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Research article

Solutions to some generalized Fermat-type differential-difference equations

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Abstract: The main purpose of this article is to study Fermat-type complex differential-difference equations $f^{(k)}(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = R(z)$. Our results improve some results due to Wang–Xu–Tu [AIMS. Mathematics, 2020], Zhang [Bull. Korean. Math. Soc, 2018], and Long–Qin [Applied Mathematics-A Journal of Chinese Universities, 2024]. Moreover, we provide some examples to show the existence of the solutions.

Keywords: meromorphic functions; Fermat-type equation; differential polynomials; Nevanlinna theory; differential-difference equation
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1. Introduction and results

We first assume that the reader is familiar with the basic results and notations of the Nevanlinna theory and difference Nevanlinna theory with one complex variable, which can be found in [3,7,10,22]. In the past thirty years, there were lots of research focusing on the solutions of Fermat-type differential-difference equations; readers can refer to [1,4–6,9,11,12,15–21].

A. Wiles [13] in 1995 pointed out: The Fermat equation $x^m + y^m = 1$ does not admit nontrivial solutions in rational numbers as $m \geq 3$, and this equation possesses nontrivial rational solutions as $m = 2$. About sixty years ago, Gross [2] investigated the existence of solutions for the Fermat-type functional equation $f^m + g^m = 1$ and obtained: For $m = 2$, the entire solutions are $f = \cos\alpha(z)$, $g = \sin\alpha(z)$, where $\alpha(z)$ is an entire function; for $m > 2$, there are no nonconstant entire solutions. In 2009, Liu [7] proved that the Fermat-type equation $f(z)^2 + [f(z+c) - f(z)]^2 = a^2$ has no nonconstant entire solutions of finite order, where a is a nonzero constant. In 2012, Liu [8] studied that the Fermat-type equation $f'(z)^2 + [f(z+c) - f(z)]^2 = 1$ has the transcendental entire solutions with finite order. In 2018, Zhang [23] generalized Liu's [7,8] theorem, and obtained

Theorem 1.1. *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$.*

Then f can not be a solution of the difference equation

$$f(z)^2 + [f(z+c) - f(z)]^2 = R(z),$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant.

Theorem 1.2. *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$. If f is a solution of the differential-difference equation*

$$f'(z)^2 + [f(z+c) - f(z)]^2 = R(z),$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant, then $R(z)$ is a nonzero constant, and f is of the form

$$f(z) = c_1 e^{2iz} + c_2 e^{-2iz} + b, c = k\pi + \pi/2,$$

where c_1, c_2 are two nonzero constants such that $16c_1c_2 = R(z)$, b is a constant, and k is an integer.

In 2020, Wang et al. [14] promoted Zhang's [23] form and obtained

Theorem 1.3. *Let $\alpha(\neq 0)$, $\beta \in \mathbb{C}$, k be an integer. Let f be a transcendental meromorphic solution of difference-differential equation of Fermat type*

$$f'(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = R(z),$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant. If f is of finite order and has finitely many poles, then $i\alpha c = \pm 1$, and $R(z)$ is a nonconstant polynomial with $\deg_z R \leq 2$, or $R(z)$ is a nonzero constant. Furthermore
(I) If $R(z)$ is a nonconstant polynomial and $\deg_z R \leq 2$, then f is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $R(z) = (as_1(z) + m_1)(-as_2(z) + m_2)$, $a \neq 0$, $b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a = -i(\alpha + \beta)$, $c = \frac{(2k+1)\pi}{a}i$, $i\alpha c = -1$ or $a = i(\alpha - \beta)$, $c = \frac{2k\pi}{a}i$, $i\alpha c = 1$, where $s_j = m_j z + n_j$, m_j, $n_j \in \mathbb{C}(j = 1, 2)$;
(II) If $R(z)$ is a nonzero constant, then f is of the form

$$f(z) = \frac{n_1 e^{az+b} + n_2 e^{-(az+b)}}{2} + d,$$

$R(z) = -a^2 n_1 n_2$, $a \neq 0$, $b \in \mathbb{C}$, and a, b, c, α, β satisfy the following cases:
(II_1) if $\alpha = \beta$, then $a = -2\alpha i$, $c = \frac{(2k+1)\pi}{a}i$ and $d \in \mathbb{C}$;
(II_2) if $\alpha = -\beta$, then $a = 2\alpha i$, $c = \frac{2k\pi}{a}i$ and $d = 0$;
(II_3) if $\alpha \neq \pm\beta$, then $d = 0$ and $a = -i(\alpha + \beta)$, $c = \frac{(2k+1)\pi}{a}i$ or $a = i(\alpha - \beta), c = \frac{2k\pi}{a}i$.

Theorem 1.4. *Let $\alpha(\neq 0)$, $\beta \in \mathbb{C}$, k be an integer. Let f be a transcendental meromorphic solution of difference-differential equation of Fermat type*

$$f''(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = R(z), \tag{1.1}$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant.
(I) If $\alpha = \pm\beta$, then Eq (1.1) has no finite-order transcendental meromorphic solution with finitely many poles;
(II) If $\alpha \neq \pm\beta$, and Eq (1.1) has a finite-order transcendental meromorphic solution f with finitely many poles, then $R(z)$ must be a nonzero polynomial with $\deg_z R \leq 1$. Furthermore,

(II_1) if $R(z)$ is a nonzero polynomial of degree one, then $f(z)$ is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + n_2 e^{-(az+b)}}{2},$$

where $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \dfrac{\log \frac{a^2+i\beta}{i\alpha} + 2k\pi i}{a}$, $e^{ac} = \frac{2a}{i\alpha c} \neq \pm 1$ and $R(z) = a^3 n_2[as_1(z) + 2m_1]$, $s_1(z) = m_1 z + n_1$, or $f(z)$ is of the form

$$f(z) = \frac{n_1 e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \dfrac{\log \frac{-a^2+i\beta}{i\alpha} + (2k+1)\pi i}{a}$, $e^{ac} = \frac{i\alpha c}{2a} \neq \pm 1$ and $R(z) = a^3 n_1[as_2(z) - 2m_2]$, $s_2(z) = m_2 z + n_2$;

(II_2) if $R(z)$ is a nonzero constant, then $f(z)$ is of the form

$$f(z) = \frac{c_1 e^{az+b} + c_2 e^{-(az+b)}}{2},$$

where $a, b, c, \alpha, \beta, c_1, c_2, R(z)$ satisfy $a^4 = \alpha^2 - \beta^2, b \in \mathbb{C}$, $c = \dfrac{\log \frac{a^2+i\beta}{i\alpha} + 2k\pi i}{a}$ and $R(z) = a^4 c_1 c_2$.

Inspired by [14], can f' and f'' in Theorems 1.3 and 1.4 be replaced by $f^{(k)}$? In this paper, we consider this question. Our results are listed as follows.

Theorem 1.5. *Let $\alpha(\neq 0)$, $\beta \in \mathbb{C}$, $k \in \mathbb{Z}^+$, and k be an odd number. Let f be a transcendental meromorphic solution of difference-differential equation of Fermat type*

$$f^{(k)}(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = R(z), \tag{1.2}$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant.

If f is of finite order and has finitely many poles, then $i\alpha c = \pm 1$, and $R(z)$ is a polynomial with $\deg_z R = 1$, or $R(z)$ is a nonzero constant. Let $s_j(z) = m_j z + n_j$, m_j, $n_j \in \mathbb{C} (j = 1, 2)$.
***Case I*: If $R(z)$ is a polynomial with $\deg_z R = 1$, then f is of the form**

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $R(z) = -n_2 a^{2k-1}[as_1(z) + km_1]$, $m_1 \neq 0$, $m_2 = 0$, $a \neq 0$, $b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a^k = -i(\alpha + \beta)$, $c = \frac{(2l+1)\pi}{a} i$, $l \in \mathbb{Z}$, $i\alpha c = -ka^{k-1} = -1$ or $a^k = i(\alpha - \beta)$, $c = \frac{2l\pi}{a} i$, $l \in \mathbb{Z}$, $i\alpha c = ka^{k-1} = 1$; or $R(z) = n_1 a^{2k-1}[-as_2(z) + km_2]$, $m_1 = 0$, $m_2 \neq 0$, $a \neq 0$, $b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a^k = -i(\alpha + \beta)$, $c = \frac{(2l+1)\pi}{a} i$, $l \in \mathbb{Z}$, $i\alpha c = ka^{k-1} = -1$ or $a^k = i(\alpha - \beta)$, $c = \frac{2l\pi}{a} i$, $l \in \mathbb{Z}$, $i\alpha c = -ka^{k-1} = 1$.

Case II*: *If $R(z)$ is a nonzero constant, then $m_j = 0 (j = 1, 2)$, f is of the form

$$f(z) = \frac{n_1 e^{az+b} + n_2 e^{-(az+b)}}{2} + c_0,$$

where $R(z) = -a^{2k} n_1 n_2$, $a \neq 0, b \in \mathbb{C}$, and a, b, c, α, β satisfy the following cases:

(II_1) if $\alpha = \beta$, then $a^k = -2\alpha i$, $c = \frac{(2l+1)\pi}{a} i$ and $c_0 \in \mathbb{C}$, $l \in \mathbb{Z}$;

(II_2) if $\alpha = -\beta$, then $a^k = 2\alpha i$, $c = \frac{2l\pi}{a} i$ and $c_0 = 0$, $l \in \mathbb{Z}$;

(II_3) if $\alpha \neq \pm\beta$, then $c_0 = 0$ and $a^k = -i(\alpha + \beta)$, $c = \frac{(2l+1)\pi}{a} i$, $l \in \mathbb{Z}$ or $a^k = i(\alpha - \beta)$, $c = \frac{2l\pi}{a} i$, $l \in \mathbb{Z}$.

Remark 1. *When $k = 1$, Theorem 1.5 becomes Theorem 1.3.*

Remark 2. *In [14], we find that Case (I) of Theorem 1.3 is not accurate, including the corresponding examples. On page 695, line 11 of [14], substituting (5.10) into the first equation in (5.3), we have $R_2(z) = -as_2(z) + m_2$. Meanwhile, substituting (5.10) into the second equation in (5.3), we have $R_2(z) = -as_2(z) - m_2$. From this, we have $m_2 = 0$, which is a contradiction. Therefore, if $f'(z)^2 + [\alpha f(z + c) - \beta f(z)]^2 = R(z)$ has a solution, then* $\deg_z R \leq 1$.

Next, we will provide some examples to explain the existence of solutions to Eq (1.2) in the above situation.

Example 1.1. *For Case I, the transcendental entire function*

$$f(z) = \frac{(z + 1)e^{\frac{\sqrt{3}}{3}z+b} + e^{-(\frac{\sqrt{3}}{3}z+b)}}{2}$$

satisfies

$$f^{(3)}(z)^2 + [\frac{1}{\sqrt{3}\pi} f(z + c) - (\frac{\sqrt{3}}{9} i - \frac{1}{\sqrt{3}\pi}) f(z)]^2 = -\frac{\sqrt{3}}{9}[\frac{\sqrt{3}}{9}(z + 1) + 1],$$

where $k = 3$, $s_1(z) = z + 1$, $s_2(z) = 1$, $c = \sqrt{3}\pi i$, $a = \frac{\sqrt{3}}{3}$, $b \in \mathbb{C}$, $\alpha = \frac{1}{\sqrt{3}\pi}$, $\beta = \frac{\sqrt{3}}{9} i - \frac{1}{\sqrt{3}\pi}$ and $R(z) = -\frac{\sqrt{3}}{9}[\frac{\sqrt{3}}{9}(z + 1) + 1]$.

The transcendental entire function

$$f(z) = \frac{e^{\frac{\sqrt{3}}{3}iz+b} + (z + 1)e^{-(\frac{\sqrt{3}}{3}iz+b)}}{2}$$

satisfies

$$f^{(3)}(z)^2 + [\frac{i}{\sqrt{3}\pi} f(z + c) - (\frac{\sqrt{3}}{9} - \frac{i}{\sqrt{3}\pi}) f(z)]^2 = -\frac{\sqrt{3}}{9} i[\frac{\sqrt{3}}{9} i(z + 1) - 1],$$

where $k = 3$, $s_1(z) = 1$, $s_2(z) = z + 1$, $c = \sqrt{3}\pi$, $a = \frac{\sqrt{3}}{3} i$, $b \in \mathbb{C}$, $\alpha = \frac{i}{\sqrt{3}\pi}$, $\beta = \frac{\sqrt{3}}{9} - \frac{i}{\sqrt{3}\pi}$ and $R(z) = -\frac{\sqrt{3}}{9} i[\frac{\sqrt{3}}{9} i(z + 1) - 1]$.

Example 1.2. *For Case II_1, the transcendental entire function*

$$f(z) = \frac{e^{z+b} + 2e^{-(z+b)}}{2} + c_0$$

satisfies

$$f^{(3)}(z)^2+[\frac{i}{2}f(z+c)-\frac{i}{2}f(z)]^2=-2,$$

where $k=3$, $n_1=1$, $n_2=2$, $a=1$, $b,c_0\in\mathbb{C}$, $c=\pi i$, $\alpha=\frac{i}{2}$, $\beta=\frac{i}{2}$ *and* $R(z)=-2$.

Example 1.3. *For Case* II_2*, the transcendental entire function*

$$f(z)=\frac{e^{2z+b}+e^{-(2z+b)}}{2},$$

satisfies

$$f^{(3)}(z)^2+[-4if(z+c)-4if(z)]^2=-64,$$

where $k=3$, $n_1=1$, $n_2=1$, $a=2$, $b\in\mathbb{C}$, $c=\pi i$, $\alpha=-4i$, $\beta=4i$ *and* $R(z)=-64$.

Example 1.4. *For Case* II_3*, the transcendental entire function*

$$f(z)=\frac{2e^{z+b}+e^{-(z+b)}}{2}$$

satisfies

$$f^{(3)}(z)^2+[f(z+c)-(1+i)f(z)]^2=-64,$$

where $k=3$, $n_1=2$, $n_2=1$, $a=1$, $b\in\mathbb{C}$, $c=2\pi i$, $\alpha=1$, $\beta=1+i$ *and* $R(z)=-2$.

Theorem 1.6. *Let* $\alpha(\neq 0)$, $\beta\in\mathbb{C}$, $k\in\mathbb{Z}^+$ *and* k *be an even number. Let* f *be a transcendental meromorphic solution of Eq (1.2).*

If f *is of finite order and has finitely many poles, then* $R(z)$ *is a polynomial with* $\deg_z R=1$*, or* $R(z)$ *is a nonzero constant.*

Case I**:** *If* $\alpha=\pm\beta$*, then Eq (1.2) has no-finite order transcendental meromorphic solution with finitely many poles;*

Case II**:** *If* $\alpha\neq\pm\beta$*, and Eq (1.2) has a finite-order transcendental meromorphic solution* f *with finitely many poles, then* $R(z)$ *must be a nonzero polynomial with* $\deg_z R\leq 1$*. Let* $s_j=m_jz+n_j$, m_j, $n_j\in\mathbb{C}(j=1,2)$,

(II_1) *if* $R(z)$ *is a polynomial with* $\deg_z R=1$*, then* $f(z)$ *is of the form*

$$f(z)=\frac{s_1(z)e^{az+b}+n_2e^{-(az+b)}}{2},$$

where $a^{2k}=\alpha^2-\beta^2, b\in\mathbb{C}$, $c=\dfrac{\log\frac{a^k+i\beta}{i\alpha}+2l\pi i}{a}$, $l\in\mathbb{Z}$, $e^{ac}=\frac{ka^{k-1}}{i\alpha c}\neq\pm1$ *and* $R(z)=n_2a^{2k-1}[as_1(z)+km_1]$, *or* $f(z)$ *is of the form*

$$f(z)=\frac{n_1e^{az+b}+s_2(z)e^{-(az+b)}}{2},$$

where $a^{2k}=\alpha^2-\beta^2$, $b\in\mathbb{C}$, $c=\dfrac{\log\frac{a^k+i\beta}{i\alpha}+2l\pi i}{a}$, $l\in\mathbb{Z}$, $e^{ac}=\frac{i\alpha c}{ka^{k-1}}\neq\pm1$ *and* $R(z)=n_1a^{2k-1}[as_2(z)-km_2]$.

(II_2) *if* $R(z)$ *is a nonzero constant, then* $f(z)$ *is of the form*

$$f(z)=\frac{n_1e^{az+b}+n_2e^{-(az+b)}}{2},$$

where $a, b, c, \alpha, \beta, c_1, c_2, R(z)$ *satisfy* $a^{2k} = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $l \in \mathbb{Z}$, $c = \dfrac{\log \frac{a^k + i\beta}{i\alpha} + 2l\pi i}{a}$ *and* $R(z) = a^{2k} n_1 n_2$.

Remark 3. *When* $k = 2$*, Theorem 1.6 becomes Theorem 1.4.*

Next, we will provide some examples to explain the existence of solutions to Eq (1.2) in the above situation.

Example 1.5. *For Case* II_1*, let* c_0 *be a solution of equation* $e^{2c}(2 - c) = 2$*, the transcendental entire function*

$$f(z) = \frac{ze^{z+b} + e^{-(z+b)}}{2}$$

satisfies

$$f^{(4)}(z)^2 + [\frac{4}{ic_0 e^{c_0}} f(z + c) - \frac{4 - c_0}{ic_0} f(z)]^2 = -64,$$

where $k = 4$, $s_1(z) = z$, $n_2 = 1$, $a = 1$, $b \in \mathbb{C}$, $\alpha = \dfrac{4}{ic_0 e^{c_0}}$, $\beta = \dfrac{4 - c_0}{ic_0}$, $\alpha^2 - \beta^2 = 1$, $c_0 = \log \dfrac{1 + i\beta}{i\alpha}$, $e^{ac_0} = \dfrac{4}{i\alpha c_0}$ *and* $R(z) = z + 4$.

Example 1.6. *For Case* II_2*, the transcendental entire function*

$$f(z) = \frac{e^{z+b} + e^{-(z+b)}}{2}$$

satisfies

$$f^{(4)}(z)^2 + [\frac{i}{2} f(z + c) - \frac{\sqrt{5}}{2} i f(z)]^2 = 1,$$

where $k = 4$, $n_1 = 1$, $n_2 = 1$, $e^c = \sqrt{5} - 2$, $a = 1$, $b \in \mathbb{C}$, $\alpha = \frac{i}{2}$, $\beta = \frac{\sqrt{5}}{2} i$, $\alpha^2 - \beta^2 = 1$, $c = \log(\sqrt{5} - 2)$ *and* $R(z) = 1$.

In 2024, Long and Qin [9] studied this equation

$$f^{(k)}(z)^2 + P(z)^2 f(z + c)^2 = Q(z),$$

and obtained

Theorem 1.7. *There is no transcendental entire solution with finite order of the equation*

$$f^{(k)}(z)^2 + P(z)^2 f(z + c)^2 = Q(z), \tag{1.3}$$

where $P(z)$ *is a non-constant polynomial, and* $Q(z)$ *is a non-zero polynomial.*

Motivating from Theorems 1.5 and 1.6, we replace $f(z + c)$ by the $\alpha f(z + c) - \beta f(z)$ in Theorem 1.7 and obtain

Theorem 1.8. *There is no transcendental entire solution with finite order of the equation*

$$f^{(k)}(z)^2 + P(z)^2 (\alpha f(z + c) - \beta f(z))^2 = Q(z), \tag{1.4}$$

where $k \in \mathbb{Z}^+$, $\alpha (\neq 0)$, $\beta \in \mathbb{C}$, $P(z)$ *is a non-constant polynomial and* $Q(z)$ *is a non-zero polynomial.*

2. Some lemmas

We can use these lemmas to prove our theorems.

Lemma 2.1. *Let c, a, α be three nonzero constants, $k \in \mathbb{Z}^+$ be an odd number, and $ac \neq \pm(k-1)$. If R_1, R_2 are two nonzero rational functions satisfying the following differential-difference equations*

$$\begin{cases} i\alpha[R_1(z+c) - R_1(z)] = \sum\limits_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i, \\ i\alpha[R_2(z+c) - R_2(z)] = \sum\limits_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i. \end{cases} \tag{2.1}$$

Then $i\alpha c = ka^{k-1}$ and R_i are nonzero polynomials with $\deg_z R_i \leq 1$ $(i = 1, 2)$.

Wang et al. [14] proved the case of $k = 1$, below we prove the case of $k \geq 3$.

Proof. First, we prove that $R_1(z)$ has no poles. On the contrary, suppose that z_0 is a pole of $R_1(z)$. We can write (2.1) in a new form

$$\begin{cases} R_1(z+c) = \dfrac{\sum\limits_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i}{i\alpha} + R_1(z), \\ R_2(z+c) = \dfrac{\sum\limits_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i}{i\alpha} + R_2(z). \end{cases} \tag{2.2}$$

It is easy to see that $z_0 + c$ is also a pole of $R_1(z)$ by comparing the order of pole z_0 on both sides of (2.2). By recycling this operation, we obtain a sequence of poles of $R_1(z)$ that are $z_0 + 2c$, $z_0 + 3c$, $\cdots$, $z_0 + tc$, this is impossible since $R_1(z)$ is a nonzero rational function. Then, $R_1(z)$ is a polynomial. Similarly, it can be inferred that $R_2(z)$ also is a polynomial.

Therefore, $R_1(z)$ and $R_2(z)$ are two nonzero polynomials. Let

$$R_1(z) = a_p z^p + a_{p-1} z^{p-1} + \cdots + a_1 z + a_0, \quad R_2(z) = b_t z^t + b_{t-1} z^{t-1} + \cdots + b_1 z + b_0,$$

where $a_j \in \mathbb{C}$, $j \in \{0, 1, \cdots, p\}$, $b_j \in \mathbb{C}$, $j \in \{0, 1, \cdots, t\}$, $a_p \neq 0$, $b_t \neq 0$. Substituting $R_1(z)$ and $R_2(z)$ into (2.1) and comparing the coefficients of z^{p-1}, z^{p-2}, z^{t-1}, and z^{t-2} on both sides of these two equations, we have

$$\begin{cases} i\alpha a_p pc = C_k^1 a^{k-1} a_p p, \\ i\alpha b_t tc = C_k^1 a^{k-1} b_t t, \\ i\alpha[a_p C_p^2 c^2 + a_{p-1}(p-1)c] = C_k^1 a^{k-1}(p-1)a_{p-1} + C_k^2 a^{k-2} p(p-1) a_p, \\ i\alpha[b_t C_t^2 c^2 + b_{t-1}(t-1)c] = C_k^1 a^{k-1}(t-1)b_{t-1} - C_k^2 a^{k-2} t(t-1) b_t, \end{cases} \tag{2.3}$$

which means

$$\begin{cases} i\alpha c = ka^{k-1}, \\ (\frac{C_k^1 ac}{2} - C_k^2)p(p-1) = 0, \\ (\frac{C_k^1 ac}{2} + C_k^2)t(t-1) = 0. \end{cases} \tag{2.4}$$

It follows that $i\alpha c = ka^{k-1}$, $p = 0$ or 1 and $t = 0$ or 1. Therefore, this completes the proof of Lemma 2.1. □

Lemma 2.2. *Let c, a, α be three nonzero constants and $k \in \mathbb{Z}^+$ is an even number. If R_1, R_2 are two nonzero rational functions satisfying the following differential-difference equations:*

$$\begin{cases} i\alpha e^{ac}[R_1(z+c) - R_1(z)] = \sum_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i, \\ i\alpha e^{-ac}[R_2(z+c) - R_2(z)] = \sum_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i. \end{cases} \tag{2.5}$$

Then $e^{ac} = \pm 1$ and R_i are nonzero polynomials with $\deg_z R_i \leq 1$ $(i = 1, 2)$.

Proof. First, we prove that $R_1(z)$ has no poles. On the contrary, suppose that z_0 is a pole of $R_1(z)$. We can write (2.5) in a new form

$$\begin{cases} R_1(z+c) = \dfrac{\sum_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i}{i\alpha e^{ac}} + R_1(z), \\ R_2(z+c) = \dfrac{\sum_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i}{i\alpha e^{ac}} + R_2(z). \end{cases} \tag{2.6}$$

Similar to the proof of Lemma 2.1, we can get $R_1(z)$ and $R_2(z)$ are two nonzero polynomials. Substituting $R_1(z)$ and $R_2(z)$ into (2.5) and comparing the coefficients of z^{p-1}, z^{p-2}, z^{t-1} and z^{t-2} on both sides of such two equations, it yields

$$\begin{cases} i\alpha e^{ac} a_p pc = C_k^1 a^{k-1} a_p p, \\ i\alpha e^{-ac} b_t tc = C_k^1 a^{k-1} b_t t, \\ i\alpha e^{ac}[a_p C_p^2 c^2 + a_{p-1}(p-1)c] = C_k^1 a^{k-1}(p-1)a_{p-1} + C_k^2 a^{k-2} p(p-1)a_p, \\ i\alpha e^{-ac}[b_t C_t^2 c^2 + b_{t-1}(t-1)c] = C_k^1 a^{k-1}(t-1)b_{t-1} - C_k^2 a^{k-2} t(t-1)b_t, \end{cases} \tag{2.7}$$

which means

$$\begin{cases} e^{ac} = \pm 1, \\ (\frac{C_k^1 ac}{2} - C_k^2)p(p-1) = 0, \\ (\frac{C_k^1 ac}{2} + C_k^2)t(t-1) = 0. \end{cases} \tag{2.8}$$

It follows that $e^{ac} = \pm 1$, $p = 0$ or 1, and $t = 0$ or 1. Therefore, this completes the proof of Lemma 2.2. □

Lemma 2.3. [14] *Let R be a nonconstant rational function and $p(z) = az + b (a \neq 0)$. Denote $A_1 = R' + Rp'$, $A_n = A'_{n-1} + A_{n-1}p'$, $B_1 = R' - Rp'$, $B_n = B'_{n-1} + B_{n-1}(-p)'$. Then*

$$\lim_{|z|\to\infty} \frac{A'_n}{R} = 0, \quad \lim_{|z|\to\infty} \frac{A_n}{R} = a^n, \quad \lim_{|z|\to\infty} \frac{B'_n}{R} = 0, \quad \lim_{|z|\to\infty} \frac{B_n}{R} = (-a)^n.$$

Lemma 2.4. [22] *Suppose that $f_1, f_2, \cdots, f_n (n \geq 2)$ are meromorphic functions and $g_1, g_2, \cdots, g_n$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$*;*

(ii) $g_j - g_k$ are not constants for $1 \leq j < k \leq n$;

(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o\{T(r, e^{g_h - g_k})\} (r \to \infty, r \notin E)$, where E is a set of $(0, \infty)$ *with finite linear measure.*

Then $f_j \equiv 0$ $(j = 1, 2, \cdots, n)$.

Lemma 2.5. [22] *Let f be a meromorphic function of finite order $\rho(f)$. Write*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots, (c_k \neq 0)$$

near $z = 0$ and let $\{a_1, a_{2,\dots}\}$ and $\{b_1, b_{2,\dots}\}$ be the zeros and poles of f in $\mathbb{C}\backslash\{0\}$, respectively. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are the canonical products of f formed with the non-null zeros and poles of f, respectively, and $Q(z)$ is a polynomial of the degree $\leq \rho(f)$.

Lemma 2.6. [22] *Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$, $(n \geq 3)$ are meromorphic functions that are not constants except for $f_n(z)$. Furthermore, let*

$$\sum_{j=1}^{n} f_j(z) = 1.$$

If $f_n(z) \not\equiv 0$ and

$$\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n-1) \sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1)) T(r, f_k),$$

where $\lambda < 1$ and $k = 1, 2, \cdots, n-1$, then $f_n(z) \equiv 1$.

3. Proof of Theorem 1.5

Proof. Suppose that Eq (1.2) admits a finite order transcendental meromorphic solution $f(z)$ with finitely many poles. We can rewrite Eq (1.2) in the following form:

$$[f^{(k)}(z) + i(\alpha f(z+c) - \beta f(z))][f^{(k)}(z) - i(\alpha f(z+c) - \beta f(z))] = R(z). \tag{3.1}$$

Since $f(z)$ has finitely many poles and $R(z)$ is a nonzero rational function, then $f^{(k)}(z) + i(\alpha f(z+c) - \beta f(z))$ and $f^{(k)}(z) - i(\alpha f(z+c) - \beta f(z))$ both have finitely many poles and zeros. Thus, in view of Lemma 2.5, (3.1) can be written as

$$\begin{cases} f^{(k)}(z) + i(\alpha f(z+c) - \beta f(z)) = R_1 e^{p(z)}, \\ f^{(k)}(z) - i(\alpha f(z+c) - \beta f(z)) = R_2 e^{-p(z)}, \end{cases} \tag{3.2}$$

where R_1, R_2 are two nonzero rational functions such that $R_1 R_2 = R$ and $p(z)$ is a nonzero polynomial. By solving the above equations system, we have

$$\begin{cases} f^{(k)}(z) = \dfrac{R_1 e^{p(z)} + R_2 e^{-p(z)}}{2}, \\ \alpha f(z+c) - \beta f(z) = \dfrac{R_1 e^{p(z)} - R_2 e^{-p(z)}}{2i}, \end{cases} \tag{3.3}$$

In view of the second equation of (3.3), it follows that

$$\alpha f^{(k)}(z+c) - \beta f^{(k)}(z) = \frac{A_k e^{p(z)} - B_k e^{-p(z)}}{2i}, \tag{3.4}$$

where $A_1 = R_1' + R_1 p'$, $B_1 = R_2' - R_2 p'$, $A_k = A_{k-1}' + A_{k-1} p'$ and $B_k = B_{k-1}' - B_{k-1} p'$. Substituting the first equation of system (3.3) into (3.4), it yields that

$$\begin{aligned} &e^{p(z)}[i\alpha R_1(z+c)e^{p(z+c)-p(z)} - i\beta R_1(z) - A_k(z)] \\ &\quad + e^{-p(z)}[i\alpha R_2(z+c)e^{-p(z+c)+p(z)} - i\beta R_2(z) + B_k(z)] = 0. \end{aligned} \tag{3.5}$$

By Lemma 2.4, it follows from (3.5) that

$$\begin{cases} i\alpha R_1(z+c)e^{p(z+c)-p(z)} - i\beta R_1(z) - A_k(z) = 0, \\ i\alpha R_2(z+c)e^{-p(z+c)+p(z)} - i\beta R_2(z) + B_k(z) = 0. \end{cases} \tag{3.6}$$

Since R_1, R_2 are two nonzero rational functions, which implies that $p(z)$ is a polynomial of degree one. Let $p(z) = az + b$, $a \neq 0$, $b \in \mathbb{C}$. Substituting $p(z)$, A_k, and B_k into (3.6), and letting $|z| \to \infty$, thus, we can conclude from Lemma 2.3 that

$$\begin{cases} \lim\limits_{|z|\to\infty} i(\alpha \dfrac{R_1(z+c)}{R_1(z)} e^{p(z+c)-p(z)} - \beta) = i(\alpha e^{ac} - \beta) = \lim\limits_{|z|\to\infty} \dfrac{A_k(z)}{R_1(z)} = a^k, \\ \lim\limits_{|z|\to\infty} i(\alpha \dfrac{R_2(z+c)}{R_2(z)} e^{-p(z+c)+p(z)} - \beta) = i(\alpha e^{-ac} - \beta) = \lim\limits_{|z|\to\infty} \dfrac{-B_k(z)}{R_2(z)} = -(-a)^k. \end{cases} \tag{3.7}$$

Two equations of (3.7), which mean that

$$\begin{cases} i(\alpha e^{ac} - \beta) = a^k, \\ i(\alpha e^{-ac} - \beta) = a^k. \end{cases} \tag{3.8}$$

Hence, it yields $e^{ac} = \pm 1$.

If $e^{ac} = 1$, then $a^k = i\alpha - i\beta$. Thus, we can rewrite (3.6) in the following form:

$$\begin{cases} i\alpha[R_1(z+c) - R_1(z)] = \sum\limits_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i, \\ i\alpha[R_2(z+c) - R_2(z)] = \sum\limits_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i. \end{cases} \tag{3.9}$$

If $R_j (j = 1, 2)$ are two nonzero rational functions, then in view of Lemma 2.1, it follows that $i\alpha c = ka^{k-1}$ and R_i are nonzero polynomials with $\deg_z R_i \leq 1$ $(i = 1, 2)$. In view of $R = R_1 R_2$, thus R is a nonzero polynomial with $\deg_z R \leq 2$.

If $e^{ac} = -1$, then $a^k = -i\alpha - i\beta$. Thus, we can rewrite (3.6) in the following form

$$\begin{cases} -i\alpha[R_1(z+c) - R_1(z)] = \sum\limits_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i, \\ -i\alpha[R_2(z+c) - R_2(z)] = \sum\limits_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i. \end{cases} \tag{3.10}$$

Similar to the discussion above, we can obtain that $i\alpha c = -ka^{k-1}$ and R_i are nonzero polynomials with $\deg_z R_i \leq 1$ $(i = 1, 2)$. In view of $R = R_1 R_2$, thus R is a nonzero polynomial with $\deg_z R \leq 2$.

Hence, we can obtain that $i\alpha c = \pm ka^{k-1}$, R is a nonzero polynomial with $\deg_z R \leq 2$.

Suppose that $R(z)$ is a nonzero polynomial with $\deg_z R \leq 2$, then in view of the first equation of (3.3), it follows that $f(z)$ is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2} + c_{k-1}z^{k-1} + \cdots + c_0, \tag{3.11}$$

where $s_j(z) = m_j z + n_j$, $m_j, n_j \in \mathbb{C}(j = 1, 2)$ and $c_0, \cdots, c_{k-1}$ are constants.

If $\deg_z R = 2$, then it follows that $m_j \neq 0 (j = 1, 2)$.

If $i\alpha c = ka^{k-1}$ and $a^k = i(\alpha - \beta)$, then $e^{ac} = 1$, i.e., $c = \frac{2l\pi i}{a}$, $l \in \mathbb{Z}$. According to (3.8), if $\alpha = \beta$, we have $a = 0$, it is a contradiction. If $\alpha = -\beta$, then $a^k = 2i\alpha$. Combining $i\alpha c = ka^{k-1}$, $a^k = 2i\alpha$ and $e^{ac} = 1$, we have $1 = e^{ac} = e^{2k}$, it is a contradiction. Hence, $\alpha \neq \pm\beta$. Substituting (3.11) into the second equation of (3.3), it follows that $c_0 = \cdots = c_{k-1} = 0$, we have

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2}. \tag{3.12}$$

Substituting (3.12) into the first equation of (3.3), it yields

$$R_1(z) = a^k s_1(z) + ka^{k-1} m_1 \quad and \quad R_2(z) = (-a)^k s_2(z) + ka^{k-1} m_2. \tag{3.13}$$

Substituting (3.12) into the second equation of (3.3), it yields

$$R_1(z) = a^k s_1(z) + m_1 \quad and \quad R_2(z) = (-a)^k s_2(z) - m_2. \tag{3.14}$$

Comparing (3.13) and (3.14), we have $ka^{k-1} = 1$ and $ka^{k-1} = -1$, it is a contradiction.

If $i\alpha c = -ka^{k-1}$ and $a^k = -i(\alpha + \beta)$, then $e^{ac} = -1$, i.e., $c = \frac{(2l+1)\pi i}{a}$, $l \in \mathbb{Z}$. Similar to the discussion above, we can obtain a contradiction. Therefore, there are two categories below:

***Case I*:** If $\deg_z R = 1$, one of m_1 and m_2 is zero, without loss of generality, assume $m_2 = 0$. Substituting (3.12) into (3.3), it follows that R_1 is a polynomial of degree one and R_2 is a constant, where $i\alpha c = ka^{k-1} = 1$ and $a^k = i(\alpha - \beta)$ or $i\alpha c = -ka^{k-1} = -1$ and $a^k = -i(\alpha + \beta)$. Similar to the discussion above, it is easy to prove that $\alpha \neq \pm\beta$ and $c_0 = \cdots = c_{k-1} = 0$.

Therefore, $f(z)$ is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + n_2 e^{-(az+b)}}{2},$$

where $R(z) = -n_2 a^{2k-1}[as_1(z) + km_1]$, $m_1 \neq 0$, $a \neq 0, b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a^k = -i(\alpha + \beta)$, $c = \frac{(2l+1)\pi i}{a}$, $l \in \mathbb{Z}$, $i\alpha c = -ka^{k-1} = -1$, or $a^k = i(\alpha - \beta)$, $c = \frac{2l\pi i}{a}$, $l \in \mathbb{Z}$, $i\alpha c = ka^{k-1} = 1$.

If $m_1 = 0$, similar to the discussion above, it is easy to prove that $f(z)$ is of the form

$$f(z) = \frac{n_1 e^{az+b} + s_2 e^{-(az+b)}}{2},$$

where $R(z) = n_1 a^{2k-1}[-as_2(z) + km_2]$, $m_2 \neq 0$, $a \neq 0, b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a^k = -i(\alpha + \beta)$, $c = \frac{(2l+1)\pi i}{a}$, $l \in \mathbb{Z}$, $i\alpha c = ka^{k-1} = -1$, or $a^k = i(\alpha - \beta)$, $c = \frac{2l\pi i}{a}$, $l \in \mathbb{Z}$, $i\alpha c = -ka^{k-1} = 1$.

***Case II*:** If $R(z)$ is a nonzero constant, by using the first equation of (3.3), it follows that $f(z)$ is of the form

$$f(z) = \frac{n_1 e^{az+b} + n_2 e^{-(az+b)}}{2} + c_{k-1} z^{k-1} + \cdots + c_0, \tag{3.15}$$

where $n_1, n_2 \in \mathbb{C}$ and $c_0, \cdots, c_{k-1} \in \mathbb{C}$. Substituting (3.15) into the second equation of (3.3), it yields $R = -a^{2k} n_1 n_2$.

(II_1) If $\alpha = \beta$, in view of (3.8), it follows that $e^{ac} = \pm 1$. If $e^{ac} = 1$, then $a = 0$, as $i\alpha(e^{ac} - 1) = a^k$, a contradiction. Thus, $e^{ac} = -1$. Hence, it follows that $c = \frac{(2l+1)\pi i}{a}$, $l \in \mathbb{Z}$, $a^k = -2i\alpha$, and $c_0 \in \mathbb{C}$,

$c_1 = \cdots = c_{k-1} = 0$.

(II_2) If $\alpha = -\beta$, in view of (3.8), it follows that $e^{ac} = \pm 1$. If $e^{ac} = -1$, then $a = 0$, as $i\alpha(e^{ac} + 1) = a^k$, a contradiction. Thus, $e^{ac} = 1$. Hence, it follows that $c = \frac{2l\pi i}{a}$, $l \in \mathbb{Z}$, $a^k = 2i\alpha$, and $c_0 = \cdots = c_{k-1} = 0$.

(II_3) If $\alpha \neq \pm\beta$, substituting (3.15) into the second equation of (3.3), it yields $c_0 = \cdots = c_{k-1} = 0$. In view of (3.8), it follows that $e^{ac} = \pm 1$. If $e^{ac} = 1$, it follows that $c = \frac{2l\pi i}{a}$ and $a^k = i(\alpha - \beta)$, $l \in \mathbb{Z}$. If $e^{ac} = -1$, it follows that $c = \frac{(2l+1)\pi}{a}$ and $a^k = -i(\alpha + \beta)$, $l \in \mathbb{Z}$. Therefore, this completes the proof of Theorem 1.5. □

4. Proof of Theorem 1.6

Proof. Similar to the method of proving Theorem 1.5, we can obtain the expression (3.8).

When k is an even number, two equations of (3.7), which mean that

$$\begin{cases} i(\alpha e^{ac} - \beta) = a^k, \\ i(\alpha e^{-ac} - \beta) = -a^k. \end{cases} \tag{4.1}$$

Hence, it follows $a^{2k} = \alpha^2 - \beta^2$.

***Case I*:** If $\alpha = \pm\beta$, this is a contradiction with $a \neq 0$.

***Case II*:** If $\alpha \neq \pm\beta$. Substituting $p(z) = az + b$ and (4.1) into (3.6), it yields

$$\begin{cases} i\alpha e^{ac}[R_1(z + c) - R_1(z)] = \sum\limits_{i=0}^{k-1} C_k^i R_1^{(k-i)} a^i, \\ i\alpha e^{-ac}[R_2(z + c) - R_2(z)] = \sum\limits_{i=0}^{k-1} (-1)^{i+1} C_k^i R_2^{(k-i)} a^i. \end{cases} \tag{4.2}$$

Suppose that R_1, R_2 are nonzero rational functions; in view of Lemma 2.2, we can conclude that $e^{ac} = \pm 1$ and R_i are nonzero polynomials with $\deg_z R_i \leq 1$ ($i = 1, 2$). In view of $R = R_1 R_2$, thus R is a nonzero polynomial with $\deg_z R \leq 2$. Set $\deg_z R_1 = p$ and $\deg_z R_2 = t$.

When $p = 1$ and $t = 1$, if $e^{ac} = 1$, then from (4.1), it follows that $i\alpha - i\beta = a^k$ and $i\alpha - i\beta = -a^k$, a contradiction. If $e^{ac} = -1$, then from (4.1), it follows that $-i\alpha - i\beta = a^k$ and $-i\alpha - i\beta = -a^k$, a contradiction. Hence, there is at most a polynomial of degree one in R_1 and R_2.

(II_1) Suppose that $p = 1$, $t = 0$. In view of (3.3), it follows that f is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + n_2 e^{-(az+b)}}{2} + P(z), \tag{4.3}$$

where $a \neq 0$, $b \in \mathbb{C}$, $s_1(z) = m_1 z + n_1$, $m_1(\neq 0)$, n_1, $n_2 \in \mathbb{C}$, and $P(z)$ is a polynomial of degree $k - 1$. Since $\alpha \neq \beta$, then it yields from the second equation of (3.3) that $P(z) \equiv 0$. And by using the first equation in (4.2), it follows that $i\alpha e^{ac} c = ka^{k-1}$. Hence, $f(z)$ is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + n_2 e^{-(az+b)}}{2},$$

where $a^{2k} = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \dfrac{\log \frac{a^k + i\beta}{i\alpha} + 2l\pi i}{a}$, $l \in \mathbb{Z}$, $e^{ac} = \frac{ka^{k-1}}{i\alpha c} \neq \pm 1$ and $R = n_2 a^{2k-1}[a s_1(z) + k m_1]$.

Suppose that $p = 0$, $t = 1$. Similar to the above argument as in (II_1), we obtain

$$f(z) = \frac{n_1 e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $a^{2k}=\alpha^2-\beta^2, b\in\mathbb{C}, c=\dfrac{\log\frac{a^k+i\beta}{i\alpha}+2l\pi i}{a}, l\in\mathbb{Z}, e^{ac}=\frac{i\alpha c}{ka^{k-1}}\neq\pm1$ and $R=n_1a^{2k-1}[as_2(z)-km_2]$.

(II_2) Suppose that $p=0$, $t=0$. By using (3.3), it follows that f is of the form

$$f(z)=\frac{n_1e^{az+b}+n_2e^{-(az+b)}}{2}+P(z), \tag{4.4}$$

where $a\neq 0$, $b\in\mathbb{C}$, n_1, $n_2\in\mathbb{C}\backslash\{0\}$, and $P(z)$ is a polynomial of degree $k-1$. Since $\alpha\neq\beta$, then it yields from the second equation of (3.3) that $P(z)\equiv 0$. Hence, $f(z)$ is of the form

$$f(z)=\frac{n_1e^{az+b}+n_2e^{-(az+b)}}{2},$$

where $a^{2k}=\alpha^2-\beta^2, b\in\mathbb{C}, c=\dfrac{\log\frac{a^k+i\beta}{i\alpha}+2l\pi i}{a}, l\in\mathbb{Z}$ and $R=a^{2k}n_1n_2$. Therefore, this completes the proof of Theorem 1.6. □

5. Proof of Theorem 1.8

Proof. Suppose, on the contrary, to the assertion that there exists a transcendental entire solution f of (1.4) with finite order. We aim for a contradiction. By using a similar reason as in the proof of Theorem 1.5, we obtain

$$f^{(k)}(z)=\frac{Q_1(z)e^{h(z)}+Q_2(z)e^{-h(z)}}{2} \tag{5.1}$$

and

$$\alpha f(z+c)-\beta f(z)=\frac{Q_1(z)e^{h(z)}-Q_2(z)e^{-h(z)}}{2iP(z)}, \tag{5.2}$$

where $h(z)$ is a non-constant polynomial, $Q_1(z)$ and $Q_2(z)$ are non-zero polynomials such that $Q_1(z)Q_2(z)=Q(z)$. Combining (5.1) and (5.2), we obtain

$$\begin{aligned}&\alpha f^{(k)}(z+c)-\beta f^{(k)}(z)\\ =\;&\frac{\alpha Q_1(z+c)e^{h(z+c)}+\alpha Q_2(z+c)e^{-h(z+c)}}{2}-\frac{\beta Q_1(z)e^{h(z)}+\beta Q_2(z)e^{-h(z)}}{2}\\ =\;&\frac{h_1(z)e^{h(z)}-h_2(z)e^{-h(z)}}{2iP(z)^{k+1}},\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}h_1(z)=\;&\sum_{i=0}^{k-1}C_{k-1}^i\sum_{t=0}^{k-i}C_{k-i}^tQ_1^{(k-t-i)}[(h')^t+M_t(h,h',\cdots,h^{(t)})]P^{(i)}P^{k-1}\\ &-\sum_{i=0}^{k-1}C_{k-1}^i\sum_{t=0}^{i}C_i^tQ_1^{(h-t)}[(h')^t+M_t(h,h',\cdots,h^{(t)})]P^{(k-i)}P^{k-1}+o(h_1(z)),\end{aligned}$$

$$h_2(z)=\sum_{i=0}^{k-1}C_{k-1}^i\sum_{t=0}^{k-i}C_{k-i}^tQ_2^{(k-t-i)}[(h')^t+N_t(h,h',\cdots,h^{(t)})]P^{(i)}P^{k-1}$$

$$-\sum_{i=0}^{k-1} C_{k-1}^{i} \sum_{t=0}^{i} C_{i}^{t} Q_2^{(h-t)}[(h')^t + N_t(h, h', \cdots, h^{(t)})]P^{(k-i)}P^{k-1} + o(h_2(z)),$$

M_t and N_t are differential polynomials of $(h, h', \cdots, h^{(t)})$. Thus from (5.3), we get

$$\frac{h_1(z) + \beta i P(z)^{k+1} Q_1(z)}{\alpha i P(z)^{k+1} Q_2(z+c)} e^{h(z)+h(z+c)} - \frac{h_2(z) - \beta i P(z)^{k+1} Q_2(z)}{\alpha i P(z)^{k+1} Q_2(z+c)} e^{h(z+c)-h(z)} - \frac{Q_1(z+c)}{Q_2(z+c)} e^{2h(z+c)} \equiv 1. \tag{5.4}$$

It is easy to see that both $\frac{h_1(z) + \beta i P(z)^{k+1} Q_1(z)}{\alpha i P(z)^{k+1} Q_2(z+c)} e^{h(z)+h(z+c)}$ and $\frac{Q_1(z+c)}{Q_2(z+c)} e^{2h(z+c)}$ are not constants. Using Lemma 2.6, we obtain $-[h_2(z) - \beta i P(z)^{k+1} Q_2(z)]e^{h(z+c)-h(z)} = \alpha i P(z)^{k+1} Q_2(z+c)$, so $h(z) = Az + B$, A is a non-zero constant, and B is a constant. Thus, we obtain

$$iP(z)^{k+1}[\beta Q_2(z)e^{Ac} - \alpha Q_2(z+c)] = h_2(z)e^{Ac}. \tag{5.5}$$

Set $\deg(P(z)) = p$, $\deg(Q(z)) = q$, $\deg(Q_1(z)) = q_1$, $\deg(Q_2(z)) = q_2$ and $\deg(h(z)) = h$. By comparing the degree of both sides of (5.5), it is not difficult to find that the degree of the left hand-side is $(k+1)p + q_2$ or $(k+1)p + q_2 - 1$, and the degree of right-hand side is $kp + q_2 - 1$; this is a contradiction. Therefore, this completes the proof of Theorem 1.8. □

Author contributions

Zhiyong Xu: Conceptualization, Methodology, Writing-original draft; Junfeng Xu: Supervision, Writing-review and editing. Both of the authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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