lemma 2, we have

$$\begin{aligned} \varepsilon_1^k(v_N) &= (F_t^{\alpha,\lambda(t)} u_N^k - F_t^{\alpha,\lambda(t)} \pi_N u(t_k), v_N)_N \\ &= ((I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)} u(t_k) - R_\tau^{k,\alpha}), v_N)_N \\ &\leq |((I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)} u(t_k) - R_\tau^{k,\alpha}), v_N)| + \tilde{c} N^{-1}\|(I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)} u(t_k) - R_\tau^{k,\alpha})\|_1 \|v_N\|_{0,N} \\ &\leq \Big[\|(I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)} u(t_k) - R_\tau^{k,\alpha})\|_{0,N} + \tilde{c} N^{-1}\|(I_d - \pi_N)(\partial_t^{\alpha,\lambda(t)} u(t_k) - R_\tau^{k,\alpha})\|_1\Big] \|v_N\|_{0,N}. \end{aligned} \tag{4.17}$$

According to Lemma 4, for $l = 0, 1$, we have

$$\|(I_d - \pi_N) R_\tau^{k,\alpha}\|_l \leq \tilde{c}_{\alpha,T} \max_{0 \leq t \leq t_k} \|(I_d - \pi_N)\partial_t^2 u(t)\|_l \tau^{2-\alpha} + \tilde{c}_{\alpha,T}\varepsilon \max_{0 \leq t \leq t_k} \|(I_d - \pi_N)\partial_t^2 u(t)\|_l.$$

Combing (3.2), Eq (4.17) produces

$$|\varepsilon_1^k(v_N)| \leq \tilde{c}\left(N^{-m}\|\partial_t^{\alpha,\lambda(t)} u\|_{L^\infty(H^m)} + N^{-m}\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(H^m)} + \varepsilon N^{-m}\|u\|_{L^\infty(H^m)}\right)\|v_N\|_{0,N}.$$

According to the definition of the $\|\cdot\|_{0,N}$ and projection operator π_N, we have

$$(\nabla \pi_N u(t_k), \nabla v_N)_N = (\nabla \pi_N u(t_k), \nabla v_N) = (\nabla u(t_k), \nabla v_N), \forall v_N \in V_N^0(\Omega). \tag{4.18}$$

Applying (1.1), (4.16), and (4.18), we obtain

$$\begin{aligned}\varepsilon_2^k(v_N) &= -(F_t^{\alpha,\lambda(t)}u(t_k), v_N)_N - (\nabla\pi_N u(t_k), \nabla v_N)_N \\ &= (F_t^{\alpha,\lambda(t)}u(t_k), v_N) - (F_t^{\alpha,\lambda(t)}u(t_k), v_N)_N + (R_\tau^{k,\alpha}, v_N).\end{aligned}$$

Furthermore, using Lemmas 2 and 4, we obtain

$$\begin{aligned}|\varepsilon_2^k(v_N)| &\le (\tilde{c}N^{-m}\|\partial_t^{\alpha,\lambda(t)}u(t_k) - R_\tau^{k,\alpha}\|_m + \|R_\tau^{k,\alpha}\|_{0,N})\|v_N\|_{0,N} \\ &\le \tilde{c}_{\alpha,T}\Big(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + N^{-m}\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(H^m)} + \varepsilon N^{-m}\|u\|_{L^\infty(H^m)} \\ &+ \tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(L^2)} + \varepsilon\|u\|_{L^\infty(L^2)}\Big)\|v_N\|_{0,N}.\end{aligned}$$

Combining the estimates of $|\varepsilon_1^k(v_N)|$, $|\varepsilon_2^k(v_N)|$ and the Poincaré inequality, we have

$$\begin{aligned}|\varepsilon_1^k(v_N)| + |\varepsilon_2^k(v_N)| &\le \tilde{c}_{\alpha,T}\Big(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + N^{-m}\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(H^m)} + \varepsilon N^{-m}\|u\|_{L^\infty(H^m)} \\ &+ \tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(L^2)} + \varepsilon\|u\|_{L^\infty(L^2)}\Big)\|\nabla v_N\|_{0,N}.\end{aligned}$$

With the similar argument given in Theorem 4, we obtain

$$\begin{aligned}\|e_N^k\| + \sqrt{\alpha_0 c_0^{-1}}\|\nabla e_N^k\| &\le \tilde{c}_{\alpha,T}\Big(N^{-m}\|\partial_t^{\alpha,\lambda(t)}u\|_{L^\infty(H^m)} + N^{-m}\tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(H^m)} + \varepsilon N^{-m}\|u\|_{L^\infty(H^m)} \\ &+ \tau^{2-\alpha}\|\partial_t^2 u\|_{L^\infty(L^2)} + \varepsilon\|u\|_{L^\infty(L^2)}\Big).\end{aligned}$$

By the triangle inequality $\|u(t_k) - u_N^k\|_{1,N} \le \|u(t_k) - \pi_N u(t_k)\|_{1,N} + \|u_N^k - \pi_N u(t_k)\|_{1,N}$, we finally obtain our conclusion. □

5. Numerical experiments

In this section, we will test our theory results given in previous sections. The error in time direction at terminal time $T = 1$ is measured by the point-wise maximum norm. In the following numerical experiments, we used two examples with the exponential kernel function $\lambda(t) = e^{-t}$, which modify the numerical examples given in reference [5]. In our numerical experiments, all the algorithms are implemented by MATLAB R2022b, which were run on a 2.90 GHz PC with 32 GB of RAM and the Windows 11 operating system.

Example 1. *First, we consider problem* (1.1) *with exact solution* $u(x, y, t) = (1 + (6 - (6 + 6t + 3t^2 + t^3)e^{-t})\sin(\pi x)\sin(\pi y)$. *The source term and initial values are given by the exact solution.*

In Table 1, we list the errors and convergence orders of the present numerical scheme (3.4). Here, we fix the degree of polynomial $N_x = N_y = 20$. It can be observed that the time direction convergence order is $O(\tau^{2-\alpha})$, which is consistent with our theoretical analysis. To test the convergence order in spatial direction, we set $\tau = 1/1000$ for $\alpha = 0.3, 0.5$, and 0.9. The relationship between the errors and polynomial degree N is shown in the semi-log graph in Figure 2. We find that the numerical solution has the exponential accuracy.

Table 1. Maximum errors and convergence orders at $T = 1$ with $N_x = N_y = 20$.

	α=0.3			α=0.5			α=0.9	
τ	Error	Order	τ	Error	Order	τ	Error	Order
1/10	9.3238e-05	*	1/10	2.3796e-04	*	1/10	1.1903e-03	*
1/20	2.9882e-05	1.6416	1/20	8.6493e-05	1.4601	1/20	5.6183e-04	1.0831
1/40	9.4754e-06	1.6570	1/40	3.1118e-05	1.4748	1/40	2.6352e-04	1.0922
1/80	2.9837e-06	1.6671	1/80	1.1129e-05	1.4834	1/80	1.2326e-04	1.0962

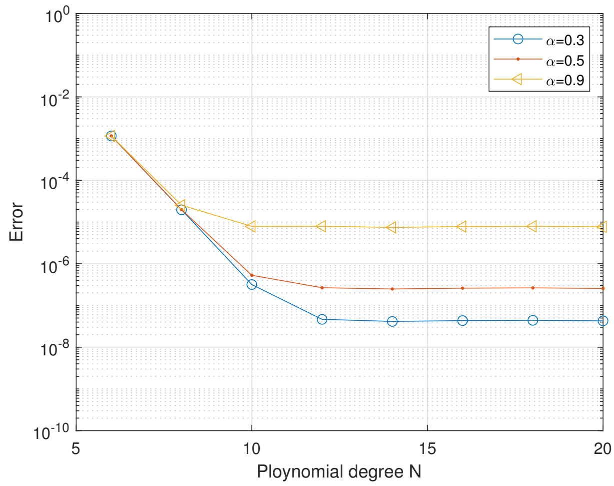

Figure 2. The numerical errors of the smooth solution with the polynomial degree $N_x = N_y = 20$.

Example 2. *To test the effectiveness of the fast difference/spectral collocation scheme* (4.13)*, we consider problem* (1.1) *with exact solution* $u(x, y, t) = (3 - (2 + 2t + t^2)e^{-t})\sin(\pi x)\sin(2\pi y)$, *and the source term and initial values are calculated by the exact solution.*

The numerical results of schemes (3.4) and (4.13) are listed in Tables 2 and 3. In these tables, we compared the errors, convergence orders, and CPU times of the two schemes in Tables 2 and 3 for $\alpha = 0.5$ and $\alpha = 0.7$, respectively. In the implementation of fast difference/spectral collocation, we take $\varepsilon = 1e - 9$, and it can be seen that the time convergence order of fast solver is $O(\tau^{2-\alpha})$, which is consistent with our theoretical results. In this example, the CPU(s) are measured, and the total times of numerical errors for the time steps τ vary in $[1/40, 1/80, 1/160, 1/320, 1/640, 1/1280]$. We also observe that the fast $L1$ scheme/spectral collocation has lower complexity and achieves the same accuracy as the $L1$ scheme ones.

Table 2. The comparison of maximum errors and convergence orders at $T = 1$ for the fast scheme and the direct scheme with $N_x = N_y = 20$, α=0.5.

	Direct scheme (3.4)			Fast scheme (4.13)	
τ	Error	Order	τ	Error	Order
1/40	6.7179e-06	*	1/40	6.5977e-06	*
1/80	2.3550e-06	1.5123	1/80	2.3258e-06	1.5042
1/160	8.2788e-07	1.5082	1/160	8.2074e-07	1.5027
1/320	2.9156e-07	1.5056	1/320	2.8981e-07	1.5018
1/640	1.0281e-07	1.5038	1/640	1.0237e-07	1.5013
1/1280	3.6276e-08	1.5029	1/1280	3.6269e-08	1.5010
CPU(s)	657.5		CPU(s)	42.6	

Table 3. The comparison of maximum errors and convergence orders at $T = 1$ for the fast scheme and the direct scheme with $N_x = N_y = 20$, α=0.7.

	Direct scheme (3.4)			Fast scheme (4.13)	
τ	Error	Order1	τ	Error	Order1
1/40	1.9589e-05	*	1/40	1.9405e-05	*
1/80	7.8911e-06	1.3117	1/80	7.8473e-06	1.3061
1/160	3.1894e-06	1.3069	1/160	3.1789e-06	1.3036
1/320	1.2916e-06	1.3041	1/320	1.2891e-06	1.3021
1/640	5.2369e-07	1.3023	1/640	5.2313e-07	1.3011
1/1280	2.1246e-07	1.3015	1/1280	2.1237e-07	1.3005
CPU(s)	659.0		CPU(s)	61.7	

6. Conclusions

In this paper, the two-dimensional fractional diffusion equation with generalized memory kernel is analyzed and approximated in time and space. We used the spectral collocation method in the spatial direction, and the $L1$ formula and the fast $L1$ formula are used in time to approximate the fractional derivative. Finally, we obtained estimates of the temporal and spatial errors of the present schemes, the time and space error estimates are $O(\tau^{2-\alpha} + O(N^{1-m})$. By comparing the $L1$ scheme with the fast $L1$ scheme, we found that the fast $L1$ scheme can significantly reduce storage costs and computational costs. There were also many improvements in this article, such as the use of graded grids instead of uniform grids (e.g., Stynes et al. [11]) for time grids when the initial singularity of solution is considered. Our future work will focus on this issue with the help of the existing techniques developed in references [10, 27, 28].

Author contributions

Zunyuan Hu: Methodology, software, writing-original draft, writing-review & editing; Can Li: Conceptualization, methodology, writing-original draft, writing-review & editing; Shimin Guo: Methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

It is declared that none of the authors have any competing interests in this manuscript.

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Appendix A: Proof of Lemma 5

(1) According to the definitions of $\rho_{k-j}^{k,\alpha}$ and c_0, we obtain

$$\begin{aligned} &\rho_{k-j}^{k,\alpha} = (c_{j-1} - c_j)c_0^{-1}, \rho_0^{k,\alpha} = c_{k-1}c_0^{-1}, \\ &c_0^{-1} = \frac{1}{\lambda_{1/2}a_0 + (\lambda_0 - \lambda_1)b_0} = \frac{1}{e^{-b\tau/2} + (1 - e^{-b\tau})(\frac{1}{2-\alpha} - \frac{1}{2})} > 0. \end{aligned}$$

By simple calculation, and using Lemma 1, we have

$$c_{j-1} - c_j = \lambda_{j-1/2}a_{j-1} - \lambda_{j+1/2}a_j + (\lambda_{j-1} - \lambda_j)b_{j-1} - (\lambda_j - \lambda_{j+1})b_j > 0.$$

(2) After calculating term by term, we obtain

$$\begin{aligned} \sum_{j=1}^{k} \rho_{k-j}^{k,\alpha} &= \rho_0^{k,\alpha} + \rho_1^{k,\alpha} + \rho_2^{k,\alpha} + \ldots + \rho_{k-1}^{k,\alpha} \\ &= c_0^{-1}\Big[(c_0 - c_1) + (c_1 - c_2) + (c_2 - c_3) + \ldots + c_{k-1}\Big] \\ &= c_0^{-1}c_0 = 1. \end{aligned}$$

(3) Using Lemma 1 once again, we have

$$\begin{aligned} \rho_0^{k,\alpha} &= c_{k-1}c_0^{-1} \\ &= \frac{\lambda_{k-1/2}a_{k-1} + (\lambda_{k-1} - \lambda_k)b_{k-1}}{\lambda_{1/2}a_0 + (\lambda_0 - \lambda_1)b_0} \\ &> \frac{\tau^\alpha\Gamma(2-\alpha)\lambda_{k-1/2}}{\Gamma(1-\alpha)t_k^\alpha} \cdot \frac{1}{\lambda_{1/2}a_0 + (\lambda_0 - \lambda_1)b_0} \\ &> \frac{\lambda_{k-1/2}(1-\alpha)}{k^\alpha} \cdot \frac{1}{\lambda_{1/2}a_0 + (\lambda_0 - \lambda_1)b_0} > \frac{2(1-\alpha)\lambda_{k-1/2}}{3k^\alpha}. \end{aligned}$$

(4) Since (1) provides that $\rho_{k-1}^{k,\alpha} > 0$, then $\rho_{k-1}^{k,\alpha} = (c_0 - c_1)c_0^{-1} = 1 - \frac{c_1}{c_0} < 1$.

(5) Note that

$$\begin{aligned} c_0 &= \lambda_{1/2}a_0 + (\lambda_0 - \lambda_1)b_0 \\ &= e^{-\frac{1}{2}b\tau}\left[1 - e^{-\frac{1}{2}b\tau}(\frac{1}{2-\alpha} - \frac{1}{2})\right] + \left(\frac{1}{2-\alpha} - \frac{1}{2}\right) \\ &= e^{-\frac{1}{2}b\tau}\left[1 - e^{-\frac{1}{2}b\tau}Z\right] + Z, \end{aligned}$$

where $Z = \frac{1}{2-\alpha} - \frac{1}{2}$, $0 \le Z \le \frac{1}{2}$. We may rewrite c_0 as follows

$$c_0 = e^{-\frac{1}{2}b\tau}\left[1 - e^{-\frac{1}{2}b\tau}Z\right] + Z := g(\tau, Z).$$

By checking the partial derivatives of g with respect to variable Z, we obtain $\frac{\partial g(\tau,Z)}{\partial Z} = 1 - e^{-b\tau} \ge 0$, which means $g(\tau, Z)$ is monotone increasing function with respect to variable Z, thus we get $c_0 \le e^{-\frac{1}{2}b\tau}\left[1 - \frac{1}{2}e^{-\frac{1}{2}b\tau}\right] + \frac{1}{2}$, and $c_0 \ge e^{-\frac{1}{2}b\tau}$. Furthermore, we define $\tilde{g}(\tau) = e^{-\frac{1}{2}b\tau}\left[1 - \frac{1}{2}e^{-\frac{1}{2}b\tau}\right] + \frac{1}{2}$ and take the derivative of $\tilde{g}(\tau)$, we obtain

$$\begin{aligned} \frac{\partial \tilde{g}(\tau)}{\partial \tau} &= -\frac{1}{2}be^{-\frac{1}{2}b\tau}\left[1 - \frac{1}{2}e^{-\frac{1}{2}b\tau}\right] + \frac{1}{4}be^{-b\tau} \\ &= \frac{1}{4}be^{-b\tau} - \frac{1}{2}be^{-\frac{1}{2}b\tau} + \frac{1}{4}be^{-b\tau} \\ &= be^{-\frac{1}{2}b\tau}\left[\frac{1}{2}e^{-\frac{1}{2}b\tau} - \frac{1}{2}\right] \le 0. \end{aligned}$$

Hence, $\tilde{g}(\tau) \le 1$, $0 < c_0 = g(\tau, Z) \le \tilde{g}(\tau) \le 1$, it provides that $c_0^{-1} = \frac{1}{g(\tau,Z)} \ge \frac{1}{\tilde{g}(\tau)} \ge 1$. Finally we get $1 \le c_0^{-1} \le e^{\frac{1}{2}b\tau}$.

Appendix B: Proof of Lemma 6

(1) From Lemma 5, for $j = 1, 2, ..., k$, we have $\rho_{k-j}^{k,\alpha} > 0$, thus $\sigma = \frac{1}{2}\rho_{k-1}^{k,\alpha} > 0$. By using $\bar{\rho}_{k-j}^{k,\alpha} = \sum_{l=1}^{j} \sigma^{j-l}\bar{\rho}_{k-l}^{k,\alpha} - \sigma^j$, we further have

$$\begin{aligned} \bar{\rho}_{k-j}^{k,\alpha} &= \sum_{l=1}^{j} \sigma^{j-l}\bar{\rho}_{k-l}^{k,\alpha} - \sigma^j \\ &= (2\sigma^j + \sigma^{j-2}\bar{\rho}_{k-2}^{k,\alpha} + \sigma^{j-3}\bar{\rho}_{k-3}^{k,\alpha} + ... + \sigma\bar{\rho}_{k-j+l}^{k,\alpha} + \bar{\rho}_{k-j}^{k,\alpha}) - \sigma^j \\ &= \sigma^j + \sigma^{j-2}\bar{\rho}_{k-2}^{k,\alpha} + \sigma^{j-3}\bar{\rho}_{k-3}^{k,\alpha} + ... + \sigma\bar{\rho}_{k-j+l}^{k,\alpha} + \bar{\rho}_{k-j}^{k,\alpha} > 0. \end{aligned}$$

(2) According to Lemma 5, we have $\frac{1}{\rho_0^{k,\alpha}} < \frac{3k^\alpha}{2(1-\alpha)\lambda_{k-1/2}}$. Moreover, note that $\rho_0^{k,\alpha} < \bar{\rho}_0^{k,\alpha}$, we have $\frac{1}{\bar{\rho}_0^{k,\alpha}} < \frac{1}{\rho_0^{k,\alpha}} < \frac{3k^\alpha}{2(1-\alpha)\lambda_{k-1/2}}$.

(3) Note that $\frac{1}{\bar{\rho}_0^{k,\alpha}} < \frac{3k^\alpha}{2(1-\alpha)\lambda_{k-1/2}}$, we may check that

$$\begin{aligned}\sum_{j=1}^{k}\bar{\rho}_{k-j}^{k,\alpha} =&\rho_{k-1}^{k,\alpha}(1+\sigma+\sigma^2+\sigma^3\ldots+\sigma^{k-1})+\rho_{k-2}^{k,\alpha}(1+\sigma+\sigma^2+\sigma^3\ldots+\sigma^{k-2})\\&+\ldots+\rho_2^{k,\alpha}(1+\sigma+\sigma^2)+\rho_1^{k,\alpha}(1+\sigma)+\rho_0^{k,\alpha}-(\sigma+\sigma^2+\sigma^3\ldots+\sigma^k)\\&=\frac{1}{1-\sigma}[(\rho_{k-1}^{k,\alpha}+\rho_{k-2}^{k,\alpha}+\rho_{k-3}^{k,\alpha}+\ldots+\rho_1^{k,\alpha}+\rho_0^{k,\alpha})-\sigma(1-\sigma^k)\\&-\sigma(\rho_{k-1}^{k,\alpha}\sigma^{k-1}+\rho_{k-2}^{k,\alpha}\sigma^{k-2}+\ldots+\rho_1^{k,\alpha}\sigma+\rho_0^{k,\alpha})]\\&=1-\frac{\sigma}{1-\sigma}\bar{\rho}_0^{k,\alpha}\le 1-\frac{2\sigma(1-\alpha)\lambda_{k-1/2}}{3(1-\sigma)k^\alpha}.\end{aligned}$$

(4) In view of $|\zeta_{k-j}^{k,\alpha}| \le 2\tau^\alpha\varepsilon$, we obtain $|\bar{\zeta}_{k-j}^{k,\alpha}| = |\sum_{l=0}^{j}\sigma^{j-l}\zeta_{k-j}^{k,\alpha}| \le 2\tau^\alpha\varepsilon\sum_{l=0}^{j}\sigma^l \le 4\tau^\alpha\varepsilon$.
(5) From Lemma 5, we have $\sigma = \frac{1}{2}\rho_{k-1}^{k,\alpha} < \frac{1}{2}$.

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