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Research article

A new optimal control approach to uncertain Euler-Lagrange equations: H_∞ disturbance estimator and generalized H_2 tracking controller

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Abstract: This paper proposed a new optimal control method for uncertain Euler-Lagrange systems, focusing on estimating model uncertainties and improving tracking performance. More precisely, a linearization of the nonlinear equation was achieved through the inverse dynamic control (IDC) and an H_∞ optimal estimator was designed to address model uncertainties arising in this process. Subsequently, a generalized H_2 optimal tracking controller was obtained to minimize the effect of the estimation error on the tracking error in terms of the induced norm from L_2 to L_∞. Necessary and sufficient conditions for the existences of these two optimal estimator and controller were characterized through the linear matrix inequality (LMI) approach, and their synthesis procedures can be operated in an independent fashion. To put it another way, this developed approach allowed us to minimize not only the modeling error between the real Euler-Lagrange equations and their nominal models occurring from the IDC approach but also the maximum magnitude of the tracking error by solving some LMIs. Finally, the effectiveness of both the H_∞ optimal disturbance estimator and the generalized H_2 tracking controller were demonstrated through some comparative simulation and experiment results of a robot manipulator, which was one of the most representative examples of Euler-Lagrange equations.

Keywords: Euler-Lagrange equations; inverse dynamic control; H_∞ optimal estimator; generalized H_2 optimal controller; linear matrix inequalities
Mathematics Subject Classification: 93B18, 93B36, 93B50, 93B53

1. Introduction

Uncertain elements could make real systems unstable if their effects on the systems are not concerned with. They occur from various sources such as modeling errors, actual parameter variations, changes in operating points, unexpected external disturbances, and so on, as discussed in [1–3]. The issue of dealing with unknown elements is also a significant problem in Euler-Lagrange systems [4–6], and thus there have been a number of studies on developing robust [7–9] and/or optimal [10–12] control approaches to uncertain Euler-Lagrange equations. These studies could be also classified by two approaches in the following aspects.

1.1. Related studies on uncertain Euler-Lagrange systems

The first approach is concerned with ensuring the stability [13–15] and/or the optimality [16–18] with respect to norm-bounded uncertain elements. Even though this first approach can be applied extensively to Euler-Lagrange systems, it is still quite difficult to characterize norm properties of uncertain elements in real systems. To put it another way, determining a set of all possible uncertain elements in real Euler-Lagrange systems is a nontrivial task, and, thus, an intrinsically different approach might be required.

To tackle these limitations in the first approach, the second approach aims at estimating and compensating uncertain elements at every moment, not obtaining all the possible ranges of them. One of the most representative methods in the second approach is to design disturbance observers (DOBs) [19–21], in which uncertain elements are obtained by using the real control input for the difference of the outputs between the real system and the nominal system. Because the inverse of the nominal system is used in the DOB-based method, the so-called Q-filter is proposed to deal with the case of strictly proper nominal systems. To avoid some involved arguments on designing Q-filters as in [22–24] (e.g., high-order requirements and direct current-gain constraints), relatively straightforward methods for estimating uncertain elements are also developed in [8, 9] by using the differences between nominal parameters from their filtered and delayed values, respectively. However, all the estimation methods mentioned above do depend on the prespecified tracking controllers. In other words, the corresponding DOBs and estimators are determined after the closed-loop systems consisting of the nominal systems and the tracking controllers are obtained, thus, enhancing both the estimation and tracking accuracy at the same time are quite difficult in the above methods.

On the other hand, as a novel study on dealing with model uncertainties for nonlinear systems in the presence of dynamic model uncertainties and input constraints, the so-called triple event trigger mechanism is recently introduced in [25] to establish the finite-time convergence. However, the relevant performance measures are confined to quadratic forms and it should also be prespecified norm properties of uncertain elements.

To put it another way, there is no discussion on reducing both the magnitudes of uncertain elements and the trajectory tracking errors at the same time in all the aforementioned methods. With respect to this, it would be also worthwhile to note that the tracking errors are often desired to be suppressed in the time-domain bounds for the safety issue such as the collision avoidance. In the same line, the L_∞ norm is taken in [26–28] as the measure for evaluating the output of various systems, but no discussion on extending this norm to the second approach is provided in the literature. To summarize, solving the following issues would lead to significant improvements of treating uncertain Euler-Lagrange equations:

- A new framework for dealing with estimating uncertain elements and reducing tracking errors in an independent fashion.
- Adequate performance measures for both the estimation and tracking errors.

- Optimal synthesis procedures for both the corresponding estimator and the tracking controller.
- Analysis and synthesis in terms of the time-domain bounds on signals (i.e., the L_∞ norms) for the safety issue.

1.2. Contributions and organization of this paper

To facilitate the application of robust and optimal control techniques [1–3], we initially employ the inverse dynamics control (IDC) approach [29–31], which transforms the nonlinear Euler-Lagrange equation into a linear time-invariant system with unknown signals. To tackle the two tasks of estimating uncertain elements and trajectory tracking control in an independent fashion, we propose a new structural framework based on the IDC approach. More precisely, we design a disturbance estimator for minimizing the L_2-induced norm (i.e., the H_∞ norm) from unknown signals to estimation errors by noting the fact that the unknown signals can be regarded as involved in the L_2 space. Because it is required to suppress the time-domain bounds of the trajectory tracking errors for the disturbance estimation errors in the L_2 space, we consider a tracking controller for minimizing the induced norm from L_2 to L_∞ (i.e., the generalized H_2 norm) of the mapping from the estimation errors to the tracking errors. Beyond the aforementioned practical aspects, the rationale taking these two performance measures can be interpreted as that the minimization problems are feasible and the corresponding controller synthesis procedures can be carried out through the linear matrix inequality (LMI) approach. In connection with this, the contributions of this paper can be summarized as follows:

- The new structure proposed in this paper allows us to design a disturbance estimator and a tracking controller independently of each other.
- The disturbance estimation accuracy is optimized in terms of the L_2-induced norm in the time domain, i.e., the H_∞ norm in the frequency domain.
- The tracking accuracy of uncertain Euler-Lagrange systems is optimized in the time-domain bounds for the first time, through the generalized H_2 optimal tracking controller minimizing the induced norm from L_2 to L_∞.

This paper is structured as follows: The IDC approach to uncertain Euler-Lagrange systems and motivating issues are reviewed in Section 2. The main results of this paper, i.e., the new control framework with the H_∞ optimal estimator and the generalized H_2 optimal tracking controller, are derived in Section 3. Some simulations and experiments are provided in Section 4 to demonstrate the effectiveness and validity of the overall arguments. Finally, conclusions are given in Section 5.

The notations used in this paper are as follows: The notations $\mathbb{R}^\nu$ and I_ν are taken to denote the set of ν-dimensional real vectors and the ν-dimensional identity matrix, respectively. For a real vector or a real matrix, the notation $|\cdot|_2$ denotes its 2-norm, i.e.,

$$|v|_2 := (v^T v)^{1/2}, \quad |A|_2 := \sup_{v \neq 0} \frac{|Av|_2}{|v|_2} = \lambda_{\max}^{1/2}(A^T A),$$

where $\lambda_{\max}(\cdot)$ means the maximum eigenvalue of a real symmetric matrix $(\cdot)$. For a real vector-valued signal, we use the notations $\|\cdot\|_2$ and $\|\cdot\|_\infty$ to mean the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms, respectively, i.e.,

$$\|f(\cdot)\|_2 := \left(\int_0^\infty |f(t)|_2^2 dt \right)^{1/2}, \quad \|f(\cdot)\|_\infty := \operatorname*{ess\,sup}_{0 \le t < \infty} |f(t)|_2.$$

For symmetric matrices A and B, the notation $A > B$ implies that $A - B$ is positive definite.

2. Motivating issues and problem formulation

Consider the Euler-Lagrange equation described as

$$M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \tau, \tag{2.1}$$

where $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $V(q, \dot{q}) \in \mathbb{R}^n$ is the Coriolis and centrifugal torque vector, $G(q)$ is the gravity vector, $\tau \in \mathbb{R}^n$ is the torque input, and $q \in \mathbb{R}^n$ is the vector of generalized coordinates.

For a predetermined reference trajectory $q_d(t) \in \mathbb{R}^n$, this paper aims at reducing the corresponding error in the time-domain bound. We define the position error as

$$e(t) := q_d(t) - q(t), \tag{2.2}$$

and are concerned with reducing the L_∞ norm of a tracking error function associated with both the error e and its rate $\dot{e}$ as small as possible. With respect to minimizing such L_∞ norms in practical environments, it would be required to establish an optimal control framework. However, it is quite difficult to ensure optimality in the nonlinear differential equation of (2.1) itself. Thus, we adopt the IDC approach [29–31], which reformulates the Euler-Lagrange equation into a linear time-invariant system through a specific torque input given by

$$\tau = \hat{M}(q)(\ddot{q}_d - u) + \hat{V}(q, \dot{q}) + G(q), \tag{2.3}$$

where $\hat{(\cdot)}$ implies the nominal value of $(\cdot)$* and u serves as an auxiliary input (which will be determined by outer control loops). More precisely, substituting (2.3) into (2.1) leads to

$$\ddot{e} = u + \hat{M}^{-1}(q)\{\tilde{M}(q)\ddot{q} + \tilde{V}(q, \dot{q})\} =: u + w, \tag{2.4}$$

with the model uncertainties defined as

$$\tilde{M}(q) := M(q) - \hat{M}(q), \quad \tilde{V}(q, \dot{q}) = V(q, \dot{q}) - \hat{V}(q, \dot{q}). \tag{2.5}$$

We call w in (2.4) the total uncertainty throughout the paper.

For the ideal case of the above IDC without model uncertainties (i.e., $w \equiv 0$), we can readily make the tracking error e converge to 0 through a simple proportional-derivative controller, e.g.,

$$u = -K_p e - K_d \dot{e}$$

with $K_p, K_d > 0$. However, there are often model uncertainties and their effects on the tracking error cannot be ignored in many real Euler-Lagrange systems. In this sense, one could raise another approach to adequately defining the regulated output z and reducing an induced norm from w to z in a certain level. With this in mind, let us note the following example:

*No gravity compensation error can be often assumed in various real Euler-Lagrange systems.

Example 1. *We first take an IDC approach to a 6-dimensional Euler-Lagrange systems with model uncertainties generated by random signals establishing*

$$(|\tilde{M}(q)|_2 + |\tilde{V}(q,\dot{q})|_2)/(|M(q)|_2 + |V(q,\dot{q})|_2) \approx 0.05,$$

as shown in Figure 1.

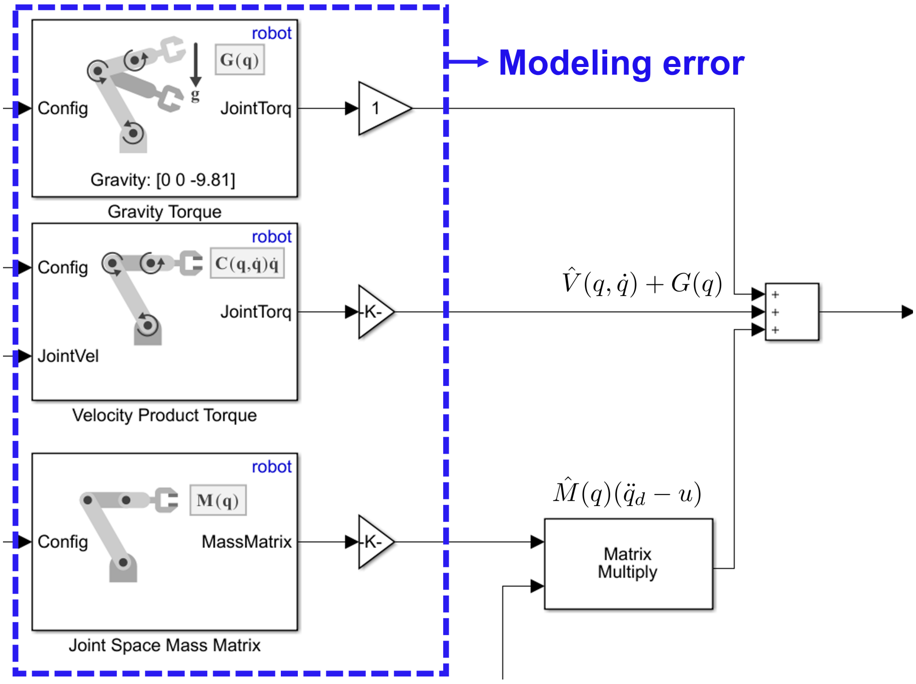


Figure 1. Simulation environment of the uncertain robot manipulator.

More precisely, the simulations are conducted in MATLAB Simulink such that the nominal values $\hat{M}(q)$ *and* $\hat{V}(q,\dot{q})$ *are set to be in the range satisfying*

$$|\tilde{M}(q)/M(q)|_2, |\tilde{V}(q,\dot{q})/V(q,\dot{q})|_2 < 0.05,$$

by which the above condition is ensured. Based on Eq (2.4)*, we next design the generalized* H_2 *optimal controller by using the arguments in [32–34] (the details will also be discussed in Subsection 3.2) with respect to the measured output*

$$y := \begin{bmatrix} e^T & \dot{e}^T \end{bmatrix}^T$$

and the regulated output

$$z := 10e + \dot{e}.$$

Then, the tracking error e is not bounded as shown in Figure 2, although the generalized H_2 *controller is implemented to ensure the internal stability for the nominal system.*

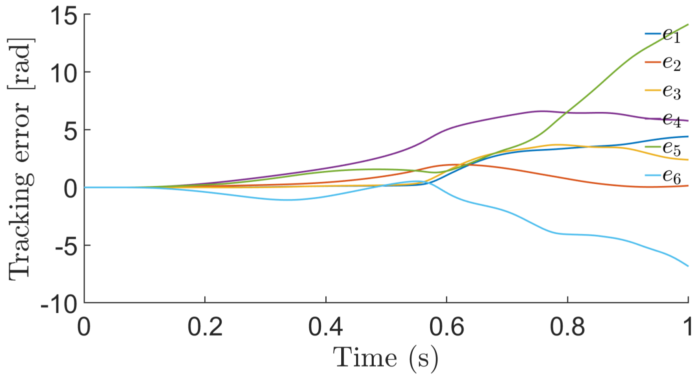


Figure 2. Unstable trajectories despite the generalized H_2 optimal control.

As clarified from Example 1, there could exist a case such that the generalized H_2 optimal controller cannot lead to stabilizing uncertain Euler-Lagrange systems. This can be interpreted as occurring from the fact that the generalized H_2 optimal controller [32–34] aims at minimizing the induced norm from $w(\in L_2)$ to $z(\in L_\infty)$, assuming that w is an external signal independent of the internal signals. However, w in (2.4) is a function of $(q, \dot{q}, \ddot{q})$. To put it another way, treating w as an external disturbance and simply designing a sort of disturbance rejection controller might not lead to suppressing the position error e in a desired level, and, thus, a new control framework for dealing with model uncertainties in Euler-Lagrange systems should be established. With respect to this, this paper proposes a direct consideration of the total uncertainty w to improve tracking accuracy in terms of the L_∞ norm, and formulates the problem definition as follows:

Problem 1. *In the IDC approach to uncertain Euler-Lagrange systems described by* (2.4)*, construct a new optimal control framework with the H_∞ optimal disturbance estimator and the generalized H_2 optimal tracking controller.*

3. Main results

This section develops a new optimal control framework for uncertain Euler-Lagrange systems by developing the H_∞ optimal disturbance estimator and the generalized H_2 optimal tracking controller.

3.1. Structure of new optimal control approach

As a preliminary step to establish such a new control framework, we first introduce some important properties introduced in [4–6] associated with the Euler-Lagrange systems in (2.1) as follows:

Property 1. *The following assertions hold:*

(1) $M(q)$ is a positive definite matrix for all $q \in \mathbb{R}^n$.

(2) There exist positive constants $\underline{m}$ and $\overline{m}$ such that

$$\underline{m} I_n \le M(q) \le \overline{m} I_n, \quad \forall q \in \mathbb{R}^n.$$

(3) There exists a positive constant v such that

$$|V(q, \dot{q})|_2 \le v |\dot{q}|_2^2, \quad \forall q, \dot{q} \in \mathbb{R}^n.$$

Because the conditions in Property 1 are ensured for the parameters in real Euler-Lagrange equations, i.e., $M(q)$ and $V(q,\dot{q})$ in (2.1), their nominal values $\hat{M}(q)$ and $\hat{V}(q,\dot{q})$ should be determined to establish the same conditions. In connection with this, we provide the following property:

Property 2. *There exist positive constants* $\underline{\hat{m}}$, $\overline{\tilde{m}}$, *and* $\tilde{v}$ *such that*

$$\underline{\hat{m}} \leq |\hat{M}(q)|_2, \quad |\tilde{M}(q)|_2 \leq \overline{\tilde{m}}, \quad |\tilde{V}(q,\dot{q})| \leq \tilde{v}|\dot{q}|_2^2, \quad \forall q,\dot{q} \in \mathbb{R}^n. \tag{3.1}$$

On the other hand, we remark that all the signals in the Euler-Lagrange equation, especially for robot manipulators, could be regarded as bounded functions without loss of generality because there should exist hardware limitations, and, thus, all the signals have their upper and lower bounds. In addition, the trajectory tracking process is carried out in a finite-time interval and/or robot manipulators are required to achieve their stationary states at a sufficiently large time $t >> 0$. In connection with this, all the signals of uncertain robot manipulators are assumed in [35–37] to be bounded in the L_2 sense, and we provide the following assumption:

Assumption 1. *All the signals are bounded in terms of the* L_2 *norm, i.e.,*

$$\left\| \begin{bmatrix} q_d^T & \dot{q}_d^T & \ddot{q}_d^T & q^T & \dot{q}^T & \ddot{q}^T & \tau^T \end{bmatrix} \right\|_2 < \infty. \tag{3.2}$$

Based on Property 2 and Assumption 1, we can readily obtain the following result.

Lemma 1. *The total uncertainty* w *(defined in* (2.4)*) is bounded in terms of the* L_2 *norm.*

Proof. Note from (2.4) that

$$\begin{aligned} ||w||_2 &= ||\hat{M}^{-1}(q)\{\tilde{M}(q)\ddot{q} + \tilde{V}(q,\dot{q})\}||_2 \\ &\leq \frac{1}{\underline{\hat{m}}}(\bar{\tilde{m}}||\ddot{q}||_2 + \tilde{v}||\dot{q}||_2) \\ &< \infty. \end{aligned} \tag{3.3}$$

This completes the proof. □

It is obvious from Lemma 1 that taking the L_2 space as the underlying space for the total uncertainty w is theoretically meaningful. By noting this together with the fact that the L_∞ norm is taken for the tracking accuracy, the following parts are devoted to providing a new framework for dealing with the total uncertainty w.

We propose a two-step optimal control approach to uncertain Euler-Lagrange systems as shown in Figure 3, and the details can be described as follows:

(i) Design the H_∞ optimal disturbance estimator minimizing the L_2-induced norm from the total disturbance w to an associated estimation error function.
(ii) Design the generalized H_2 optimal tracking controller minimizing the induced norm from L_2 to L_∞ associated with from the estimation error function to a tracking error function.

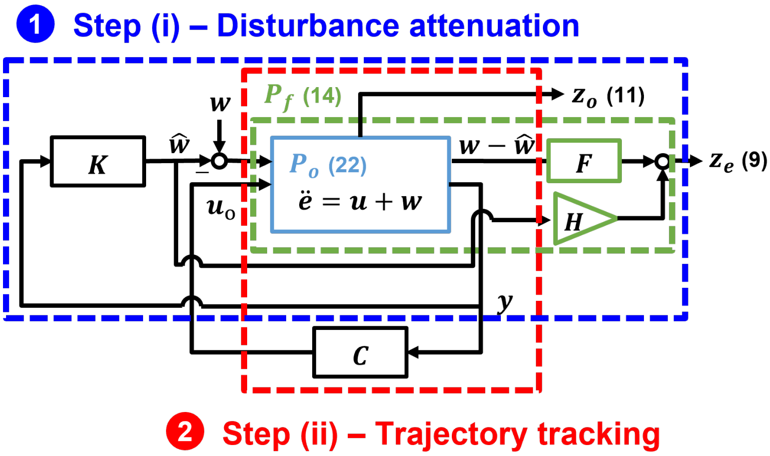


Figure 3. Proposed framework for 2-step optimal estimator and controller.

In the context of the above framework, the equation of error dynamics in (2.4) can be reinterpreted by

$$\ddot{e} = u + w =: u_o + w - \hat{w}, \tag{3.4}$$

where u_o is obtained from the generalized H_2 optimal tracking controller C and $\hat{w}$ is an estimated value of w obtained through the H_∞ optimal estimator K.

The rationale behind taking these two different system norms can be described in the following aspects. Because w is in L_2, its estimate as well as an associated estimation error should be regarded as elements in L_2. In this sense, it should be required for the associated optimal estimator to minimize the L_2-induced norm (i.e., the H_∞ norm) in the Step (i). After the H_∞ optimal disturbance estimation, the tracking accuracy can be further improved by designing an optimal tracking controller minimizing the effect of the estimation error function to a relevant tracking error function. In connection with this, if we note that the estimation error function is in L_2 while the tracking error function is regarded as an element in L_∞, it is natural to design the optimal tracking controller minimizing the induced norm from L_2 to L_∞, i.e., the generalized H_2 norm, for the Step (ii).

On the other hand, it is important to note that the design processes for the H_∞ optimal disturbance estimator and the generalized H_2 optimal tracking controller considered in the Steps (i) and (ii), respectively, do not depend on each other, although the latter is suggested to be designed after the former is obtained. The details could be clarified in the following subsection, which addresses the associated synthesis procedures in depth.

3.2. Optimal synthesis for estimator and controller

With respect to a disturbance estimator, we consider the estimation error function defined as

$$z_e := F(s)(w - \hat{w}) + H\hat{w}, \tag{3.5}$$

where $F(s)$ is an $n \times n$-dimensional transfer function matrix with the real coefficients and H is an $n \times n$-dimensional weighting matrix, i.e., $H \in \mathbb{R}^{n\times n}$, and they are determined according to desired performance specifications. Based on the representation (3.5), we aim at designing the H_∞ optimal disturbance estimator minimizing the L_2-induced norm defined as

$$\sup_{w\neq 0} \frac{\|z_e\|_2}{\|w\|_2}. \tag{3.6}$$

Regarding a tracking controller, on the other hand, the tracking error function is given by

$$z_o := E_1 e + E_2 \dot{e} + D_{12} u_o, \tag{3.7}$$

with some weighting matrices E_1, E_2, and D_{12}, and their selection does depend on required specifications of tracking performances. For the disturbance estimation error defined as

$$d := w - \hat{w}, \tag{3.8}$$

and the tracking error function given by (3.7), we design the generalized H_2 optimal tracking controller minimizing the induced norm from d to z_o defined as

$$\sup_{d\neq 0} \frac{\|z_o\|_\infty}{\|d\|_2}. \tag{3.9}$$

On the other hand, it would be worthwhile to discuss the rationale behind taking $F(s)$ and H as mentioned above. First of all, it is often required to consider a specific frequency range of w and not the total range $[0, \infty)$ in practical systems. Hence, it is reasonable to take $F(s)$ by a strictly proper transfer function matrix. This also makes the synthesis problem of the H_∞ disturbance estimator to be practically meaningful, and the details will be also discussed in the arguments around Theorem 2. For a selection of H, if H is set to 0, then the H_∞ disturbance estimator often leads to a high-gain, which is not desired in practical systems. In this sense, we have taken a nonzero matrix H, and such a selection does not affect the feasibility of the synthesis of the H_∞ optimal disturbance estimator.

For the synthesis of an H_∞ optimal disturbance estimator, we first derive the relevant plant. To simplify the arguments, assume that the measured output y and $F(s)$ in (3.5) are taken by $\begin{bmatrix} e^T & \dot{e}^T \end{bmatrix}^T$ and $\frac{1}{1+\tau s} I_n$, respectively (other cases can also be proceeded in an equivalent fashion to the following arguments). With the state vector $f(t)$ of $F(s)$, defining

$$x_f(t) := \begin{bmatrix} e^T(t) & \dot{e}^T(t) & f^T(t) \end{bmatrix}^T$$

leads to the state-space representation of P_f in Figure 3 described by

$$P_f : \begin{bmatrix} \dot{x}_f \\ z_e \\ y \end{bmatrix} = \begin{bmatrix} A^{[f]} & B_1^{[f]} & B_2^{[f]} \\ C_1^{[f]} & 0 & D_{12}^{[f]} \\ C_2^{[f]} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_f \\ w \\ \hat{w} \end{bmatrix}, \tag{3.10}$$

where

$$A^{[f]} = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau} I_n \end{bmatrix} \in \mathbb{R}^{3n\times 3n}, \quad B_1^{[f]} = \begin{bmatrix} 0 \\ I_n \\ \frac{1}{\tau} I_n \end{bmatrix} \in \mathbb{R}^{3n\times n}, \quad B_2^{[f]} = \begin{bmatrix} 0 \\ -I_n \\ -\frac{1}{\tau} I_n \end{bmatrix} \in \mathbb{R}^{3n\times n},$$

$$C_1^{[f]} = \begin{bmatrix} 0 & 0 & I_n \end{bmatrix} \in \mathbb{R}^{n\times 3n}, \quad C_2^{[f]} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 3n}, \quad D_{12}^{[f]} = H \in \mathbb{R}^{n\times n}.$$

With respect to the representation (3.10), we are concerned with designing an H_∞ optimal disturbance estimator such that

$$\min_{K(s)} \sup_{w\neq 0} \frac{||z_e||_2}{||w||_2}, \tag{3.11}$$

where $K(s)$ is described by the state-space representation

$$K : \begin{bmatrix} \dot{x}_k \\ \hat{w} \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} x_k \\ y \end{bmatrix}. \tag{3.12}$$

To put it another way, we aim at obtaining the estimation parameters

$$A_k \in \mathbb{R}^{3n\times 3n}, \quad B_k \in \mathbb{R}^{3n\times 2n}, \quad C_k \in \mathbb{R}^{n\times 3n} \quad \text{and} \quad D_k \in \mathbb{R}^{n\times 2n}$$

in (3.12) corresponding to the optimal problem given by (3.11). In connection with this, taking the arguments in [34] leads to the following theorem.

Theorem 1. *For a prespecified $\gamma_{2/2} > 0$, there exists an estimation parameter (A_k, B_k, C_k, D_k) ensuring the internal stability of the closed-loop system and the H_∞ performance given by*

$$\sup_{w\neq 0} \frac{||z_e||_2}{||w||_2} < \gamma_{2/2}, \tag{3.13}$$

if and only if there exist decision variables $X,\ Y,\ J,\ L,\ M,\ N$ such that the LMIs

$$\begin{cases} \mathcal{X} \succ 0, \\ \begin{bmatrix} \mathcal{A}^T + \mathcal{A} & \mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T & -\gamma_{2/2} I_n & 0 \\ \mathcal{C} & 0 & -\gamma_{2/2} I_n \end{bmatrix} \prec 0, \end{cases} \tag{3.14}$$

are feasible, where

$$\mathcal{A} = \begin{bmatrix} A^{[f]}Y + B_2^{[f]}M & A^{[f]} + B_2^{[f]}NC_2^{[f]} \\ J & A^{[f]}X + LC_2^{[f]} \end{bmatrix} \in \mathbb{R}^{6n\times 6n}, \quad \mathcal{B} = \begin{bmatrix} B_1^{[f]} \\ XB_1^{[f]} \end{bmatrix} \in \mathbb{R}^{6n\times n},$$

$$\mathcal{C} = \begin{bmatrix} C_1^{[f]}Y + D_{12}^{[f]}M & C_1^{[f]} + D_{12}^{[f]}NC_2^{[f]} \end{bmatrix} \in \mathbb{R}^{n\times 6n}, \quad \mathcal{X} = \begin{bmatrix} Y & I_{3n} \\ I_{3n} & X \end{bmatrix} \in \mathbb{R}^{6n\times 6n}.$$

Furthermore, if these LMI-based conditions are feasible, then there exist non-singular matrices U and V such that

$$UV^T = I - XY$$

and the estimation parameter (A_k, B_k, C_k, D_k) is determined by

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = S_1^{-1} S_2 S_3^{-1}, \tag{3.15}$$

where

$$S_1 = \begin{bmatrix} U & XB_2^{[f]} \\ 0 & I_n \end{bmatrix} \in \mathbb{R}^{4n\times 4n}, \quad S_2 = \begin{bmatrix} J - XAY & L \\ M & N \end{bmatrix} \in \mathbb{R}^{4n\times 5n}, \quad S_3 = \begin{bmatrix} V^T & 0 \\ C_2^{[f]}Y & I_{2n} \end{bmatrix} \in \mathbb{R}^{5n\times 5n}.$$

Remark 1. *The first LMI in* (3.14) *(i.e., $\mathcal{X} > 0$) corresponds to a necessary and sufficient condition for the internal stability of the closed-loop system, while the second LMI in* (3.14) *is equivalent to a necessary and sufficient condition for ensuring that the H_∞ norm of the closed-loop systems is less than $\gamma_{2/2}$. Furthermore, it immediately follows that S_1 and S_3 are invertible since U and V are non-singular.*

Theorem 1 provides a solution to the optimal control problem of (3.11) since this theorem establishes a necessary and sufficient condition for the existence of an H_∞ estimator $K(s)$ with respect to a pre-given performance level $\gamma_{2/2}$. To put it another way, minimizing $\gamma_{2/2}$ in the LMI-based constraints (3.14) leads to the H_∞ optimal disturbance estimator. In a practical aspect, it would be also worthwhile to note that the order of the proposed H_∞ disturbance estimator coincides with that of the plant P_f, independently of that of a tracking controller.

In connection with taking $F(s)$ by a strictly proper transfer function matrix, we remark that there exists a nonzero feedthrough term $D_{11}^{[f]} (\in \mathbb{R}^{n\times n})$ in (3.10) if $F(s)$ is not strictly proper. This leads to modifying the second inequality in (3.14) by

$$\begin{bmatrix} \mathcal{A}^T + \mathcal{A} & \mathcal{B} & C^T \\ \mathcal{B}^T & -\gamma_{2/2} I_n & (D_{11}^{[f]})^T \\ C & D_{11}^{[f]} & -\gamma_{2/2} I_n \end{bmatrix} \prec 0. \tag{3.16}$$

In terms of (3.16), we develop the following results.

Theorem 2. *The LMI of* (3.16) *is feasible only if*

$$\gamma_{2/2} > |D_{11}^{[f]}|_2.$$

Proof. Note that

$$\begin{bmatrix} \mathcal{A}^T + \mathcal{A} & \mathcal{B} & C^T \\ \mathcal{B}^T & -\gamma_{2/2} I_n & (D_{11}^{[f]})^T \\ C & D_{11}^{[f]} & -\gamma_{2/2} I_n \end{bmatrix} \prec 0 \Rightarrow \begin{bmatrix} -\gamma_{2/2} I_n & (D_{11}^{[f]})^T \\ D_{11}^{[f]} & -\gamma_{2/2} I_n \end{bmatrix} \prec 0 \Leftrightarrow \gamma_{2/2} > |D_{11}^{[f]}|_2. \tag{3.17}$$

This completes the proof. □

Theorem 2 implies that the L_2-induced norm defined as

$$\sup_{w\neq 0} \frac{||w - \hat{w}||_2}{||w||_2}$$

cannot be smaller than $|D_{11}^{[f]}|_2$ if $F(s)$ is not strictly proper. In contrast, no such a constraint is required for solving the LMIs given in Eq (3.14), and, thus, taking a strictly proper transfer function matrix $F(s)$ allows us to obtain a wider range of achievable H_∞ performance with respect to $K(s)$.

After the H_∞ optimal estimator $K(s)$ is obtained via the arguments in Theorem 1, regardless of the choice of $K(s)$, we next consider the synthesis of an outer-loop tracking controller $C(s)$. The main objective in Step (ii) is to minimize the effect of the residual disturbance (i.e., the estimation error defined as (3.8)) on the tracking error function z_o (defined as (3.7)). With respect to this, defining

$$x(t) := \begin{bmatrix} e^T(t) & \dot{e}^T(t) \end{bmatrix}^T$$

results in the form for the generalized plant given by

$$P_o : \begin{bmatrix} \dot{x} \\ z_o \\ y \end{bmatrix} = \begin{bmatrix} A^{[o]} & B_1^{[o]} & B_2^{[o]} \\ C_1^{[o]} & 0 & D_{12}^{[o]} \\ C_2^{[o]} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ u_o \end{bmatrix}, \tag{3.18}$$

where

$$A^{[o]} = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad B_1^{[o]} = B_2^{[o]} = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \in \mathbb{R}^{2n \times n},$$
$$C_1^{[o]} = \begin{bmatrix} E_1 & E_2 \end{bmatrix} \in \mathbb{R}^{n \times 2n}, \quad C_2^{[o]} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \in \mathbb{R}^{2n \times 2n}.$$

Based on the representation described by (3.18), we aim at designing the generalized H_2 controller such that

$$\min_{C(s)} \sup_{d \neq 0} \frac{\|z_o\|_\infty}{\|d\|_2}, \tag{3.19}$$

where $C(s)$ is described by the state-space representation

$$C : \begin{bmatrix} \dot{x}_c \\ u_o \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}. \tag{3.20}$$

This paper considers designing the control parameters

$$A_c \in \mathbb{R}^{2n \times 2n}, \quad B_c \in \mathbb{R}^{2n \times 2n}, \quad C_c \in \mathbb{R}^{n \times 2n}$$

and

$$D_c \in \mathbb{R}^{n \times 2n}$$

in (3.20) for ensuring the optimal problem described by (3.19). With taking the arguments in [34], such a control parameter can be characterized by the following theorem.

Theorem 3. *For a given*

$$\gamma_{\infty/2} > 0,$$

there exists a control parameter (A_c, B_c, C_c, D_c) *ensuring the internal stability of the closed-loop system and the generalized* H_2 *performance given by*

$$\sup_{d \neq 0} \frac{\|z_o\|_\infty}{\|d\|_2} < \gamma_{\infty/2}, \tag{3.21}$$

if, and only if, there exist decision variables $X,\ Y,\ J,\ L,\ M,\ N$ *such that the LMIs*

$$\begin{cases} \begin{bmatrix} \mathcal{A}^T + \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & -\gamma_{\infty/2} I_n \end{bmatrix} \prec 0, \\ \begin{bmatrix} \mathcal{X} & \mathcal{C}^T \\ \mathcal{C} & \gamma_{\infty/2} I_n \end{bmatrix} \succ 0, \end{cases} \tag{3.22}$$

are feasible, where $\mathcal{A}$, $\mathcal{B}$, C, $\mathcal{X}$ are defined as in Theorem 1 by replacing $(\cdot)^{[f]}$ with $(\cdot)^{[o]}$. Furthermore, if these LMIs are feasible, then the control parameter (A_c, B_c, C_c, D_c) is also determined as in (3.15), *i.e.,*

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = S_1{}^{-1} S_2 S_3{}^{-1}. \tag{3.23}$$

Remark 2. *If the second LMI in* (3.22) *is established, then we obtain that $\mathcal{X} > 0$. This ensures that the closed-loop system is internally stable as discussed in Remark 1. However, in contrast to Theorem 1, establishing both the first and second LMIs in* (3.22) *leads to a necessary and sufficient condition for the generalized H_2 norm of the closed-loop system to be less than a prespecified $\gamma_{\infty/2}$.*

Similarly to Theorem 1, we can obtain the generalized H_2 optimal controller with respect to (3.19) by minimizing $\gamma_{\infty/2}$ in the LMI-based constraints (3.22) since this theorem leads to a necessary and sufficient condition for the existence of a generalized H_2 tracking controller $C(s)$ achieving a pre-given performance level $\gamma_{\infty/2}$. The order of the proposed generalized H_2 tracking controller also is equal to that of the plant P_o. More importantly, it should be stressed that the LMI-based conditions in Theorem 3 do not depend on the H_∞ optimal disturbance estimator obtained in Theorem 1 and thus we can confirm once again that the synthesis procedures in these theorems are independent of each other.

On the other hand, it would be worthwhile to note that the feedthrough term $D_{11}^{[o]}$ is assumed to be zero in (3.18), and this assumption is necessary for the generalized H_2 norm (i.e., the induced norm from L_2 to L_∞) to be well-defined and bounded. Thus, regardless of the choice of $C(s)$, the transfer function matrix from d to z_o should be strictly proper whenever the generalized H_2 norm is taken as the corresponding performance measure. This turns out that the transfer function matrix from w to z_o is also strictly proper and thus its frequency response gain decreases as the considered frequency becomes larger. In other words, the generalized H_2 optimal control might also be interpreted in the frequency domain as dealing with low ranges rather than the overall bound $(-\infty, \infty)$. In this sense, it is natural to take a strictly proper transfer function matrix $F(s)$ when we are concerned with reducing w itself.

To summarize, the overall process of the proposed control framework and its characteristics can be described as follows:

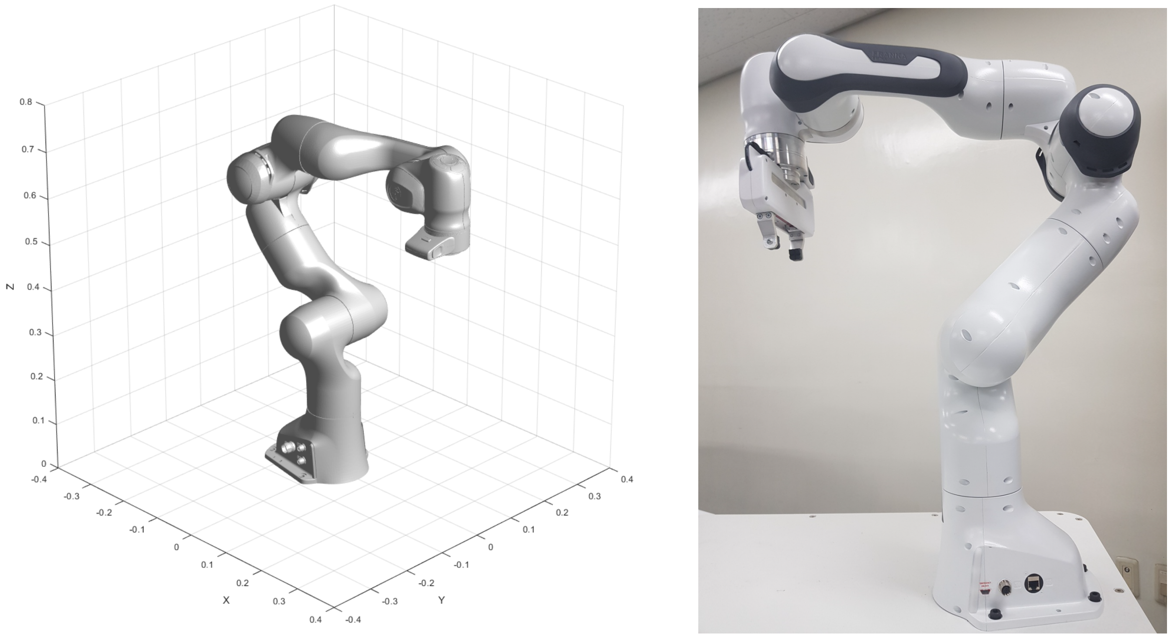
- The total uncertainty w (given in (2.4)) is estimated and its magnitude is reduced, especially for low frequency ranges, by designing the H_∞ optimal disturbance estimator.
- The effect of the estimation error

$$d = w - \hat{w}$$

  on the tracking function z_o (defined as (3.7)) is minimized in terms of the induced norm from L_2 to L_∞ by designing the generalized H_2 optimal tracking controller.
- Even though the generalized H_2 tracking controller is proposed to be designed after the H_∞ disturbance estimator is obtained, their LMI-based synthesis conditions do not depend on each other.

4. Illustrative examples

This section evaluates the validity of the overall arguments presented in the preceding section by means of simulation and experimental results. We employ the 6-degree of freedom robot manipulators shown in Figure 4 with the reference trajectories shown in Figure 5 for both simulations and experiments.



(a) Simulated robot (b) Real robot

Figure 4. Robot manipulators used in simulations and experiments.

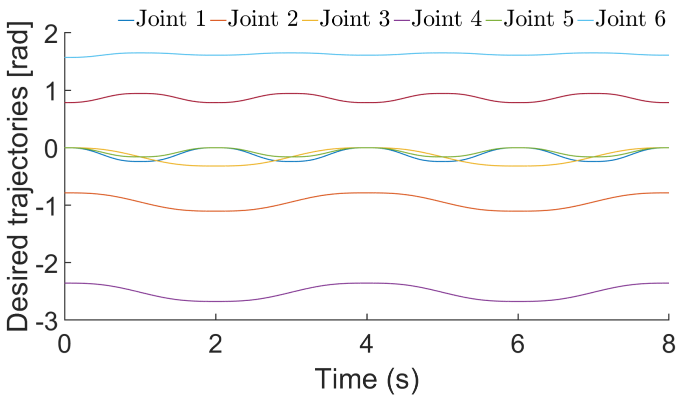


Figure 5. Reference trajectories for simulations and experiments.

Comparative simulations and experiments to the recent results of the enhanced unknown system dynamics estimator-based sliding mode control (EUSDE-SMC) [8] and the time-delay based disturbance estimation (TDE) [9] are conducted. The rationale behind taking these two existing studies is that they are sorts of the second approach mentioned in Subsection 1.1 and their control architectures are similar to that of the proposed method (PM); this fact makes the comparison between the PM to the EUSDE-SMC and the TDE fair.

More precisely, the superiority of the PM over the existing two methods in estimating uncertainties is demonstrated through simulations, while that in trajectory tracking is validated through experiments. For both simulations and experiments, we take the following parameters for P_f and P_o:

$$F(s) := \frac{1}{s+1} I_6, \quad H = \frac{1}{2} I_6, \quad C_1^{[o]} = \begin{bmatrix} 10I_6 & I_6 \end{bmatrix}$$

and

$$D_{12}^{[o]} = \frac{1}{2} I_6.$$

4.1. Simulation results

Similarly to Example 1, let us consider the case such that model uncertainties are generated by embedding random signals as shown in Figure 1, and comparative analysis on estimating uncertain elements is considered with respect to the PM, EUSDE-SMC, and TDE; all the three methods derive time-series signals for all the joints in real-time as the estimated values of uncertainties. The overall simulations are conducted through MATLAB, and the LMI conditions required for obtaining the H_∞ optimal disturbance estimator and the generalized H_2 tracking controller are tackled by using the function "mincx" in the MATLAB LMI solver.

With respect to solving the LMIs in (3.14), the minimum value of $\gamma_{2/2}$ can also be obtained by taking the arguments in [15, 18] relevant to the H_∞ optimal controller synthesis, and such a value is given by 0.5010.

On the other hand, we consider the total uncertainty w as shown in Figure 6, and the simulation results for the estimation errors with respect to the three methods are shown in Figure 7. First of all, we can observe from Figure 7 that the estimation error with the PM is closer to 0 than those with the existing EUSDE-SMC and TDE methods under the same joint. Here, these estimation errors are expected to show significant oscillations because the total uncertainty w consists of high frequency elements as observed from Figure 6. However, the estimation errors with the PM are quite smoother than those with the existing two methods for all the 6 joints. This observation implies that the strictly proper transfer function matrix $F(s)$ in (3.5) is adequately taken to reduce the high frequency elements in w.

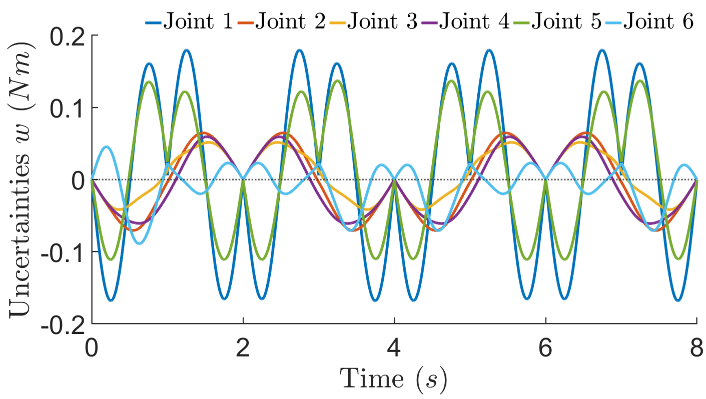
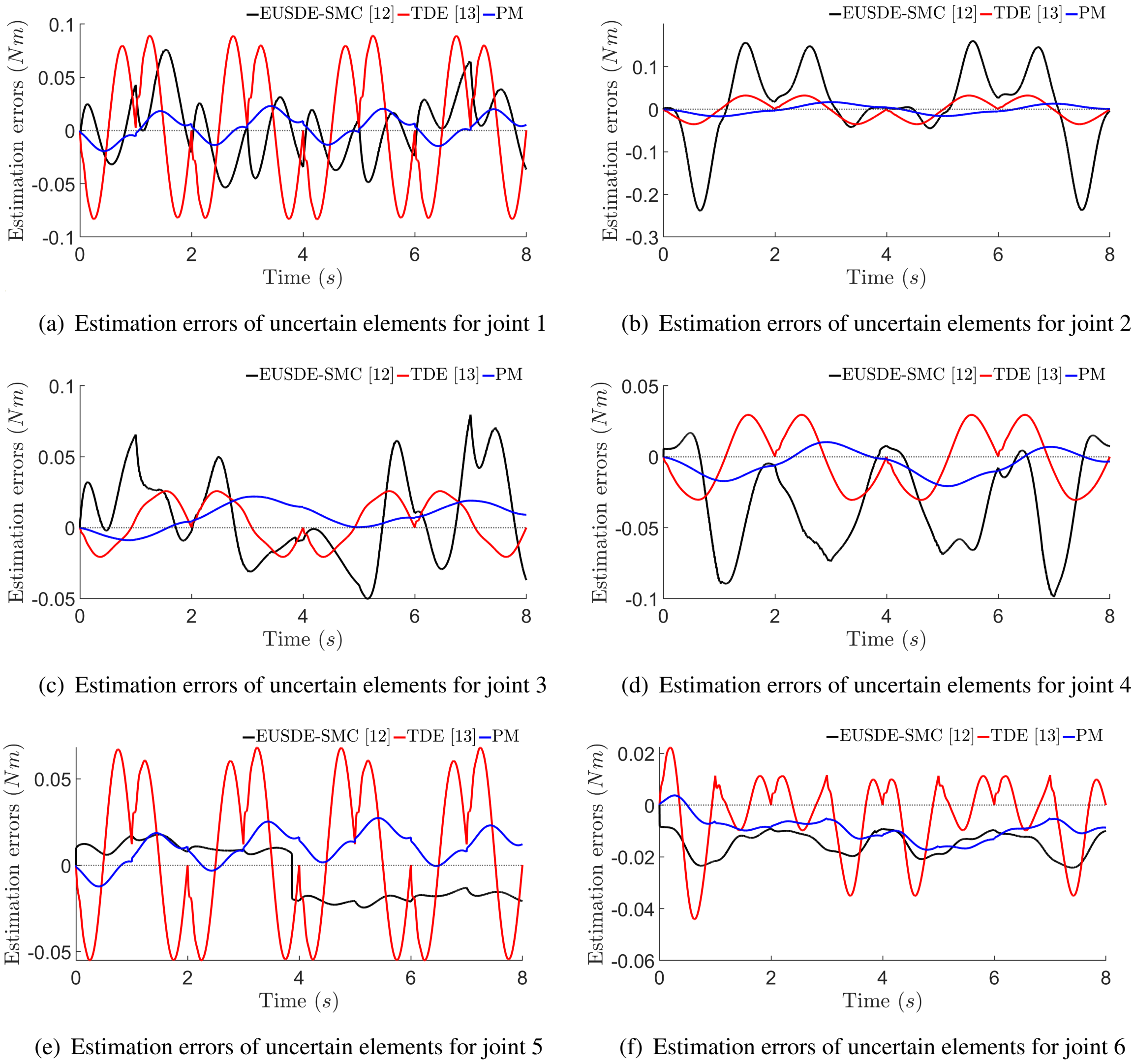


Figure 6. Total uncertainty w taken in simulations.



(a) Estimation errors of uncertain elements for joint 1
(b) Estimation errors of uncertain elements for joint 2
(c) Estimation errors of uncertain elements for joint 3
(d) Estimation errors of uncertain elements for joint 4
(e) Estimation errors of uncertain elements for joint 5
(f) Estimation errors of uncertain elements for joint 6

Figure 7. Simulation results for estimation errors of uncertain elements with three methods.

To make the comparison between the PM and the two existing methods clearer in a quantitative aspect, the simulation results for the root mean square (RMS) value of the estimation errors are shown in Table 1.

Table 1. RMS values of estimation errors of uncertain elements in simulations [NM].

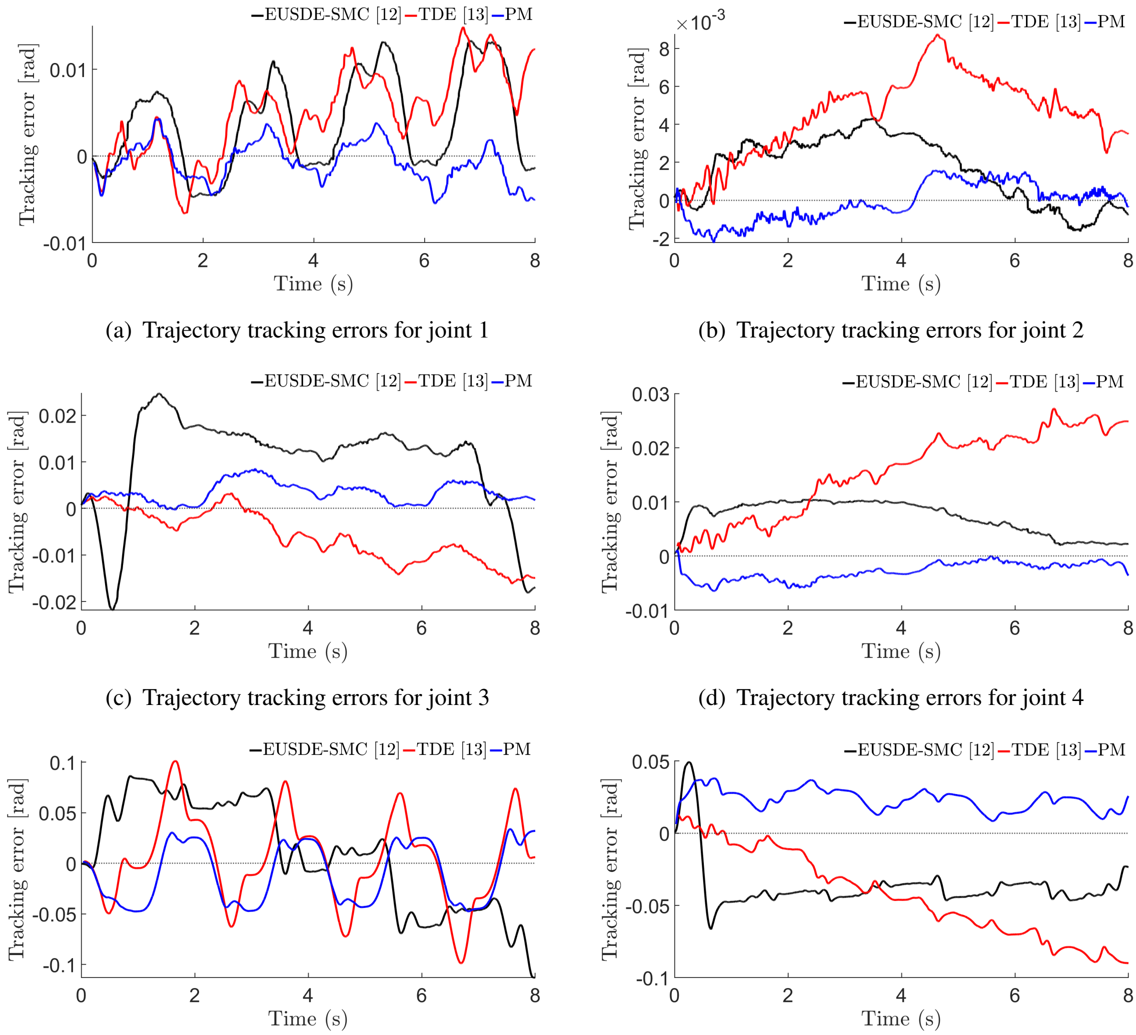
	EUSDE-SMC [8]	TDE [9]	PM
Joint 1	0.0292	0.0594	0.0117
Joint 2	0.0950	0.0231	0.0100
Joint 3	0.0320	0.0164	0.0119
Joint 4	0.0462	0.0212	0.0100
Joint 5	0.0157	0.0431	0.0140
Joint 6	0.0157	0.0155	0.0103

We can observe from Table 1 that the RMS values obtained through the PM are quite smaller than those of the two existing methods for all the 6 joints, and, thus, the PM can be regarded as outperforming the existing EUSDE-SMC and TDE methods in estimating uncertain elements of robot manipulators.

These observations validate the effectiveness of the developed arguments relevant to designing the H_∞ optimal disturbance estimator.

4.2. Experiment results

The experimental results related to the joint position errors are shown in Figure 8. We can observe from Figure 8 that the maximum magnitude of the tracking error achieved by PM is always lower compared to those of the EUSDE-SMC and TDE methods for the same joint. This finding indicates that the PM exhibits superior tracking performance relative to the two conventional approaches.



(a) Trajectory tracking errors for joint 1
(b) Trajectory tracking errors for joint 2
(c) Trajectory tracking errors for joint 3
(d) Trajectory tracking errors for joint 4
(e) Trajectory tracking errors for joint 5
(f) Trajectory tracking errors for joint 6

Figure 8. Experiments results for trajectory tracking errors with three methods.

For a more quantitative analysis relevant to the experiment results, we are in a position to compare the L_∞ norm (i.e., the maximum magnitude) of the joint position errors, as shown in Table 2. We can see from Table 2 that the L_∞ norms of the joint position errors obtained through the PM are quite smaller than those through the EUSDE-SMC and the TDE, with respect to all the 6 joints.

Table 2. L_∞ norms of trajectory tracking errors [rad].

	EUSDE-SMC [8]	TDE [9]	PM
Joint 1	0.5956	0.6201	0.2163
Joint 2	0.2118	0.4429	0.0883
Joint 3	1.2881	0.7234	0.3618
Joint 4	0.6978	1.5404	0.2889
Joint 5	4.7868	3.8190	2.7415
Joint 6	3.6274	4.6619	2.1596

Regarding a different aspect of the practical effectiveness, the experiment results for the total torque powers are shown in Table 3. We can observe from Table 3 that the torque power required by the PM is lower than that required by both the EUSDE-SMC and TDE.

Table 3. Torque powers required for three methods [W].

	EUSDE-SMC [8]	TDE [9]	PM
Power	14.9473	15.6809	14.4772

To summarize, the experimental analysis above clearly highlights the practical advantages of the PM over the existing EUSDE-SMC and TDE in reducing tracking errors for uncertain robot manipulators. Specifically, it shows that the PM achieves the smallest L_∞ norms across all six joints while requiring the least torque power among the three control methods. These experiment observations also clearly validate the effectiveness of designing the generalized H_2 optimal tracking controller for reducing the tracking error in the time-domain bounds, and, thus, the safety issue on robot manipulators can be expected to be solved by taking the PM.

5. Conclusions

Motivated by the fact that estimating uncertain elements at every moment is practically useful and it is required to suppress the relevant tracking errors in the time-domain bounds for the safety issue such as the collision avoidance, this paper developed a new control framework for uncertain Euler-Lagrange systems. Based on this new control framework, we next proposed two dynamic compensators called the H_∞ optimal disturbance estimator and the generalized H_2 optimal tracking controller, respectively. More precisely, the L_2-induced norm of the mapping from the unknown disturbance to the corresponding estimation error function is minimized through the H_∞ optimal disturbance estimator, while the generalized H_2 norm (i.e., the induced norm from L_2 to L_∞) of the mapping from the estimation error function to the relevant tracking error function is minimized by the generalized H_2 optimal tracking controller. It was also shown in this paper that the H_∞ disturbance estimator and the generalized H_2 optimal tracking controller can be readily obtained through the linear matrix inequality-based approach. Even though the H_∞ optimal disturbance estimator and the generalized H_2 optimal tracking controller were explained in the step (i) and the step (ii), respectively, it should be remarked that their synthesis procedures could be carried out independently of each other. The effectiveness of the H_∞ optimal disturbance estimator was demonstrated through some comparative simulations to the conventional methods in [8, 9]. The practical superiority of the proposed method, which combines the H_∞ optimal disturbance estimator and the generalized H_2 optimal tracking controller, over conventional methods was also validated through comparative experimental results.

Finally, it would be worthwhile to note that the arguments developed in this paper (i.e., the new structural framework, the H_∞ optimal disturbance estimator, and the generalized H_2 optimal tracking controller) could be applied in a parallel fashion to different linearized systems such as electric motors, power systems, and so on, although this paper dealt with only uncertain Euler-Lagrange equations. However, it is still unclear how to obtain performance limitations (i.e., lower bounds on the H_∞ norm and the generalized H_2 norm) before the optimal estimator and controller synthesis. Thus, it might be possible to require quite long computation times for minimizing $\gamma_{2/2}$ and $\gamma_{\infty/2}$ in Theorems 1 and 3, respectively, when the initial conditions are very far from the optimal values. This issue is left for an interesting but difficult future study.

Author contributions

Taewan Kim: original draft preparation, methodology, investigation, and formal analysis; Jung Hoon Kim: supervision, validation, manuscript-review, project administration, funding acquisition, and revisions. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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