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Research article

Quadric surfaces of finite Chen II-type

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Abstract: In this paper we studied quadric surfaces in the Euclidean 3-space that were of finite type with respect to the second fundamental form II. The main result presented in this article was that spheres were the only quadric surfaces of finite type. This indicated a specific and notable classification within the broader category of quadric surfaces based on their finite type characteristics in relation to the second fundamental form.

Keywords: surfaces in 3-space; finite Chen-type surfaces; second Beltrami operator; quadric surfaces
Mathematical Subject Classification: 53A05, 53A45

1. Introduction

The theory of sub-manifolds of finite type has led to significant insights and results in differential geometry, helping to identify and characterize sub-manifolds with special geometric properties.

In [1], Chen mentioned the concept of Euclidean immersions of finite type concerning the first fundamental form I of a surface $\aleph$. According to Chen's theory, a surface $\aleph$ is said to be of finite type if its coordinate functions can be expressed as a finite sum of eigenfunctions of the Beltrami operator Δ^I.

For instance, Chen [2] posed the problem of classifying finite type sub-manifolds in 3-dimensional Euclidean space E^3. This initiated a comprehensive study of the spectral properties of the Laplacian on these sub-manifolds, leading to the classification of minimal surfaces, spheres, and circular cylinders as specific examples of finite-type sub-manifolds.

If we consider the surface $\aleph$ in E^3, its position vector

$$\boldsymbol{X} = \boldsymbol{X}(v^1,\ v^2)$$

can be written as:

$$\boldsymbol{X} = \sum_{i=1}^{n} F_i(v^1,\ v^2)\boldsymbol{e}_i,$$

where $F_i(v^1,\ v^2)$ are eigenfunctions of the Beltrami operator Δ^I, and $\boldsymbol{e}_i$ are constant vectors in E^3.

For a surface to be of finite II-type, its shape operator (related to the second fundamental form) must also have a similar decomposition into a finite sum of eigenfunctions of the Beltrami operator.

To understand the implications of a surface $\aleph$ being of finite type l, we start by considering the relevant equation involving the second fundamental form, Δ^{II}, which is the Laplacian operator applied to the components of the second fundamental form.

When $\aleph$ is of finite type l, there exists a monic polynomial

$$F(x) \neq 0,$$

such that

$$F(\Delta^{II})(\boldsymbol{X} - \boldsymbol{c}) = \boldsymbol{0}.$$

Here, $\boldsymbol{X}$ represents the position vector of a point on the surface, and $\boldsymbol{c}$ is a constant vector.

Suppose the polynomial

$$F(x) = x^l + \gamma_1 x^{l-1} + ... + \gamma_{l-1} x + \gamma_l.$$

Then the coefficients γ_i are determined by the specific relationship between the eigenfunctions of Δ^{II}, and the position vector components. These coefficients γ_i are related to the eigenvalues μ_i of Δ^{II} acting on the coordinate functions of $\boldsymbol{X}$. In detail, γ_i are typically given in terms of symmetric polynomials of these eigenvalues. Specifically, they can be expressed as follows:

$$\begin{aligned}
\gamma_1 &= -(\mu_1 + \mu_2 + ... + \mu_l),\\
\gamma_2 &= (\mu_1\mu_2 + \mu_1\mu_3 + ... + \mu_1\mu_l + \mu_2\mu_3 + ... + \mu_2\mu_l + ... + \mu_{l-1}\mu_l),\\
\gamma_3 &= -(\mu_1\mu_2\mu_3 + ... + \mu_{l-2}\mu_{l-1}\mu_l),\\
&\cdots\\
\gamma_l &= (-1)^l \mu_1\mu_2 ... \mu_l.
\end{aligned}$$

Therefore the position vector $\boldsymbol{X}$ satisfies the following equation (see [3]):

$$(\Delta^{II})^l \boldsymbol{X} + \sigma_1 (\Delta^J)^{l-1} \boldsymbol{X} + ... + \sigma_l (\boldsymbol{X} - \boldsymbol{c}) = \boldsymbol{0}. \quad (1.1)$$

Finite-type immersions involve studying sub-manifolds whose coordinate functions are finite sums of eigenfunctions of the Laplace-Beltrami operator. This notion provides a way to classify sub-manifolds based on the spectral properties of the Laplacian acting on the coordinate functions.

These classifications help in understanding the geometric and topological properties of sub-manifolds. For instance, the result that spheres are the only quadric surfaces of finite II-type in E^3 provides a clear distinction between spheres and other quadric surfaces like ellipsoids, hyperboloids, and paraboloids, based on their curvature properties.

A recent study in [4] authors investigated the Hasimoto surfaces according to their finite Chen type, while in [5, 6] interesting researches were done by studying the class of translation surfaces according to it's finite Chen III-type once in E^3, and on the other hand in Sol_3.

Takahashi in [7] mentioned that a surface M^2 whose position vector $\boldsymbol{X}$ satisfies

$$\Delta^I \boldsymbol{X} = \mu \boldsymbol{X}$$

is either a minimal with $\mu = 0$ or M^2 lies in an ordinary sphere S^2 with a fixed nonzero eigenvalue.

Garay in his article [8] made a generalization of Takahashi's condition. In his study, he considered surfaces in E^3 satisfying

$$\Delta^I \boldsymbol{X}_i = \mu_i X_i, \quad i = 1, 2, 3,$$

where (X_1, X_2, X_3) are the coordinate functions of the position vector $\boldsymbol{X}$ and μ_i, are different eigenvalues. Garay's work expands on the problem of identifying surfaces in E^3 that satisfy this eigenvalue condition. The coordinate functions of these surfaces are expressed as eigenfunctions of the Laplace-Beltrami operator associated with distinct eigenvalues, contributing to the understanding of surfaces of finite type in a more general context.

Another related general problem was presented in [9], which investigated surfaces in E^3 satisfying

$$\Delta^I \boldsymbol{X} = K\boldsymbol{X} + L, \tag{1.2}$$

where K is a 3×3 matrix; L is a 3×1 matrix. It was proven that minimal surfaces, spheres, and circular cylinders are the only surfaces in E^3 satisfying Eq (1.2). Surfaces meeting this criterion are said to be of coordinate finite type.

As an application, the alignment of molecules in relation to quadric surfaces has meaningful applications in understanding molecular orientations, interactions, and behaviors under external influences. Quadric surfaces, such as ellipsoids, hyperboloids, and paraboloids, serve as mathematical representations of properties like potential energy distributions, molecular shapes, and field effects. The shapes of molecules can often be described using quadric surfaces such as ellipsoids which is common for anisotropic molecules like liquid crystal rods or elongated organic molecules, or spheres which represent isotropic molecules such as noble gases or symmetric compounds like $CH4$. These quadric shapes help model how molecules orient or align in space (see [10, 11]).

We consider a (connected) surface $\aleph$ in a Euclidean 3-space E^3 referred to any system of coordinates v^1, v^2, whose Gaussian curvature never vanishes. Let B_{st} be the components of the second fundamental form

$$II = B_{st} dv^s dv^t$$

of $\aleph$. For any two sufficiently differentiable functions $\varphi(v^1, v^2)$ and $\psi(v^1, v^2)$ on $\aleph$, the first Beltrami operator with respect to the second fundamental form of $\aleph$ is given by

$$\nabla^{II}(\psi, \varphi) := B^{st} \psi_{/s} \varphi_{/t},$$

where

$$\psi_{/s} := \frac{\partial \psi}{\partial v^s},$$

and (B^{st}) denotes the inverse tensor of (B_{st}).

The second Beltrami operator regarding the second fundamental form of $\aleph$ is defined by [12]

$$\Delta^{II} \varphi := -\frac{1}{\sqrt{B}} (\sqrt{B} B^{st} \varphi_{/s})_{/t},$$

where

$$B := Det(B_{st}).$$

In [13], authors proved that, for the position vector

$$\boldsymbol{X} = \boldsymbol{X}(v^1, v^2)$$

of $\aleph$, the relation is

$$\Delta^{II}\boldsymbol{X} = \frac{1}{2K}\nabla^{III}K - 2\boldsymbol{G},$$

where $\boldsymbol{G}$ is the Gauss map, K the Gauss curvature, and H the mean curvature of $\aleph$.

The main result in this study presents the following as detailed below.

Theorem 1. *Among all quadric surfaces in E^3, the only one that satisfies the finite II-type condition is the sphere.*

This result highlights a unique geometric property of spheres compared to other quadric surfaces such as ellipsoids, hyperboloids, and paraboloids.

2. Quadric surfaces

Let $\aleph$ be a C^r quadric surface in $\mathbb{E}^3$ defined on a region $\mathcal{U} \subset \mathbb{R}^2$. Then, $\aleph$ is one of the following three kinds [14, 15]:

$1^{st}Kind$: $\aleph$ is a ruled surface.

$2^{nd}Kind$: $\aleph$ is of the form

$$z^2 = \gamma + \alpha\, X^2 + \beta\, Y^2, \quad \alpha,\beta,\gamma \in \mathbb{R}, \quad \alpha\beta \neq 0, \quad \gamma > 0.$$

$3^{rd}Kind$: $\aleph$ is of the form

$$z = \frac{\alpha}{2}X^2 + \frac{\beta}{2}Y^2, \quad \alpha,\beta \in \mathbb{R}, \quad \alpha,\beta > 0.$$

The class of ruled surfaces has been studied in [16], so we will complete our study by investigating the second and third kinds of surfaces mentioned above in terms of their finite Chen type.

2.1. Quadrics of the second kind

A parametrization of a part of a quadric of this kind is [15]

$$\boldsymbol{x}(\upsilon, \nu) = \left(\upsilon, \nu, \sqrt{\alpha\,\upsilon^2 + \beta\,\nu^2 + \gamma}\right), \quad \alpha\,\upsilon^2 + \beta\,\nu^2 + \gamma > 0. \tag{2.1}$$

For simplicity, we use

$$\alpha\,\upsilon^2 + \beta\,\nu^2 + \gamma := \omega.$$

The metrics I, II of $\aleph$, are respectively,

$$I = (\frac{\alpha^2\upsilon^2}{\omega} + 1)d\upsilon^2 + \frac{2\alpha\,\beta\upsilon\nu}{\omega}d\upsilon\, d\nu + (\frac{\beta^2\nu^2}{\omega} + 1)d\nu^2,$$
$$II = \frac{1}{\omega\sqrt{T}}(\alpha(\gamma + \beta\nu^2)d\upsilon^2 - 2\alpha\,\beta\upsilon\nu d\upsilon\, d\nu + \beta(\gamma + \alpha\upsilon^2)d\nu^2),$$

where

$$T = \gamma + \beta(\beta+1)\nu^2 + \alpha(\alpha+1)\upsilon^2.$$

The Laplacian $\triangle^{II}$ of $\aleph$ can be expressed as follows:

$$\triangle^{II} = -\frac{\sqrt{T}}{\gamma}\left[\frac{\gamma+\alpha\upsilon^2}{\alpha}\frac{\partial^2}{\partial\upsilon^2} + 2\upsilon\nu\frac{\partial^2}{\partial\upsilon\partial\nu} + \frac{\gamma+\beta\nu^2}{\beta}\frac{\partial^2}{\partial\nu^2} + 2\upsilon\frac{\partial}{\partial\upsilon} + 2\nu\frac{\partial}{\partial\nu}\right]. \tag{2.2}$$

For a function $\varphi(\upsilon) \in C^\infty(\mathcal{U})$, on account of Eq (2.2), we get

$$\triangle^{II}\varphi = -\frac{\sqrt{T}}{\gamma}\left[\frac{\gamma+\alpha\upsilon^2}{\alpha}\frac{\partial^2\varphi}{\partial\upsilon^2} + 2\upsilon\frac{\partial\varphi}{\partial\upsilon}\right]. \tag{2.3}$$

On use of Eq (2.2), it can be easily proved:

Lemma 1. *The relation*

$$\triangle^{II}\Big(\frac{\alpha^r(\alpha+1)^r\upsilon^m}{T^n}\Big) = -\frac{\alpha^{r+2}(\alpha+1)^{r+2}[m(m+1)+4n(n-m)-2n]\upsilon^{m+4}}{\gamma T^{n+\frac{3}{2}}} + \frac{1}{\gamma T^{n+\frac{3}{2}}}F(\upsilon,\nu)$$

holds true, where $F(\upsilon,\nu)$ is a polynomial of degree at most $m+4$, and when $\nu = 0$, $deg(F(\upsilon,0))$ is at most $m+2$.

We denote by (x_1, x_2, x_3) the components of $\boldsymbol{x}(\upsilon,\nu)$. On account of Eq (2.3), we have

$$\triangle^{II}x_1 = \triangle^{II}\upsilon = -\frac{2\upsilon\sqrt{T}}{\gamma}. \tag{2.4}$$

Applying Eq (2.2) for the relation (2.4), we find

$$(\triangle^{II})^2x_1 = (\triangle^{II})^2\upsilon = \frac{2}{\gamma^2T}[6\alpha^2(\alpha+1)^2\upsilon^5 + f_2(\upsilon,\nu)], \tag{2.5}$$

where

$$\begin{aligned} f_2(\upsilon,\nu) =& \alpha\gamma(\alpha+1)(\gamma(\beta+1)+2\alpha+11)\upsilon^3 + \alpha\beta(\alpha+1)(\beta+1)(\gamma+12)\upsilon^3\nu^2 \\ &+ \beta^2(\beta+1)^2(\gamma+6)\upsilon\nu^4 + \beta\gamma(\beta+1)(\beta\gamma+2\gamma+3\alpha+11)\upsilon\nu^2 \\ &+ \gamma^2(\gamma(\beta+1)+3\alpha+5)\upsilon. \end{aligned}$$

Note that $f_2(\upsilon,\nu)$ is a polynomial of degree at most 5, and if we put $\nu = 0$, then $f_2(\upsilon,0)$ is a polynomial in υ of degree at most 3.

From Lemma 1, we get

$$(\triangle^{II})^3\upsilon = -\frac{2}{\gamma^3T^{\frac{5}{2}}}[72\alpha^4(\alpha+1)^4\upsilon^9 + f_3(\upsilon,\nu)], \tag{2.6}$$

where $f_3(\upsilon,\nu)$ is a polynomial of degree at most 9, with

$$deg(f_3(\upsilon,0)) \leq 7.$$

We will also prove:

Lemma 2. *The relation*

$$\left(\triangle^{II}\right)^{l} \upsilon = (-1)^{l}\left(\prod_{i=1}^{l} i(i+1)\right)\left(\frac{\alpha^{2l-2}\,(\alpha+1)^{2l-2}\,\upsilon^{4l-3} + P_{l}(\upsilon,\nu)}{\gamma^{l} T^{\frac{3}{2}l-2}}\right),$$

holds true, where

$$\deg(P_{l}(\upsilon,0)) \leq 4l-5.$$

Proof. The proof goes by induction on l.

Base case: For $l = 1$, the formula comes true from Eq (2.4) applied to $\varphi = \upsilon$.

Inductive step: Assume that the lemma is true for $l-1$. So,

$$\left(\triangle^{II}\right)^{l-1} \upsilon = (-1)^{l-1}\left(\prod_{i=1}^{l-1} i\,(i+1)\right)\left(\frac{\alpha^{2l-4}\,(\alpha+1)^{2l-4}\,\upsilon^{4l-7} + P_{l-1}(\upsilon,\nu)}{\gamma^{l-1} T^{\frac{3}{2}l-\frac{7}{2}}}\right).$$

Proof for l. Taking into account $\nu = 0$, relation (2.3), and Lemma 1, we obtain

$$\begin{aligned}
\left(\triangle^{II}\right)^{l} \upsilon &= \triangle^{II}\Big(\left(\triangle^{II}\right)^{l-1} \upsilon\Big) \\
&= (-1)^{l-1}\left(\prod_{i=1}^{l-1} i\,(i+1)\right)\left(\frac{1}{\gamma^{l-1}}\right)\left(\triangle^{II}\left(\frac{\alpha^{2l-4}\,(\alpha+1)^{2l-4}\,\upsilon^{4l-7}}{T^{\frac{3}{2}l-\frac{7}{2}}}\right)\right) + \triangle^{II}\left(\frac{P_{l-1}(\upsilon,\nu)}{\gamma^{l-1} T^{\frac{3}{2}l-\frac{7}{2}}}\right) \\
&= \frac{(-1)^{l-1}}{\gamma^{l-1}}\left(\prod_{i=1}^{l-1} i\,(i+1)\right)\left(-\frac{l(l+1)\alpha^{2l-4+2}\,(\alpha+1)^{2l-4+2}\,\upsilon^{4l-7+4} + P_{l}(\upsilon,\nu)}{\gamma T^{\frac{3}{2}l-\frac{7}{2}+\frac{3}{2}}}\right) \\
&= \frac{(-1)^{l}}{\gamma^{l}}\left(l(l+1)\prod_{i=1}^{l-1} i\,(i+1)\right)\left(\frac{\alpha^{2l-2}\,(\alpha+1)^{2l-2}\,\upsilon^{4l-3} + P_{l}(\upsilon,\nu)}{T^{\frac{3}{2}l-2}}\right) \\
&= (-1)^{l}\left(\prod_{i=1}^{l} i(i+1)\right)\left(\frac{\alpha^{2l-2}\,(\alpha+1)^{2l-2}\,\upsilon^{4l-3} + P_{l}(\upsilon,\nu)}{\gamma^{l} T^{\frac{3}{2}l-2}}\right).
\end{aligned}$$

This completes the proof. □

For the second component x_2, we have

$$\triangle^{II} x_2 = \triangle^{II} \nu = -\frac{2\nu\sqrt{T}}{\gamma}. \tag{2.7}$$

Also

$$(\triangle^{II})^2 x_2 = (\triangle^{II})^2 \nu = \frac{2}{\gamma^2 T}[6\beta^2(\beta+1)^2\nu^5 + g_2(\upsilon,\nu)], \tag{2.8}$$

and

$$\begin{aligned}
g_2(\upsilon,\nu) =& \beta\gamma(\beta+1)(\gamma(\alpha+1)+2\beta+11)\nu^3 + \alpha\beta(\alpha+1)(\beta+1)(\gamma+12)\upsilon^2\nu^3 \\
&+ \alpha^2(\alpha+1)^2(\gamma+6)\nu\upsilon^4 + \alpha\gamma(\alpha+1)(\alpha\gamma+2\gamma+3\beta+11)\upsilon^2\nu \\
&+ \gamma^2(\gamma(\alpha+1)+3\beta+5)\nu.
\end{aligned}$$

Similarly, $g_2(\upsilon,\nu)$ is a polynomial of degree at most 5, and if we put $\upsilon = 0$, then $g_2(0,\nu)$ is a polynomial in ν of degree at most 3.

Lemma 3. *The relation*

$$\triangle^{II}\left(\frac{\beta^r(\beta+1)^r\nu^m}{T^n}\right) = -\frac{\beta^{r+2}(\beta+1)^{r+2}[m(m+1)+4n(n-m)-2n]\nu^{m+4}}{\gamma T^{n+\frac{3}{2}}} + \frac{1}{\gamma T^{n+\frac{3}{2}}}G(\upsilon,\nu),$$

holds true, where $G(\upsilon,\nu)$ is a polynomial of degree at most $m+4$, and when $\upsilon = 0$, then $deg(G(0,\nu))$ is at most $m+2$.

So, using the above lemma, one can find that

$$(\triangle^{II})^3\nu = -\frac{2}{\gamma^3 T^{\frac{5}{2}}}[72\beta^4(\beta+1)^4\nu^9 + g_3(\upsilon,\nu)], \tag{2.9}$$

where $g_3(\upsilon,\nu)$ is a polynomial of degree at most 9, with

$$deg(g_3(0,\nu)) \leq 7.$$

By induction, one can also obtain:

Lemma 4. *The relation*

$$\left(\triangle^{II}\right)^l \nu = (-1)^l\left(\prod_{i=1}^{l} i(i+1)\right)\left(\frac{\beta^{2l-2}(\beta+1)^{2l-2}\nu^{4l-3} + Q_l(\upsilon,\nu)}{\gamma^l T^{\frac{3}{2}l-2}}\right),$$

is valid, and

$$\deg(Q_l(0,\nu)) \leq 4l-5.$$

Let now $\aleph$ be of finite II-type l. Then, there exist real numbers c_i, $i = 1,\dots,l$ such that

$$\left(\triangle^{II}\right)^{l+1}\boldsymbol{x} + c_1\left(\triangle^{II}\right)^{l}\boldsymbol{x} + \ldots + c_l\triangle^{II}\boldsymbol{x} = \boldsymbol{0}. \tag{2.10}$$

Applying Eq (2.10) to the coordinate functions $x_1 = \upsilon$ and $x_2 = \nu$ of the position vector (2.1) of $\aleph$, we obtain

$$\left(\triangle^{II}\right)^{l+1}\upsilon + c_1\left(\triangle^{II}\right)^{l}\upsilon + \cdots + c_l\triangle^{II}\upsilon = 0, \tag{2.11}$$

$$\left(\triangle^{II}\right)^{l+1}\nu + c_1\left(\triangle^{II}\right)^{l}\nu + \cdots + c_l\triangle^{II}\nu = 0. \tag{2.12}$$

From Lemma 2, relations (2.4)–(2.6), and (2.11), it follows that

$$\begin{aligned}
&(-1)^{l+1}\left(\prod_{i=1}^{l+1} i(i+1)\right)\left(\frac{\alpha^{2l}(\alpha+1)^{2l}\upsilon^{4l+1} + P_{l+1}(\upsilon,\nu)}{\gamma^{l+1}T^{\frac{3}{2}l-\frac{1}{2}}}\right)\\
&+ c_1(-1)^l\left(\prod_{i=1}^{l} i(i+1)\right)\left(\frac{\alpha^{2l-2}(\alpha+1)^{2l-2}\upsilon^{4l-3} + P_l(\upsilon,\nu)}{\gamma^l T^{\frac{3}{2}l-2}}\right) + \cdots\\
&+ c_{l-1}\frac{2}{\gamma^2 T}(6\alpha^2(\alpha+1)^2\upsilon^5 + P_2(\upsilon,\nu)) + c_l\frac{2\upsilon\sqrt{T}}{\gamma} = 0,
\end{aligned}$$

which can be written as

$$
\begin{aligned}
&(-1)^{l+1}\left(\prod_{i=1}^{l+1} i(i+1)\right)\left(\frac{\alpha^{2l}\,(\alpha+1)^{2l}\,\upsilon^{4l+1}}{\gamma^{l+1}}\right)+P_{l+1}(\upsilon,\nu)\\
&+c_1\,(-1)^{l}\left(\prod_{i=1}^{l} i(i+1)\right)\left(\frac{\alpha^{2l-2}\,(\alpha+1)^{2l-2}\,\upsilon^{4l-3}T^{\frac{3}{2}}P_l(\upsilon,\nu)}{\gamma^{l}}\right)+\cdots\\
&+c_{l-1}\frac{12\alpha^2(\alpha+1)^2\upsilon^5T^{\frac{3}{2}(l-1)}}{\gamma^2}+c_{l-1}T^{\frac{3}{2}(l-1)}P_2(\upsilon,\nu)+c_l\frac{2\upsilon T^{\frac{3}{2}l}}{\gamma}=0. \qquad (2.13)
\end{aligned}
$$

Inserting $\nu = 0$ in (2.13), we obtain a nontrivial polynomial in υ with constant coefficients. Therefore, the above equation can be rewritten as

$$
(-1)^{l+1}\left(\prod_{i=1}^{l+1} i(i+1)\right)\left(\frac{\alpha^{2l}\,(\alpha+1)^{2l}\,\upsilon^{4l+1}}{\gamma^{l+1}}\right)+P(\upsilon,\nu)=0, \qquad (2.14)
$$

with

$$
deg(P(\upsilon,\nu))\leq 4l.
$$

Since $\alpha \neq 0$, the relation (2.14) implies that α must be equal -1.

Following the same procedure for the second component x_2, by using relations (2.7)–(2.9), (2.12), and Lemma 4, we get

$$
(-1)^{l+1}\left(\prod_{i=1}^{l+1} i(i+1)\right)\left(\frac{\beta^{2l}\,(\beta+1)^{2l}\,\nu^{4l+1}}{\gamma^{l+1}}\right)+Q(\upsilon,\nu)=0, \qquad (2.15)
$$

where $Q(\upsilon,\nu)$ is a polynomial of degree at most $4l$. Putting $\upsilon = 0$, then Eq (2.15) is a nontrivial polynomial in ν with constant coefficients. However $\beta \neq 0$, so from (2.15) β equals -1. Therefore, $\aleph$ is a sphere.

Let

$$
\alpha=\beta=-1.
$$

Then,

$$
T=\gamma.
$$

Thus, relation (2.2) reduces to

$$
\triangle^{II}=-\frac{1}{\sqrt{\gamma}}\left[(\upsilon^2-\gamma)\frac{\partial^2}{\partial\upsilon^2}+2\upsilon\nu\frac{\partial^2}{\partial\upsilon\partial\nu}+(\nu^2-\gamma)\frac{\partial^2}{\partial\nu^2}+2\upsilon\frac{\partial}{\partial\upsilon}+2\nu\frac{\partial}{\partial\nu}\right].
$$

So, relations (2.4) and (2.7) become

$$
\triangle^{II}\upsilon=-\frac{2}{\sqrt{\gamma}}\upsilon.
$$

$$
\triangle^{II}\nu=-\frac{2}{\sqrt{\gamma}}\nu.
$$

For the third coordinate

$$x_3 = \sqrt{\omega} = \sqrt{\gamma - \upsilon^2 - \nu^2},$$

after simple calculation, we conclude

$$\triangle^{II} \sqrt{\omega} = -\frac{2}{\sqrt{\gamma}} \sqrt{\omega}.$$

Thus, we find that

$$\triangle^{II} \boldsymbol{x} = -\frac{2}{\sqrt{\gamma}} \boldsymbol{x}.$$

That is, spheres are the only quadric surfaces of the kind (2) of finite II-Chen type.

2.2. Quadrics of the third kind

A parametrization of a part of a quadric of this kind is

$$\boldsymbol{x}(\upsilon, \nu) = \Big(\upsilon, \nu, \frac{\alpha}{2}\upsilon^2 + \frac{\beta}{2}\nu^2\Big). \tag{2.16}$$

The matrix of the components of the first fundamental form of $\aleph$ is the following:

$$(g_{ij}) = \begin{bmatrix} 1 + \alpha^2\upsilon^2 & \alpha\beta\upsilon\nu \\ \alpha\beta\upsilon\nu & 1 + \beta^2\nu^2 \end{bmatrix}.$$

Denote

$$\mathfrak{g} := Det\Big(g_{ij}\Big) = 1 + (\alpha\,\upsilon)^2 + (\beta\,\nu)^2$$

The matrix of the components of the second fundamental form II is given as follows:

$$(b_{ij}) = \begin{bmatrix} \frac{\alpha}{\sqrt{\mathfrak{g}}} & 0 \\ 0 & \frac{\beta}{\sqrt{\mathfrak{g}}} \end{bmatrix}.$$

Thus, $\triangle^{II}$ of $\aleph$ becomes

$$\triangle^{II} = -\sqrt{\mathfrak{g}}\Big(\frac{1}{\alpha}\frac{\partial^2}{\partial\upsilon^2} + \frac{1}{\beta}\frac{\partial^2}{\partial\nu^2}\Big). \tag{2.17}$$

By applying the operator $\triangle^{II}$ to the components

$$x_1 = \upsilon \quad \text{and} \quad x_2 = \nu,$$

we get

$$\triangle^{II} x_1 = \triangle^{II} x_2 = 0.$$

For the third coordinate

$$x_3 = \frac{\alpha}{2}\upsilon^2 + \frac{\beta}{2}\nu^2,$$

we find

$$\triangle^{II}\Big(\frac{\alpha}{2}\upsilon^2 + \frac{\beta}{2}\nu^2\Big) = -2\sqrt{\mathfrak{g}}. \tag{2.18}$$

Applying Eq (2.17) for the relation (2.18), we find

$$(\triangle^{II})^2 x_3 = \frac{2(\alpha+\beta)}{\mathfrak{g}} + \frac{2\alpha\beta}{\mathfrak{g}} f_1(\upsilon,\nu), \tag{2.19}$$

where

$$f_1(\upsilon,\nu) = \alpha\upsilon^2 + \beta\nu^2,$$

with

$$deg(f_1) = 2.$$

On one hand, using (2.17), one can find:

Lemma 5. *For $n > 0$, we find*

$$\triangle^{II}(\mathfrak{g}^{-n}) = 2n(\alpha+\beta)\mathfrak{g}^{-n-\frac{1}{2}} - 4n(n+1)\mathfrak{g}^{-n-\frac{3}{2}}(\alpha^3\upsilon^2 + \beta^3\nu^2), \tag{2.20}$$

and, on the other hand, we have:

Lemma 6.

$$\begin{aligned}\triangle^{II}\Big(\frac{\alpha^r\upsilon^t + \beta^r\nu^t}{\mathfrak{g}^n}\Big) = &-\Big(\frac{t(t-1)(\alpha^{r-1}\upsilon^{t-2} + \beta^{r-1}\nu^{t-2})}{\mathfrak{g}^{n-\frac{1}{2}}}\Big)\\ &+\Big(\frac{2n(2t+1)(\alpha^{r+1}\upsilon^t + \beta^{r+1}\nu^t) + 2n\alpha\beta(\alpha^{r-1}\upsilon^t + \beta^{r-1}\nu^t)}{\mathfrak{g}^{n+\frac{1}{2}}}\Big)\\ &-\Big(\frac{4n(n+1)(\alpha^{r+3}\upsilon^{t+2} + \beta^{r+3}\nu^{t+2})}{\mathfrak{g}^{n+\frac{3}{2}}}\Big)\\ &-\Big(\frac{4n(n+1)\alpha^3\beta^3\upsilon^2\nu^2(\alpha^{r-3}\upsilon^{t-2} + \beta^{r-3}\nu^{t-2})}{\mathfrak{g}^{n+\frac{3}{2}}}\Big).\end{aligned} \tag{2.21}$$

From Lemma 6, we obtain that

$$\triangle^{II}\Big(\frac{\alpha^r\upsilon^t + \beta^r\nu^t}{\mathfrak{g}^n}\Big) = -\Big(\frac{1}{\mathfrak{g}^{n+\frac{3}{2}}}\Big)G(\upsilon,\nu), \tag{2.22}$$

where $deg(G(\upsilon,\nu))$ is at most $t+2$.

Taking into account (2.20), and (2.22), we get

$$(\triangle^{II})^3 x_3 = \frac{4(\alpha+\beta)^2}{\mathfrak{g}^{\frac{3}{2}}} + \frac{1}{\mathfrak{g}^{\frac{5}{2}}} G_2(\upsilon,\nu), \tag{2.23}$$

where $G_2(\upsilon,\nu)$ is a polynomial in υ,ν of degree at most 4. Similarly, we get

$$(\triangle^{II})^4 x_3 = \frac{12(\alpha+\beta)^3}{\mathfrak{g}^2} + \frac{1}{\mathfrak{g}^3} G_3(\upsilon,\nu), \tag{2.24}$$

where $G_3(\upsilon,\nu)$ is a polynomial in υ,ν of degree at most 6. In general, we have

Lemma 7. *The relation*

$$\left(\triangle^{II}\right)^{l} x_3 = \frac{2(l-1)!(\alpha+\beta)^{l-1}}{\mathfrak{g}^{\frac{1}{2}l}} + \frac{1}{\mathfrak{g}^{1+\frac{1}{2}l}} G_{l-1}(u,v),$$

holds true for $l > 2$, with

$$\deg(G_{l-1}(u,v)) \leq 2l-2.$$

On account of relation (1.1) applied to the component x_3, we have

$$\left(\triangle^{II}\right)^{l+1} x_3 + c_1 \left(\triangle^{II}\right)^{l} x_3 + \cdots + c_l \triangle^{II} x_3 = 0.$$

From (2.18), (2.19), (2.23), (2.24), and Lemma 7, we get

$$\frac{2(l)!(\alpha+\beta)^{l}}{\mathfrak{g}^{\frac{1}{2}(l+1)}} + \frac{1}{\mathfrak{g}^{\frac{3}{2}+\frac{1}{2}l}} G_l(u,v) + \frac{2c_1(l-1)!(\alpha+\beta)^{l-1}}{\mathfrak{g}^{\frac{1}{2}l}} + \frac{c_1}{\mathfrak{g}^{1+\frac{1}{2}l}} G_{l-1}(u,v) + \cdots + 2c_l\sqrt{\mathfrak{g}} = 0$$

or

$$2(l)!(\alpha+\beta)^{l} + 2c_1(l-1)!(\alpha+\beta)^{l-1}\mathfrak{g}^{\frac{1}{2}} + 2c_2(l-2)!(\alpha+\beta)^{l-2}\mathfrak{g}^{\frac{3}{2}} + \cdots + 2c_l\mathfrak{g}^{1+\frac{1}{2}l} + \frac{1}{\mathfrak{g}} G(u,v) = 0, \quad (2.25)$$

where

$$\deg(G(u,v)) \leq 2l.$$

Relation (2.25) must hold true for all (υ, v). This is clearly impossible since the first term of (2.25), which is the constant term of (2.25), must equal 0, something that cannot be satisfied since $\alpha, \beta > 0$.

3. Conclusions

This research article was divided into three sections, where after the introduction, the needed definitions and relations regarding this interesting field of study were given. Then, a formula for the Laplace operator corresponding to the second fundamental form II was proved once for the position vector and another for the Gauss map of a surface. Finally, we classify the quadric surfaces of finite Chen type regarding the second fundamental form. An interesting study can be drawn if this type of study can be applied to other classes of surfaces that have not been investigated yet, such as spiral surfaces, or tubular surfaces.

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Conflict of interest

The author declares that he has no conflict of interest.

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