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Research article

Parallel one forms on special Finsler manifolds

Salah G. Elgendi*

Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia

* **Correspondence:** Email: selgendi@iu.edu.sa, salahelgendi@yahoo.com.

Abstract: In this paper, we investigated the existence of parallel 1-forms on specific Finsler manifolds. We demonstrated that Landsberg manifolds admitting a parallel 1-form had a mean Berwald curvature of rank of at most $n-2$. As a result, Landsberg surfaces with parallel 1-forms were necessarily Berwaldian. We further established that the metrizability freedom of the geodesic spray for Landsberg metrics with parallel 1-forms was at least 2. We figured out that some special Finsler metrics did not admit a parallel 1-form. Specifically, no parallel 1-form was admitted for any Finsler metrics of nonvanishing scalar curvature, among them the projectively flat metrics with nonvanishing scalar curvature. Furthermore, neither the general Berwald's metric nor the non-Riemannian spherically symmetric metrics admited a parallel 1-form. Consequently, we observed that certain (α,β)-metrics and generalized (α,β)-metrics did not admit parallel 1-forms.

Keywords: parallel 1-form; Landsberg manifolds; scalar curvature; general Berwald's metric; spherically symmetric metrics
Mathematics Subject Classification: 53B40, 53C60

1. Introduction

Parallel 1-forms find diverse applications in both Finsler (or Riemannian) geometry and physics, particularly in general relativity. In Finsler geometry, parallel 1-forms play a significant role. For instance, within the class of (α,β)-metrics, if the 1-form β is parallel with respect to the Levi-Civita connection of the Riemannian metric α, then the Riemannian metric α and the (α,β)-metric share the same geodesic spray. Consequently, the (α,β)-metric becomes a Berwald metric. Additionally, if β is parallel, the Levi-Civita connection and the Cartan connection of the (α,β)-metric coincide (see [1,2]). From an application standpoint, in general relativity, if a metric g admits a parallel vector field and satisfies the Einstein equations, then the energy-momentum tensor vanishes (see [3]).

In Riemannian geometry, a vector field is parallel if and only if its associated 1-form is parallel. This equivalence stems from the metricity of the Levi-Civita connection, which implies that the covariant derivative of the metric tensor vanishes. However, in Finsler geometry, the situation is more complex, especially when Finsler connection is not metrical.

In [4], Kozma and Elgendi delved into the concept of parallel 1-forms on Finsler manifolds. Specifically, considering a Berwald connection attached to a Finsler space (M, F), a 1-form $\beta = b_i(x)y^i$ is termed horizontally parallel (or simply, parallel) if, and only if, the Berwald horizontal covariant derivative of b_i vanishes, i.e., $b_{i|j} = 0$. For a Finsler space (M, F), they explored the connection between the metrizability freedom of the geodesic spray of the Finsler structure F and the existence of parallel 1-forms on (M, F). Furthermore, they employed Finslerian tools to discuss the presence of parallel 1-forms on both Riemannian and Finslerian manifolds.

In this paper, we investigate the existence of parallel 1-forms on certain special Finsler spaces. First, we consider the Landsberg spaces admitting parallel 1-forms. If (M, F) is a Landsberg metric and provides a parallel 1-form, then the rank of the mean Berwald curvature is at most $n - 2$. As by-product, a Landsberg surface that admits a parallel 1-form is Berwaldian. Moreover, if (M, F) is a Landsberg manifold whose geodesic spray is S, then the metrizability freedom of S is at least 2.

The Finsler metrics with scalar curvature are the second type of special Finsler manifolds that we address in the present study. We prove that there is no parallel 1-form existing on Finsler manifolds with nonvanishing scalar curvature. We consequently infer that no parallel 1-form can be admitted for any projectively flat Finsler metrics of nonvanishing scalar curvature. It is not enough for a Finsler metric F to provide a parallel 1-form merely to have a vanishing scalar curvature. Consider the following projectively flat metric with zero flag curvature, which is investigated and provided by Shen [5]

$$F(x, y) = \left\{1 + \langle a, x\rangle + \frac{\langle a, y\rangle - |x|^2\langle a, y\rangle}{\sqrt{|y|^2 - |x|^2|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle}\right\} \times \frac{\left(\sqrt{|y|^2 - |x|^2|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\right)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - |x|^2|y|^2 + \langle x, y\rangle^2}}.$$

This metric admits no parallel 1-forms, where $|\cdot|$ (resp., $\langle\cdot,\cdot\rangle$) refers to the standard Euclidean norm (resp., inner product) on $\mathbb{R}^n$. By the way, since this metric generalizes Berwald's metric [6], then we call it the general Berwald's metric.

Finally, we turn our attention to one of the most significant and diverse classes in Finsler geometry: The class of spherically symmetric metrics. This class has several applications in both Finsler geometry and physics. Now, let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric admitting a parallel 1-form, then we consider its geodesic spray is characterized by the following special formulae of the functions P and Q:

$$P = P(r, s), \quad Q = \frac{s^2 f'(r)}{2r^3 f(r)} - \frac{sP}{r^2} + \frac{1}{2r^2},$$

where $b_i = f(r)x_i$, $f' := \frac{df}{dr}$, and $f(r)$ is a smooth function of r. Then, a question arises, precisely, is this spray metrizable? We show that this spray is only Riemann metrizable. That is, there is no non-Riemannian spherically symmetric metric that provides a parallel 1-form.

The class of spherically symmetric metrics is an example of generalized (α,β)-metrics, while the general Berwald metric is an example of (α,β)-metrics. In conclusion, we demonstrate the existence of (α,β)-metrics and generalized (α,β)-metrics that do not admit parallel 1-forms.

2. Preliminaries

Let M be a smooth manifold of n dimensions, let its tangent bundle be (TM, π_M, M), and let $(\mathcal{T}M, \pi, M)$ be a sub-bundle of nonzero tangent vectors. We use (x^i, y^i) to represent the induced coordinates of TM, where (x^i) is the local coordinate of a base point $x \in M$ and (y^i) represents the tangent vectors $y \in T_xM$, where T_xM is tangent space at x. The tangent structure J of TM is a vector 1-form defined locally by $J = \frac{\partial}{\partial y^i} \otimes dx^i$, where $\otimes$ is the tensor product of $\frac{\partial}{\partial y^i}$ and dx^i. The Liouville or canonical vector field C is a vector field on TM and is defined by $C = y^i \frac{\partial}{\partial y^i}$.

A spray is a vector field S given on the tangent bundle TM with the properties $JS = C$, and $[C, S] = S$. It can be written locally as in the following expression:

$$S = y^j \frac{\partial}{\partial x^j} - 2G^j \frac{\partial}{\partial y^j}, \tag{2.1}$$

where the functions $G^j = G^j(x, y)$ are called the spray coefficients. These functions are smooth and 2-homogeneous in y.

A nonlinear connection is defined by an n-dimensional distribution H on $\mathcal{T}M$, which is the complement of the vertical distribution $V\mathcal{T}M$. So, for each $z \in \mathcal{T}M$, we have the following direct sum

$$T_z(\mathcal{T}M) = H_z(\mathcal{T}M) \oplus V_z(\mathcal{T}M).$$

Each spray S can be associated by a canonical nonlinear connection with a horizontal and vertical projectors given as follows

$$h = \frac{1}{2}(Id + [J, S]), \quad v = \frac{1}{2}(Id - [J, S]). \tag{2.2}$$

The horizontal projector h and the vertical projector v are expressed, locally, by the formulae

$$h = \frac{\delta}{\delta x^k} \otimes dx^k, \qquad v = \frac{\partial}{\partial y^k} \otimes \delta y^k,$$

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^i_k(x, y)\frac{\partial}{\partial y^i}, \quad \delta y^k = dy^k + N^k_i(x, y)dx^i, \quad N^h_i(x, y) = \frac{\partial G^h}{\partial y^i},$$

where N^k_i are the components of the nonlinear connection.

Let K be a vector k-form on M, that is, $K : (\mathfrak{X}(M))^k \longrightarrow \mathfrak{X}(M)$. Each vector k-form K induces graded derivations of the Grassmann algebra of M, namely, i_K and d_K as follows:

$$i_K\varphi = 0, \quad i_K d\varphi = d\varphi \circ K,$$

$$d_K := [i_K, d] = i_K \circ d - (-1)^{k-1} d i_K,$$

where $\varphi \in C^\infty(M)$, $d\varphi$ represents the differential of $\varphi \in C^\infty(M)$. As a special case, for a vector field $\xi \in \mathfrak{X}(M)$, we have the Lie derivative $\mathcal{L}_\xi$ with respect to ξ and the interior product i_ξ by ξ.

The Jacobi endomorphism (or, Riemann curvature) [7] is defined by

$$\Phi = v \circ [S, h] = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j = \left(2\frac{\partial G^i}{\partial x^j} - S(N^i_j) - N^i_k N^k_j\right) \frac{\partial}{\partial y^i} \otimes dx^j.$$

The curvature R of S is defined by

$$R = \frac{1}{2}[h,h] = \frac{1}{2}R^h_{jk}\frac{\partial}{\partial y^h}\otimes dx^j \wedge dx^k, \qquad R^h_{jk} = \frac{\delta G^h_j}{\delta x^k} - \frac{\delta G^h_k}{\delta x^j}.$$

One can see that $R^h_i = R^h_{ij}y^j$. For more details, refer to [8].

We adopt the notations

$$\partial_i := \frac{\partial}{\partial x^i}, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}, \quad \delta_i := \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i(x,y)\frac{\partial}{\partial y^j}.$$

The Berwald connection's coefficients G^h_{ij} [9] are given by $G^h_{ij} = \dfrac{\partial G^h_j}{\partial y^i}$.

Definition 2.1. *A pair (M,F) is termed a Finsler manifold (or, Finsler space), wherein M denotes a smooth n-dimensional manifold and $F : TM \to \mathbb{R}$ with the properties:*

(a) *F is strictly positive and smooth on $\mathcal{T}M$.*

(b) *F is positively 1-homogeneous in y.*

(c) *The metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$ has rank n.*

The function F is known as a Finsler function (or structure, or metric).

The Berwald curvature tensor $\mathcal{G}$ and the Landsberg curvature tensor $\mathcal{L}$ are defined, respectively, by

$$\mathcal{G} = G^h_{ijk}dx^i \otimes dx^j \otimes dx^k \otimes \dot{\partial}_h, \tag{2.3}$$

$$\mathcal{L} = L_{ijk}dx^i \otimes dx^j \otimes dx^k, \tag{2.4}$$

where $L_{ijk} = -\frac{1}{2}FG^h_{ijk}\dot{\partial}_h F$, $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$; see [10]. The mean Berwald curvature E_{jk} of a spray S is defined by [9, Definition 6.1.2] as follows:

$$E_{jk} = \frac{1}{2}G^i_{ijk} = \frac{1}{2}\frac{\partial^3 G^i}{\partial y^i \partial y^j \partial y^k}.$$

Definition 2.2. *A Berwald space is a Finsler space (M,F) where the components G^h_{ijk} of Berwald curvature tensor vanishes identically. Similarly, a Landsberg space is a Finsler space (M,F) in which the components L_{jkh} of the Landsberg curvature tensor vanishes identically.*

3. Parallel 1-forms on Landsberg manifolds

In [4], Kozma and the author of this article investigated and studied the concept of parallel 1-forms on Riemannian and Finsler manifolds. They characterized the existence of parallel 1-forms in general. Here, in this section, we begin to study the presence of parallel 1-forms on some specific Finsler spaces of interest. Precisely, we start with Landsberg metrics. Let us provide the definition of a parallel 1-form.

Definition 3.1. *[4] Let $\beta = b_i(x)y^i$ be a 1-form on a Finsler space (M, F). Then, β is said to be a horizontally parallel (or simply parallel) 1-form with respect to the attached Berwald connection to F if $b_{i|j} = 0$, where the symbol $|$ denotes the Berwald horizontal covariant derivative.*

Considering a parallel 1-form $\beta = b_i(x)y^i$, according to [11], we observe that β is a holonomy-invariant function on the slit tangent bundle $\mathcal{T}M$, meaning $d_h\beta = 0$. As a compatibility condition, β must satisfy the property $d_R\beta = 0$. Furthermore, considering that β is a function on $\mathcal{T}M$ and homogeneous of degree 1 in y, we have $d_C\beta = \beta$. Summarizing these facts, we conclude that the existence of a parallel 1-form $\beta = b_i(x)y^i$ on a Finsler space (M, F) can be characterized by the system

$$d_h\beta = 0, \quad d_C\beta = \beta,$$

and additionally, the compatibility condition $d_R\beta = 0$, where $d_h\beta(X) = hX(\beta)$ for all $X \in \mathfrak{X}(\mathcal{T}M)$, $d_R\beta = R(\beta)$, and $d_C\beta = C(\beta)$.

The following lemma is required for subsequent use.

Lemma 3.2. *Consider a 1-form $\beta = b_i(x)y^i$ on a Finsler space (M, F), then the covector defined by*

$$m_j = b_j - \frac{\beta}{F}\ell_j,$$

where $\ell_j := \dot{\partial}_i F$, is nonvanishing on $\mathcal{T}M$, that is, $m_j \neq 0$.

Proof. The proof is proceeded by contradiction. Let $m_j = 0$, then we have

$$b_j - \frac{\beta}{F}\ell_j = 0.$$

Taking the derivative with respect to y^k, we get

$$-\frac{1}{F}b_k\ell_j - \frac{\beta}{F}\ell_{jk} + \frac{\beta}{F^2}\ell_j\ell_k = 0.$$

Substituting by $b_j = \frac{\beta}{F}\ell_j$ implies

$$-\frac{\beta}{F^2}\ell_k\ell_j - \frac{\beta}{F}\ell_{jk} + \frac{\beta}{F^2}\ell_j\ell_k = 0.$$

Therefore, we conclude the angular metric $h_{ij} = F\ell_{ij} = 0$, where $\ell_{ij} = \dot{\partial}_j\dot{\partial}_i F$. Contracting $h_{ij} = g_{ij} - \ell_i\ell_j = 0$ by the components of the inverse metric tensor yields $g^{ij}h_{ij} = g^{ij}(g_{ij} - \ell_i\ell_j) = n - 1 = 0$, that is, $n = 1$, which is a contradiction. □

Theorem 3.3. *Let (M, F) be a Landsberg space providing a parallel 1-form, then the rank of the mean Berwald curvature is at most $n - 2$.*

Proof. Let (M, F) be a Landsberg space providing a parallel 1-form $\beta = b_iy^i$. Then, by [4], we have

$$\ell_h G^h_{ijk} = 0, \quad b_h G^h_{ijk} = 0.$$

Since for a Landsberg manifold the Berwald curvature satisfies the property that $G_{hijk} := g_{\ell h} G^{\ell}_{ijk}$ is completely symmetric, the property $b_h G^h_{ijk} = 0$ implies that

$$b^i G^h_{ijk} = 0.$$

As the mean Berwald curvature is $E_{ij} = \frac{1}{2} G^h_{hij}$, we get

$$b^i E_{ij} = 0.$$

Now, we show that y^i and b^i are independent. Assume the combination

$$\mu y^i + \lambda b^i = 0,$$

for some functions μ and λ on $\mathcal{T}M$. By contracting the above equation by h_{ij}, we have

$$\lambda(b_j - \frac{\beta}{F}\ell_j) = 0.$$

By making use of Lemma 3.2, we obtain that $\lambda = 0$ and hence $\mu = 0$. That is, y^i and b^i are independent and, thus, the rank of E_{ij} is at most $n-2$. □

By utilizing the above theorem together with [12, Theorem A], we get the following result.

Theorem 3.4. *A Landsberg surface that admits a parallel 1-form is Berwaldian.*

The following result is a generalized version of [4, Proposition 3.8].

Proposition 3.5. *Consider a 1-form $\beta = b_i(x)y^i$ on a Finsler manifold (M, F). Then, the Finsler functions F and $\overline{F} = F\varphi(s)$, $s := \frac{\beta}{F}$, defined on M, are locally functionally independent where φ is an appropriate nonconstant, positive, and smooth function on $\mathbb{R}$.*

Proof. Let F and $\overline{F} = \varphi(\mathbf{s})F$ be two functionally dependent functions. Then, the 2-form $d\overline{F} \wedge dF$ vanishes, that is, we have

$$d\overline{F} \wedge dF = \frac{\partial \varphi}{\partial \mathbf{s}} \frac{1}{F} d\beta \wedge dF = 0.$$

Since φ is nonconstant, then $\frac{\partial \varphi}{\partial \mathbf{s}} \neq 0$, and thus $d\beta \wedge dF = 0$. Moreover, we have

$$0 = d\beta \wedge dF = \partial_i \beta\, \partial_j F\, dx^i \wedge dx^j + \dot{\partial}_i \beta\, \dot{\partial}_j F\, dy^i \wedge dy^j + \left(\partial_i \beta\, \dot{\partial}_j F - \partial_i F\, \dot{\partial}_j \beta\right) dx^i \wedge dy^j.$$

The above equation holds if and only if each term vanishes. In particular, the combination or the term $\dot{\partial}_i \beta\, \dot{\partial}_j F\, dy^i \wedge dy^j$. This combination vanishes if and only if $\dot{\partial}_i \beta\, \dot{\partial}_j F$ is symmetric in i and j, hence we have

$$\dot{\partial}_i \beta\, \dot{\partial}_j F - \dot{\partial}_j \beta\, \dot{\partial}_i F = \ell_i b_j - \ell_j b_i = 0, \quad \ell_i := \dot{\partial}_i F.$$

Contracting the above equation by y^i implies $F b_j - \beta \ell_j = 0$. Then, taking the derivative with respect to y^k and keeping the fact that $\ell_i b_j = \ell_j b_i$ in mind, we get

$$F\beta h_{jk} = 0,$$

where h_{jk} is the angular metric attached to the Finsler structure F. Since all of the objects h_{jk}, F, and β are nonvanishing, then a contradiction is attained. □

Theorem 3.6. *Consider a Landsberg space (M, F) with the geodesic spray S and admitting a parallel 1-form. Then, S has metrizability freedom at least* 2.

Proof. Let (M, F) be a Landsberg space that admits a horizontally parallel 1-form $\beta = b_i y^i$. Then we have

$$\ell_h G^h_{ijk} = 0, \quad b_h G^h_{ijk} = 0,$$

which reads that

$$(\ell_h + b_h) G^h_{ijk} = 0.$$

One considers

$$\overline{\ell}_r := \ell_r + b_r.$$

Contracting the above equation by y^r implies $\overline{F} = F + \beta$. That is, we get a Randers change for F by the parallel form β; moreover, $\overline{F}$ has the same geodesic spray as F because β is parallel, that is, $\overline{G}^r_{ijk} = G^r_{ijk}$. So, we get a Randers metric $\overline{F}$ in which $\overline{\ell}_r \overline{G}^r_{ijk} = 0$, which means that $\overline{F}$ is Landsberg. Moreover, by Proposition 3.5, F and $\overline{F}$ are functionally independent. Then, by [13], the result follows. □

4. Parallel 1-forms on Finsler metrics of scalar curvature

Flag curvature is an important object in Finsler geometry, analogous to sectional curvature in Riemannian geometry. Finsler metrics with scalar flag curvature are of particular interest. In this section, we explore the existence of parallel 1-forms on Finsler metrics of scalar flag curvature. Let's present the definition of a Finsler space of scalar flag curvature as follows:

Definition 4.1. *[9] A Finsler space (M, F) has scalar curvature if its Riemann curvature tensor is given by*

$$R^h_i = K(F^2 \delta^h_i - y_i y^h), \tag{4.1}$$

where $K(x, y)$ is a smooth function on $\mathcal{T}M$.

Now, we have the following result.

Theorem 4.2. *All Finsler spaces of nonvanishing scalar curvature admits no parallel 1-forms.*

Proof. Let (M, F) be a Finsler space admitting a parallel 1-form β, then we have

$$d_R\beta = 0 \Longrightarrow R(\beta) = R^h_{jk} \dot{\partial}_b \beta = R^h_{jk} b_h = 0 \Longrightarrow y^k R^h_{jk} b_h = R^h_j b_h = 0.$$

Now, by making use of (4.1), we obtain

$$R^h_j b_h = KF^2 \left(b_j - \frac{\beta}{F^2} y_j \right) = KF^2 m_j = 0.$$

Since both K and F are nonzero, then $m_j = 0$. However, by Lemma 3.2, we get a contradiction and consequently, the Finsler manifold (M, F) does not provide a parallel 1-form. □

Since every projectively flat Finsler metric is of scalar curvature, then we have the following corollary.

Corollary 4.3. *A projectively flat Finsler metric with nonzero scalar curvature does not admit parallel 1-forms.*

Two examples of projectively flat metrics with vanishing curvature are presented below. While the second example does not provide a parallel 1-form, the first one does. This illustrates that the presence of a parallel 1-form does not necessitate the vanishing of the flag curvature.

Example 1. Let $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and inner product on $\mathbb{R}^n$, respectively. Consider the Finsler metric $F(x,y)$ on the unit ball $\mathbb{B}^n$ defined by:

$$F(x,y)=\frac{\sqrt{1-|a|^2}}{(1+\langle a,x\rangle)^2}\sqrt{|y|^2-\frac{2\langle a,y\rangle\langle x,y\rangle}{1+\langle a,x\rangle}-\frac{(1-|x|^2)\langle a,y\rangle^2}{1+\langle a,x\rangle}},$$

where $y\in T_xB^n=\mathbb{R}^n$, $a=(a_1,a_2,...,a_n)\in\mathbb{R}^n$ is a fixed vector with $|a|<1$. The spray coefficients G^i of the geodesic spray of F are

$$G^i=-\frac{\langle a,y\rangle}{1+\langle a,x\rangle}y^i.$$

By [4], F is a projectively flat metric with zero curvature and admits a parallel one form $\beta=b_i(x)y^i$ defined by the component b_i as follows:

$$b_1(x)=\frac{c+c_\mu x^\mu}{(1+\langle a,x\rangle)^2},\quad b_\mu(x)=\frac{a_\mu b_1}{a_1}-\frac{c_\mu(1+\langle a,x\rangle)}{a_1(1+\langle a,x\rangle)^2},$$

where $\mu=2,...,n$.

Example 2. Consider the class of projectively flat metrics with zero flag curvature studied by Shen [5] and given by

$$\begin{aligned}F(x,y)=&\left\{1+\langle a,x\rangle+\frac{\langle a,y\rangle-|x|^2\langle a,y\rangle}{\sqrt{|y|^2-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle}\right\}\\&\times\frac{\left(\sqrt{|y|^2-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}},\end{aligned}\tag{4.2}$$

with the geodesic spray coefficients

$$G^i=\mathcal{P}y^i=\frac{\sqrt{|y|^2-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle}{1-|x|^2}y^i,$$

where $\mathcal{P}$ is the projective factor given by

$$\mathcal{P}=\frac{\sqrt{|y|^2-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle}{1-|x|^2}.$$

This metric does not provide a parallel 1-form as it is shown below.

Remark 4.4. Since by choosing $a=0$ in (4.2), we get the Berwald's metric [6]

$$F=\frac{(\sqrt{|y|^2-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{|y|^2-|x|^2|y|^2+\langle x,y\rangle^2}},\tag{4.3}$$

then we call the class (4.2) the *general Berwald's metric*.

Proposition 4.5. *The Berwald curvature* G^h_{ijk} *of the general Berwald's metric* (4.2) *is given by*

$$\begin{aligned}
G^h_{ijk} =& \frac{1}{L}\frac{1}{1-|x|^2}\left(\delta_{ij}\delta^h_k + \delta_{jk}\delta^h_i + \delta_{ki}\delta^h_j\right) - \frac{1}{L^3}\frac{1}{(1-|x|^2)^2}(y_i\delta_{jk} + y_j\delta_{ki} + y_k\delta_{ij})y^h \\
&- \frac{1}{L^3}\frac{1}{(1-|x|^2)^2}(y_iy_j\delta^h_k + y_jy_k\delta^h_i + y_ky_i\delta^h_j) - \frac{1}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^3}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})y^h \\
&+ \left(-\frac{1}{L^3}\frac{\langle x,y\rangle^2}{(1-|x|^2)^4} + \frac{1}{L}\frac{1}{(1-|x|^2)^2}\right)(x_ix_j\delta^h_k + x_jx_k\delta^h_i + x_kx_i\delta^h_j) \\
&- \frac{1}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^3}\left((x_iy_j + x_jy_i)\delta^h_k + (x_jy_k + x_ky_j)\delta^h_i + (x_ky_i + x_iy_k)\delta^h_j\right) \\
&+ \left(-\frac{3}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^4} + \frac{3}{L^5}\frac{\langle x,y\rangle^3}{(1-|x|^2)^6}\right)x_ix_jx_ky^h + \frac{3}{L^5}\frac{1}{(1-|x|^2)^3}y_iy_jy_ky^h \\
&+ \frac{3}{L^5}\frac{\langle x,y\rangle}{(1-|x|^2)^4}\left(y_iy_jx_k + y_jy_kx_i + y_ky_ix_j\right)y^h \\
&+ \left(\frac{3}{L^5}\frac{\langle x,y\rangle^2}{(1-|x|^2)^5} - \frac{1}{L^3}\frac{1}{(1-|x|^2)^3}\right)\left(y_ix_jx_k + y_jx_kx_i + y_kx_ix_j\right)y^h.
\end{aligned} \tag{4.4}$$

Proof. For simplicity, let's write the projective factor $\mathcal{P}$ in the form

$$\mathcal{P} = L + \frac{\langle x,y\rangle}{1-|x|^2}, \quad L := \frac{\sqrt{|y|^2 - |x|^2|y|^2 + \langle x,y\rangle^2}}{1-|x|^2}.$$

Now, taking the derivative of the projective factor $\mathcal{P}$ with respect to y^i, we have

$$\mathcal{P}_i := \dot{\partial}_i\mathcal{P} = \frac{1}{L}\left(\frac{y_i}{1-|x|^2} + \frac{\langle x,y\rangle x_i}{(1-|x|^2)^2}\right) + \frac{x_i}{1-|x|^2}.$$

Similarly, taking the derivative of $\mathcal{P}_i$ with respect to y^j, we get

$$\mathcal{P}_{ij} := \dot{\partial}_i\mathcal{P}_j = -\frac{1}{L^3}\left(\frac{y_iy_j}{(1-|x|^2)^2} + \frac{\langle x,y\rangle(x_iy_j + x_jy_i)}{(1-|x|^2)^3} + \frac{\langle x,y\rangle^2x_ix_j}{(1-|x|^2)^4}\right) + \frac{1}{L}\left(\frac{\delta_{ij}}{1-|x|^2} + \frac{x_ix_j}{(1-|x|^2)^2}\right).$$

Furthermore, taking the derivative of $\mathcal{P}_{ij}$ with respect to y^k, we obtain the formula

$$\begin{aligned}
\mathcal{P}_{ijk} = \dot{\partial}_k\mathcal{P}_{ij} =& \frac{3}{L^5}\frac{\langle x,y\rangle}{(1-|x|^2)^4}\left(y_iy_jx_k + y_jy_kx_k + y_ky_ix_j\right) \\
&+ \left(\frac{3}{L^5}\frac{\langle x,y\rangle^2}{(1-|x|^2)^5} - \frac{1}{L^3}\frac{1}{(1-|x|^2)^3}\right)\left(y_ix_jx_k + y_jx_kx_k + y_kx_ix_j\right) \\
&- \frac{1}{L^3}\frac{1}{(1-|x|^2)^2}(y_i\delta_{jk} + y_j\delta_{ki} + y_k\delta_{ij}) - \frac{1}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^3}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij}) \\
&+ \left(-\frac{3}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^4} + \frac{3}{L^5}\frac{\langle x,y\rangle^3}{(1-|x|^2)^6}\right)x_ix_jx_k + \frac{3}{L^5}\frac{1}{(1-|x|^2)^3}y_iy_jy_k.
\end{aligned}$$

By substitution by the above formulae of $\mathcal{P}_{ij}$ and $\mathcal{P}_{ijk}$ into the Berwald curvature

$$G^h_{ijk} = \mathcal{P}_{ijk}y^h + \mathcal{P}_{ij}\delta^h_k + \mathcal{P}_{jk}\delta^h_i + \mathcal{P}_{ki}\delta^h_j,$$

the result follows. □

Theorem 4.6. *The general Berwald's metric* (4.2) *does not provide a parallel 1-form.*

Proof. Let $\beta = b_i y^i$ be a parallel 1-form. Then, we have the condition [4] $G^h_{ijk} b_h = 0$. By (4.4), we have

$$\begin{aligned}
0 &= G^h_{ijk} b_h \\
&= \frac{1}{L}\frac{1}{1-|x|^2}\left(\delta_{ij}b_k + \delta_{jk}b_i + \delta_{ki}b_j\right) - \frac{1}{L^3}\frac{1}{(1-|x|^2)^2}(y_i\delta_{jk} + y_j\delta_{ki} + y_k\delta_{ij})\beta \\
&- \frac{1}{L^3}\frac{1}{(1-|x|^2)^2}(y_iy_jb_k + y_jy_kb_i + y_ky_ib_j) - \frac{1}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^3}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})\beta \\
&+ \left(-\frac{1}{L^3}\frac{\langle x,y\rangle^2}{(1-|x|^2)^4} + \frac{1}{L}\frac{1}{(1-|x|^2)^2}\right)(x_ix_jb_k + x_jx_kb_i + x_kx_ib_j) \\
&- \frac{1}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^4}\left((x_iy_j + x_jy_i)b_k + (x_jy_k + x_ky_j)b_i + (x_ky_i + x_iy_k)b_j\right) \\
&+ \left(-\frac{3}{L^3}\frac{\langle x,y\rangle}{(1-|x|^2)^4} + \frac{3}{L^5}\frac{\langle x,y\rangle^3}{(1-|x|^2)^6}\right)x_ix_jx_k\beta + \frac{3}{L^5}\frac{1}{(1-|x|^2)^3}y_iy_jy_k\beta \\
&+ \frac{3}{L^5}\frac{\langle x,y\rangle}{(1-|x|^2)^4}\left(y_iy_jx_k + y_jy_kx_k + y_ky_ix_j\right)\beta \\
&+ \left(\frac{3}{L^5}\frac{\langle x,y\rangle^2}{(1-|x|^2)^5} - \frac{1}{L^3}\frac{1}{(1-|x|^2)^3}\right)\left(y_ix_jx_k + y_jx_kx_k + y_kx_ix_j\right)\beta.
\end{aligned}$$

By contracting the above equation by δ^{ij} and combining like terms, we get the following:

$$\begin{aligned}
0 &= G^h_{ijk} b_h \delta^{ij} \\
&= \frac{1}{A^4L^3}\left((n+2)A^3L^2 - A^2u^2 + r^2A^2L^2 - r^2\langle x,y\rangle^2 - 2\langle x,y\rangle^2\right)b_k \\
&+ \frac{1}{A^5L^5}(-(n+4)A^3L^2\beta - 2A\langle x,b\rangle L^2\langle x,y\rangle + 2A^2\beta u^2 + 6A\beta\langle x,y\rangle^2 + r^2\beta(3\langle x,y\rangle^2 - A^2L^2))y_k \\
&+ \frac{1}{A^6L^5}(-(n+2)A^3L^2\beta\langle x,y\rangle + 2A^4L^4\langle x,b\rangle - 2A^2L^2\langle x,b\rangle\langle x,y\rangle^2 - 2A^2\beta L^2\langle x,y\rangle \\
&+ 3r^2\beta\langle x,y\rangle^3 - 3r^2A^2L^2\beta\langle x,y\rangle + 3A^2\beta u^2\langle x,y\rangle + 6A\beta\langle x,y\rangle^3 - 2A^3L^2\beta\langle x,y\rangle)x_k,
\end{aligned}$$

where we use the notations

$$A = 1 - |x|^2, \quad r = |x|, \quad u = |y|, \quad b = (b_1, b_2, \cdots, b_n).$$

Again, by contracting by x^k, we have

$$\begin{aligned}
0 &= G^h_{ijk} b_h \delta^{ij} x^k \\
&= \frac{1}{A^4L^3}\left((n+2)A^3L^2 - A^2u^2 + r^2A^2L^2 - r^2\langle x,y\rangle^2 - 2\langle x,y\rangle^2\right)\langle x,b\rangle \\
&+ \frac{1}{A^5L^5}(-(n+4)A^3L^2\beta - 2A\langle x,b\rangle L^2\langle x,y\rangle + 2A^2\beta u^2 + 6A\beta\langle x,y\rangle^2 + r^2\beta(3\langle x,y\rangle^2 - A^2L^2))\langle x,y\rangle \\
&+ \frac{1}{A^6L^5}(-(n+2)A^3L^2\beta\langle x,y\rangle + 2A^4L^4\langle x,b\rangle - 2A^2L^2\langle x,b\rangle\langle x,y\rangle^2 - 2A^2\beta L^2\langle x,y\rangle \\
&+ 3r^2\beta\langle x,y\rangle^3 - 3r^2A^2L^2\beta\langle x,y\rangle + 3A^2\beta u^2\langle x,y\rangle + 6A\beta\langle x,y\rangle^3 - 2A^3L^2\beta\langle x,y\rangle)r^2.
\end{aligned}$$

Then, we have the following algebraic equation:

$$A_1\langle x,b\rangle u^4 + (A_2\beta + A_3\langle x,b\rangle\langle x,y\rangle)\langle x,y\rangle u^2 + A_4\langle x,b\rangle\langle x,y\rangle^4 + A_5\beta\langle x,y\rangle^3 = 0, \tag{4.5}$$

where

$$\begin{aligned}
A_1 &:= ((n-2)r^6 - 3(n-1)r^4 + 3nr^2 - (n+1)),\\
A_2 &:= (-3r^6 + (n+3)r^4 - 2(n+1)r^2 + n + 2),\\
A_3 &:= (-2nr^4 + (4n-1)r^2 - (2n-1)),\\
A_4 &:= ((n+2)r^2 - (n-2)),\\
A_5 &:= (2r^4 - (n-2)r^2 + n - 2).
\end{aligned}$$

The Eq (4.5) represents a polynomial of degree 4 in the y^i. Since this polynomial holds for all values of the y^i, all coefficients must vanish, including the coefficients of $y_1^4, y_2^4, \ldots, y_n^4$, which are listed respectively, by:

$$\begin{aligned}
A_1\langle x,b\rangle + A_2 b_1 x_1 + A_3 x_1^2\langle x,b\rangle + A_4 x_1^4\langle x,b\rangle + A_5 b_1 x_1^3 &= 0,\\
A_1\langle x,b\rangle + A_2 b_2 x_2 + A_3 x_2^2\langle x,b\rangle + A_4 x_2^4\langle x,b\rangle + A_5 b_2 x_2^3 &= 0,\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots&\\
A_1\langle x,b\rangle + A_2 b_n x_n + A_3 x_n^2\langle x,b\rangle + A_4 x_n^4\langle x,b\rangle + A_5 b_n x_n^3 &= 0.
\end{aligned}$$

Summing the above n equations, we obtain:

$$nA_1\langle x,b\rangle + A_2\langle x,b\rangle + A_3 r^2\langle x,b\rangle + A_4\rho^2\langle x,b\rangle + A_5\langle x+\xi, b\rangle = 0,$$

where $\xi = (x_1^2, x_2^2, \cdots, x_n^2)$ and $\rho = |\xi|$. Therefore, we can express the above equation as follows

$$\langle (nA_1 + A_2 + A_3 r^2 + A_4\rho^2 + A_5)x + A_5\xi, b\rangle = 0.$$

Since the above equation holds for all x^i, and the case where $A_5 = 0$ and $nA_1 + A_2 + A_3 r^2 + A_4\rho^2 + A_5 = 0$ provides polynomial in x^i that cannot hold for all x^i, we conclude that $b_i = 0$, implying that $\beta = 0$. This means that there exists no nonzero parallel 1-form, hence the proof is completed. □

5. Parallel 1-forms on spherically symmetric metrics

Spherically symmetric Finsler metrics are important in Finsler geometry and physics. We're looking at whether they can have parallel 1-forms. A Finsler structure F on $\mathbb{B}^n(r_0) \subset \mathbb{R}^n$ is spherically symmetric if it has the form

$$F(x,y) = u\phi(r,s),$$

where $r = |x|$, $u = |y|$, $s = \frac{\langle x,y\rangle}{|y|}$, and $\phi : [0, r_0) \times \mathbb{R}^n \to \mathbb{R}$.

We lower the indices in y^i and x^i using the Kronecker delta δ_{ij}, the metric tensor components attached to the Euclidean norm, as follows:

$$y_i := \delta_{ij} y^j, \quad x_i := \delta_{ij} x^j.$$

That is, y_i and x_i coincide with y^i and x^i, respectively. Additionally, we have

$$y^j y_j = u^2, \quad x^j x_j = r^2, \quad y^j x_j = x^j y_j = \langle x, y\rangle.$$

For further details, refer to, for example, [14–16].

The following derivatives will be used in subsequent calculations.

$$\frac{\partial r}{\partial x^k} = \frac{1}{r}x_k, \quad \frac{\partial u}{\partial y^k} = \frac{1}{u}y_k, \quad \frac{\partial u}{\partial x^k} = 0, \quad \frac{\partial r}{\partial y^k} = 0, \quad \frac{\partial s}{\partial x^k} = \frac{1}{u}y_k, \quad \frac{\partial s}{\partial y^k} = \frac{1}{u}(x_k - \frac{s}{u}y_k). \tag{5.1}$$

The coefficients G^i of the geodesic spray of the Finsler metric $F = u\phi(r, s)$ are expressed as follows:

$$G^i = uPy^i + u^2 Qx^i, \tag{5.2}$$

where P and Q are defined by

$$P := -\frac{Q}{\phi}(s\phi + (r^2 - s^2)\phi_s) + \frac{1}{2r\phi}(s\phi_r + r\phi_s), \tag{5.3}$$

$$Q := \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \tag{5.4}$$

where the subscripts s (resp., r) denote the derivative with respect to s (resp., r).

The components G^i_j of the nonlinear connection of F are

$$G^i_j = uP\delta^i_j + P_s x_j y^i + \frac{1}{u}(P - sP_s)\, y_j y^i + uQ_s x^i x_j + (2Q - sQ_s)x^i y_j. \tag{5.5}$$

Computing the functions P and Q enables us to determine the geodesic sprays generated by the coefficients G^i given in (5.2) for the spherically symmetric metric $F = u\phi$. Conversely, the inverse problem involves recovering the Finsler metric from specified functions P and Q. For solving the inverse problem, we have the following lemma obtained in [14].

Lemma 5.1. *[14] Given arbitrary functions $P(r, s)$ and $Q(r, s)$, let $F = u\phi(r, s)$ be a Finsler structure whose geodesic spray is determined by P and Q. Then, the function ϕ must satisfy the following two conditions:*

$$\begin{aligned} (1 + sP - (r^2 - s^2)(2Q - sQ_s))\phi_s + (sP_s - 2P - s(2Q - sQ_s))\phi &= 0,\\ \frac{1}{r}\phi_r - (P + Q_s(r^2 - s^2))\phi_s - (P_s + sQ_s)\phi &= 0. \end{aligned} \tag{5.6}$$

We shall refer to the aforementioned conditions as the ‘metrizability conditions’, as they directly determine the metrizability of the spray.

Proposition 5.2. *Assume that $F = u\phi(r, s)$ is spherically symmetric. Then, F admits a parallel 1-form $\beta = b_i y^i$ if, and only if, its geodesic spray is given by the functions*

$$P = P(r, s), \quad Q = \frac{s^2 f'(r)}{2r^3 f(r)} - \frac{sP}{r^2} + \frac{1}{2r^2}, \tag{5.7}$$

where $b_i = f(r)x_i$, $f' := \frac{df}{dr}$, and $f(r)$ is a smooth function of r.

Proof. Let F be given by $F = u\phi(r, s)$ such that F admits a parallel 1-form β. Then β can be written in the form $\beta = b_i(r)y^i$. Since β is parallel, then b_i must be gradient, that is, there is a function $h(r)$ such that $b_i = \frac{\partial h}{\partial x^i}$. Therefore, we can write

$$b_i = \frac{\partial h}{\partial x^i} = \frac{dh}{dr}\frac{\partial r}{\partial x^i} = \frac{1}{r}\frac{dh}{dr}x_i = f(r)x_i,$$

where we set $f(r) = \frac{1}{r}\frac{dh}{dr}$. Moreover, we have

$$\beta = b_i y^i = f(r)x_i y^i = f(r)\langle x, y\rangle = f(r)su.$$

Now, the condition $d_h\beta = 0$ implies

$$\delta_i\beta = \partial_j(b_i y^i) - G^i_j\dot{\partial}_i\beta = 0.$$

Using (5.1) and plugging (5.5) into the above equation, we have

$$u\left(\frac{sf'}{r} - sfP_s - fP - fr^2Q_s\right)x_i + (f - fsP + fs^2P_s - 2fr^2Q + fsr^2Q_s)y_i = 0,$$

which implies the equations

$$\frac{sf'}{r} - sfP_s - fP - fr^2Q_s = 0, \tag{5.8}$$

$$f - fsP + fs^2P_s - 2fr^2Q + fsr^2Q_s = 0. \tag{5.9}$$

Adding (5.9) to the mulitple of (5.8) by s, we obtain

$$\frac{s^2f'}{r} + f - 2fsP - 2fr^2Q = 0. \tag{5.10}$$

By making use of (5.10), we have

$$Q = \frac{s^2f'}{2r^3f} - \frac{sP}{r^2} + \frac{1}{2r^2},$$

where P is arbitrary and this completes the proof. □

Theorem 5.3. *The spray determined by the functions P and Q given in* (5.7) *is only Riemann metrizable. Therefore, the non-Riemannian spherically symmetric metrics do not admit parallel 1-forms.*

Proof. By substituting by the formula (5.7) of P and Q into the metrizability conditions (5.6), we have

$$(1 + sP - (r^2 - s^2)(2Q - sQ_s))\frac{\phi_s}{\phi} + sP_s - 2P - s(2Q - sQ_s) = 0. \tag{5.11}$$

Substituting the expressions of P and Q into the above formula, we have

$$\left(-sP_s(r^2 - s^2) + P(2r^2 - s^2) + s\right)\left(s\frac{\phi_s}{\phi} - 1\right) = 0.$$

Now, we have two cases either $\frac{\phi_s}{\phi} = \frac{1}{s}$ or

$$-sP_s(r^2 - s^2) + P(2r^2 - s^2) + s = 0.$$

If $\frac{\phi_s}{\phi} = \frac{1}{s}$, then we get $\phi = c(r)s$. Hence, the Finsler structure F takes the form

$$F = u\phi = c(r)su = c(r)\langle x, y\rangle.$$

As the inner product $\langle x, y\rangle$ is linear in y, so is the formula for F. This implies that the metric tensor g_{ij} is degenerate, consequently, the spray is non-Finsler metrizable.

Now, assume that

$$-sP_s(r^2 - s^2) + P(2r^2 - s^2) + s = 0.$$

Rewriting the above equation as follows

$$P_s - \frac{2r^2 - s^2}{s(r^2 - s^2)}P = \frac{1}{r^2 - s^2},$$

which can be seen as a linear differential equation with the solution

$$P = -\frac{s}{r^2} + \frac{c_1(r)s^2}{\sqrt{r^2 - s^2}},$$

where $c_1(r)$ is a function of r.

To determine the function ϕ, let's rewrite (5.3) as follows

$$(2(r^2 - s^2)Q - 1)r\phi_s - s\phi_r + 2r(P + sQ)\phi = 0. \tag{5.12}$$

Differentiating both sides of the preceding equation with respect to s yields

$$(2(r^2 - s^2)Q - 1)r\phi_{ss} - s\phi rs - \phi_r + 2(P + (r^2 - s^2)Q_s)r\phi_s + 2r(P_s + Q + sQ_s)\phi = 0.$$

Rewrite (5.4) as follows:

$$(2(r^2 - s^2)Q - 1)r\phi_{ss} - s\phi_{rs} + \phi_r + 2rQ\phi - 2rsQ\phi_s = 0.$$

Subtracting the above two equations, we get

$$\left(P + sQ + (r^2 - s^2)Q_s\right)r\phi_s - \phi_r + r(P_s + sQ_s)\phi = 0. \tag{5.13}$$

By substituting by ϕ_r from (5.13) into (5.12), we have

$$\left(2(r^2 - s^2)Q - 1 - s(P + sQ + (r^2 - s^2)Q_s\right)r\phi_s + (2rP + 2rsQ - rs(P_s + sQ_s))\,\phi = 0.$$

Substituting P and Q which are given in (5.7), we get

$$(-2fc_1r^2s^3 + (rs^2f' + r^2f + 2s^2f)\sqrt{r^2 - s^2})\phi_s = 0.$$

If $-2fc_1r^2s^3 + (rs^2f' + r^2f + 2s^2f)\sqrt{r^2 - s^2}f = 0$, then we have

$$-2fc_1r^2s^3 = -(rs^2f' + r^2f + 2s^2f)\sqrt{r^2 - s^2}.$$

Squaring both sides of the preceding equation and collecting like terms yields

$$(4c_1r^4f^2 + r^2f'^2 + 4rff' + 4f^2)s^6 - r^3(rf'^2 + 2ff')s^4 - 4rfr^4(2rf' + 3f)s^2 - r^6f^2 = 0,$$

which is a polynomial of degree 6 in s and satisfied for all values of s. That is, we get $f = 0$ and, hence, $\beta = 0$, which is the trivial case. If $\phi_s = 0$, then $\phi = \psi(r)$. Thus, the Finsler function F is given by

$$F = u\phi = \psi(r)u,$$

which is Riemannian. □

6. Conclusions

We end this work with the following observations:

• As discussed at the beginning of Section 3, the presence of a parallel 1-form β on a Finsler space (M, F) is characterized by the system:

$$d_h\beta = 0, \quad d_C\beta = \beta,$$

as well as the compatibility condition $d_R\beta = 0$ Alternatively, we can consider the parallel property with respect to the horizontal covariant derivative of the Berwald connection. It's worth noting that we obtain the same concept even if we use the horizontal covariant derivative with respect to the Cartan, Chern, or Hashiguchi connections. This is because the fundamental four connections have the same components R^h_{ij} of h(v)-torsion; see [17, Table 1].

• The class of (α,β)-metrics comprises a Riemannian structure α and a 1-form β. If β is parallel with respect to α (the Levi-Civita connection), then any (α,β)-metric F inherits certain geometric properties from the background metric α. For example, α and F share the same geodesic spray and connection coefficients. Therefore, one can conclude that a parallel 1-form β allows for the construction of numerous Finsler metrics that share the same geodesic spray.

• The presence of a parallel 1-form β on a Riemannian space (M, α) not only preserves certain geometric properties but also has its own impact on the Finsler space constructed by any (α,β)-metrics. For instance, Landsberg manifolds admitting a parallel 1-form exhibit a mean Berwald curvature of rank of at most $n-2$. As a result, Landsberg surfaces with parallel 1-forms are necessarily Berwaldian. Additionally, the metrizability freedom of the geodesic spray for Landsberg metrics with parallel 1-forms is at least 2.

• We figure out that some special Finsler metrics do not admit a parallel 1-form. Specifically, no parallel 1-form is admitted for any Finsler metrics of nonvanishing scalar curvature, among them the projectively flat metrics with nonvanishing scalar curvature. Furthermore, neither the general Berwald's metric nor non-Riemannian spherically symmetric metrics admit a parallel 1-form. Consequently, we observe that certain (α,β)-metrics and generalized (α,β)-metrics do not admit parallel 1-forms.

Conflict of interest

The author declares no conflict of interest.

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