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Research article

Lump-type kink wave phenomena of the space-time fractional phi-four equation

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Abstract: This article examines the space-time fractional phi-four (PF) model which is an improvement of the Klein-Fock-Gordon (KFG) model that is relevant in quantum mechanics, transmission of ultra-short pulses in optical fibers, de Broglie wave duality, and compound particles in spinless relativistic systems. New solitary wave solutions of the fractional phi-four equation were found with the help of the Riccati-Bernoulli sub-ODE method together with the Bäcklund transformation. The conformable fractional derivative played the role of improving the modeling of the complex dynamical systems. The obtained solutions, which were illustrated by giving 2D, 3D MATLAB plots and contour illustrations, were s eful in nuclear, particle physics and fluid mechanics. Thus, the current work demonstrates the applicability of the proposed approach in handling the fractional order deviations and contributes to the investigation of the dynamic behavior of the PF model.

Keywords: Bäcklund transformation; fractional phi-four (PF); Riccatti equation; symbolic computation; solitary wave solutions
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

Over the past decade or so, there has been an increasing interest among scientists in the study of a specific type of PDE known as nonlinear partial differential equations (NLPDEs). These equations are used to model many phenomena like fluid dynamics, operation of optical fibers, biological networks, nonlinear optics, superconductivity, and plasmas, amongst other things. Theoretical studies have concentrated on elaborate procedures to find soliton solutions for nonlinear wave equations that are critical for the analysis of these model systems [1–3]. To address such nonlinear issues, researchers have developed a number of complex mathematical tools during the last few decades to provide both qualitative and quantitative approaches. The numerical techniques are within those in the analytical techniques. Among the emergent techniques, some of them are the Hamiltonian method [4], He's variational iteration technique [5], the generalized exponential rational function technique [6], the direct algebraic technique [7]. Furthermore, relatively complex methods, like the Jacobi elliptic function method, have been employed to analyze the structure of the nonlinear systems, and uncover some exact solutions [8–10].

It has been seen that in recent years the research community has shown much fascination toward the application of sophisticated mathematical tools associated with the fundamentals of fractional-order differential equations because of their usefulness in modeling the practical problems across divers areas spanning from quantum mechanics to fluid mechanics and optical fibers [11–13]. It is important to note that there are several numerical techniques for addressing inverse problems, namely the homotopy method, the multigrid method, the multigrid-homotopy method, and the wavelet multiscale method [14–16]. On this basis, the present work is devoted to the analysis of the space-time fractional phi-four (PF) model and plays a significant role in quantum mechanics, ultrashort pulse transmission in optical fibers, and the wave-particle problem. In this work, the technique of the Riccati-Bernoulli sub-ODE method and Bäcklund transformation are employed, and new solitary wave solutions of the space-time fractional PF model are derived. These solutions, employing conformable fractional derivatives, provide fresh perspectives on fractional-order variations that are beneficial in nuclear and particle physics and fluid mechanics [17–19]. It should be pointed out that the proposed methodology reveals a genuine advantage of fractional-order analysis as a PF model, which is illustrated by the graphs. This work benefited from progressing with the recent techniques while showing new views on the modeling and analysis of fractional-order dynamics [20–22].

In the latest stages, the fractional calculus has become a central aspect of the fractional-order systems within a broad range of disciplines related to, for instance, physics, chemistry, biology, engineering, nonlinear optics, and fractional dynamics [23–25]. Its application is widespread in the fields of control theory, ecology, signal processing, systems identification, and fluid mechanics [26–30]. Fractional calculus provides a better understanding of other complex physical processes through an extended modeling of memory and hereditary effects that are present in numerous natural systems. It has therefore become mandatory to investigate exact solutions of different FDEs in the course of developing the applied science. As it has been pointed out that fractional models are gaining importance in the representation of different phenomena, a number of analytical and numerical techniques have been established to solve the FDEs. These techniques include the tanh-sech method [31], first integral method [32], (G'/G)-expansion method [33], and the newly developed modified Kudryashov method [34]. These methods are useful for some of the elaborate behaviors of fractional systems and assist researchers in solving a host of new real-world problems with a higher degree of accuracy and precision.

This article considers the nonlinear fractional phi-four model that has been discussed in [35]:

$$D_t^{2\alpha}(f) - D_x^{2\alpha}(f) + \lambda^2(f) + \sigma(f)^3 = 0, \qquad 0 < \alpha \leq 1, \tag{1.1}$$

where (α) is the order of the conformable fractional derivative, and (σ) and (λ) stand for the linear and nonlinear parts, respectively. The phi-four model is one of the variations of the Klein-Fock-Gordon model, which reflects core processes in nuclear and partical physics, the interaction of kink and anti-kink solitary waves in particular. The KFG model is deeply connected with the Schrödinger equation and was introduced by Oskar Klein and Walter Gordon. This model has been used in numerous scopes of applied research. The ability to simulate solitonic wave characteristics entails its application in a classical and quantum viewpoint of wave phenomena from quantum field theory to condensed matter physics. This model has a fractional extensions, which is more general, that enables the study of systems with memory and nonlocal interactions.

In recent years, the more complex nonlinear space-time fractional phi-four equation has been solved by numerous methods to obtain solitary wave solutions. All these methods provide sound approaches to investigate the interactions within this system. Some of them include the $exp(-\phi)$ method and the modified Kudryashov technique [36], and the generalized Kudryashov method [37]. Furthermore, the new extended direct algebraic scheme [38], Weierstrass elliptic function method [39], mapping technique [40], and unified method [41] have been utilized. Some other refined methods, including the modified simple equation method [42] and the tanh function method [43], also help to evaluate the exact solutions for the fractional phi-four model, which shows the increasing complexity in solving fractional nonlinear equations.

The main objective of this current work is to obtain some additional closed-form wave solutions of the space-time fractional phi-four equation, with the help of the Riccati-Bernoulli sub-ODE method in association with the Bäcklund transformation. These results are then compared to the results derived using the generalized (G'/G)-expansion method [33]. In this regard, we also capture the influence of fractional derivate order in wave solutions. Thus, the obtained solutions will be discussed in view of some specific parameter values relevant in physical applications. In contrast to the current studies, our research presents a wider range of more accurate solutions to the indicated fractional order for explaining various aspects of wave behavior while giving stronger versatility to describe other nonlinear physical phenomena in the framework of the fractional phi-four equation. The Riccati-Bernoulli sub-ODE method is an efficient method for the analysis of nonlinear PDEs for fractional order systems. One of its strengths is that it is capable of producing exacting analytical solutions, which are trigonometric, hyperbolic, and rational forms, and it helps to understand the physical processes modeled by the equations. Unlike other numerical methods, such as the homotopy perturbation method or multigrid method, which may need operational settings or boundary conditions, the Riccati-Bernoulli is quite easy and time effective. Also, it can be used for a vast class of nonlinear equations, which allows broadening of the understanding of wave properties, including lump-type kink solutions, under integer as well as fractional conditions. These are the characteristics of this package that make it a powerful tool for analysing the behavior of space-time fractional models as presented in this study.

In addition, the operator integrating α-derivatives of powers agrees exactly with the idea of conformable fractional derivatives [44].

$$D_{\phi}^{\alpha}W(\phi) = \lim_{i\to 0}\frac{W(i(\phi)^{i-\alpha} - W(\phi))}{i}, \quad 0 < \alpha \leq 1. \tag{1.2}$$

$$\begin{cases} D_{\phi}^{\alpha}\phi^{p} = p\phi^{p-\alpha}, \\ D_{\phi}^{\alpha}\left(p_1\eta(\phi) \pm p_2 t(\phi)\right) = p_1 D_{\phi}^{\alpha}(\eta(\phi)) \pm m_2 D_{\phi}^{\alpha}(t(\phi)), \\ D_{\phi}^{\alpha}\left[f \circ g\right] = \phi^{1-\alpha} g\left(\phi\right) D_{\phi}^{\alpha} f\left(g(\phi)\right). \end{cases} \tag{1.3}$$

2. Methodology

In this section, we focus on studying the given nonlinear fractional partial differential equation with the help of the systematic Riccati-Bernoulli sub-ODE method [45–47]. The proposed methodology elaborates on the information based on the principles produced from the Riccati equation $\frac{d\phi}{d\xi} = \varrho+\phi(\psi)^2$ and is optimal for the fractional-order systems. The approach is applied to the generalized form of a nonlinear fractional PDE, structured as follows:

$$U\left(u, D_t^{\alpha}u, D_x^{\alpha}u, D_t^{2\alpha}u, D_x^{2\alpha}u, \ldots\right) = 0. \tag{2.1}$$

First, a change of variable of the form $u(t, r_1, r_2, r_3, \ldots, r_s) = F(\psi)$ is set where the character (ψ) can have different meanings depending on the context. This transformation simplifies Eq (2.1), converting it into the following nonlinear ordinary differential equation (NODE):

$$V\left(u, \frac{du}{d\psi}, \frac{d^2u}{d\psi^2}, \frac{d^3u}{d\psi^3}, \ldots\right) = 0. \tag{2.2}$$

After this, we use the series expansion to rewrite the solution of the equation given by Eq (2.2),

$$f(x,t) = F(\psi) = \sum_{i=-m}^{m} y_i T(\psi)^i. \tag{2.3}$$

This approach presents the solution in the form of a series and can be in the form of a power series or any other form of representation depending on the problem at hand. By substituting this expansion into Eq (2.2), a system of algebraic equations is obtained. The systematic numerical solving of these equations establishes the coefficients of the series and essentially transforms the original nonlinear equation. The solution of these algebraic systems yields a precise or an approximate solution that may give clearer information regarding the dynamic behavior of the system governed by the equation of interest.

In this case, a function $T(\psi) = \frac{-\varrho R_2+R_1\phi(\psi)}{R_1+R_2\phi(\psi)}$ is derived through the use of Bäcklund transformation. The next crucial step includes the presenting of a $\phi(\psi)$ formula, where (ψ) is obtained from the traveling wave transformation. It can also be seen that $\phi(\psi)$ must satisfy the Riccati equation so that the given transformation as well as the structure of the solution fit well within the context of the nonlinear framework, where (ϱ), (R_1), and (R_2) are the constants to be determined. The positive integer (m) is a balance so that the coefficient of the highest order of the derivative equals the coefficient of the nonlinear terms in Eq (2.2). Substituting Eq (2.3) into Eq (2.2) and equating the coefficients of $\phi(\psi)$ to zero, we get a set of algebraic equations. Solving this system gives the exact values of the constants in the linear equation known as (y_i), (ϱ), (a), and (b). Also the Ricatti equation has different types of solutions [48], which give possibilities to define different types of wave solutions derived by this method.

$$\begin{aligned}
\phi(\psi) &= \begin{cases} -\sqrt{-\varrho}\tanh(\sqrt{-\varrho}\psi), & \text{as } \varrho < 0, \\ -\sqrt{-\varrho}\coth(\sqrt{-\varrho}\psi), & \text{as } \varrho < 0, \end{cases} \\
\phi(\psi) &= -\frac{1}{\psi}, \quad \text{as } \varrho = 0, \\
\phi(\psi) &= \begin{cases} \sqrt{\varrho}\tan(\sqrt{\varrho}\psi), & \text{as } \varrho > 0, \\ -\sqrt{\varrho}\cot(\sqrt{\varrho}\psi), & \text{as } \varrho > 0. \end{cases}
\end{aligned} \tag{2.4}$$

3. Execution of the problem

The phi-four model may be considered as a particular case of the special relativistic Klein-Fock-Gordon (KFG) equation derived by Oskar and Walter Gordon. This equation is based on the Schrödinger equation and includes relativities. Therefore, it is crucial in quantum mechanics. This model is one of the main models in condensate matter physics and is successfully applied for phase transmutations and spontaneous symmetry breakdown. Moreover, it is widely used as a probe for studying quantum effects in the field of particle and nuclear physics, wave-corpuscle dualism, and the formation of solitons in collisionless plasma. These applications demonstrate the ability of the model to handle field behavioral profiles that are both traditional and quantum in physical systems. Thus using the wave transformation $f(x,t) = F(\psi) = a\frac{x^\alpha}{\alpha} - b\frac{t^\alpha}{\alpha}$ transforms Eq (1.1) into the following expression:

$$\left(b^2 - a^2\right)\frac{d^2F}{d\psi^2} + \lambda^2 F(\psi) + \sigma\left(f(\psi)\right)^3 = 0. \tag{3.1}$$

If the balancing procedure is made between $\frac{d^2F}{d\psi^2}$ and (F^3), then $(m = 1)$ is obtained. By substituting Eqs (2.3) and (2.2) into Eq (3.1), and then equating the coefficients of various powers of $\phi(\psi)$ to be zero, a set of algebraic equations results. It helps to reach a conclusion of the most suitable unknown parameters for a given system, which provides the exact solutions for the same. The equations obtained in the course of this procedure form a system of algebraic equations which can be easily solved by the means of Maple, Mathematica, MATLAB, or other analogous programs. This enables the identification of multiple distinct solution cases, outlined as follows:

$$\begin{aligned}
&\phi^0 : -\sigma {y_{-1}}^3{R_2}^6 + 2a^2y_{-1}{R_2}^6\varrho^2 - 2b^2y_{-1}{R_2}^6\varrho^2 = 0,\\
&\phi^1 : 3\sigma {y_{-1}}^2{R_2}^6y_0\varrho = 0,\\
&\phi^2 : -2b^2y_{-1}{R_2}^6\varrho^3 + 2a^2y_{-1}{R_2}^6\varrho^3 - \lambda^2y_{-1}{R_2}^6\varrho^2 - 3\sigma y_{-1}{R_2}^6{y_0}^2\varrho^2 - 3\sigma {y_{-1}}^2{R_2}^6y_1\varrho^2 = 0,\\
&\phi^3 : 6\sigma y_{-1}{R_2}^6y_0y_1\varrho^3 + \lambda^2y_0\varrho^3{R_2}^6 + \sigma {y_0}^3\varrho^3{R_2}^6 = 0,\\
&\phi^4 : -\lambda^2y_1\varrho^4{R_2}^6 - 2b^2y_1{R_2}^6\varrho^5 + 2a^2y_1{R_2}^6\varrho^5 - 3\sigma {y_0}^2y_1\varrho^4{R_2}^6 - 3\sigma y_{-1}{R_2}^6{y_1}^2\varrho^4 = 0,\\
&\phi^5 : 3\sigma y_0{y_1}^2\varrho^5{R_2}^6 = 0,\\
&\phi^6 : -2b^2y_1{R_2}^6\varrho^6 + 2a^2y_1{R_2}^6\varrho^6 - \sigma {y_1}^3\varrho^6{R_2}^6 = 0.
\end{aligned} \tag{3.2}$$

Case 1.

$$y_0 = 0, y_1 = 0, y_{-1} = \sqrt{\left(2\sigma a^2 - 2\sigma b^2\right)^{-1}}\lambda^2, a = a, b = b, \varrho = 1/2\,\frac{\lambda^2}{(a-b)(a+b)}. \tag{3.3}$$

Case 2.

$$y_0 = 0, y_1 = -1/4\,\frac{\lambda^2}{\sigma y_{-1}}, y_{-1} = y_{-1}, a = 1/8\,\sqrt{-\frac{-2\lambda^4 - 64\sigma {y_{-1}}^2b^2}{\sigma}}{y_{-1}}^{-1}, b = b, \varrho = 4\,\frac{\sigma {y_{-1}}^2}{\lambda^2}. \tag{3.4}$$

In this connection, we have the above two cases that were developed from the algebraic system stated in one of the previous sections. From these cases, we expect to find the families of solutions relevant to (ϱ) in order to provide the various behaviors of the solution in various scenarios as the conclusion. These families of solutions will provide the detail regarding the time evolution of the system and physical realization of a system, which will possibly exhibit rich behavior for the variation in the values of (ϱ).

Here also, we provide the solutions for Case 1 given particular values of (ϱ) and illustrate how such decisions give various solution sets of Eq (1.1). These solutions explain the probability distribution of the system outputs given certain inputs, because every choice of (ϱ) configuration yields a clearly defined solution meeting the equation and condition imposed earlier.

Solutions set 1. ($\varrho < 0$).

$$f_1(x,t) = \frac{\sqrt{(2\sigma a^2 - 2\sigma b^2)^{-1}}\lambda^2\left(R_1 - 1/2\,R_2\sqrt{-2\,\frac{\lambda^2}{(a-b)(a+b)}}\tanh\left(1/2\,\sqrt{-2\,\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)\right)}{\left(-1/2\,\frac{\lambda^2 R_2}{(a-b)(a+b)} - 1/2\,R_1\sqrt{-2\,\frac{\lambda^2}{(a-b)(a+b)}}\tanh\left(1/2\,\sqrt{-2\,\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)\right)}, \tag{3.5}$$

or

$$f_2(x,t) = \frac{\lambda^2\left(R_1 - \frac{1}{2}R_2\sqrt{-2\lambda^2/\left((a-b)(a+b)\right)}\coth\left(\frac{1}{2}\sqrt{-2\lambda^2/\left((a-b)(a+b)\right)}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} - \frac{1}{2}R_1\sqrt{-2\lambda^2/\left((a-b)(a+b)\right)}\coth\left(\frac{1}{2}\sqrt{-2\lambda^2/\left((a-b)(a+b)\right)}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)}. \tag{3.6}$$

Solutions set 2.

$$f_3(x,t) = \frac{\lambda^2\left(R_1 + \frac{1}{2}R_2\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\tan\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} + \frac{1}{2}R_1\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\tan\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)}, \tag{3.7}$$

or

$$f_4(x,t) = \frac{\sqrt{\frac{1}{2\sigma a^2 - 2\sigma b^2}}\cdot\lambda^2\left(R_1 - \frac{1}{2}R_2\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\cot\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} - \frac{1}{2}R_1\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\cot\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)\right)}. \tag{3.8}$$

Solutions set 3. ($\varrho = 0$).

$$f_5(x,t) = \frac{\sqrt{2\sigma a^2 - 2\sigma b^2}\cdot\lambda^2\left(R_1 - R_2\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)^{-1}\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} - R_1\left(\frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}\right)^{-1}}. \tag{3.9}$$

Here we show associated solutions for Case 2 values of (ϱ), illustrating how each selection yields unique solution sets for Eq (1.1).

Solutions set 4. ($\varrho < 0$), $a = 1/8\,\sqrt{-\frac{-2\,\lambda^4 - 64\,\sigma\,{y_{-1}}^2 b^2}{\sigma}}{y_{-1}}^{-1}$ and $\psi = \frac{ax^\alpha}{\alpha} - \frac{bt^\alpha}{\alpha}$.

$$f_6(x,t) = y_{-1}\frac{\left(R_1 - 2R_2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\tanh\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - 2R_1\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\tanh\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)} - \frac{\lambda^2}{4}\frac{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - 2R_1\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\tanh\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\sigma y_{-1}\left(R_1 - 2R_2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\tanh\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}, \tag{3.10}$$

or

$$f_7(x,t) = y_{-1}\frac{\left(R_1 - 2R_2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\coth\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - 2R_1\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\coth\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)} - \frac{\lambda^2}{4}\frac{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - 2R_1\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\coth\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\sigma y_{-1}\left(R_1 - 2R_2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\coth\left(2\sqrt{-\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}. \tag{3.11}$$

Solutions set 5. ($\varrho < 0$).

$$f_8(x,t) = y_{-1}\frac{\left(R_1 + 2R_2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\tan\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} + 2R_1\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\tan\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)} - \frac{\lambda^2}{4}\frac{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} + 2R_1\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\tan\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\sigma y_{-1}\left(R_1 + 2R_2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\tan\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}, \tag{3.12}$$

or

$$f_9(x,t) = y_{-1}\frac{\left(R_1 - 2R_2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\cot\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - 2R_1\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\cot\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)} - \frac{\lambda^2}{4}\frac{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - 2R_1\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\cot\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}{\sigma y_{-1}\left(R_1 - 2R_2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\cot\left(2\sqrt{\frac{\sigma y_{-1}^2}{\lambda^2}}\psi\right)\right)}. \tag{3.13}$$

Solutions set 6. ($\varrho = 0$).

$$f_{10}(x,t) = \frac{y_{-1}\left(R_1 - \frac{R_2}{\psi}\right)}{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - \frac{R_1}{\psi}\right)} - \frac{\lambda^2}{4}\cdot\frac{\left(-4\frac{\sigma y_{-1}^2 R_2}{\lambda^2} - \frac{R_1}{\psi}\right)}{\sigma y_{-1}\left(R_1 - \frac{R_2}{\psi}\right)}. \tag{3.14}$$

The comparison table (Table 1) emphasizes more on the proposed Riccati-Bernoulli sub-ODE method coupled with Bäcklund transformation than the (G'/G)-expansion method for solving, more efficiently the space-time fractional phi-four model. The merits of the present method include its flexibility of obtaining valid solutions for every case including the critical cases where $\varrho = 0$ obviously missing in the (G'/G)-expansion method. When $\varrho < 0$, the constructed method yields complex solutions with hyperbolic functions and tracks a richer variety of behavior compared to the offered simpler forms according to the alternative method. Likewise, the present method provides explicitly trigonometric representations with $\varrho > 0$, and more general relationships between the parameters. In light of this work, the (G'/G)-expansion method yields solutions with restricted, fewer imaginary elements. The flexibility and detail of the present method underscore its advantage to capture those rich dynamics of wave and fractional order system modeling.

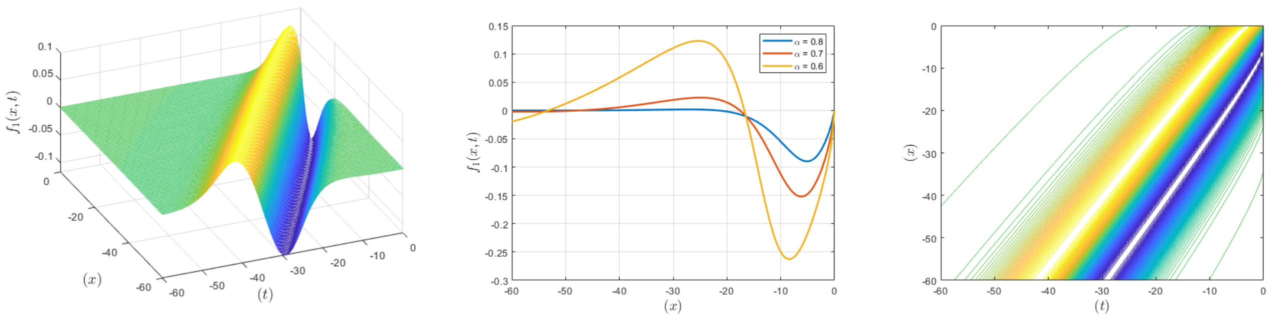
Table 1. Comparison of the current approach with the alternative approach, specifically the (G'/G)-expansion method [33].

Case I: $\varrho < 0$ Present method
$f_2(x,t) = \dfrac{\lambda^2\left(R_1 - \frac{1}{2}R_2\sqrt{-2\lambda^2/((a-b)(a+b))}\coth\left(\frac{1}{2}\sqrt{-2\lambda^2/((a-b)(a+b))}\left(\frac{ax^\alpha}{\alpha}-\frac{bt^\alpha}{\alpha}\right)\right)\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} - \frac{1}{2}R_1\sqrt{-2\lambda^2/((a-b)(a+b))}\coth\left(\frac{1}{2}\sqrt{-2\lambda^2/((a-b)(a+b))}\left(\frac{ax^\alpha}{\alpha}-\frac{bt^\alpha}{\alpha}\right)\right)}.$
Case I: $b < 0$ (G'/G)-expansion method
$u_1 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right),$
$u_2 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right).$
Case II: $\varrho > 0$ Present method
$f_4(x,t) = \dfrac{\sqrt{\frac{1}{2\sigma a^2-2\sigma b^2}}\cdot\lambda^2\left(R_1 - \frac{1}{2}R_2\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\cot\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha}-\frac{bt^\alpha}{\alpha}\right)\right)\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} - \frac{1}{2}R_1\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\cot\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{\lambda^2}{(a-b)(a+b)}}\left(\frac{ax^\alpha}{\alpha}-\frac{bt^\alpha}{\alpha}\right)\right)}.$
Case II: $\varrho > 0$ (G'/G)-expansion method
$u_3 = \mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}\tan\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right),$
$u_4 = \pm i\sqrt{\frac{\rho}{Q^2+4S\psi}}\cot\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right).$
Case III: $\varrho = 0$ Present method
$f_5(x,t) = \dfrac{\sqrt{2\sigma a^2-2\sigma b^2}\cdot\lambda^2\left(R_1 - R_2\left(\frac{ax^\alpha}{\alpha}-\frac{bt^\alpha}{\alpha}\right)^{-1}\right)}{-\frac{1}{2}\frac{\lambda^2 R_2}{(a-b)(a+b)} - R_1\left(\frac{ax^\alpha}{\alpha}-\frac{bt^\alpha}{\alpha}\right)^{-1}}.$
Case III: $b = 0$ (G'/G)-expansion method
There are no solutions corresponding to $\varrho = 0$ in this method.

4. Results and discussion

The goal in this paper is to introduce an effective analytical technique for future studies of essential nuclear and particle physics and fluid dynamics problems employing exact solutions of nonlinear PDEs. With the help of the Bäcklund transformation, these PDEs are effectively reduced to the ODEs which makes the derivation of the solutions easier when compared with other transformations. The Riccati-Bernoulli sub-ODE system is essential for the determination of the coefficients of the series and for providing more understanding on the integrated forms of the systems. In this way, we found out the different types of hyperbolic, rational, and trigonometric solutions, each of the which are associated with a particular structural configuration of the system. The hyperbolic solutions are associated with solitary wave solutions, characterizing stable, non-dispersive waves, which are typical for significant nonlinear structures and proportions where the balance is traceable between the dispersion and nonlinearity. These solutions are important for simulating localized energy transport in schemes as quantum fields or fiber optics. The rational solutions, algebraically decaying at large as lump-type structures, reflect localized processes such as rogue waves or peak-like structures to illustrate their engagements with other waves or fields. The trigonometric solutions depict periodic wave forms; they are oscillatory or periodic in nature in the system that is associated with wave particle duality and ultra-short pulse generation. Each kind of solution gives an understanding of the characteristic dynamics of the configurations of the space-time fractional phi-four model, which is enriched with fractional-order information. Surprisingly, it yields the opportunity to construct the analytical solutions with the required accuracy, and it can expand the existing paradigms of nuclear physics, particle physics, fluid dynamics, and many other fields. In particular, this framework is helpful for describing and analyzing such objects as lump-type kink solitons, which play important roles in the theory of topological solitons in particle physics and dislocations in material science. By associating these solutions with such real-world phenomena, this work demonstrates how the proposed framework can be applied to solve actual problems in these disciplines, including the nature of high-energy particles and behavior of fluid systems.

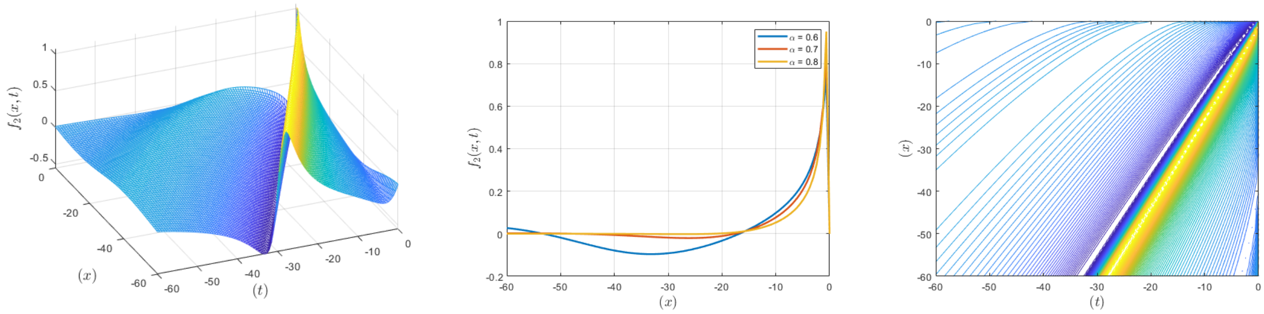
The plot in Figure 1 is a lump-type kink solution, which is the most significant soliton structure concerning the physically important application. The lump-type kink solution, with increasing amplitude as the fractional parameter (α), has been shown for the proposed fractional system, which manifests the improved stability and energy confinement of the system. This is especially advantageous for numerical solutions to real-life applications in nuclear physics, particle interactions, and fluid dynamics in which such solutions can represent high energy processes, durable solitons, and wave behaviors. The usage of a fractional order therefore brings additional freedom in the management of the character of such systems, which in turn enables the researchers to model and study more exciting and energetic phenomena in different physical fields.



(a) **(b)** **(c)**

Figure 1. Figure showing $3D$, contour, and $2D$ fractional-order variations of the function $f_1(x,t)$.

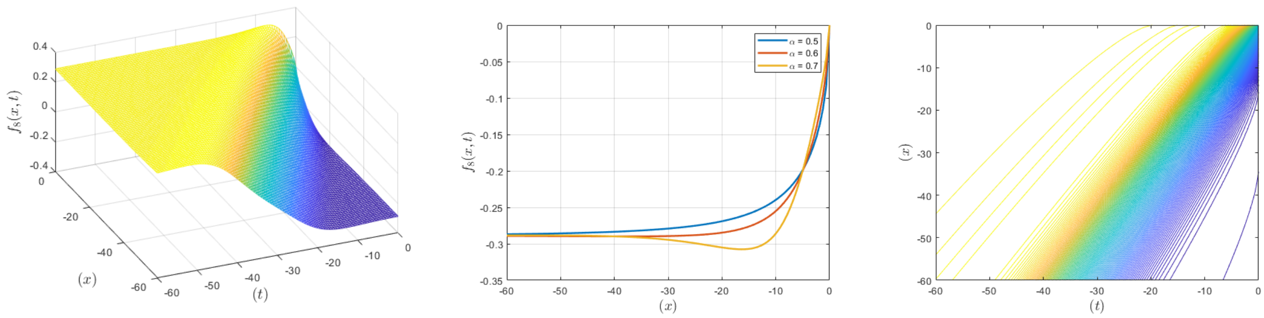
Figure 2 depicts a typical kink solution with increasing the fractional order (α), where the waveform oscillations are observed. Practical consequences of such behavior are found in wave phenomena, turbulence, particle-particle and particle-matrix interactions, and material defects. The perturbations can represent real physical systems, in which instabilities or oscillations are essential, and provide fresh perspectives and solution methodologies for high-dimensional problems in areas such as fluid dynamics, plasma and solid state physics, and quantum field theory. With the help of the given fractional order, it is possible to investigate and control a certain degree of perturbation, which makes this model highly effective with nonlinear dynamics at both the stable and unstable states.



(a) **(b)** **(c)**

Figure 2. Figure showing $3D$, contour, and $2D$ fractional-order variations of the function $f_2(x, t)$.

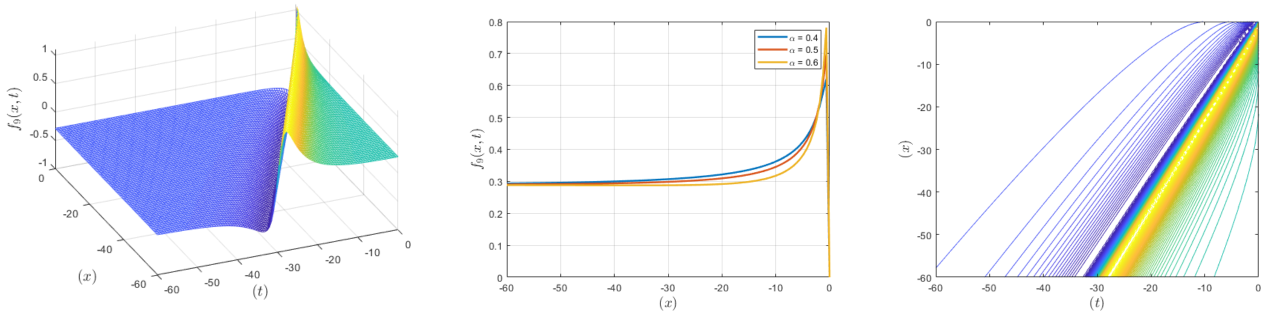
Figure 3 presents a lump-type kink solution, and it is seen that a greater fractional order results in higher stability and less oscillations as compared to the perturbations observed in Figure 2. This behavior has application in wave mechanics, particle physics, fluid dynamics, and material science because the stability and integrity of soliton structures is valuable. With an increase of the fractional order, systems are capable of sustaining solitonic solutions immune to perturbations which in turn makes this particular model essential in situations where predictable and stable waveforms or field solutions are desirable across a range of arenas in the physical sciences.



(a) **(b)** **(c)**

Figure 3. Figure showing $3D$, contour, and $2D$ fractional-order variations of the function $f_8(x, t)$.

A prominent kink solution that sharpens as the fractional parameter increases is shown in Figure 4. This behavior appears especially convenient for all those applications that need cut-off and localized or concentrated signals, waves, or energy densities, as for instance in particle physics, nonlinear optics, fluids mechanics, or solid state physics. By changing the parameter (α), the model offers an excellent tool for tuning the localization. It also describes with high accuracy the localized solitons, shocks, or any topological field configurations in the different physical systems.



(a) **(b)** **(c)**

Figure 4. Figure showing $3D$, contour, and $2D$ fractional-order variations of the function $f_9(x, t)$.

For this study, we have obtained the analytic solutions to the optical fractional differential equations for wave phenomena and it is worth mentioning that despite handling the theoretical formulas, there are practical applications of the concepts. This means that the approaches are based on solutions exhibiting different waves features, including solitons and kink waves, and can be used in conditions described in fluid dynamics, material science, and nonlinear wave excitation. Future work will indeed involve the attempt to compare these theoretical results with experimental data or observed behaviors in such systems for additional confirmation and to ascertain their utility. This integration of theory and experiment will further reduce the distance between mathematical modeling and physical reality, making the value of our discoveries more apparent in application. On this regard, though this research is informative in modeling the analytical solutions of fractional differential equations, there are certain constraints that need to be considered. First, our approach is mainly based on exact solutions, which implies a lot of theoretical heuristics being very useful in obtaining exact analytical solutions, but it is not devoid of much realism, as numerical simulations and experimental data are critically important for heuristics being very useful in complicated real-world processes. As well, it is noteworthy that the applicability of the derived solutions to definite physical processes may need further validation by experimental methods or comparison with experimental evidence, which is outside the present work.

5. Conclusions

Thus, the present investigation accurately implemented the Ricatti-Bernoulli sub-ODE method with the help of Bäcklund transformation to obtain the lump-type kink and anti-kink solutions for the fractional nonlinear wave equation. We also illustrated how fractional-order variations produce localized or expanded wave behaviors compared to integer-order solutions depicted in $3D$ and $2D$ diagrams. Extending from the prior findings, we went further to analyze influences of fractional derivatives on the localization of waves and transitions. Such implications are especially important for the modeling of various dynamical systems in hydrodynamics, plasma physics, and nonlinear optics, where the control of the wave behavior and wave-wave interaction is especially important.

Moreover, the Riccati-Bernoulli sub-ODE approach, which yields the exact solutions of the polynomial nonlinear ODEs, can be not very robust when applied to more complicated, higher-order nonlinear equations or to the multi-dimensional systems. These aspects will form subjects of future researches. Future work is to apply the present method for other nonlinear fractional differential equations with other forms of boundary conditions or in multi-dimensions. Also, the comparing of the analytical solutions with numerical simulations will allow the evaluation of the effectiveness and computational cost of the obtained solutions. The experimental verification of the given solutions, especially in some applied sciences like fluid dynamics or wave propagation, will be a worthy direction for further studies.

Author contributions

Khudhayr A. Rashedi: Formal analysis, visualization, funding, data curation, validation; Musawa Yahya Almusawa: Conceptualization, project administration, writing-review & editing; Hassan Almusawa: Data curation; Tariq S. Alshammari: Investigation, software, resources; Adel Almarashi: Resources, project administration, writing-review & editing. All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflict of interest.

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