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*Research article*

# A novel approach to Lyapunov stability of Caputo fractional dynamic equations on time scale using a new generalized derivative

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**Abstract:** In this work, we introduced a generalized concept of Caputo fractional derivatives, specifically the Caputo fractional delta derivative (FrΔD) and Caputo fractional delta Dini derivative (FrΔDiD) of order $\alpha \in (0, 1)$, on an arbitrary time domain $\mathbb{T}$, which was a closed subset of $\mathbb{R}$. By bridging the gap between discrete and continuous time domains, this unified framework enabled a more thorough approach to stability and asymptotic stability analysis on time scales. A key contribution of this work was the new definition of the Caputo FrΔD for a Lyapunov function, which served as the basis for establishing comparison results and stability criteria for Caputo fractional dynamic equations. The proposed framework extended beyond the limitations of traditional integer-order calculus, offering a more flexible and generalizable tool for researchers working with dynamic systems. The inclusion of fractional orders enabled the modeling of more complex dynamics that occur in real-world systems, particularly those involving both continuous and discrete time components. The results presented in this work contributed to the broader understanding of fractional calculus on time scales, enriching the theoretical foundation of dynamic systems analysis. Illustrative examples were included to demonstrate the effectiveness, relevance, and practical applicability of the established stability and asymptotic stability results. These examples highlighted the advantage of our definition of fractional-order derivative over integer-order approaches in capturing the intricacies of dynamic behavior.

**Keywords:** stability; Caputo derivative; Lyapunov function; fractional dynamic equation; time scale
**Mathematics Subject Classification:** 34A08, 34A34, 34D20, 34N05

## 1. Introduction

Research has shown that fractional calculus is highly effective for capturing complex dynamics and accurately modeling real-life problems [9, 40]. Fractional calculus extends integer-order derivatives and integrals, also known as differentiation and integration, to arbitrary orders [32], proving to be a powerful tool for understanding intricate systems. Numerous researchers have utilized the Lyapunov second method, or Lyapunov direct method, to analyze the qualitative and quantitative characteristics of dynamical systems. The Lyapunov direct method is particularly beneficial because it does not require knowledge of the differential equation's solution [35]. In [1–4], various fractional derivatives (FrDs) of Lyapunov functions (LF) were used in stability investigations, including the Caputo FrD, Dini FrD, and Caputo fractional Dini derivative. The most preferred is the Caputo FrD:

$$ {}^{C}_{t_0}D^{\alpha}_{t}\mathcal{L}(t,\nu(t)) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\frac{d}{ds}(\mathcal{L}(s,\nu(s)))ds, \quad t\in[t_0,T),\ \alpha\in(0,1). $$

This derivative is easier to handle and more applicable, though it requires the function $\mathcal{L}(t,\nu(t)) \in [\mathbb{R}\times\mathbb{R}^n,\mathbb{R}_+]$ to be continuously differentiable, which can be challenging. Other LF derivatives do not have this limitation, allowing sufficient conditions for these derivatives using a continuous LF that does not need to be continuously differentiable. The Dini FrD

$$ D^{\alpha}_{+}\mathcal{L}(t,\nu;t_0) = \limsup_{\kappa\to 0+}\frac{1}{\kappa^{\alpha}}\Big\{\mathcal{L}(t,\nu) - \sum_{\mathfrak{r}=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{\mathfrak{r}+1}\,{}^{\alpha}C_{\mathfrak{r}}\mathcal{L}(t-\mathfrak{r}\kappa,\nu-\kappa^{\alpha}\mathfrak{f}(t,\nu))\Big\}, $$

where $\mathcal{L}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_+$, is continuous, $\mathfrak{f}\in C[\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n]$, $\kappa$ is a positive number and ${}^{\alpha}C_{\mathfrak{r}} = \frac{\alpha(\alpha-1)\ldots(\alpha-\mathfrak{r}+1)}{\mathfrak{r}!}$, maintains the concept of FrDs, depending on both the present point $(t)$ and the initial point $(t_0)$ but not on the initial state $\mathcal{L}(t_0,\nu_0)$. This led to a more suitable definition

$$ \begin{aligned} {}^{C}_{t_0}D^{\alpha}_{+}\mathcal{L}(t,\nu(t)) &= \limsup_{\kappa\to 0+}\frac{1}{\kappa^{\alpha}}\Big\{\mathcal{L}(t,\nu(t)) - \mathcal{L}(t_0,\nu(t_0)) - \sum_{\mathfrak{r}=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{\mathfrak{r}+1}\,{}^{\alpha}C_{\mathfrak{r}}[\mathcal{L}(t-\mathfrak{r}\kappa,\nu(t) \\ &\quad -\kappa^{\alpha}\mathfrak{f}(t,\nu(t))) - \mathcal{L}(t_0,\nu(t_0))]\Big\}, \end{aligned} \tag{1.1} $$

to be considered.

Several forms of stability for Caputo fractional differential equations (FrDE) with continuous domain have been investigated using this Caputo fractional Dini derivative (1.1) [1]. As noted in [7] and [25], a more holistic examination of stability can be achieved across time domains. In [1–3, 12, 13, 18], stability results were obtained for continuous time, ignoring discrete details, while in [8,23,26,28], discrete domains were considered. However, some systems undergo smooth and abrupt changes almost simultaneously, with multiple time scales or frequencies. Modeling such phenomena is better represented as dynamic systems that include continuous and discrete times, known as time scales or measure chains, denoted by $\mathbb{T}$ [14,17]. Dynamic equations on time scales, defined on discrete, continuous (connected), or combined domains, provide a broader analysis of difference and differential systems [16]. In order to extend stability properties from the classical to the fractional-order sense, we focus on the Lyapunov stability analysis of the Caputo fractional dynamic equations on time scale

(FrDET) using a novel definition for the delta derivative of a LF, known as the Caputo FrΔD on a time scale.

The study of fractional dynamic systems on time scales is recent and ongoing due to its advantages in modeling, mechanics, and population dynamics (see [39]). Recent literature on fractional dynamic systems on time scales focuses on the existence and uniqueness of solutions of FrDET, with Caputo-type derivatives being given more recognition ( [10, 11, 20, 29, 30, 33]). However, in [24], the stability of fractional dynamic systems on time scales with applications to population dynamics were examined. Although the stability results were interesting, they applied Hyers-Ulam type stability, which is restrictive compared to Lyapunov stability, which has a broader application scope ( [19, 27, 37]). In [36, 38], methods for solving discrete time scales in Caputo FrDET were developed.

Building on the existence and uniqueness results for Caputo-type FrDET established in [4], we extend the stability results in [21] to fractional order and the Lyapunov stability results for Caputo FrDE in [1] to a more generalized domain (time scale). This unification of continuous and discrete calculus gives rise to fractional difference equations (FrDfE) in discrete time, FrDE in continuous time, and fractional calculus on time scales in combined continuous and discrete time.

For this work, we consider the Caputo fractional dynamic system of order $\alpha$, with $0 < \alpha < 1$

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}\nu^{\Delta} &= \Upsilon(t,\nu),\ t \in \mathbb{T}, \\ \nu(t_0) &= \nu_0,\ t_0 \geq 0, \end{aligned} \tag{1.2}$$

where $\Upsilon \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $\Upsilon(t, 0) \equiv 0$, and ${}^{C\mathbb{T}}D^{\alpha}\nu^{\Delta}$ is the Caputo FrΔD of $\nu \in \mathbb{R}^n$ of order $\alpha$ with respect to $t \in \mathbb{T}$. Let $\nu(t) = \nu(t, t_0, \nu_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$ be a solution of (1.2), assuming the solution exists and is unique ( [4, 24]), this work aims to investigate the stability and asymptotic stability of the system (1.2).

To do this, we shall use the dynamic system of the form

$$ {}^{C\mathbb{T}}D^{\alpha}\chi^{\Delta} = \Xi(t,\chi),\ \chi(t_0) = \chi_0 \geq 0, \tag{1.3}$$

where $\chi \in \mathbb{R}_+$, $\Xi : \mathbb{T}\times\mathbb{R}_+ \to \mathbb{R}_+$ and $\Xi(t, 0) \equiv 0$. System (1.3) is called the comparison system. For this work, we will assume that the system (1.3) with $\chi(t_0) = \chi_0$ has a solution $\chi(t) = \chi(t; t_0, \chi_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}_+]$ which is unique ( [4]).

In the next section (Section 2), we examine key terminologies, remarks, and a fundamental lemma laying the groundwork for the subsequent contributions. New definitions and crucial remarks are also introduced. In Section 3, we present Lemmas 3.1 and 3.2, which are essential components for proving the major results. In Section 4, practical examples demonstrate the significance and applicability of the newly introduced definitions and the established stability and asymptotic stability theorem. In Section 5, we provide the conclusion, summarizing the major findings and implications of the investigation.

## 2. Preliminaries, definitions, and notations

The foundational principles of dynamic equations, encompassing derivatives and integrals, can be extended to non-integer orders by applying fractional calculus. This generalization to non-integer orders becomes particularly relevant when exploring dynamic equations on time scales, allowing for a versatile and comprehensive analysis of system behavior across continuous and discrete time domains. See [5, 24, 31, 34]. In this section, we set the foundation, introduce notations, and definitions.

**Definition 2.1.** *[7] For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

*and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

*The following conditions hold:*

*(i) If $\sigma(t) > t$, then $t$ is termed right-scattered (rs).*
*(ii) If $\rho(t) < t$, then $t$ is termed left-scattered (ls).*
*(iii) If $t < \max \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense (rd).*
*(iv) If $t > \min \mathbb{T}$ and $\rho(t) = t$, then $t$ is called left-dense (ld).*

**Definition 2.2.** *[7] The graininess function $\mu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined by*

$$\mu(t) = \sigma(t) - t,$$

*where $\sigma(t)$ is the forward jump operator.*

**Definition 2.3.** *[7] Let $\mathfrak{p} : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The delta derivative $\mathfrak{p}^\Delta$ also known as the Hilger derivative is defined as:*

$$\mathfrak{p}^\Delta(t) = \lim_{s \to t} \frac{\mathfrak{p}(\sigma(t)) - \mathfrak{p}(s)}{\sigma(t) - s}, \quad s \neq \sigma(t),$$

*provided the limit exist.*

*If $t$ is rd, the delta derivative of $\mathfrak{p}$, becomes*

$$\mathfrak{p}^\Delta(t) = \lim_{s \to t} \frac{\mathfrak{p}(t) - \mathfrak{p}(s)}{t - s},$$

*and if $t$ is rs, the Delta derivative becomes*

$$\mathfrak{p}^\Delta(t) = \frac{\mathfrak{p}^\sigma(t) - \mathfrak{p}(t)}{\mu(t)},$$

*where $\mathfrak{p}^\sigma$ denotes $\mathfrak{p}(\sigma(t))$.*

**Definition 2.4.** *[15] $\mathfrak{p} : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it remains continuous at each rd point within $\mathbb{T}$ and possesses finite left-hand limits at ld points in $\mathbb{T}$. The collection of all such rd-continuous functions is denoted as*

$$C_{rd} = C_{rd}(\mathbb{T}).$$

**Definition 2.5.** *[7] Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{T}$ and $\mathfrak{p} \in C_{rd}$, then, the integration on the time scale $\mathbb{T}$ is defined as:*

*(i)*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{p}(t)\Delta t = \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{p}(t)dt,$$

*if* $\mathbb{T} = \mathbb{R}$.

*(ii) If the interval* $[\mathfrak{a}, \mathfrak{b}]_{\mathbb{T}}$ *contains only isolated points, then*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{p}(t)\Delta t = \begin{cases} \sum_{t\in[\mathfrak{a},\mathfrak{b})} \mu(t)\mathfrak{p}(t) & \text{if} \quad \mathfrak{a} < \mathfrak{b} \\ 0 & \text{if} \quad \mathfrak{a} = \mathfrak{b} \\ -\sum_{t\in[\mathfrak{a},\mathfrak{b})} \mu(t)\mathfrak{p}(t) & \text{if} \quad \mathfrak{a} > \mathfrak{b}. \end{cases}$$

*(iii) If there exists a point* $\sigma(t) > t$, *then*

$$\int_{t}^{\sigma(t)} \mathfrak{p}(s)\Delta s = \mu(t)\mathfrak{p}(t).$$

**Definition 2.6.** *[15] A function* $\phi : [0, r] \to [0, \infty)$ *is of class* $\mathcal{K}$ *if it is continuous, and strictly increasing on* $[0, r]$ *with* $\phi(0) = 0$.

**Definition 2.7.** *[15]* $\mathcal{L} \in C[\mathbb{R}^n, \mathbb{R}]$ *with* $\mathcal{L}(0) = 0$ *is called positive definite(negative definite) on the domain D if* $\exists$ *a function* $\phi \in \mathcal{K} : \phi(|\chi|) \leq \mathcal{L}(\chi)$ $(\phi(|\chi|) \leq -\mathcal{L}(\chi))$ *for* $\chi \in D$.

**Definition 2.8.** *[15]* $\mathcal{L} \in C[\mathbb{R}^n, \mathbb{R}]$ *with* $\mathcal{L}(0) = 0$ *is called positive semidefinite (negative semi-definite) on D if* $\mathcal{L}(\chi) \geq 0$ $(\mathcal{L}(\chi) \leq 0)$ $\forall\, \chi \in D$ *and it can also vanish for some* $\chi \neq 0$.

**Definition 2.9.** *[21] Assume* $\mathcal{L} \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $\Upsilon \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $\mu(t)$ *is the graininess function, then the dini derivative of* $\mathcal{L}(t, \nu)$ *is defined as:*

$$D_{-}\mathcal{L}^{\Delta}(t, \nu) = \liminf_{\mu(t)\to 0} \frac{\mathcal{L}(t, \nu) - \mathcal{L}(t - \mu(t), \nu - \mu(t)\Upsilon(t, \nu))}{\mu(t)}, \tag{2.1}$$

$$D^{+}\mathcal{L}^{\Delta}(t, \nu) = \limsup_{\mu(t)\to 0} \frac{\mathcal{L}(t + \mu(t), \nu + \mu(t)\Upsilon(t, \nu)) - \mathcal{L}(t, \nu)}{\mu(t)}. \tag{2.2}$$

*If* $\mathcal{L}$ *is differentiable, then* $D_{-}\mathcal{L}^{\Delta}(t, \nu) = D^{+}\mathcal{L}^{\Delta}(t, \nu) = \mathcal{L}^{\Delta}(t, \nu)$.

**Definition 2.10.** *[6] Consider* $\alpha \in (0, 1)$, *with* $[\mathfrak{a}, \mathfrak{b}]$ *being an interval on* $\mathbb{T}$, *and let* $\Xi$ *be a function that is integrable over* $[\mathfrak{a}, \mathfrak{b}]$. *The fractional integral of* $\Xi$, *w.r.t the order* $\alpha$, *is expressed as follows:*

$${}_{\mathfrak{a}}^{\mathbb{T}}I_t^{\alpha}\Xi^{\Delta}(t) = \int_{\mathfrak{a}}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}\Xi(s)\Delta s.$$

**Definition 2.11.** *[4] Let* $t \in \mathbb{T}$, $0 < \alpha < 1$, *and* $\Xi : \mathbb{T} \to \mathbb{R}$. *The Caputo FrD of order* $\alpha$ *of* $\Xi$ *is expressed as follows:*

$${}_{\mathfrak{a}}^{\mathbb{T}}D_t^{\alpha}\Xi^{\Delta}(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{\mathfrak{a}}^{t} (t - s)^{-\alpha}\Xi^{\Delta^n}(s)\Delta s.$$

**Lemma 2.1.** *[22] Let* $\mathbb{T}$ *represent a time scale with a minimal element* $t_0 \geq 0$. *Assume that for each* $t \in \mathbb{T}$, *there is a proposition* $\mathbf{S}(t)$ *such that the following conditions are satisfied:*

(i) $\mathbf{S}(t_0)$ *holds;*
(ii) *if $t$ is rs and $\mathbf{S}(t)$ holds, then $\mathbf{S}(\sigma(t))$ also holds;*
(iii) *for any rd $t$, there is a neighborhood $\mathcal{U}$ such that if $\mathbf{S}(t)$ is true, then $\mathbf{S}(t^*)$ is true for every $t^* \in \mathcal{U}$ with $t^* \geq t$;*
(iv) *for left-dense $t$, if $\mathbf{S}(t^*)$ holds for all $t^* \in [t_0, t)$, then $\mathbf{S}(t)$ also holds.*

*Therefore, $\mathbf{S}(t)$ is true for all $t \in \mathbb{T}$.*

**Remark 2.1.** *When $\mathbb{T} = \mathbb{N}$, Lemma 2.1 simplifies to the principle of mathematical induction. Specifically:*

(1) $\mathbf{S}(t_0)$ *being true corresponds to the statement holding for $n = 1$;*
(2) $\mathbf{S}(t) \Rightarrow \mathbf{S}(\sigma(t))$ *corresponds to: if the statement holds for $n = k$, then it also holds for $n = k + 1$.*

Now, we give the following definitions and remarks.

**Definition 2.12.** *Let $h \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$, the G-L FrΔD is given by*

$$ {}^{GL\mathbb{T}}D^{\alpha}_{0} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{2.3} $$

*and the G-L FrΔDiD is given by*

$$ {}^{GL\mathbb{T}}D^{\alpha}_{0^+} h^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{2.4} $$

*where $0 < \alpha < 1$, ${}^{\alpha}C_r = \frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!}$, and $[\frac{(t-t_0)}{\mu}]$ represents the integer part of the fraction $\frac{(t-t_0)}{\mu}$.*

*Observe that if the domain is $\mathbb{R}$, then (2.4) becomes*

$$ {}^{GL\mathbb{T}}D^{\alpha}_{0^+} h^{\Delta}(t) = \limsup_{\kappa \to 0+} \frac{1}{\kappa^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\kappa}]} (-1)^{r\alpha} C_r [h(t - r\kappa)], \quad t \geq t_0. $$

**Remark 2.2.** *It is necessary to note that the relationship between the Caputo FrΔD and the G-L FrΔD is given by*

$$ {}^{C\mathbb{T}}D^{\alpha}_{0} h^{\Delta}(t) = {}^{GL\mathbb{T}}D^{\alpha}_{0} [h(t) - h(t_0)]^{\Delta}, \tag{2.5} $$

*substituting (2.3) into (3.11) we have that the Caputo FrΔD becomes*

$$ {}^{C\mathbb{T}}D^{\alpha}_{0} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)], \quad t \geq t_0, $$

$$ {}^{C\mathbb{T}}D^{\alpha}_{0} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \Big\{ h(\sigma(t)) - h(t_0) + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \Big\}, \tag{2.6} $$

*and the Caputo FrΔDiD becomes*

$$^{C\mathbb{T}}D^{\alpha}_{0^+}h^{\Delta}(t) = \limsup_{\mu\to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t) - r\mu) - h(t_0)], \quad t \geq t_0, \tag{2.7}$$

*which is equivalent to*

$$^{C\mathbb{T}}D^{\alpha}_{0^+}h^{\Delta}(t) = \limsup_{\mu\to 0+} \frac{1}{\mu^{\alpha}} \Big\{ h(\sigma(t)) - h(t_0) + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t) - r\mu) - h(t_0)] \Big\}. \tag{2.8}$$

For notation simplicity, we represent the Caputo FrΔD of order $\alpha$ as $^{C\mathbb{T}}D^{\alpha}$ and the Caputo FrΔDiD of order $\alpha$ as $^{C\mathbb{T}}D^{\alpha}_{+}$.

**Definition 2.13.** *The trivial solution of (1.2) is called:*

- $S_1$ *Stable if for every* $\epsilon > 0$ *and* $t_0 \in \mathbb{T}$, $\exists\ \delta = \delta(\epsilon, t_0) > 0$ : *for any* $\nu_0 \in \mathbb{R}^n$, $\|\nu_0\| \leq \delta \implies \|\nu(t; t_0, \nu_0)\| < \epsilon$ *for* $t \geq t_0$.
- $S_2$ *Asymptotically stable if it is stable and locally attractive, that is* $\exists$ *a* $\delta_0 = \delta_0(t_0) > 0 : \|\nu(t_0)\| < \delta_0 \implies \lim_{t\to\infty} \|\nu(t)\| = 0$ *for* $t_0, t \in \mathbb{T}$.

We now give the definition of the derivative of LF using the FrΔDiD of $h(t)$, as provided in Eq (2.7).

**Definition 2.14.** *The Caputo FrΔDiD of the Lyapunov function* $\mathcal{L}(t, \nu) \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$ *(which is locally Lipschitzian with respect to its second argument and* $\mathcal{L}(t, 0) \equiv 0$*) along the trajectories of solutions of system (1.2) is defined as:*

$$^{C\mathbb{T}}D^{\alpha}_{+}\mathcal{L}^{\Delta}(t, \nu) = \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}} \Big[ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^{r} ({}^{\alpha}C_r)[\mathcal{L}(\sigma(t) - r\mu, \nu(\sigma(t)) - \mu^{\alpha}\Upsilon(t, \nu(t))) - \mathcal{L}(t_0, \nu_0)] \Big],$$

*and can be expanded as*

$$\begin{aligned} ^{C\mathbb{T}}D^{\alpha}_{+}\mathcal{L}^{\Delta}(t, \nu) &= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}} \Big\{ \mathcal{L}(\sigma(t), \nu(\sigma(t)) - \mathcal{L}(t_0, \nu_0) \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[\mathcal{L}(\sigma(t) - r\mu, \nu(\sigma(t)) - \mu^{\alpha}\Upsilon(t, \nu(t))) - \mathcal{L}(t_0, \nu_0)] \Big\}, \end{aligned} \tag{2.9}$$

*where* $t \in \mathbb{T}$, *and* $\nu, \nu_0 \in \mathbb{R}^n$, $\mu = \sigma(t) - t$ *and* $\nu(\sigma(t)) - \mu^{\alpha}\Upsilon(t, \nu) \in \mathbb{R}^n$.

If $\mathbb{T}$ represents a discrete time scale and $\mathcal{L}(t, \nu(t))$ remains continuous at $t$, the Caputo FrΔDiD of the LF for discrete times is expressed as:

$$^{C\mathbb{T}}D^{\alpha}_{+}\mathcal{L}^{\Delta}(t, \nu) = \frac{1}{\mu^{\alpha}} \Big[ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^{r} ({}^{\alpha}C_r)(\mathcal{L}(\sigma(t), \nu(\sigma(t))) - \mathcal{L}(t_0, \nu_0)) \Big], \tag{2.10}$$

and if $\mathbb{T}$ is continuous, that is $\mathbb{T} = \mathbb{R}$, and $\mathcal{L}(t, \nu(t))$ is continuous at $t$, we have that

$$\begin{aligned} {}^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t,\nu) &= \limsup_{\kappa\to 0^{+}} \frac{1}{\kappa^{\alpha}}\Big\{\mathcal{L}(t,\nu(t)) - \mathcal{L}(t_0,\nu_0) \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}({}^{\alpha}C_r)[\mathcal{L}(t-r\kappa,\nu(t)) - \kappa^{\alpha}\Upsilon(t,\nu(t)) - \mathcal{L}(t_0,\nu_0)]\Big\}. \end{aligned} \tag{2.11}$$

Notice that (2.11) is the same in [1] where $\kappa > 0$.

Given that $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ where $\alpha \in (0,1)$, and $\lim_{\mu\to 0^{+}}[\frac{(t-t_0)}{\mu}] = \infty$ therefore it is evident that

$$\lim_{\mu\to 0^{+}} \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha}C_r = -1, \tag{2.12}$$

Also, based on Eq (2.7) and given that the Caputo and R-L definitions are equivalent when $h(t_0) = 0$ (see [1]), we can conclude that:

$${}^{C\mathbb{T}}D_{+}^{\alpha}h^{\Delta}(t) = {}^{RL\mathbb{T}}D_{+}^{\alpha}h^{\Delta}(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)], \quad t \geq t_0, \tag{2.13}$$

setting $h(\sigma(t) - r\mu = 1$ we obtain

$${}^{C\mathbb{T}}D_{+}^{\alpha}h^{\Delta}(t) = \limsup_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r = {}^{RL\mathbb{T}}D^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \geq t_0. \tag{2.14}$$

## 3. Results

**Lemma 3.1.** *Assume* $\mathfrak{h}$ *and* $\mathfrak{m} \in C_{rd}(\mathbb{T},\mathbb{R})$. *Suppose* $\exists\, t_1 > t_0$, *where* $t_1 \in \mathbb{T}$, : $\mathfrak{h}(t_1) = \mathfrak{m}(t_1)$ *and* $\mathfrak{h}(t) < \mathfrak{m}(t)$ *for* $t_0 \leq t < t_1$. *Then, if the Caputo Fr∆DiD of* $\mathfrak{h}$ *and* $\mathfrak{m}$ *exist at* $t_1$, *the inequality* ${}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{h}^{\Delta}(t_1) > {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{m}^{\Delta}(t_1)$ *holds.*

*Proof.* Applying (2.7), we have

$${}^{C\mathbb{T}}D_{+}^{\alpha}(\mathfrak{h}(t)-\mathfrak{m}(t))^{\Delta} = \limsup_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r\alpha}C_r[\mathfrak{h}(\sigma(t)-r\mu) - \mathfrak{m}(\sigma(t)-r\mu)] - [\mathfrak{h}(t_0) - \mathfrak{m}(t_0)]\Big\},$$

$${}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{h}^{\Delta}(t) - {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{m}^{\Delta}(t) = \limsup_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r\alpha}C_r[\mathfrak{h}(\sigma(t)-r\mu) - \mathfrak{m}(\sigma(t)-r\mu)] - [\mathfrak{h}(t_0) - \mathfrak{m}(t_0)]\Big\},$$

at $t_1$, we have

$${}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{h}^{\Delta}(t_1) = -\limsup_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=0}^{[\frac{t_1-t_0}{\mu}]}(-1)^{r\alpha}C_r[\mathfrak{h}(t_0) - \mathfrak{m}(t_0)]\Big\} + {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{m}^{\Delta}(t_1). \tag{3.1}$$

Applying (2.14) to (3.1), we have

$$ {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{h}^{\Delta}(t_1) = -\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[\mathfrak{h}(t_0)-\mathfrak{m}(t_0)] + {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{m}^{\Delta}(t_1), $$

however, based on the Lemma's statement, we know that

$$ \begin{aligned} &\mathfrak{h}(t) < \mathfrak{m}(t), \text{ for } t_0 \le t < t_1, \\ \Longrightarrow\ &\mathfrak{h}(t) - \mathfrak{m}(t) < 0, \text{ for } t_0 \le t < t_1, \end{aligned} $$

then, we obtain

$$ -\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[\mathfrak{h}(t_0)-\mathfrak{m}(t_0)] > 0, $$

implying

$$ {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{h}^{\Delta}(t_1) > {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{m}^{\Delta}(t_1). $$

**Lemma 3.2.** *Assume that:*

*(1)* $\nu^*(t;t_0,\nu_0)$*, with* $\nu^* \in C_{rd}^{\alpha}([t_0,T]_{\mathbb{T}},\mathbb{R}^n)$*, represents a solution to the system* (1.2)*.*
*(2)* $\mathcal{L}(t,\nu^*) \in C_{rd}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$ *and for any* $t \in [t_0,\infty)_{\mathbb{T}},\ \nu^* \in \mathbb{R}^n$*,*

$$ {}^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t,\nu^*) \le -\phi(\|\nu^*(t)\|), \tag{3.2} $$

*where* $\phi \in \mathcal{K}$*.*
*Then for* $t \in [t_0,T]_{\mathbb{T}}$*, the inequality*

$$ \mathcal{L}(t,\nu^*(t)) \le \mathcal{L}(t_0,\nu_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|\nu^*(t)\|)\Delta s $$

*holds.*

*Proof.* Let

$$ \begin{aligned} \mathfrak{p}(t) &= \mathcal{L}(t,\nu^*(t)), \\ \text{with}\quad \mathfrak{p}(t_0) &= \mathcal{L}(t_0,\nu_0), \end{aligned} \tag{3.3} $$

and

$$ \mathfrak{W}(t) = \phi(\|\nu^*(t)\|). \tag{3.4} $$

Then, from (3.2) we have

$$ {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{p}^{\Delta}(t) = {}^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t,\nu^*(t)) \le -\phi(\|\nu^*(t)\|) = -\mathfrak{W}(t), \quad \text{for } t \in [t_0,T]_{\mathbb{T}}. \tag{3.5} $$

Consider the system

$$ {}^{C\mathbb{T}}D^{\alpha}\chi^{\Delta}(t) = -\mathfrak{W}(t), \quad \chi(t_0) = \mathfrak{p}_{\omega}(t_0), \quad \text{where } \mathfrak{p}_{\omega}(t_0) = \mathfrak{p}(t_0) + \omega. \tag{3.6} $$

A solution $\chi(t) = \chi(t, t_0, \chi_0)$ of (3.6) will also satisfy the Volterra delta integral equation

$$\chi(t) = \mathfrak{p}_\omega(t_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathfrak{W}(s) \Delta s. \tag{3.7}$$

We claim that

$$\mathfrak{p}(t) < \chi(t), \quad t \in [t_0, T]_{\mathbb{T}}. \tag{3.8}$$

In the event that this claim is untrue, there is a time $t_1 \in (t_0, T]_{\mathbb{T}}$:

$$\mathfrak{p}(t_1) = \chi(t_1) \quad \text{and} \quad \mathfrak{p}(t) < \chi(t) \quad \text{for} \quad t \in [t_0, t_1)_{\mathbb{T}}. \tag{3.9}$$

Lemma 3.1 is applied to (3.9) to obtain

$$\begin{aligned} {}^{C\mathbb{T}}D_+^\alpha \mathfrak{p}^\Delta(t_1) >^{C\mathbb{T}} D_+^\alpha \chi^\Delta(t_1) =^{C\mathbb{T}} D^\alpha \chi^\Delta(t_1) = -\mathfrak{W}(t_1), \\ \Longrightarrow \; {}^{C\mathbb{T}}D_+^\alpha \mathfrak{p}^\Delta(t_1) > -\mathfrak{W}(t_1). \end{aligned} \tag{3.10}$$

Thus, based on (3.10) and for $t \in [t_0, T]_{\mathbb{T}}$, we obtain:

$${}^{C\mathbb{T}}D_+^\alpha \mathfrak{p}^\Delta(t) > -\mathfrak{W}(t). \tag{3.11}$$

Clearly, (3.11) contradicts (3.5), so we conclude that the claim in (3.8) holds.

Combining (3.3),(3.4),(3.7) and (3.8) we get

$$\mathcal{L}(t, \nu^*(t)) \;=\; \mathfrak{p}(t) < \mathfrak{p}_\omega(t_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \phi(\|\nu^*(s)\|) \Delta s, \tag{3.12}$$

$\Longrightarrow$

$$\mathcal{L}(t, \nu^*(t)) \;\leq\; \mathcal{L}(t_0, \nu_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \phi(\|\nu^*(s)\|) \Delta s. \tag{3.13}$$

If $\mathbb{T}$ contains right scattered points, and $\mathcal{L}(t, \nu^*)$ is continuous, then (3.13) becomes

$$\mathcal{L}(t, \nu^*(t)) \leq \mathcal{L}(t_0, \nu_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\sigma(t)} (t-s)^{\alpha-1} \phi(\|\nu^*(s)\|) ds.$$

**Theorem 3.1.** *Assume that:*

($i$) $\Xi \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $\Xi(t, \chi)\mu^\alpha$ *is non-decreasing in* $\chi$.

($ii$) $\mathcal{L} \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$ *be locally Lipschitz in the second variable such that*

$${}^{C\mathbb{T}}D_+^\alpha \mathcal{L}^\Delta(t, \nu) \leq \Xi(t, \mathcal{L}(t, \nu)), (t, \nu) \in \mathbb{T} \times \mathbb{R}^n. \tag{3.14}$$

($iii$) $z(t) = z(t; t_0, \chi_0)$ *existing on* $\mathbb{T}$ *is the maximal solution of (1.3).*

*Then,*

$$\mathcal{L}(t, \nu(t)) \leq z(t), \quad t \geq t_0, \tag{3.15}$$

*provided that*

$$\mathcal{L}(t_0, \nu_0) \leq \chi_0, \tag{3.16}$$

*where* $\nu(t) = \nu(t; t_0, \nu_0)$ *is any solution of* (1.2), $t \in \mathbb{T}, \quad t \geq t_0$.

*Proof.* Utilizing the principle of induction as outlined in Lemma 2.1 for the assertion

$$\mathbf{S(t)} : \mathcal{L}(t, \nu(t)) \leq z(t), \quad t \in \mathbb{T},\ t \geq t_0,$$

(i) $\mathbf{S(t_0)}$ is true since $\mathcal{L}(t_0, \nu_0) \leq \chi_0$;
(ii) Let $t$ be rs and $\mathbf{S(t)}$ be true. We need to show that $\mathbf{S}(\sigma(t))$ is true; that is

$$\mathcal{L}(\sigma(t), \nu(\sigma(t))) \leq z(\sigma(t)), \tag{3.17}$$

set $\mathfrak{p}(t) = \mathcal{L}(t, \nu(t))$, then, $\mathfrak{p}(\sigma(t)) = \mathcal{L}(\sigma(t), \nu(\sigma(t)))$, but from (2.7), we have

$${}^{C\mathbb{T}}D_+^\alpha \mathfrak{p}^\Delta(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [\mathfrak{p}(\sigma(t) - r\mu) - \mathfrak{p}(t_0)], \quad t \geq t_0.$$

Also,

$${}^{C\mathbb{T}}D_+^\alpha z^\Delta(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)], \quad t \geq t_0,$$

so that,

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^\alpha z^\Delta(t) - {}^{C\mathbb{T}} D_+^\alpha \mathfrak{p}^\Delta(t) &= \limsup_{\mu \to 0+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)] \\
&\quad - \limsup_{\mu \to 0+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [\mathfrak{p}(\sigma(t) - r\mu) - \mathfrak{p}(t_0)], \\
{}^{C\mathbb{T}}D_+^\alpha z^\Delta(t) - {}^{C\mathbb{T}} D_+^\alpha \mathfrak{p}^\Delta(t) &= \limsup_{\mu \to 0+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha} C_r \Big[ [z(\sigma(t) - r\mu) - z(t_0)] \\
&\quad - [\mathfrak{p}(\sigma(t) - r\mu) - \mathfrak{p}(t_0)] \Big], \\
\Big( {}^{C\mathbb{T}}D_+^\alpha z^\Delta(t) - {}^{C\mathbb{T}} D_+^\alpha \mathfrak{p}^\Delta(t) \Big) \mu^\alpha &= \limsup_{\mu \to 0+} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha} C_r \Big[ [z(\sigma(t) - r\mu) - z(t_0)] \\
&\quad - [\mathfrak{p}(\sigma(t) - r\mu) - \mathfrak{p}(t_0)] \Big], \\
\Big( {}^{C\mathbb{T}}D_+^\alpha z^\Delta(t) - {}^{C\mathbb{T}} D_+^\alpha \mathfrak{p}^\Delta(t) \Big) \mu^\alpha &\leq [z(\sigma(t)) - \mathfrak{p}(\sigma(t))] - [z(t_0) - \mathfrak{p}(t_0)]
\end{aligned}$$

$$[\mathfrak{p}(\sigma(t)) - z(\sigma(t))] \leq \Big({}^{C\mathbb{T}}D_+^{\alpha}\mathfrak{p}^{\Delta}(t) - {}^{C\mathbb{T}}D_+^{\alpha}z^{\Delta}(t)\Big)\mu^{\alpha} + [\mathfrak{p}(t_0) - z(t_0)]$$
$$\leq \Big(\Xi(t,\mathfrak{p}(t)) - \Xi(t,z(t))\Big)\mu^{\alpha} + [\mathfrak{p}(t_0) - z(t_0)].$$

Given that $\Xi(t,\chi)\mu^{\alpha}$ is non-decreasing in $\mathfrak{u}$ and $\mathbf{S(t)}$ holds, it follows that $\mathfrak{p}(\sigma(t)) - z(\sigma(t)) \leq 0$, ensuring that (3.17) is satisfied.

(iii) Let $t$ be rd and $\mathcal{N}$ denote the right neighborhood of $t \in \mathbb{T}$. We demonstrate that $\mathbf{S(t^*)}$ holds for $t^* \in \mathcal{N}$. This can be established by applying the comparison theorem for Caputo FrDEs, since at every rd-point $t^* \in \mathcal{N}$, $\sigma(t^*) = t^*$.

We shall make this proof in 3 parts. In Part 1, we show that the LF, $\mathcal{L}(t^*, \nu^*(t^*))$, is maximized by a solution of the comparison system; in Part 2, we show that the family of solutions of the comparison system is uniformly bounded and equi-continuous and therefore by the Arzela-Ascoli theorem, there would exist a sub-sequence that converges uniformly to a function $z(t)$; in Part 3, we show that this function $z(t)$ is indeed the maximal solution. This three parts put together concludes that $\mathcal{L}(t^*, \nu^*(t^*)) \leq z(t^*)$ for $t^* \in \mathcal{N}$.

Let $\omega$ be a small enough arbitrary positive number such that $\omega \leq B_{\mathbb{T}}$ (where $B_{\mathbb{T}}$ is a small enough number), for the IVP

$${}^{C\mathbb{T}}D^{\alpha}\chi^{\Delta} = \Xi(t^*,\chi) + \omega, \quad \chi(t_0) = \chi_0 + \omega, \tag{3.18}$$

where $t^* \in \mathcal{N}$, the function $\chi_{\omega}(t^*) = \chi(t^*) + \omega$ is a solution of (3.18) if and only if it satisfies:

$$\chi_{\omega}(t^*) = \chi_0 + \omega + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^* - s)^{\alpha-1}(\Xi(s,\chi_{\omega}(s)) + \omega)\Delta s, \quad t^*, s \in \mathcal{N}. \tag{3.19}$$

**Part 1**

Let $\mathfrak{p}(t^*) \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ be such that $\mathfrak{p}(t^*) = \mathcal{L}(t^*, \nu^*(t^*))$.
We show that

$$\mathfrak{p}(t^*) < \chi_{\omega}(t^*), \quad \text{for } t^* \in \mathcal{N}, \tag{3.20}$$

the inequality (3.20) holds for $t^* = t_0$ since

$$\mathfrak{p}(t_0) = \mathcal{L}(t_0, \nu_0) \leq \chi_0 < \chi_{\alpha}(t_0).$$

Assuming that the inequality (3.20) is false, then, $\exists$ a point $t_1^* > t_0$ :

$$\mathfrak{p}(t_1^*) = \chi_{\omega}(t_1^*) \quad \text{and} \quad \mathfrak{p}(t^*) < \chi_{\omega}(t^*), \quad \text{for} \quad t_0 \leq t^* < t_1^*.$$

From Lemma (3.1) it follows that

$${}^{C\mathbb{T}}D_+^{\alpha}\mathfrak{p}^{\Delta}(t_1^*) > {}^{C\mathbb{T}}D_+^{\alpha}\chi_{\omega}^{\Delta}(t_1^*),$$

so,

$${}^{C\mathbb{T}}D_+^{\alpha}\mathcal{L}^{\Delta}(t_1^*, \nu^*(t_1^*)) > {}^{C\mathbb{T}}D_+^{\alpha}\chi_{\omega}^{\Delta}(t_1^*),$$

and using (3.18), we arrive at

$${}^{C\mathbb{T}}D_+^{\alpha}\mathcal{L}^{\Delta}(t_1^*, \nu^*(t_1^*)) > \Xi(t_1^*, \chi_{\omega}(t_1^*)) + \omega) > \Xi(t_1^*, \mathfrak{p}(t_1^*)).$$

Therefore,

$$ {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{p}^{\Delta}(t_1^*) > \Xi(t_1^*, \mathfrak{p}(t_1^*)). \tag{3.21} $$

Now, for $t^* \in \mathcal{N}$.

$$
\begin{aligned}
{}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{p}^{\Delta}(t^*) &= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Big\{\mathfrak{p}(t^*) - \mathfrak{p}(t_0) - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[\mathfrak{p}(t^*-r\mu) - \mathfrak{p}(t_0)]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\{\mathcal{L}(t^*, \nu^*(t^*)) - \mathcal{L}(t_0, \nu_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[\mathcal{L}(t^*-r\mu, \nu^*(t^*-r\mu)) - \mathcal{L}(t_0, \nu_0)]\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Big\{\mathcal{L}(t^*, \nu^*(t)) - \mathcal{L}(t_0, \nu_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)\Big[[\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*))) - \mathcal{L}(t_0, \nu_0)] \\
&\quad -[\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*))) - \mathcal{L}(t_0, \nu_0)] \\
&\quad +[\mathcal{L}(t^*-r\mu, \nu^*(t^*-r\mu)) - \mathcal{L}(t_0, \nu_0)]\Big]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Big\{\mathcal{L}(t^*, \nu^*(t^*)) - \mathcal{L}(t_0, \nu_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)\Big[[\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*))) - \mathcal{L}(t_0, \nu_0)] \\
&\quad -[\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*)))] + \mathcal{L}(t^*-r\mu, \nu^*(t^*-r\mu))\Big]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Big\{\mathcal{L}(t^*, \nu^*(t^*)) - \mathcal{L}(t_0, \nu_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)\Big[[\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*))) - \mathcal{L}(t_0, \nu_0)] \\
&\quad +[\mathcal{L}(t^*-r\mu, \nu^*(t^*-r\mu)) - [\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*)))]\Big]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Big\{\mathcal{L}(t^*, \nu^*(t^*)) - \mathcal{L}(t_0, \nu_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r[\mathcal{L}(t^*-r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*))) - \mathcal{L}(t_0, \nu_0)]\Big\} \\
&\quad - \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Big\{ \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[\mathcal{L}(t^*-r\mu, \nu^*(t^*-r\mu))
\end{aligned}
$$

$$-\mathcal{L}(t^* - r\mu, \nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*)))]\Big\},$$

but $\mathcal{L}(t^*, \nu)$ is Lipschitz in the second variable, so,

$$\begin{aligned} {}^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{p}^{\Delta}(t^*) &\leq {}^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t^*, \nu^*(t^*)) \\ &\quad + L\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\|\nu^*(t^*-r\mu) - (\nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*)))\|, \end{aligned}$$

where $L > 0$ is the Lipschitz constant.
As $\mu \to 0$, $\|\nu^*(t^* - r\mu) - (\nu^*(t^*) - \mu^{\alpha}\Upsilon(t^*, \nu^*(t^*)))\| \to 0$, so that from (3.14) we have

$$^{C\mathbb{T}}D_{+}^{\alpha}\mathfrak{p}^{\Delta}(t^*) = {}^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t^*, \nu^*(t^*)) \leq \Xi(t^*, \mathcal{L}(t^*, \nu^*(t^*))) = \Xi(t^*, \mathfrak{p}(t^*)). \tag{3.22}$$

Now, (3.22) with $t^* = t_1^*$ contradicts (3.21), hence (3.20) is true.

**Part 2**

For $t^* \in \mathcal{N}$, we now show that whenever $\omega_1 < \omega_2$, then

$$\chi_{\omega_1}(t^*) < \chi_{\omega_2}(t^*). \tag{3.23}$$

Notice that (3.23) holds for $t^* = t_0$ since $\chi(t_0) + \omega_1 < \chi(t_0) + \omega_2 \implies \omega_1 < \omega_2$. If the inequality (3.23) is false, there would exist a time $t_1^*$ such that $\chi_{\omega_1}(t_1^*) = \chi_{\omega_2}(t_1^*)$ and $\chi_{\omega_1}(t^*) < \chi_{\omega_2}(t^*)$ for $t_0 \leq t^* < t_1^*$, $t^* \in \mathcal{N}$.
From Lemma (3.1), it follows that

$$^{C\mathbb{T}}D_{+}^{\alpha}\chi_{\omega_1}^{\Delta}(t_1^*) > {}^{C\mathbb{T}}D_{+}^{\alpha}\chi_{\omega_2}^{\Delta}(t_1^*).$$

However,

$$\begin{aligned} {}^{C\mathbb{T}}D_{+}^{\alpha}\chi_{\omega_1}^{\Delta}(t_1^*) - {}^{C\mathbb{T}}D_{+}^{\alpha}\chi_{\omega_2}^{\Delta}(t_1^*) &= \Xi(t_1^*, \chi_{\omega_1}(t_1^*)) + \omega_1 - [\Xi(t_1^*, \chi_{\omega_2}(t_1^*)) + \omega_2] \\ &= \omega_1 - \omega_2 < 0, \end{aligned}$$

which is a contradiction, and so (3.23) is true. Now, from (3.23), and since $\omega \leq B_{\mathbb{T}}$, we determine that

$$\chi_{\omega_1}(t^*) < \chi_{\omega_2}(t^*) < \cdots < \chi(t^*) + \omega_i \leq |\chi(t^*) + B_{\mathbb{T}}| \leq M,$$

and therefore we can say that the family of solutions $\{\chi_{\omega_i}(t^*)\}$ is uniformly bounded with bound $M > 0$ on $\mathbb{T}$. This means that $|\chi_{\omega_i}(t^*)| \leq M$ for $t^* \in \mathcal{N}$ and $\omega \in (0, B_{\mathbb{T}}]$.
We will now demonstrate that the family $\{\chi_{\omega_i}(t^*)\}$ is equicontinuous on $\mathbb{T}$. Suppose $\mathcal{S} = \sup\{|\Xi(t^*, \nu^*)| : (t^*, \nu^*) \in \mathcal{N} \times [-M, M]\}$. Next, consider $\{\omega_i\}_{i=1}^{\infty}(t^*)$ as a decreasing sequence with $\lim_{i\to\infty}\omega_i = 0$, and a sequence of functions $\chi_{\omega_i}(t^*)$. Let $t_1^*, t_2^* \in \mathcal{N}$ with $t_1^* < t_2^*$, then the estimation that follows is valid:

$$|\chi_{\omega_i}(t_2^*) - \chi_{\omega_i}(t_1^*)| = \Big|\chi_0 + \omega_i + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2^*}(t_2^* - s)^{\alpha-1}(\Xi(s, \chi_{\omega_i}(s)) + \alpha_i)\Delta s$$

$$
\begin{aligned}
&-(\chi_0+\omega_i+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1^*}(t_1^*-s)^{\alpha-1}(\Xi(s,\chi_{\omega_i}(s))+\omega_i))\Delta s\Big| \\
=\;& \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{t_2^*}(t_2^*-s)^{\alpha-1}(\Xi(s,\chi_{\omega_i}(s)))\Delta s-\int_{t_0}^{t_1^*}(t_1^*-s)^{\alpha-1}(\Xi(s,\chi_{\omega_i}(s)))\Delta s\Big| \\
\le\;& \frac{1}{\Gamma(\alpha)}\Big[\Big|\int_{t_0}^{t_2^*}(t_2^*-s)^{\alpha-1}\Big|\Big|(\Xi(s,\chi_{\omega_i}(s)))\Big|\Delta s+\Big|\int_{t_0}^{t_1^*}(t_1^*-s)^{\alpha-1}\Big|\Big|(\Xi(s,\chi_{\omega_i}(s)))\Big|\Delta s\Big] \\
\le\;& \frac{\mathcal{S}}{\Gamma(\alpha)}\Big[\Big|\int_{t_0}^{t_2^*}(t_2^*-s)^{\alpha-1}\Delta s\Big|+\Big|\int_{t_0}^{t_1^*}(t_1^*-s)^{\alpha-1}\Delta s\Big|\Big] \\
=\;& \frac{\mathcal{S}}{\Gamma(\alpha)}\Big[\Big|\int_{t_0}^{t_1^*}(t_2^*-s)^{\alpha-1}\Delta s+\int_{t_1^*}^{t_2^*}(t_2^*-s)^{\alpha-1}\Delta s\Big|+\Big|\int_{t_0}^{t_1^*}(t_1^*-s)^{\alpha-1}\Delta s\Big|\Big] \\
=\;& \frac{\mathcal{S}}{\Gamma(\alpha)}\Big[\Big|-\frac{(t_2^*-t_1^*)^\alpha}{\alpha}+\frac{(t_2^*-t_0)^\alpha}{\alpha}+\frac{(t_2^*-t_1^*)^\alpha}{\alpha}\Big|+\Big|\frac{(t_1^*-t_0)^\alpha}{\alpha}\Big|\Big] \\
=\;& \frac{\mathcal{S}}{\Gamma(\alpha)}\Big[\Big|\frac{(t_2^*-t_0)^\alpha}{\alpha}\Big|+\Big|\frac{(t_1^*-t_0)^\alpha}{\alpha}\Big|\Big] \\
=\;& \frac{\mathcal{S}}{\Gamma(\alpha+1)}\Big[(t_2^*-t_0)^\alpha+(t_1^*-t_0)^\alpha\Big] \\
\le\;& \frac{2\mathcal{S}}{\Gamma(\alpha+1)}\Big[(t_2^*-t_0)^\alpha\Big].
\end{aligned}
$$

A family of solutions $\{\chi_{\omega_i}(t^*)\}$ is said to be equicontinuous if given $\epsilon>0$, we can find $\delta>0$ such that $|\chi_{\omega_i}(t_2^*)-\chi_{\omega_i}(t_1^*)|<\epsilon$ whenever $|t_2^*-t_1^*|<\delta$,
implying that $|\chi_{\omega_i}(t_2^*)-\chi_{\omega_i}(t_1^*)|\le\frac{2\mathcal{S}}{\Gamma(\alpha+1)}\Big[(t_2^*-t_0)^\alpha\Big]<\epsilon$ provided $|t_2^*-t_1^*|<\delta$.

Now, we choose $\delta=\left(\frac{\epsilon\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}}$, $\left(\frac{\epsilon\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}}>\left(\frac{2\mathcal{S}(t_2^*-t_0)^\alpha}{\Gamma(\alpha+1)}\times\frac{\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}}=(t_2^*-t_0)$ but $(t_2^*-t_0)>|t_2^*-t_1^*|$ so since $(t_2^*-t_0)<\delta$, then $|t_2^*-t_1^*|<\delta$. Proving that the family of solutions $\{\chi_\omega(t^*)\}$ is equicontinuous. According to the Arzelà-Ascoli theorem, the family $\{\chi_{\omega_i}(t^*)\}$ contains a subsequence $\{\chi_{\omega_{i_j}}(t^*)\}$ that converges uniformly to a function $z(t^*)$ on $\mathbb{T}$.

**Part 3**

We then show that $z(t^*)$ is a solution of (1.3). Equation (3.19) becomes

$$\chi_{\omega_{i_j}}(t^*)=\chi_0+\omega_{i_j}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Xi(s,\chi_{\omega_{i_j}}(s))+\omega_{i_j})\Delta s. \tag{3.24}$$

Taking the limit as $i_j\to\infty$, then $\chi_{\omega_{i_j}}(t^*)\to z(t^*)$ on $\mathbb{T}$. Now (3.24) yields

$$z(t^*)=\chi_0+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Xi(s,z(t^*)))\Delta s. \tag{3.25}$$

Thus, $z(t^*)$ is a solution of (1.3) on $\mathbb{T}$. Since $\lim_{j\to\infty}\chi_{\omega_{i_j}}(t^*)=z(t)$ exists, then for any $\chi_{\omega_i}$ that satisfies the dynamic equation (1.3), $\chi_\omega(t^*)\le z(t^*)$. So from (3.20), we have that $\mathfrak{p}(t^*)<\chi_\omega(t^*)\le z(t^*)$ on $\mathbb{T}$. Therefore by induction principle, the statement $\mathbf{S(t)}$ is true. Completing the proof.

**Remark 3.1.** *Although comparison theorems for FrDE, FrDfE, and FrDET focus on understanding the behavior of solutions using a simpler comparison system, they differ in their time domains: FrDE has a continuous time domain, FrDfE has a discrete domain, and FrDET combines both. FrDE applies to continuous time systems, while FrDfE applies to discrete time systems. Theorem 3.1 examines the behavior of the LF concerning the maximal solution of the comparison system* (1.3)*, considering an arbitrary time domain with a jump operator* $\sigma(t)$ *that can be discrete or continuous. This is illustrated in conditions (ii) and (iii) of Lemma 2.1 in the proof of Theorem 3.1. This suggests that the comparison theorems found in the literature [1,2] address only a specific case (case iii) of Theorem 3.1, particularly condition (iii), when* $\mathbb{T} = \mathbb{R}$.

**Theorem 3.2.** *(Stability) Assume that:*

*(1)* $\mathcal{L}(t,\nu) \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $\mathcal{L}(t,\nu(t))$ *is locally Lipschitz with respect to* $\nu$, $\mathcal{L}(t,0) \equiv 0$, *and*

$$\phi(\|\nu\|) \le \mathcal{L}(t,\nu(t)) \tag{3.26}$$

*holds* $\forall\ (t,\nu) \in \mathbb{T} \times \mathbb{R}^n$ *and* $\phi \in \mathcal{K}$.

*(2)* $\Xi \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ *is nondecreasing with respect to the second variable at all* $t \in \mathbb{T}$, $\Xi(t,0) \equiv 0$, *and*

$$^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t,\nu(t)) \le \Xi(t,\mathcal{L}(t,\nu(t))).$$

*(3) The zero solution of the comparison equation (1.3) is stable.*

Then, the zero solution of the system (1.2) is stable.

*Proof.* By the assumption of stability of the zero solution of (1.3), let $\epsilon > 0$ be given, and for $\phi(\epsilon)$ and $t_0 \in \mathbb{T}$, there exists $\lambda = \lambda(t_0,\epsilon) > 0$ :

$$z(t) < \phi(\epsilon) \quad \text{for all } t \ge t_0, \tag{3.27}$$

whenever $\chi_0 < \lambda$, where $z(t) = z(t,t_0,\chi_0)$ is the maximal solution of (1.3).

Given that $\mathcal{L}(t,0) = 0$ and $\mathcal{L} \in C_{rd}$, which implies continuity of $\mathcal{L}$ at the origin, then given $\lambda > 0$, we can find a $\delta = \delta(t_0,\lambda) > 0$ : for $\nu_0 \in \mathbb{R}^n$, if $\|\nu_0\| < \delta$, then $\mathcal{L}(t_0,\nu_0) < \lambda$.

Claim that $\|\nu_0\| < \delta$ implies $\|\nu(t)\| < \epsilon$, $\forall\, t \in \mathbb{T}$, where $\nu(t) = \nu(t,t_0,\nu_0)$ is any solution of (1.2). If this claim is incorrect, then there would exist a time $t_1 \in \mathbb{T}$, $t_1 > t_0$ : the solution $\nu(t)$ of the dynamic system (1.2) at the instant time $t_1$ leaves the $\epsilon-$ neighborhood of the zero solution. That is $\|\nu(t)\| < \epsilon$ at $t_0 \le t < t_1$ and

$$\|\nu(t_1)\| \ge \epsilon. \tag{3.28}$$

However, we know from Theorem 3.1 that

$$\mathcal{L}(t,\nu(t)) \le z(t), \quad t_0 \le t \le t_1, \tag{3.29}$$

provided $\mathcal{L}(t_0,\nu_0) \le \chi_0$, where $z(t)$ is the maximal solution of system (1.3).

Combining (3.26), (3.27), (3.29), and (3.28) for $t = t_1$, we obtain

$$\phi(\|\nu(t_1)\|) \le \mathcal{L}(t_1,\nu(t_1)) \le z(t_1) < \phi(\epsilon) \le \phi(\|\nu(t_1)\|),$$

$$\Longrightarrow \phi(\|\nu(t_1)\|) < \phi(\|\nu(t_1)\|). \tag{3.30}$$

The contradiction (3.30) shows that $t_1 \notin \mathbb{T}$ and therefore $\|\nu(t)\| < \epsilon\ \forall\, t \in \mathbb{T}$ whenever $\|\nu_0\| < \delta$, and so the zero solution (1.2) is stable.

**Theorem 3.3** (Asymptotic Stability)**.** *Assume the following:*

*(1)* $\mathcal{L}(t,\nu) \in C_{rd}[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *is locally Lipschitz in* $\nu$ *for each* $t \in \mathbb{T}$ *and* $\mathcal{L}(t,0) \equiv 0$.
*(2) For* $t \geq t_0$,

$$b(\|\nu(t)\|) \leq \mathcal{L}(t,\nu), \quad \textit{where } b \in \mathcal{K}.$$

*(3) The inequality*

$${}^{C\mathbb{T}}D_+^{\alpha}\mathcal{L}^{\Delta}(t,\nu) \leq -\phi(\|\nu(t)\|) \textit{ holds for } (t,\nu) \in \mathbb{T} \times \mathbb{R}^n \textit{ and } \phi \in \mathcal{K}.$$

*Then, the zero solution* $\nu = 0$ *of the fractional dynamic system* (1.2) *is asymptotically stable.*

*Proof.* By Theorem 3.2, the zero solution $\nu = 0$ of (1.2) is stable. It remains to show that

$$\lim_{t\to\infty} \|\nu(t)\| = 0. \tag{3.31}$$

We shall make this proof in two phases.

**Phase 1.** Assuming (3.31) is not true, such that $\liminf_{t\to\infty} \|\nu(t)\| \neq 0$, then there would exist $T > 0$ such that for a given $\epsilon > 0$,

$$\|\nu(t)\| \geq \epsilon, \quad \text{for } t = \sigma(T) > T. \tag{3.32}$$

By condition 3, we deduce that $\mathcal{L}(\nu(t))$ is monotone decreasing and

$$\lim_{t\to\infty} \mathcal{L}(t,\nu(t)) = \mathcal{L}_0 \geq 0, \tag{3.33}$$

since $\mathcal{L}(t,\nu(t))$ is positive definite and only 0 at $\nu = 0$.

By utilizing Lemma 3.2 in relation to condition 3, we get

$$\mathcal{L}(t,\nu(t)) \leq \mathcal{L}(t_0,\nu_0) - \frac{1}{\Gamma(\alpha)} \int_T^t (t-s)^{\alpha-1} \phi(\|\nu(s)\|)\Delta s, \quad \text{for } t > T,$$

$$\mathcal{L}(t,\nu(t)) \leq \mathcal{L}(t_0,\nu_0) - \frac{\phi(\epsilon)}{\alpha\Gamma(\alpha)}(t-T)^{\alpha},$$

$$0 \leq \mathcal{L}(t_0,\nu_0) - \frac{\phi(\epsilon)}{\alpha\Gamma(\alpha)}(t-T)^{\alpha}. \tag{3.34}$$

As $t \to \infty$, the RHS of (3.34) approaches $-\infty$. This is a contradiction, so $\liminf_{t\to\infty} \|\nu(t)\| = 0$.

**Phase 2.** If $\limsup_{t\to\infty} \|\nu(t)\| \neq 0$, then given $\eta > 0$, there is a divergent sequence $\{t_k\}, : \|\nu(t_k)\| \geq \eta$.Each $t_k \in \mathbb{T}$ could potentially be related to one of the following:

(i) $t_k$ is rs and ls (isolated points).
(ii) $t_k$ is rs and ld.
(iii) $t_k$ is rd and ls.
(iv) $t_k$ is rd and ld (dense points).

Suppose $\exists$ a divergent subsequence $\{t_i\}$ of $\{t_k\}$, where each $t_i$ falls into one of the four cases mentioned above. Then, by Lemma 3.2 and Definition 2.5, we have that for case (i),

$$\mathcal{L}(t_i, \nu(t_i)) \leq \mathcal{L}(t_0, \nu_0) - \frac{1}{\Gamma(\alpha)} \int_t^{\sigma(t)} r(s_i)\phi(\|\nu(s_i)\|)\Delta s,$$

$$0 \leq \mathcal{L}(t_i, \nu(t_i)) \leq \mathcal{L}(t_0, \nu_0) - \frac{\phi(\eta)}{\Gamma(\alpha)} \sum_{j=1}^{i} \mu(t_j)r(t_j),$$

for all $t_i, t_j \in \mathbb{T}$ and $r(t) = (t-s)^{\alpha-1}$.

This leads to a contradiction as $i \to \infty$, since $\mu(t_i)$ remains constant for each $i$. Therefore, $\limsup_{t\to\infty} \|\nu(t)\| = 0$.

For case (ii),

$$\mathcal{L}(t_i, \nu(t_i)) \leq \mathcal{L}(t_0, \nu_0) - \frac{1}{\Gamma(\alpha)} \int_t^{\sigma(t)} r(s_i)\phi(\|\nu(s_i)\|)\Delta s,$$

$$0 \leq \mathcal{L}(t_i, \nu(t_i)) \leq \mathcal{L}(t_0, \nu_0) - \frac{\phi(\eta)}{\Gamma(\alpha)} \mu(t_i)r(t_j),$$

for $r(t) = (t-s)^{\alpha-1}$. This results in a contradiction as $i \to \infty$, since $\mu(t_i)$ is a constant for each $i$. So $\limsup_{t\to\infty} \|\nu(t)\| = 0$.

For cases (iii) and (iv)

$$\mathcal{L}(t_i, \nu^*(t_i)) \leq \mathcal{L}(t_0, \chi_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_i} (t-s)^{\alpha-1}\phi(\|\nu^*(s_i)\|)\Delta s,$$

$$0 \leq \mathcal{L}(t_i, \nu^*(t_i)) \leq \mathcal{L}(t_0, \nu_0) - \frac{\sigma(\eta)(t_i - t_0)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{3.35}$$

As $t_i \to \infty$, the right-hand side of (3.35) approaches $-\infty$ also contradicting the definition of $\mathcal{L}(t, \nu(t))$; $\Longrightarrow \limsup_{t\to\infty} \|\nu(t)\| = 0$.

Since $\liminf_{t\to\infty} \|\nu(t)\| = \limsup_{t\to\infty} \|\nu(t)\| = 0$, so, (3.31) holds. Then the zero solution $\nu = 0$ of (1.2) is asymptotically stable.

**Remark 3.2.** *Theorems 3.2 and 3.3 represent a significant advancement in fractional calculus and stability analysis, building on research in FrDfE, FrDE, and integer-order dynamic equations. Although similar to the stability analysis in [1–3, 23, 24, 26, 28, 33, 35, 36, 38], which addresses the stability of the zero solution of fractional dynamic systems, they differ by generalizing the time domain of system* (1.2)*. This allows for stability analysis on various time scales, including discrete, continuous, and mixed. Moreover, the results extend to non-integer orders, enabling a more comprehensive analysis of system* (1.2)*'s behavior. Theorems 3.2 and 3.3 are better suited for analyzing the behavior of solutions in complex systems with multiple time scales and non-uniform grids, which can be challenging with FrDE and FrDfE.*

## 4. Discussion

Let us examine the dynamic system

$$\begin{aligned} \nu_1^\Delta(t) = &\quad \frac{\nu_1}{\cos^2 t} - (\nu_2+\nu_1)\frac{\sin^2 t}{\cos^2 t} + \nu_2\frac{\cos^2 t}{\sin^2 t}, \\ \nu_2^\Delta(t) = &\quad 2(\nu_1-\nu_2) + \frac{\nu_2}{\sin^2 t} - 2\nu_1\cos^2 t, \end{aligned} \tag{4.1}$$

for $t \geq t_0$, with initial conditions

$$\nu_1(t_0) = \nu_{10} \quad \text{and} \quad \nu_2(t_0) = \nu_{20},$$

where $\nu = (\nu_1, \nu_2)$ and $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$.

Consider $\mathcal{L}(t, \nu_1, \nu_2) = |\nu_1| + |\nu_2|$, for $t \in \mathbb{T}$ and $(\nu_1, \nu_2) \in \mathbb{R}^2$. Then we compute the Dini derivative for $\mathcal{L}(t, \nu_1, \nu_2) = |\nu_1| + |\nu_2|$ as follows; from (2.2), we have that

$$\begin{aligned} D^+\mathcal{L}^\Delta(t,\nu) &= \limsup_{\mu(t)\to 0} \frac{\mathcal{L}(t+\mu(t), \nu+\mu(t)\Upsilon(t,\nu)) - \mathcal{L}(t,\nu)}{\mu(t)} \\ &= \limsup_{\mu(t)\to 0} \frac{|\nu_1+\mu(t)\Upsilon_1(t,\nu)| + |\nu_2+\mu(t)\Upsilon_2(t,\nu)| - [|\nu_1|+|\nu_2|]}{\mu(t)} \\ &\leq \limsup_{\mu(t)\to 0} \frac{|\nu_1| + |\mu(t)\Upsilon_1(t,\nu)| + |\nu_2| + |\mu(t)\Upsilon_2(t,\nu)| - |\nu_1| - |\nu_2|}{\mu(t)} \\ &= \limsup_{\mu(t)\to 0} \frac{\mu(t)[|\Upsilon_1(t,\nu)| + |\Upsilon_2(t,\nu)|]}{\mu(t)} \\ &\leq |\Upsilon_1(t,\nu)| + |\Upsilon_2(t,\nu)| \\ &= \left| \frac{\nu_1}{\cos^2 t} - (\nu_2+\nu_1)\frac{\sin^2 t}{\cos^2 t} + \nu_2\frac{\cos^2 t}{\sin^2 t} \right| + \left| 2(\nu_1-\nu_2) + \frac{\nu_2}{\sin^2 t} - 2\nu_1\cos^2 t \right| \\ &= \left| \nu_1\left(\frac{1}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}\right) - \nu_2\left(\frac{\sin^2 t}{\cos^2 t} - \frac{\cos^2 t}{\sin^2 t}\right)\right| + \left| 2\nu_1(1-\cos^2 t) - \nu_2\left(2 - \frac{1}{\sin^2 t}\right) \right| \\ &\leq \left| \nu_1\left(\frac{\cos^2 t}{\cos^2 t}\right) - \nu_2\left(\frac{\sin^2 t - \cos^2 t)(\sin^2 t + \cos^2 t)}{\cos^2 t \sin^2 t}\right) \right| + 2|\nu_1| + 3|\nu_2| \\ &\leq |\nu_1| + |\nu_2| \left| \left( \frac{\sin^2 t - \cos^2 t}{\cos^2 t \sin^2 t} \right) \right| + 2|\nu_1| + 3|\nu_2| \\ &= |\nu_1| + |\nu_2| \left| \left( \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t} \right) \right| + 2|\nu_1| + 3|\nu_2| \\ &\leq 3|\nu_1| + |\nu_2| \left( \left| \frac{1}{\cos^2 t} \right| + \left| \frac{1}{\sin^2 t} \right| \right) + 3|\nu_2| \\ &\leq 3|\nu_1| + 5|\nu_2| \leq 5[|\nu_1| + |\nu_2|] \\ D^+\mathcal{L}^\Delta(t,\nu) &\leq 5\mathcal{L}(t,\nu_1,\nu_2) = \Xi(t,\mathcal{L}). \end{aligned}$$

Now, consider the consider the comparison equation

$$D^+\chi^\Delta = 5\chi > 0, \ \chi(0) = \chi_0, \tag{4.2}$$

with solution

$$\chi(t) = \chi_0 e^{5t}. \tag{4.3}$$

Even though conditions (i)–(iii) of [21] are satisfied that is $\mathcal{L} \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $D^+\mathcal{L}^{\Delta}(t, \nu_1, \nu_2) \leq \Xi(t, \mathcal{L}(t, \nu))$ and $\sqrt{\nu_1^2 + \nu_2^2} \leq |\nu_1| + |\nu_2| \leq 2(\nu_1^2 + \nu_2^2)$, for $\mathfrak{b}(\|\nu\|) = r$ and $\mathfrak{a}(\|\nu\|) = 2r^2$, it is obvious to see that the solution (4.3) of the comparison system (4.2) is not stable, so we can not deduce the stability properties of the system (4.1) by applying the basic definition of the Dini-derivative to the LF $\mathcal{L}(t, \nu_1, \nu_2) = |\nu_1| + |\nu_2|$.

Now, we will apply our new definition on the same system but as a Caputo fractional dynamic system

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}\nu_1^{\Delta}(t) = &\quad \frac{\nu_1}{\cos^2 t} - (\nu_2 + \nu_1)\frac{\sin^2 t}{\cos^2 t} + \nu_2\frac{\cos^2 t}{\sin^2 t}, \\ {}^{C\mathbb{T}}D^{\alpha}\nu_2^{\Delta}(t) = &\quad 2(\nu_1 - \nu_2) + \frac{\nu_2}{\sin^2 t} - 2\nu_1\cos^2 t, \end{aligned} \tag{4.4}$$

for $t \geq t_0$, with initial conditions

$$\nu_1(t_0) = \nu_{10} \quad \text{and} \quad \nu_2(t_0) = \nu_{20},$$

where $\nu = (\nu_1, \nu_2)$ and $\Upsilon = (\Upsilon_1, \Upsilon_2)$.

Consider $\mathcal{L}(t, \nu_1, \nu_2) = |\nu_1| + |\nu_2|$, for $t \in \mathbb{T}$ and $(\nu_1, \nu_2) \in \mathbb{R}^2$. Then condition 1 of Theorem (3.2) is satisfied, for $\phi = \frac{1}{2}r$, where $\phi \in \mathcal{K}$, with $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$, so that the associated norm $\|\nu\| = \sqrt{\nu_1^2 + \nu_2^2}$.

Since,

$$\mathcal{L}(t, \nu_1, \nu_2) = |\nu_1| + |\nu_2|,$$

then, $\phi(\|\nu\|) \leq \mathcal{L}(t, \nu_1, \nu_2)$. From (2.9), we compute the Caputo Fr∆DiD for $\mathcal{L}(t, \nu_1, \nu_2) = |\nu_1| + |\nu_2|$ as follows:

$$\begin{aligned} {}^{C\mathbb{T}}D_+^{\alpha}\mathcal{L}^{\Delta}(t, \nu) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}}\Big\{\mathcal{L}(\sigma(t), \nu(\sigma(t)) - \mathcal{L}(t_0, \nu_0) \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[\mathcal{L}(\sigma(t) - r\mu, \nu(\sigma(t)) - \mu^{\alpha}\Upsilon(t, \nu(t))) - \mathcal{L}(t_0, \nu_0)]\Big\} \\ &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}}\Big\{(|\nu_1(\sigma(t))| + |\nu_2(\sigma(t))|) - (|\nu_{10}| + |\nu_{20}|) \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r}({}^{\alpha}C_r)[|\nu_1(\sigma(t)) - \mu^{\alpha}\Upsilon_1(t, \nu_1)| + |\nu_2(\sigma(t)) - \mu^{\alpha}\Upsilon_2(t, \nu_2)| - (|\nu_{10}| + |\nu_{10}|)]\Big\} \\ &\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}}\Big\{(|\nu_1(\sigma(t))| + |\nu_2(\sigma(t))|) - (|\nu_{10}| + |\nu_{20}|) \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r}({}^{\alpha}C_r)[|\nu_1(\sigma(t))| + |\mu^{\alpha}\Upsilon_1(t; \nu_1)| + |\nu_2(\sigma(t))| + |\mu^{\alpha}\Upsilon_2(t; \nu_2)| - (|\nu_{10}| + |\nu_{10}|)]\Big\} \\ &\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}}\Big\{(|\nu_1(\sigma(t))| + |\nu_2(\sigma(t))|) - (|\nu_{10}| + |\nu_{20}|) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\Big[|\nu_1(\sigma(t))|+|\nu_2(\sigma(t))|\Big]+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)|\Big[|\mu^{\alpha}\Upsilon_1(t;\nu_1)|+|\mu^{\alpha}\Upsilon_2(t;\nu_2)|\Big]\\
&-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\Big[|\nu_{10}|+|\nu_{10}|\Big]\Big\}\\
\le\;& \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\Big[|\nu_1(\sigma(t))|+|\nu_2(\sigma(t))|\Big]-\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\Big[|\nu_{10}|+|\nu_{10}|\Big]\Big\}\\
&+\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)|\Big[|\Upsilon_1(t;\nu_1)|+|\Upsilon_2(t;\nu_2)|\Big].
\end{aligned}
$$

Applying (2.12) and (2.14) we obtain

$$
\begin{aligned}
=\;& \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|\nu_1(\sigma(t))|+|\nu_2(\sigma(t))|)-\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|\nu_{10}|+|\nu_{10}|)-\Big[\,|\Upsilon_1(t;\nu_1)|+|\Upsilon_2(t;\nu_2)|\,\Big]\\
\le\;& \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|\nu_1(\sigma(t))|+|\nu_2(\sigma(t))|)-\Big[\,|\Upsilon_1(t;\nu_1)|+|\Upsilon_2(t;\nu_2)|\,\Big].
\end{aligned}
$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|\nu_1(\sigma(t))|+|\nu_2(\sigma(t))|)\to 0$, then

$$
\begin{aligned}
&{}^{C\mathbb{T}}D_+^{\alpha}\mathcal{L}^{\Delta}(t;\nu_1,\nu_2)\\
\le\;& -\Big[\,|\chi_1(t;\nu_1)|+|\chi_2(t;\nu_2)|\,\Big]\\
=\;& -\left[\left|\frac{\nu_1}{\cos^2 t}-(\nu_2+\nu_1)\frac{\sin^2 t}{\cos^2 t}+\nu_2\frac{\cos^2 t}{\sin^2 t}\right|+\left|2(\nu_1-\nu_2)+\frac{\nu_2}{\sin^2 t}-2\nu_1\cos^2 t\right|\right]\\
=\;& -\left[\,\left|\left(\frac{\nu_1}{\cos^2 t}-\nu_1\frac{\sin^2 t}{\cos^2 t}\right)-\nu_2\left(\frac{\sin^2 t}{\cos^2 t}-\frac{\cos^2 t}{\sin^2 t}\right)\right|+\left|2\nu_1(1-\cos^2 t)-\nu_2\left(2-\frac{1}{\sin^2 t}\right)\right|\,\right]\\
=\;& -\left[\,\left|\nu_1\left(\frac{1}{\cos^2 t}-\frac{\sin^2 t}{\cos^2 t}\right)-\nu_2\left(\frac{\sin^2 t}{\cos^2 t}-\frac{\cos^2 t}{\sin^2 t}\right)\right|+\left|2\nu_1(\sin^2 t)-\nu_2\left(2-\frac{1}{\sin^2 t}\right)\right|\,\right]\\
\le\;& -\left[\left|\nu_1\left(\frac{\cos^2 t}{\cos^2 t}\right)-\nu_2\left(\frac{(\sin^2 t-\cos^2 t)(\sin^2 t+\cos^2 t)}{\cos^2 t\sin^2 t}\right)\right|+2\,|\nu_1|+3\,|\nu_2|\right]\\
\le\;& -\left[|\nu_1|+|\nu_2|\left|\left(\frac{\sin^2 t-\cos^2 t}{\cos^2 t\sin^2 t}\right)\right|+2\,|\nu_1|+3\,|\nu_2|\right]\\
\le\;& -3\,|\nu_1|-5\,|\nu_2|\le -3\,[|\nu_1|+|\nu_2|]\,.
\end{aligned}
$$

Therefore,

$$
{}^{C\mathbb{T}}D_+^{\alpha}\mathcal{L}^{\Delta}(t;\nu_1,\nu_2)\le -3\mathcal{L}(t,\nu_1,\nu_2). \tag{4.5}
$$

Consider the comparison system

$$
{}^{C\mathbb{T}}D_+^{\alpha}\chi^{\Delta}=\Xi(t,\chi)\le -3\chi, \tag{4.6}
$$

using the Laplace transform method

$$
{}^{C\mathbb{T}}D_+^{\alpha}\chi^{\Delta}+3\chi=0.
$$

$$X(s) = \frac{\chi_0 S^{\alpha-1}}{S^{\alpha} + 3}$$

taking the inverse Laplace transform we have

$$\chi(t) = \chi_0 E_{\alpha,1}(-3t^{\alpha}), \quad \text{for } \alpha \in (0,1), \tag{4.7}$$

where $E_{\alpha,1}(z)$ is the Mittag-Leffler function, which can be approximated as:

$$E_{\alpha,1}(-t^{\alpha}) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{n\alpha}}{\Gamma(\alpha k + 1)} = 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + ... \approx \exp\left[-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right].$$

Now, let $|\chi_0| < \delta$, then from (4.11), we have
$|\chi(t)| = \left|3\chi_0 E_{\alpha,1}(-t^{\alpha})\right| = \left|3\chi_0 \exp\left[-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right]\right| < 3\left|\exp\left[-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right]\right|\delta < \epsilon$ whenever $|\chi_0| < \delta = \frac{\epsilon}{3\left|\exp\left[-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right]\right|}$.

Therefore given $\epsilon > 0$, we can find a $\delta > 0$ such that $|\chi(t)| < \epsilon$ whenever $|\chi_0| < \delta$.

We conclude that the trivial solution of system (4.4) is stable as it satisfies all the conditions of Theorem 3.2 and the trivial solution of the comparison system (4.10) is stable.

Figure 1 below is the graphical representation of $E_{\alpha,1}(-3t^{\alpha})$. The behavior of the curve shows stability of the solution $\chi(t)$ over time.

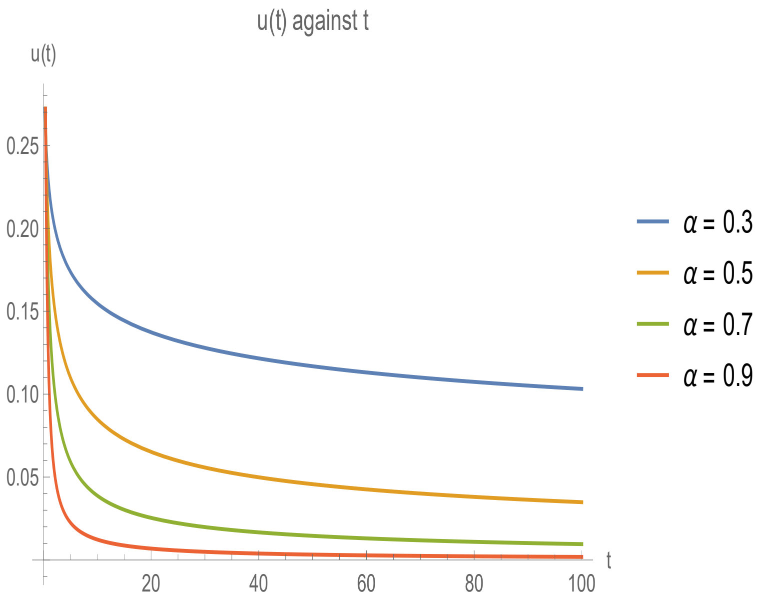


**Figure 1.** Graph of $\chi(t) = \chi_0 E_{\alpha,1}(-3t^{\alpha})$, for $\alpha \in (0,1)$ against $t$.

Consider the Caputo fractional dynamic system

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}\upsilon_1^{\Delta}(t) &= \upsilon_1 + \upsilon_2 - 3\upsilon_3, \\ {}^{C\mathbb{T}}D^{\alpha}\upsilon_2^{\Delta}(t) &= -\upsilon_1 + \upsilon_2 + \upsilon_2\upsilon_3^2, \\ {}^{C\mathbb{T}}D^{\alpha}\upsilon_3^{\Delta}(t) &= 3\upsilon_1 + \upsilon_1^2\upsilon_2^2\upsilon_3 + \upsilon_3, \end{aligned} \tag{4.8}$$

for $t \geq t_0$, with initial conditions

$$\upsilon_1(t_0) = \upsilon_{10}, \quad \upsilon_2(t_0) = \upsilon_{20}, \quad \text{and} \quad \upsilon_3(t_0) = \upsilon_{30}$$

where $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)$, $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)$.

Consider $\mathcal{L}(t, \upsilon_1, \upsilon_2, \upsilon_3) = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2$, for $t \in \mathbb{T}$ and $(\upsilon_1, \upsilon_2, \upsilon_3) \in \mathbb{R}^3$. Then, condition 1 of Theorem (3.3) is satisfied, for $b(\|\upsilon\|) \leq \mathcal{L}(t, \upsilon) \leq a(\|\upsilon\|)$, with $b(r) = r$, $a(r) = 2r^2$, $a, b \in \mathcal{K}$, where the associated norm $\|\upsilon\| = \sqrt{\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2}$.

Since

$$\mathcal{L}(t, \upsilon_1, \upsilon_2, \upsilon_3) = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2,$$

then, $\sqrt{\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2} \leq \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 \leq 2(\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2)$. From (2.9), we compute the Caputo FrΔDiD for $\mathcal{L}(t, \upsilon_1, \upsilon_2, \upsilon_3) = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2$ as follows:

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^\alpha \mathcal{L}^\Delta(t, \upsilon) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \Big\{ \Big[(\upsilon_1(\sigma(t)))^2 + (\upsilon_2^2(\sigma(t)))^2 + (\upsilon_3(\sigma(t)))^2\Big] - \Big[(\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2\Big] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right) [(\upsilon_1(\sigma(t)) - \mu^\alpha \Upsilon_1(t, \upsilon_1, \upsilon_2, \upsilon_3))^2 + (\upsilon_2(\sigma(t)) \\
&\quad - \mu^\alpha \Upsilon_2(t, \upsilon_1, \upsilon_2, \upsilon_3))^2] + [(\upsilon_3(\sigma(t)) - \mu^\alpha \Upsilon_3(t, \upsilon_1, \upsilon_2, \upsilon_3))^2 - ((\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2)] \Big\} \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \Big\{ \Big[(\upsilon_1(\sigma(t)))^2 + (\upsilon_2^2(\sigma(t)))^2 + (\upsilon_3(\sigma(t)))^2\Big] - \Big[(\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2\Big] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right) \Big[(\upsilon_1(\sigma(t)))^2 - 2\upsilon_1(\sigma(t))\mu^\alpha \Upsilon_1(t, \upsilon_1, \upsilon_2, \upsilon_3) + \mu^{2\alpha}(\Upsilon_1(t, \upsilon_1, \upsilon_2, \upsilon_3))^2\Big] \\
&\quad + (\upsilon_2(\sigma(t)))^2 - 2\upsilon_2(\sigma(t))\mu^\alpha \Upsilon_2(t, \upsilon_1, \upsilon_2, \upsilon_3) + \mu^{2\alpha}(\Upsilon_2(t, \upsilon_1, \upsilon_2, \upsilon_3))^2 \\
&\quad + (\upsilon_3(\sigma(t)))^2 - 2\upsilon_3(\sigma(t))\mu^\alpha \Upsilon_3(t, \upsilon_1, \upsilon_2, \upsilon_3) + \mu^{2\alpha}(\Upsilon_3(t, \upsilon_1, \upsilon_2, \upsilon_3))^2 \\
&\quad - \Big[(\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2\Big] \Big\} \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \Big\{ \Big[(\upsilon_1(\sigma(t)))^2 + (\upsilon_2^2(\sigma(t)))^2 + (\upsilon_3(\sigma(t)))^2\Big] - \Big[(\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2\Big] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right) \Big[(\upsilon_1(\sigma(t)))^2 - 2\upsilon_1(\sigma(t))\mu^\alpha \Upsilon_1(t, \upsilon_1, \upsilon_2, \upsilon_3) + \mu^{2\alpha}(\Upsilon_1(t, \upsilon_1, \upsilon_2, \upsilon_3))^2\Big] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right) \Big[(\upsilon_2(\sigma(t)))^2 - 2\upsilon_2(\sigma(t))\mu^\alpha \Upsilon_2(t, \upsilon_1, \upsilon_2, \upsilon_3) + \mu^{2\alpha}(\Upsilon_2(t, \upsilon_1, \upsilon_2, \upsilon_3))^2\Big] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right) [(\upsilon_3(\sigma(t)))^2 - 2\upsilon_3(\sigma(t))\mu^\alpha \Upsilon_3(t, \upsilon_1, \upsilon_2, \upsilon_3) \\
&\quad + \mu^{2\alpha}(\Upsilon_3(t, \upsilon_1, \upsilon_2, \upsilon_3))^2 - \Big[(\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2\Big] \Big\} \\
&= -\limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \left\{ \sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right) \Big[(\upsilon_{10})^2 + (\upsilon_{20})^2 + (\upsilon_{30})^2\Big] \right\}
\end{aligned}$$

$$+\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)\left[(\upsilon_1(\sigma(t)))^2+(\upsilon_2(\sigma(t)))^2+(\upsilon_3(\sigma(t)))^2\right]\right\}$$

$$-\limsup_{\mu\to 0^+}\left\{\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)[2\upsilon_1(\sigma(t))\mu^\alpha\Upsilon_1(t,\upsilon_1,\upsilon_2,\upsilon_3)\right.$$

$$\left.+2\upsilon_2(\sigma(t))\mu^\alpha\Upsilon_2(t,\upsilon_1,\upsilon_2,\upsilon_3)+2\upsilon_3(\sigma(t))\mu^\alpha\Upsilon_3(t,\upsilon_1,\upsilon_2,\upsilon_3)]\right\}.$$

Applying (2.12) and (2.14) we obtain

$$\begin{aligned}\le\ & \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_1(\sigma(t)))^2+(\upsilon_2(\sigma(t)))^2+(\upsilon_3(\sigma(t)))^2\right]\\ & -[2\upsilon_1(\sigma(t))\Upsilon_1(t,\upsilon_1,\upsilon_2,\upsilon_3)+2\upsilon_2(\sigma(t))\Upsilon_2(t,\upsilon_1,\upsilon_2,\upsilon_3)+2\upsilon_3(\sigma(t))\Upsilon_3(t,\upsilon_1,\upsilon_2,\upsilon_3)].\end{aligned}$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_1(\sigma(t)))^2+(\upsilon_2(\sigma(t)))^2+(\upsilon_3(\sigma(t)))^2\right]\to 0$, then

$$\le\ -2[\upsilon_1(\sigma(t))\Upsilon_1(t,\upsilon_1,\upsilon_2,\upsilon_3)+\upsilon_2(\sigma(t))\Upsilon_2(t,\upsilon_1,\upsilon_2,\upsilon_3)+\upsilon_3(\sigma(t))\Upsilon_3(t,\upsilon_1,\upsilon_2,\upsilon_3)],$$

applying $\upsilon(\sigma(t))\le\mu^{C\mathbb{T}}D^\alpha\upsilon(t)+\upsilon(t)$

$$\begin{aligned}=\ & -2\Big[\mu(t)\Upsilon_1^2(t,\upsilon_1,\upsilon_2,\upsilon_3)+\upsilon_1(t)\Upsilon_1(t,\upsilon_1,\upsilon_2,\upsilon_3)+\mu(t)\Upsilon_2^2(t,\upsilon_1,\upsilon_2,\upsilon_3)\\ & +\upsilon_2(t)\Upsilon_2(t,\upsilon_1,\upsilon_2,\upsilon_3)+\mu(t)\Upsilon_3^2(t,\upsilon_1,\upsilon_2,\upsilon_3)+\upsilon_3(t)\Upsilon_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big]\\ =\ & -2\Big[\mu(t)(\upsilon_1+\upsilon_2-3\upsilon_3)^2+\upsilon_1(\upsilon_1+\upsilon_2-3\upsilon_3)+\mu(t)(-\upsilon_1+\upsilon_2+\upsilon_2\upsilon_3^2)^2\\ & +\upsilon_2(-\upsilon_1+\upsilon_2+\upsilon_2\upsilon_3^2)+\mu(t)(3\upsilon_1+\upsilon_1^2\upsilon_2^2\upsilon_3+\upsilon_3)^2+\upsilon_3(3\upsilon_1+\upsilon_1^2\upsilon_2^2\upsilon_3+\upsilon_3)\Big]\\ =\ & -2\Big[\upsilon_1^2+\upsilon_2^2+\upsilon_3^2+\mu(t)(\upsilon_1+\upsilon_2-3\upsilon_3)^2+\mu(t)(-\upsilon_1+\upsilon_2+\upsilon_2\upsilon_3^2)^2+\mu(t)(3\upsilon_1+\upsilon_1^2\upsilon_2^2\upsilon_3+\upsilon_3)^2\Big]\\ & -2\Big[\upsilon_2^2\upsilon_3^2+\upsilon_1^2\upsilon_2^2\upsilon_3^2\Big]\\ \le\ & -2\Big[\upsilon_1^2+\upsilon_2^2+\upsilon_3^2+\mu(t)(\upsilon_1+\upsilon_2-3\upsilon_3)^2+\mu(t)(-\upsilon_1+\upsilon_2+\upsilon_2\upsilon_3^2)^2+\mu(t)(3\upsilon_1+\upsilon_1^2\upsilon_2^2\upsilon_3+\upsilon_3)^2\Big]\\ =\ & -2\Big[\upsilon_1^2+\upsilon_2^2+\upsilon_3^2\Big]-2\mu(t)\Big[(\upsilon_1+\upsilon_2-3\upsilon_3)^2+(-\upsilon_1+\upsilon_2+\upsilon_2\upsilon_3^2)^2+(3\upsilon_1+\upsilon_1^2\upsilon_2^2\upsilon_3+\upsilon_3)^2\Big].\end{aligned}\tag{4.9}$$

If $\mathbb{T}=\mathbb{R}$ we have that $\mu=0$, so that (4.9) becomes;

$${}^{C\mathbb{T}}D_+^\alpha\mathcal{L}^\Delta(t;\upsilon_1,\upsilon_2)\le -2\Big[\upsilon_1^2+\upsilon_2^2+\upsilon_3^2\Big].$$

Therefore,

$${}^{C\mathbb{T}}D_+^\alpha\mathcal{L}^\Delta(t;\upsilon_1,\upsilon_2)\le -2\mathcal{L}(t,\upsilon_1,\upsilon_2,\upsilon_3).$$

Consider the comparison system

$${}^{C\mathbb{T}}D_+^\alpha\chi^\Delta=\Xi(t,\chi)\le -2\chi,\tag{4.10}$$

$$^{C\mathbb{T}}D_{+}^{\alpha}\chi^{\Delta} + 2\chi = 0.$$

Applying the Laplace transform method, we obtain

$$\chi(t) = \chi_0 E_{\alpha,1}(-2t^{\alpha}), \quad \text{for } \alpha \in (0,1). \tag{4.11}$$

Now, let $\chi_0 < \delta$, then from (4.11), we have $\chi(t) = 2\chi_0 E_{\alpha,1}(-t^{\alpha}) < 2E_{\alpha,1}(-t^{\alpha}) < \epsilon$ whenever $\chi_0 < \delta = \frac{\epsilon}{2E_{\alpha,1}(-t^{\alpha})}$

Therefore given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ (independent of $t$) : $\chi(t) < \epsilon$ whenever $\chi_0 < \delta$

If $\mathbb{T} = \mathbb{N}_0$ we have that $\mu = 1$, so that (4.9) becomes;

$$= -2\Big[\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2\Big] - 2\Big[(\upsilon_1 + \upsilon_2 - 3\upsilon_3)^2 + (-\upsilon_1 + \upsilon_2 + \upsilon_2\upsilon_3^2)^2 + (3\upsilon_1 + \upsilon_1^2\upsilon_2^2\upsilon_3 + \upsilon_3)^2\Big],$$

$$^{C\mathbb{T}}D_{+}^{\alpha}\mathcal{L}^{\Delta}(t;\upsilon_1,\upsilon_2) \leq -2\Big[\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2\Big];$$

considering the same comparison system as (4.10), we also arrive at the same conclusion as (4.11). Since all the conditions of Theorem 3.3 are satisfied, and zero solution of the comparison system (4.10) is stable, then we conclude that the zero solution of system (4.8) is stable and also asymptotically stable.

Figure 2 below is the graphical representation of $\chi(t) = E_{\alpha,1}(-2t^{\alpha})$. The behavior of the curves further buttresses the stability of $\chi(t)$ over time for $\alpha \in (0,1)$.

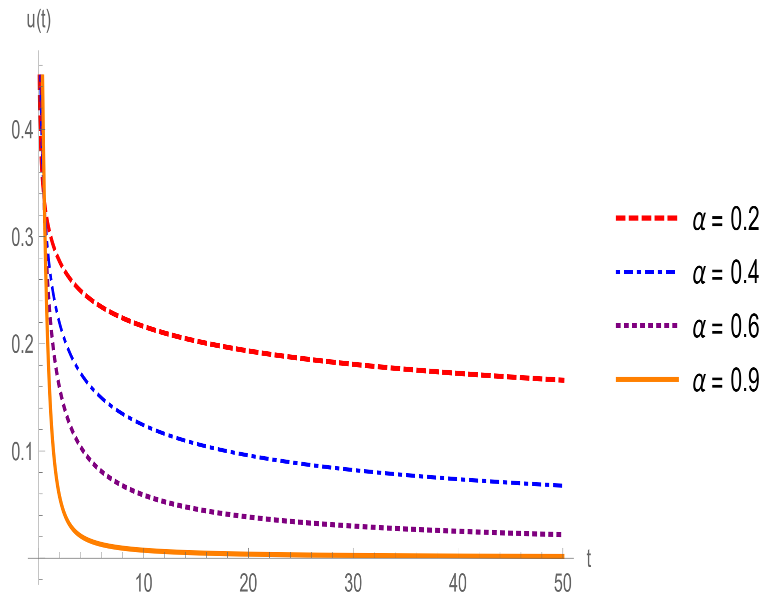


**Figure 2.** Graph of $\chi(t) = E_{\alpha,1}(-2t^{\alpha})$ against $t$.

## 5. Conclusions

In conclusion, our study significantly advances the understanding of Lyapunov stability for Caputo FrDET. The novelty of our work is in the development of a new Dini derivative (the Caputo FrΔDiD) for a Lyapunov function, which preserves the properties of FrD, requires only right dense continuity of the function and depends on the initial data $\mathcal{L}(t_0, \nu_0)$. Our novel definition generalizes existing definitions because it unifies the continuous ($\sigma(t) = t$) and discrete ($\sigma(t) > t$) time domain, as can be observed in Eqs (2.10) and (2.11). We have also shown the theoretical applicability of this definition in Theorems 3.1, 3.2, and 3.3 and the practical applicability as well as effectiveness in systems (4.4) and (4.8). Also, Figures 1 and 2 show a consistent behavior of the curves (a downward trend towards the trivial solution). This behavior reinforces the stability of the solutions obtained for systems (4.4) and (4.8), providing visual confirmation of the theoretical results. The new concept developed in this work successfully contributes to the advancement of fractional calculus in general and stability theory in particular from a continuous domain to a unified continuous and discrete domain, which is a breakthrough for modeling and other practical applications. By establishing comparison results and stability criteria, we have provided a solid theoretical foundation for analyzing the stability properties of these equations across time scales.

## Author contributions

Michael Precious Ineh: Conceptualization, Methodology, Software, Investigation, Writing original draft; Edet Peter Akpan: Conceptualization, Methodology, Investigation, Supervision; Hossam A. Nabwey: Conceptualization, Software, Investigation, Validation, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

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## Conflict of interest

The authors declare that they have no conflicts of interest.

## References

1. R. Agarwal, D. O'Regan, S. Hristova, Stability of Caputo fractional differential equations by Lyapunov functions, *Appl. Math.*, **60** (2015), 653–676. https://doi.org/10.1007/s10492-015-0116-4
2. R. Agarwal, S. Hristova, D. O'Regan, Applications of Lyapunov functions to Caputo fractional differential equations, *Mathematics*, **6** (2018), 229. https://doi.org/10.3390/math6110229
3. R. P. Agarwal, D. O'Regan, S. Hristova, Stability with initial time difference of Caputo fractional differential equations by Lyapunov functions, *Z. Anal. Anwend.*, **36** (2017), 49–77. https://doi.org/10.4171/ZAA/1579
4. A. Ahmadkhanlu, M. Jahanshahi, On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales, *Bull. Iran. Math. Soc.*, **38** (2012), 241–252.
5. N. R. O. Bastos, Fractional calculus on time scales, 2012, arXiv:1202.2960. https://doi.org/10.48550/arXiv.1202.2960

6. C. A. Benaissa, L. F. Zohra, New properties of the time-scale fractional operators with application to dynamic equations, *Math. Morav.*, **25** (2021), 123–136. https://doi.org/10.5937/MatMor2101123B
7. M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser Boston: Springer, 2001. https://doi.org/10.1007/978-1-4612-0201-1
8. J. Čermák, T. Kisela, L. Nechvátal, Stability and asymptotic properties of a linear fractional difference equation, *Adv. Differ. Equ.*, **2012** (2012), 122. https://doi.org/10.1186/1687-1847-2012-122
9. S. E. Ekoro, A. E. Ofem, F. A. Adie, J. Oboyi, G. I. Ogban, M. P. Ineh, On a faster iterative method for solving nonlinear fractional integro-differential equations with impulsive and integral conditions, *Palest. J. Math.*, **12** (2023), 477–484.
10. S. Georgiev, *Boundary value problems: Advanced fractional dynamic equations on time scales*, Cham: Springer, 2024. https://doi.org/10.1007/978-3-031-38200-0
11. B. Gogoi, U. K. Saha, B. Hazarika, Impulsive fractional dynamic equation with non-local initial condition on time scales, *Bol. Soc. Paran. Mat.*, **42** (2024), 1–13. https://doi.org/10.5269/bspm.65039
12. T. S. Hassan, C. Cesarano, R. A. El-Nabulsi, W. Anukool, Improved Hille-type oscillation criteria for second-order quasilinear dynamic equations, *Mathematics*, **10** (2022), 3675. https://doi.org/10.3390/math10193675
13. T. S. Hassan, R. A. El-Nabulsi, N. Iqbal, A. A. Menaem, New criteria for oscillation of advanced noncanonical nonlinear dynamic equations, *Mathematics*, **12** (2024), 824. https://doi.org/10.3390/math12060824
14. S. Hilger, Analysis on measure chains — A unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56. https://doi.org/10.1007/BF03323153
15. J. Hoffacker, C. C. Tisdell, Stability and instability for dynamic equations on time scales, *Comput. Math. Appl.*, **49** (2005), 1327–1334. https://doi.org/10.1016/j.camwa.2005.01.016
16. D. K. Igobi, M. P. Ineh, Results on existence and uniqueness of solutions of dynamic equations on time scale via generalized ordinary differential equations, *Int. J. Appl. Math.*, **37** (2024), 1–20. https://doi.org/10.12732/ijam.v37i1.1
17. D. K. Igobi, E. Ndiyo, M. P. Ineh, Variational stability results of dynamic equations on time-scales using generalized ordinary differential equations, *World J. Appl. Sci. Technol.*, **15** (2024), 245–254. https://doi.org/10.4314/wojast.v15i2.14
18. M. P. Ineh, J. O. Achuobi, E. P. Akpan, J. E. Ante, CDq On the uniform stability of Caputo fractional differential equations using vector Lyapunov functions, *J. Niger. Assoc. Math. Phys.*, **68** (2024), 51–64. https://doi.org/10.60787/jnamp.v68no1.416
19. S. M. Jung, Functional equation and its Hyers-Ulam stability, *J. Inequal. Appl.*, **2009** (2009), 181678. https://doi.org/10.1155/2009/181678
20. E. Karapınar, N. Benkhettou, J. E. Lazreg, M. Benchohra, Fractional differential equations with maxima on time scale via Picard operators, *Fac. Sci. Math.*, **37** (2023), 393–402. https://doi.org/10.2298/FIL2302393K

21. B. Kaymakçalan, Lyapunov stability theory for dynamic systems on time scales, *Int. J. Stoch. Anal.*, **5** (1992), 312068. https://doi.org/10.1155/S1048953392000224

22. B. Kaymakcalan, Existence and comparison results for dynamic systems on a time scale, *J. Math. Anal. Appl.*, **172** (1993), 243–255. https://doi.org/10.1006/jmaa.1993.1021

23. M. E. Koksal, Stability analysis of fractional differential equations with unknown parameters, *Nonlinear Anal. Model.*, **24** (2019), 224–240. https://doi.org/10.15388/NA.2019.2.5

24. V. Kumar, M. Malik, Existence, stability and controllability results of fractional dynamic system on time scales with application to population dynamics, *Int. J. Nonlinear Sci. Numer. Simul.*, **22** (2021), 741–766. https://doi.org/10.1515/ijnsns-2019-0199

25. V. Lakshmikantham, S. Sivasundaram, B. Kaymakçalan, *Dynamic systems on measure chains*, New York: Springer, 1996. https://doi.org/10.1007/978-1-4757-2449-3

26. N. Laledj, A. Salim, J. E. Lazreg, S. Abbas, B. Ahmad, M. Benchohra, On implicit fractional q-difference equations: Analysis and stability, *Math. Methods Appl. Sci.*, **45** (2022), 10775–10797. https://doi.org/10.1002/mma.8417

27. J. P. LaSalle, Stability theory and the asymptotic behavior of dynamical systems, *Dyn. Stab. Struct.*, 1967, 53–63. https://doi.org/10.1016/B978-1-4831-9821-7.50008-9

28. X. Liu, B. Jia, L. Erbe, A. Peterson, Stability analysis for a class of nabla $(q, h)$-fractional difference equations, *Turk. J. Math.*, **43** (2019), 664–687. https://doi.org/10.3906/mat-1811-96

29. N. K. Mahdi, A. R. Khudair, An analytical method for $q$-fractional dynamical equations on time scales, *Partial Differ. Equ. Appl. Math.*, **8** (2023), 100585. https://doi.org/10.1016/j.padiff.2023.100585

30. N. K. Mahdi, A. R. Khudair, Some analytical results on the $\Delta$-fractional dynamic equations, *TWMS J. Appl. Eng. Math.*, 2022.

31. K. Mekhalfi, D. F. Torres, Generalized fractional operators on time scales with application to dynamic equations, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3489–3499. https://doi.org/10.1140/epjst/e2018-00036-0

32. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.

33. K. S. Nisar, C. Anusha, C. Ravichandran, A non-linear fractional neutral dynamic equations: Existence and stability results on time scales, *AIMS Mathematics*, **9** (2024), 1911–1925. https://doi.org/10.3934/math.2024094

34. S. Rashid, N. M. Aslam, K. S. Nisar, D. Baleanu, G. Rahman, A new dynamic scheme via fractional operators on time scale, *Front. Phys.*, **8** (2020). https://doi.org/10.3389/fphy.2020.00165

35. S. J. Sadati, R. Ghaderi, A. Ranjbar, Some fractional comparison results and stability theorem for fractional time delay systems, *Rom. Rep. Phys.*, **65** (2013), 94–102.

36. M. R. Segi Rahmat, M. S. Md Noorani, Caputo type fractional difference operator and its application on discrete time scales, *Adv. Differ. Equ.*, **2015** (2015), 160. https://doi.org/10.1186/s13662-015-0496-5

37. B. D. Singleton, The life and work of D.H. Hyers, 1913–1997, *Nonlinear Funct. Anal. Appl.*, **11** (2006), 697–732.

38. T. T. Song, G. C. Wu, J. L. Wei, Hadamard fractional calculus on time scales, *Fractals*, **30** (2022), 2250145. https://doi.org/10.1142/S0218348X22501456

39. S. Streipert, Dynamic equations on time scales, In: *Nonlinear systems-Recent developments and advances*, 2023. https://doi.org/10.5772/intechopen.104691

40. P. Veeresha, N. S. Malagi, D. G. Prakasha, H. M. Baskonus, An efficient technique to analyze the fractional model of vector-borne diseases, *Phys. Scr.*, **97** (2022), 054004. https://doi.org/10.1088/1402-4896/ac607b

**Supplementary**

**Table 1.** Abbreviation key.

| Abbreviation | Definition |
|---|---|
| FrDE | Fractional Differential Equations |
| FrDfE | Fractional Difference Equations |
| FrDET | Fractional Dynamic Equations on Time scale |
| FrD | Fractional Derivative |
| Fr$\Delta$D | Fractional Delta Derivative |
| Fr$\Delta$DiD | Fractional Delta Dini Derivative |
| G-L | Grunwald-Letnikov |
| IVP | Initial Value Problem |
| LF | Lyapunov Function |
| rd | right dense |
| rs | right scattered |
| ls | left scattered |
| ld | left dense |

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