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Research article

Statistical analysis of stress–strength in a newly inverted Chen model from adaptive progressive type-II censoring and modelling on light-emitting diodes and pump motors

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Abstract: A system's reliability is defined as the likelihood that its strength surpasses its stress, referred to as the stress–strength index. In this work, we introduce a new stress–strength model based on the inverted Chen distribution. By analyzing the failure times of organic white light-emitting diodes and pump motors, we focus on the inferences of the stress–strength index $\Re = P(Y < X)$, where: (1) the strength (X) and stress (Y) are independent random variables following inverted Chen distributions, and (2) the data are acquired using the adaptive progressive type-II censoring plan. The inferences are based on two estimation approaches: maximum likelihood and Bayesian. The Bayes estimates are obtained with the Markov Chain Monte Carlo sampling process leveraging the squared error and LINEX loss functions. Furthermore, two approximate confidence intervals and two credible intervals are developed. A simulation study is done to examine the various estimations presented in this work. To assess the effectiveness of different point and interval estimates, some precision metrics are applied, especially root mean square error, interval length, and coverage probability. Finally, two practical problems are examined to demonstrate the significance and applicability of the given estimation approaches. The analysis demonstrates the suitability of the proposed model for examining engineering data and highlights the superiority of the Bayesian estimation approach in estimating the unknown parameters.

Keywords: inverted Chen; stress–strength; adaptive censoring; Markovian chain; likelihood and Bayesian; real-world data modelling
Mathematics Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05

1. Introduction

Due to production fluctuation, the strength of a component, say X, might fluctuate substantially and unexpectedly. When the component is put to work, it experiences stress, denoted by Y. If X is smaller than Y, the component will fail rapidly since its strength is insufficient to bear the stress. If Y is less than X, the component's strength can sustain the stress, and it functions appropriately. Since the initial study of Birnbaum and McCarty [1], the topic of evaluating the stress–strength index $\Re = P(Y < X)$ and its related deductions when X and Y are two independent random variables has received a lot of attention. Because $\Re$ indicates the link between a system's stress and strength, it can also be referred to as the system's stress–strength index. Kotz et al. [2] provided many examples of the stress–strength model in their book, including the reliability of rocket engines, earthquake resistance, and the reaction of leprosy patients to medicine. For example, the likelihood of successfully firing the engine can be represented by $\Re = P(Y < X)$, where X is the strength of the rocket chamber and Y is the maximum chamber pressure generated when a solid propellant is burned. Recent research has focused on evaluating the stress–strength model using various statistical methods. See, for example, Al-Mutairi et al. [3], Sharma et al. [4], Hassan et al. [5], Pak et al. [6], Alsadat et al. [7], Nassar et al. [8], and Quintino et al. [9].

1.1. Inverted Chen distribution

When evaluating the stress–strength index, it is crucial to choose the appropriate statistical distribution for modelling the random variables X and Y. Choosing the best model relies on accurately evaluating the stress–strength index $\Re$. In this paper, we assume that the random variables X and Y are independent and follow the inverted Chen (IC) distribution. Srivastava and Srivastava [10] were the first to introduce the IC distribution, while Agiwal [11] extensively examined its properties. For the random variable X, for instance, the probability density function (PDF) of the IC distribution takes the form

$$g(x; a, b) = abx^{-(1+b)} \exp\left[x^{-b} + a\eta(x; b)\right], x > 0, \tag{1.1}$$

where a and b are the shape parameters and $\eta(x; b) = 1 - e^{x^{-b}}$. The cumulative distribution function (CDF) related to (1.1) is expressed as follows:

$$G(x; a, b) = \exp\left[a\eta(x; b)\right]. \tag{1.2}$$

Agiwal [11] indicated that the IC distribution can be used to model the data with an upside-down bathtub hazard rate function (HRF). The form of the HRF that corresponds to the IC distribution is given by

$$h(x; a, b) = \frac{abx^{-(1+b)}e^{x^{-b}}}{\exp\left[-a\eta(x; b)\right] - 1}.$$

Figure 1 shows plots of the HRF for the IC distribution, using selected parameter values. The figure demonstrates that the IC distribution can model data with hazard rate shapes that increase or decrease, not just the upside-down bathtub shape as pointed out by Agiwal [11].

The variability in the HRF enhances the flexibility of the IC model and makes it a suitable choice for reliability and other studies.

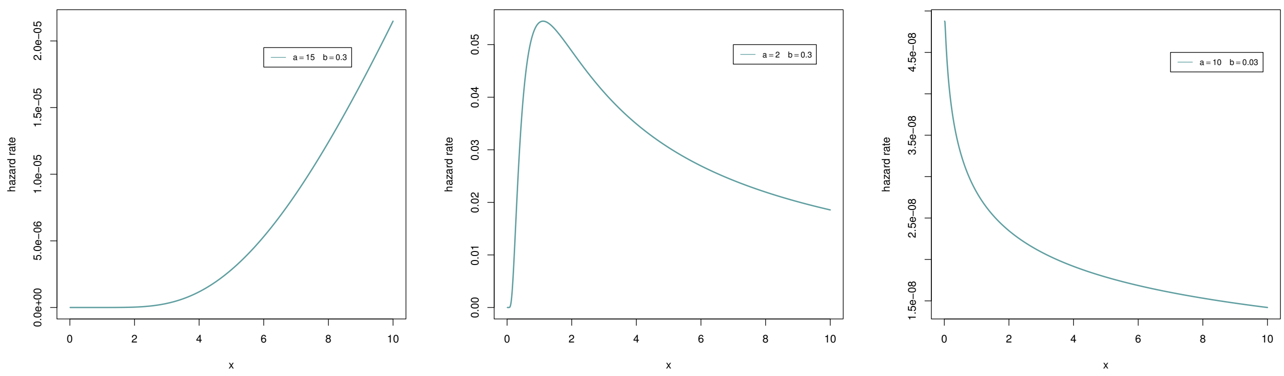


Figure 1. HRF shapes of the IC distribution.

Kumar et al. [12] conducted a study on inferences in two IC populations using joint type-II censored samples, one of the few studies to employ the IC distribution. Assume the stress and strength random variables are independent and have the IC distribution, i.e., $X \sim IC(a_1, b)$ and $Y \sim IC(a_2, b)$. Then, the stress–strength index $\Re$ takes the following form, according to Agiwal [11],

$$\Re = \int_0^\infty g(x; a_1, b)G(x; a_2, b)dx = \frac{a_1}{a_1 + a_2}. \tag{1.3}$$

1.2. Adaptive progressive type-II censoring

When performing reliability studies, it is not expected to gather data on the failure times of all tested items, particularly for highly trustworthy items. In these scenarios, the researcher gathers data from a subset of the tested units. This is referred to in the literature as censored samples. There are numerous censoring samples in the literature, including single and multi-stage censoring plans. One of the most popular censorship strategies is known as progressive type-II censoring (PTIIC). This method enables researchers to remove live items using a specified removal pattern $\underline{R} = (R_1, R_2, \ldots, R_m)$, where m represents the required number of failures taken from a random sample of size M. At the time of each failure, say $X_{i:m:n}, i = 1, \ldots, m$, R_i items are randomly chosen and eliminated from the test. Immediately, after observing the m^{th} failure, the test is terminated and all remaining items R_m are discarded. Several works considered the PTIIC plan; see, for more detail, Aggarwala and Balakrishnan [13], Balakrishnan and Lin [14], Singh et al. [15], Dey et al. [16], and Chacko and Mohan [17]. Ng et al. [18] proposed a broader censorship strategy, with the traditional PTIIC approach standing as a particular case. They termed their strategy adaptive progressive type-II censoring (APTIIC). This strategy makes certain that we will complete the test upon we have reached the specified amount of failures and that the total test length will be near the optimal prefixed time, indicated by T. This plan runs similarly to the PTIIC plan but with two choices for stopping the experiment. In the first scenario, when $X_{m:m:n} < T$, the experiment terminates at $X_{m:m:n}$, resulting in the standard PTIIC method. Otherwise, if T happens earlier than $X_{m:m:n}$, the removal pattern is adjusted by setting $R_i = 0, i = d + 1, \ldots, m - 1$, where d represents the number of detected failures recorded before T. In this situation, at $X_{m:m:n}$, all the leftover items are eliminated from the experiment, i.e., $R^* = M - m - \sum_{i=1}^{J} R_i$. Many authors incorporated the APTIIC plan; see, for example, Nassar et al. [19], Du and Gui [20], Dey et al. [21], Alotaibi et al. [22] and Lv et al. [23], Xiao et al. [24], Kumari et al. [25], and Nassar et al. [26], among others.

1.3. Problem statement and assumptions

Despite the IC distribution's usefulness due to its flexible HRF, little interest has been given in the literature to the estimations of its parameters or any function that includes these parameters, particularly when censoring samples are involved. The only study that has investigated the estimation issues of the stress–strength index is the study of Agiwal [11]. However, this study only considered the complete sample case, which is not practical due to its time-consuming nature, especially when dealing with highly reliable products. Additionally, that study focused solely on point estimation and did not provide any information regarding interval estimation for either parameters or the reliability measure. Another limitation is that they assumed the shape parameter has a non-informative prior in the Bayesian estimation framework. This assumption has drawbacks, particularly concerning the resulting improper posterior distribution. As a result, one can see that the current literature reveals a gap in research focusing on estimating the parameters as well as the stress–strength index involving the IC distribution in censoring schemes. This study aims to fill this gap by investigating both classical and Bayesian estimation approaches of the IC distribution based on the APTIIC scheme. This can be performed using the following assumptions:

(1) The strength X and stress Y are random variables that each follow the IC distribution with PDF and CDF defined in Eqs (1.1) and (1.2), respectively.
(2) The shape parameter, which is unknown, is the same for both random variables X and Y.
(3) In Bayesian estimation, the random variables a_1, a_2 and b are independent random variables and each follows a gamma distribution.

The limitations mentioned are addressed in this study as follows:

(i) Utilizing a general censoring plan, specifically the APTIIC scheme, allows for the collection of the necessary data, with the complete sample serving as a special case.
(ii) Investigating two classical confidence interval approaches and two Bayes intervals for estimating unknown parameters and the stress–strength index.
(iii) Studying the Bayesian estimation of the various parameters when the shape parameter follows the gamma prior distribution, which includes the non-informative prior as a special case.

1.4. Motivation, objectives and contributions of the study

Motivated by (1) the flexibility of the IC distribution in modelling increasing, decreasing, and upside-down bathtub data, (2) the efficiency of the APTIIC scheme in collecting data, particularly when the product under consideration is reliable, and (3) the significance of estimating the stress–strength index in various engineering and industrial applications. We encourage dealing with the estimations of the IC distribution using APTIIC samples. First, we focus on the estimations of the unknown parameters and then move on to the stress–strength index $\Re = P(Y < X)$. To achieve this, from a classical perspective, the method of maximum likelihood is used to obtain point estimates of various parameters due to its desirable properties. Approximate confidence intervals (ACIs) are derived from the asymptotic properties of these estimates. However, since the stress–strength index is in the interval $[0, 1]$, this approach may produce confidence intervals with negative lower bounds or upper bounds exceeding one. To address this issue, a normal approximation of the log-transformed estimates is also employed to get the suitable intervals. The maximum likelihood approach is known to provide inefficient estimates when dealing with small sample sizes. To address this limitation, we employ the Bayesian estimation method, which combines prior knowledge about the unknown parameters with available data to derive Bayes estimates and Bayes intervals. In this context, we utilize two loss functions to obtain the Bayes point estimates, along with the Bayes credible intervals (BCIs). These intervals are based on the assumption that the posterior distribution is symmetric; however, they may not perform well in cases of asymmetric posterior distributions. To overcome this issue, we use the highest posterior density (HPD) credible intervals. For more detail about Bayesian estimation, see Li et al. [27–29] and Ni et al. [30].

Based on the above discussion, we can summarize the main objectives of this study as follows:

(1) Getting the maximum likelihood estimates (MLEs) and two sorts of ACIs for the unknown parameters a_1, a_2 and b.
(2) Acquiring MLE and two types of ACIs employing the delta method for the stress–strength index $\Re = P(Y < X)$.
(3) Studying the Bayesian estimations of $a_1,\ a_2,\ b$, and $\Re = P(Y < X)$ leveraging both symmetric squared error (SSE) and asymmetric LINEX loss functions. The Bayes estimates are computed by drawing samples from posterior distribution via implementing the so–called Markov Chain Monte Carlo (MCMC) technique.
(4) Obtaining the BCIs and HPD credible intervals for $a_1,\ a_2,\ b$, and $\Re = P(Y < X)$.
(5) Conducting simulation research with multiple experimental designs to investigate the efficacy of classical and Bayesian estimations. The assessments are based on several common precision measurements.
(6) Investigating a pair of real-world data sets to put the estimation strategies presented into practice.

The contributions of this study can be listed below:

(1) Compare the classical point estimates of the parameters and the stress–strength index of the IC distribution with two Bayes estimates, utilizing the SSE and LINEX loss functions. This comparison will highlight how different loss functions impact the precision and bias of the parameter estimates.
(2) This study compares four interval estimates to assess their effectiveness in quantifying uncertainty in stress–strength analysis.
(3) This study examines the accuracy of different estimators under various censoring schemes and parameter settings through extensive simulation studies. The goal is to recommend the most suitable estimation approach for estimating the stress–strength index of the IC distribution.
(4) Demonstrating the applicability of the IC distribution allows for the analysis of real-world data sets and the estimation of the stress–strength index. As indicated later, the data analysis of two applications shows that the IC distribution is suitable for modelling two engineering data sets.

Following this introduction, the paper will be organised as follows: Section 2 examines the MLEs of $a_1,\ a_2$, and b, as well as the stress–strength index $\Re$ and two types of corresponding ACIs. In Section 3, we utilize the MCMC technique to obtain Bayesian estimates of the various parameters, using both the SSE and the LINEX loss functions, in addition to two types of credible intervals. Section 4 consists of a simulation study to evaluate the accuracy of the estimates obtained. Section 5 presents an analysis of two actual data sets. Finally, in Section 6, the paper is concluded.

2. Classical inference

This section focuses on the maximum likelihood estimation of the IC distribution, specifically the unknown parameters a_1, a_2 and b, as well as the stress–strength index $\Re$. The point estimates for these parameters, as well as two ACIs, are covered.

2.1. Point estimation

Let $\underline{X} = (X_{1:m_1:M_1}, \ldots, X_{m_1:m_1:M_1})$ be an APTIIC sample of size m_1 picked from the $IC(a_1, b)$ population, where $X_{1:m_1:M_1} < \cdots < X_{d_1:m_1:M_1} < T_1 < X_{d_1+1:m_1:M_1} < \cdots < X_{m_1:m_1:M_1}$, with removal pattern $\underline{R} = (R_1, \ldots, R_{d_1}, 0, \ldots, 0, R^*)$. In addition, let $\underline{Y} = (Y_{1:m_2:M_2}, \ldots, Y_{m_2:m_1:M_2})$ be an observed APTIIC sample of size m_2 taken from the $IC(a_2, b)$ population, such that $Y_{1:m_2:M_2} < \cdots < Y_{d_2:m_2:M_2} < T_2 < Y_{d_2+1:m_2:M_2} < \cdots < Y_{m_2:m_2:M_2}$, with progressive censoring plan $\underline{S} = (S_1, \ldots, S_{d_2}, 0, \ldots, 0, S^*)$. Here, d_1 and d_2 stand for the number of recorded failures before time T_1 and T_2, respectively. For brevity, we use X_i and Y_i instead of $X_{i:m_1:M_1}$ and $Y_{i:m_2:M_2}$, respectively, in this work. Let $\underline{x}$ and $\underline{y}$ be the realizations of $\underline{X}$ and $\underline{Y}$, respectively, then the joint likelihood function of the observed data can be expressed as

$$\begin{aligned} L(a_1, a_2, b|\underline{x}, \underline{y}) &= C_1 C_2 \left\{ \prod_{i=1}^{m_1} g(x_i; a_1, b) \prod_{i=1}^{d_1} [1 - F(x_i; a_1, b)]^{R_i} [1 - F(x_i; a_1, b)]^{R^*} \right\} \\ &\times \left\{ \prod_{i=1}^{m_2} g(y_i; a_2, b) \prod_{i=1}^{d_2} [1 - F(y_i; a_2, b)]^{S_i} [1 - F(y_i; a_2, b)]^{S^*} \right\}, \end{aligned} \tag{2.1}$$

where C_1 and C_2 are constants.

From (1.1), (1.2), and (2.1), the likelihood function can be formed as

$$\begin{aligned} L(a_1, a_2, b|\underline{x}, \underline{y}) &= a_1^{m_1} a_2^{m_2} b^m \exp\left\{ -(b+1)Q + \sum_{i=1}^{m_1} \left[x_i^{-b} + a_1 \eta(x_i; b) \right] + \sum_{i=1}^{m_2} \left[y_i^{-b} + a_2 \eta(y_i; b) \right] \right\} \\ &\times \exp\left\{ \sum_{i=1}^{d_1} R_i \log[\psi(x_i; a_1, b)] + R^* \log[\psi(x_m; a_1, b)] \right\} \\ &\times \exp\left\{ \sum_{i=1}^{d_2} S_i \log[\psi(y_i; a_2, b)] + S^* \log[\psi(y_m; a_2, b)] \right\}, \end{aligned} \tag{2.2}$$

where $m = m_1 + m_2$, $Q = \sum_{i=1}^{m_1} \log(x_i) + \sum_{i=1}^{m_2} \log(y_i)$ and $\psi(x_i; a_1, b) = 1 - e^{a_1 \eta(x_i; b)}$. The natural logarithm of (2.2), presented by $\mathfrak{L} = \log L(a_1, a_2, b|\underline{x}, \underline{y})$, is

$$\mathfrak{L} = m_1 \log(a_1) + m_2 \log(a_2) + m \log(b) - (b+1)Q + \sum_{i=1}^{m_1} \left[x_i^{-b} + a_1 \eta(x_i; b) \right]$$

$$+\sum_{i=1}^{m_2}\left[y_i^{-b}+a_2\eta(y_i;b)\right]+\sum_{i=1}^{d_1}R_i\log[\psi(x_i;a_1,b)]+R^*\log[\psi(x_m;a_1,b)]$$
$$+\sum_{i=1}^{d_2}S_i\log[\psi(y_i;a_2,b)]+S^*\log[\psi(y_m;a_2,b)]. \tag{2.3}$$

The three normal equations that correspond to (2.3) are given by

$$\frac{\partial\mathfrak{L}}{\partial a_1}=\tfrac{m_1}{a_1}+\textstyle\sum_{i=1}^{m_1}\eta(x_i;b)+\sum_{i=1}^{d_1}R_i\psi_1(x_i;a_1,b)+R^*\psi_1(x_m;a_1,b)=0, \tag{2.4}$$
$$\frac{\partial\mathfrak{L}}{\partial a_2}=\tfrac{m_2}{a_2}+\textstyle\sum_{i=1}^{m_2}\eta(y_i;b)+\sum_{i=1}^{d_2}S_i\psi_1(y_i;a_2,b)+S^*\psi_1(y_m;a_2,b)=0, \tag{2.5}$$

and

$$\frac{\partial\mathfrak{L}}{\partial b}=\frac{m}{b}-Q+\sum_{i=1}^{m_1}x_i^{-b}\log(x_i)\left(ae^{x_i^{-b}}-1\right)+\sum_{i=1}^{m_2}y_i^{-b}\log(y_i)\left(ae^{y_i^{-b}}-1\right)$$
$$+\sum_{i=1}^{d_1}R_i\psi_2(x_i;a_1,b)+R^*\psi_2(x_m;a_1,b)+\sum_{i=1}^{d_2}S_i\psi_2(y_i;a_2,b)+S^*\psi_2(y_m;a_2,b)=0, \tag{2.6}$$

where $\psi_1(x_i;a_1,b)=\frac{\eta(x_i;b)}{1-e^{-a_1\eta(x_i;b)}}$ and $\psi_2(x_i;a_1,b)=\frac{a_1x_i^{-b}\log(x_i)e^{x_i^{-b}}}{1-e^{-a_1\eta(x_i;b)}}$. To obtain the MLEs of (a_1,a_2,b), indicated by $(\hat{a}_1,\hat{a}_2,\hat{b})$, a computational approach is required to solve the above system of equations in (2.4)–(2.6), as there is no theoretical solution available. For this purpose, we recommend implementing the Newton-Raphson iterative method via ‘maxNR()’ function of maximization in the ‘maxLik’ package, which was proposed by Henningsen and Toomet [31].

After computing the MLEs $(\hat{a}_1,\hat{a}_2,\hat{b})$, we can use the invariance principle of the MLEs to get the MLE of the stress–strength index $\Re$ as follows:

$$\widehat{\Re}=\frac{\hat{a}_1}{\hat{a}_1+\hat{a}_2}.$$

2.2. *Interval estimation*

Here, we first derive the ACIs of the parameters a_1,a_2, and b using two ways. The first is the ACIs that use the MLEs' normal approximation (ACIs-NA). The second type uses the normal approximation of log-transformed MLEs to get the ACIs (ACIs-NT). Then, we use the same two approaches to get the required interval ranges for the stress–strength index $\Re$. The ACIs for the parameters a_1,a_2, and b are derived via the asymptotic normality of the MLEs, which implies that $(\hat{a}_1,\hat{a}_2,\hat{b})\sim N[(a_1,a_2,b),\boldsymbol{\Sigma}]$, where $\boldsymbol{\Sigma}$ is the variance-covariance matrix, which is obtained through taking the inverse of the Fisher information matrix. Practically, we estimate the variance-covariance matrix by inverting the observed Fisher information matrix as given below.

$$\widehat{\boldsymbol{\Sigma}}=\begin{bmatrix}-\frac{\partial^2\mathfrak{L}}{\partial a_1^2} & 0 & -\frac{\partial^2\mathfrak{L}}{\partial a_1\partial b}\\ & -\frac{\partial^2\mathfrak{L}}{\partial a_2^2} & -\frac{\partial^2\mathfrak{L}}{\partial a_2\partial b}\\ & & -\frac{\partial^2\mathfrak{L}}{\partial b^2}\end{bmatrix}^{-1}_{(a_1,a_2,b)=(\hat{a}_1,\hat{a}_2,\hat{b})}. \tag{2.7}$$

The second derivative elements in (2.7) are given by

$$\frac{\partial^2 \mathfrak{L}}{\partial a_1^2} = -\tfrac{m_1}{a_1^2} + \textstyle\sum_{i=1}^{d_1} R_i\psi_{11}(x_i; a_1, b) + R^*\psi_{11}(x_m; a_1, b),$$
$$\frac{\partial^2 \mathfrak{L}}{\partial a_2^2} = -\tfrac{m_2}{a_2^2} + \textstyle\sum_{i=1}^{d_2} S_i\psi_{11}(y_i; a_2, b) + S^*\psi_{11}(y_m; a_2, b),$$

$$\begin{aligned}\frac{\partial^2 \mathfrak{L}}{\partial b^2} = & -\frac{m}{b^2} + \sum_{i=1}^{m_1} x_i^{-b}\log^2(x_i)\left[1 - ae^{x_i^{-b}}(1 + x_i^{-b})\right] + \sum_{i=1}^{m_2} y_i^{-b}\log^2(y_i)\left[1 - ae^{y_i^{-b}}(1 + y_i^{-b})\right] \\ & + \sum_{i=1}^{d_1} R_i\psi_{22}(x_i; a_1, b) + R^*\psi_{22}(x_m; a_1, b) + \sum_{i=1}^{d_2} S_i\psi_{22}(y_i; a_2, b) + S^*\psi_{22}(y_m; a_2, b),\end{aligned}$$

$$\frac{\partial^2 \mathfrak{L}}{\partial a_1 \partial b} = \sum_{i=1}^{m_1} \eta^*(x_i; b) + \sum_{i=1}^{d_1} R_i\psi_{12}(x_i; a_1, b) + R^*\psi_1(x_m; a_1, b)$$

and

$$\frac{\partial^2 \mathfrak{L}}{\partial a_2 \partial b} = \sum_{i=1}^{m_2} \eta^*(y_i; b) + \sum_{i=1}^{d_2} S_i\psi_{12}(y_i; a_2, b) + S^*\psi_{12}(y_m; a_2, b),$$

where

$$\psi_{11}(x_i; a_1, b) = \frac{\eta^2(x_i; b)}{\psi(x_i; a_1, b)\left(1 - e^{-a_1\eta(x_i;b)}\right)^2},$$

$$\psi_{22}(x_i; a_1, b) = \frac{a_1\log^2(x_i)\left[e^{x_i^{-b} - a_1\eta(x_i;b)}(1 + x_i^b - a_1e^{x_i^{-b}}) - (1 + x_i^b)e^{x_i^{-b}}\right]}{x_i^{2b}\left(1 - e^{-a_1\eta(x_i;b)}\right)^2},$$

and

$$\eta^*(y_i; b) = x_i^{-b}\log(x_i)e^{x_i^{-b}} \text{ and } \psi_{12}(x_i; a_1, b) = \frac{\log(x_i)\left[e^{x_i^{-b}} + e^{x_i^{-b} - a_1\eta(x_i;b)}(a_1e^{x_i^{-b}} - a_1 - 1)\right]}{x_i^b\left(1 - e^{-a_1\eta(x_i;b)}\right)^2}.$$

Following the asymptotic normality of the MLEs, the $100(1-\alpha)\%$ ACIs-NA of a_1, a_2, and b can be computed as

$$\left(\hat{a}_1 \pm z_{\alpha/2}\sqrt{\widehat{\mathbf{\Sigma}}_{11}}\right), \quad \left(\hat{a}_2 \pm z_{\alpha/2}\sqrt{\widehat{\mathbf{\Sigma}}_{22}}\right) \quad \text{and} \quad \left(\hat{b} \pm z_{\alpha/2}\sqrt{\widehat{\mathbf{\Sigma}}_{33}}\right), \tag{2.8}$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile point of the standard normal distribution. The interval estimations in (2.8) may yield negative lower bounds. Thus, the ACIs-NT can address this issue. For any parameter, say, for example a_1, the ACI-NT can be acquired as follows:

$$\hat{a}_1 \times \exp\left(\pm z_{\alpha/2}\frac{\sqrt{\widehat{\mathbf{\Sigma}}_{11}}}{\hat{a}_1}\right).$$

On the other hand, in order to get the ACI-NA and ACI-NT of the stress–strength index $\Re$, we first need to obtain the variance of $\widehat{\Re}$. Here, we utilize the estimated variance-covariance matrix in (2.7) through the delta method to approximate the estimated variance of $\widehat{\Re}$.

$$\begin{aligned} \hat{V}_{\Re} &\simeq \left(\widehat{\Re}_{a_1}, \widehat{\Re}_{a_2}, 0\right) \widehat{\boldsymbol{\Sigma}} \begin{pmatrix} \widehat{\Re}_{a_1} \\ \widehat{\Re}_{a_2} \\ 0 \end{pmatrix} \\ &\simeq \widehat{\Re}^2_{a_1} \widehat{\boldsymbol{\Sigma}}_{11} + \widehat{\Re}^2_{a_2} \widehat{\boldsymbol{\Sigma}}_{22}, \end{aligned} \tag{2.9}$$

where

$$\widehat{\Re}_{a_1} = \frac{\hat{a}_2}{(\hat{a}_1 + \hat{a}_2)^2} \text{ and } \widehat{\Re}_{a_2} = -\frac{\hat{a}_1}{(\hat{a}_1 + \hat{a}_2)^2}.$$

Upon acquiring $\hat{V}_{\Re}$, the $100\%(1-\alpha)$ ACI-NA and ACI-NT of the stress–strength index $\Re$ can be obtained, respectively, as

$$\left(\widehat{\Re} \pm z_{\alpha/2} \sqrt{\hat{V}_{\Re}}\right) \text{ and } \left[\widehat{\Re} \times \exp\left(\pm z_{\alpha/2} \frac{\sqrt{\hat{V}_{\Re}}}{\widehat{\Re}}\right)\right].$$

3. Bayesian inference

In this part, the Bayesian estimation methodology is applied to investigate the Bayes estimates for the unknown parameters a_1, a_2, and b, as well as the stress–strength index $\Re$ of the IC distribution with APTIIC data. To begin the Bayesian analysis, we should give our knowledge regarding the unknown parameters a_1, a_2, and b using prior distributions. Because no conjugate priors are available for the IC distribution, we use the gamma distribution as the prior distribution for the three unknown parameters. The gamma distribution is flexible and can model a wide variety of shapes based on its shape parameter. It shares the same support as the true unknown parameters a_1, a_2, and b. Additionally, it is commonly used in practice, which facilitates both analytical and numerical Bayesian analysis. The closed-form expressions for the mean and variance of the gamma distribution allow for easy determination of hyperparameter values in numerical parts.

Furthermore, the closed form of its variance enables an examination of how the variation in the prior distribution affects estimation performance. For more detail about using Bayesian estimation, see Zhuang et al. [32]. Assuming the independence of the three parameters and that each follows the gamma prior distribution. Then, we can formulate the joint prior distribution as

$$\pi(a_1, a_2, b) \propto a_1^{\theta_1 - 1} \, a_2^{\theta_2 - 1} \, b^{\theta_3 - 1} \, e^{-(\beta_1 a_1 + \beta_2 a_2 + \beta_3 b)}, a_1, a_2, b > 0, \tag{3.1}$$

where $\theta_k, \beta_k > 0, k = 1, 2, 3$. By linking the likelihood function in (2.2) with the joint prior distribution in (3.1), the posterior distribution of α, β, and λ can possibly be expressed in the following way:

$$\begin{aligned} \mathfrak{P}(a_1, a_2, b | \underline{\mathrm{x}}, \underline{\mathrm{y}}) &= \frac{1}{\mathfrak{A}} a_1^{m_1 + \theta_1 - 1} a_2^{m_2 + \theta_2 - 1} b^{m + \theta_3 - 1} \\ &\quad \times \exp\left\{-b(Q + \beta_3) + \sum_{i=1}^{m_1} \left[x_i^{-b} + a_1 \eta(x_i; b)\right] + \sum_{i=1}^{m_2} \left[y_i^{-b} + a_2 \eta(y_i; b)\right]\right\} \end{aligned}$$

$$\times \exp\left\{\sum_{i=1}^{d_1} R_i \log[\psi(x_i;a_1,b)] + R^* \log[\psi(x_m;a_1,b)] - \beta_1 a_1\right\}$$
$$\times \exp\left\{\sum_{i=1}^{d_2} S_i \log[\psi(y_i;a_2,b)] + S^* \log[\psi(y_m;a_2,b)] - \beta_2 a_2\right\}, \quad (3.2)$$

where $\mathfrak{A}$ refers to the normalized constant obtained as follows

$$\mathfrak{A} = \int_0^\infty \int_0^\infty \int_0^\infty \pi(a_1,a_2,b) L(a_1,a_2,b|\underline{x},\underline{y}) da_1\, da_2\, db.$$

The loss function plays a crucial role in Bayesian analysis by capturing both overestimation and underestimation in investigations. While the symmetric loss function treats overestimation and underestimation equally, the asymmetric loss function assigns different weights to these two situations. In real-world scenarios, the asymmetric loss function is often more practical and beneficial. For instance, in reliability evaluations, overestimating the stress–strength index can have far more severe consequences than underestimating it. In our study, we consider both symmetric and asymmetric loss functions. The first is the SSE loss function, with the Bayes estimator representing the posterior mean. The second is the LINEX loss function, an asymmetric loss function that was introduced by Klebanov [33] and used by Varian [34]. From the posterior distribution in (3.2), the Bayes estimator for any parametric function, say $\xi(a_1,a_2,b)$, using the SSE and LINEX loss functions are obtained, respectively, as

$$\tilde{\xi}_{SSE}(a_1,a_2,b) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \xi(a_1,a_2,b)\pi(a_1,a_2,b) L(a_1,a_2,b|\underline{x},\underline{y}) da_1\, da_2\, db}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(a_1,a_2,b) L(a_1,a_2,b|\underline{x},\underline{y}) da_1\, da_2\, db} \quad (3.3)$$

and

$$\tilde{\xi}_{LINEX}(a_1,a_2,b) = -\frac{1}{\tau}\log\left[\frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{-\tau\xi(a_1,a_2,b)}\pi(a_1,a_2,b) L(a_1,a_2,b|\underline{x},\underline{y}) da_1\, da_2\, db}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(a_1,a_2,b) L(a_1,a_2,b|\underline{x},\underline{y}) da_1\, da_2\, db}\right], \tau \neq 0, \quad (3.4)$$

where the parameter τ indicates the direction and degree of symmetry of the LINEX loss function. It is evident from Eqs (3.3) and (3.4) that closed-form expressions for the Bayes estimators cannot be obtained due to the complicated ratio of integrals. To address this issue, we employ the MCMC technique to sample from the posterior distribution and utilize the obtained samples to compute the necessary Bayes estimates, BCIs, and HPD credible intervals. In order to implement the MCMC technique, we first need to determine the full conditional distributions of the unknown parameters. These distributions can be derived from (3.2) for the parameters a_1, a_2, and b, respectively, as

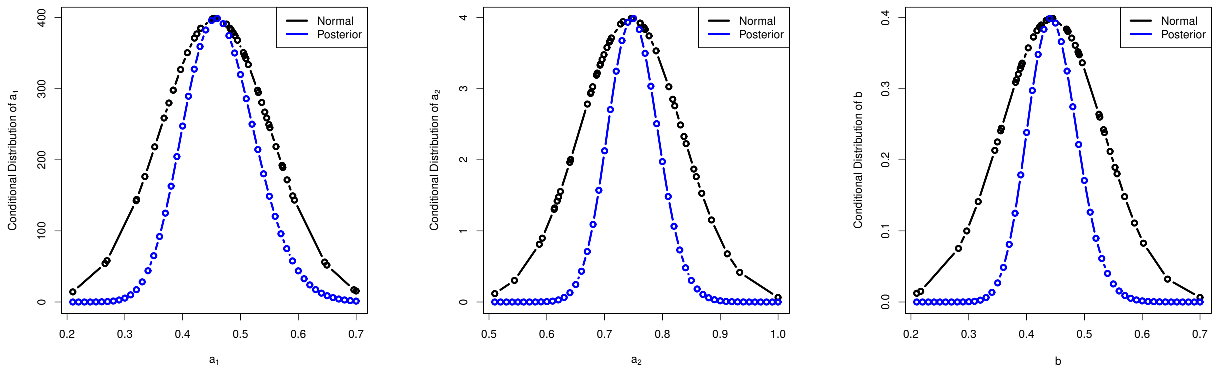
$$\mathfrak{P}_1(a_1|a_2,b,\underline{x},\underline{y}) \propto a_1^{m_1+\theta_1-1} \exp\left\{a_1\left[\sum_{i=1}^{m_1}\eta(x_i;b) - \beta_1\right]\right\}$$
$$\times \exp\left\{\sum_{i=1}^{d_1} R_i \log[\psi(x_i;a_1,b)] + R^* \log[\psi(x_m;a_1,b)]\right\}, \quad (3.5)$$

$$\mathfrak{P}_2(a_2|a_1,b,\underline{x},\underline{y}) \propto a_2^{m_2+\theta_2-1}\exp\left\{a_2\left[\sum_{i=1}^{m_2}\eta(y_i;b)-\beta_2\right]\right\} \times \exp\left\{\sum_{i=1}^{d_2} S_i \log[\psi(y_i;a_2,b)] + S^* \log[\psi(y_m;a_2,b)]\right\} \quad (3.6)$$

and

$$\mathfrak{P}_3(b|a_1,a_2,\underline{x},\underline{y}) \propto b^{m+\theta_3-1}\exp\left\{-b(Q+\beta_3)+\sum_{i=1}^{m_1}\left[x_i^{-b}+a_1\eta(x_i;b)\right]+\sum_{i=1}^{m_2}\left[y_i^{-b}+a_2\eta(y_i;b)\right]\right\} \times \exp\left\{\sum_{i=1}^{d_1} R_i \log[\psi(x_i;a_1,b)] + R^* \log[\psi(x_m;a_1,b)]\right\} \times \exp\left\{\sum_{i=1}^{d_2} S_i \log[\psi(y_i;a_2,b)] + S^* \log[\psi(y_m;a_2,b)]\right\}. \quad (3.7)$$

The initial examination of the full conditional distributions in (3.5)–(3.7) indicates that these distributions are unfamiliar and cannot be represented by any known distributions. By simulated one APTIIC random sample from $IC(a_1, a_2, b)$ population, when $(a_1, a_2, b) = (0.4, 0.8, 0.5)$, $M_i = 100$, $m_i = 50$, $T_i = 0.5$ (for $i = 1, 2$), and $R_i = S_i = 1$, $i = 1, 2, ..., m$, Figure 2 shows that the collected samples of a_1, a_2, or b from (3.5), (3.6), and (3.7), respectively, behave similar to the Gaussian density.



(a) a_1 (b) a_2 (c) b

Figure 2. The normal proposal/conditional density curves.

Therefore, we utilize the Metropolis-Hastings (MH) technique, using a normal proposal distribution (NPD), in order to obtain samples from these distributions. The next processes show how the MH algorithm produces posterior samples.

Step 1. Set $k = 1$ and put $(a_1^{(0)}, a_2^{(0)}, b^{(0)}) = (\hat{a}_1, \hat{a}_2, \hat{b})$ as initial guesses.

Step 2. From (3.5) with NPD $N(a_1^{(k-1)}, \widehat{\boldsymbol{\Sigma}}_{11})$, utilize the MH steps to simulate $a_1^{(k)}$.

Step 3. Use (3.6) and NPD $N(a_2^{(k-1)}, \widehat{\boldsymbol{\Sigma}}_{22})$ to get $a_2^{(k)}$ via the MH steps.

Step 4. Generate $b^{(k)}$ via the MH steps from (3.7) and NPD $N(b^{(k-1)}, \widehat{\mathbf{\Sigma}}_{33})$.

Step 5. Use $(a_1^{(k)}, a_2^{(k)}, b^{(k)})$ to compute the stress–strength index as

$$\Re^{(k)} = \frac{a_1^{(k)}}{a_1^{(k)} + a_2^{(k)}}.$$

Step 6. Put $k = k + 1$.

Step 7. Repeat steps 2-6, $\mathfrak{M}$ times to obtain

$$\left[a_1^{(k)}, a_2^{(k)}, b^{(k)}, \Re^{(k)}\right], k = 1, \ldots, \mathfrak{M}.$$

Before getting the Bayes estimates, BCIs, and HPD credible intervals, we remove the first $\mathfrak{B}$ created samples to guarantee convergence and erase any bias caused by starting value selection. For the stress–strength index $\Re$, as an example, and one can simply extend the next formulas to the other parameters, the Bayes estimates using the SSE and LINEX loss functions can be computed, respectively, as

$$\tilde{\Re}_{SSE} = \frac{1}{\mathfrak{M} - \mathfrak{B}} \sum_{k=\mathfrak{B}+1}^{\mathfrak{M}} \Re^{(k)}$$

and

$$\tilde{\Re}_{LINEX} = -\frac{1}{\tau} \log\left(\frac{1}{\mathfrak{M} - \mathfrak{B}} \sum_{k=\mathfrak{B}+1}^{\mathfrak{M}} e^{-\tau \Re^{(k)}}\right), \tau \neq 0.$$

To obtain the BCI and HPD credible interval of the stress–strength index $\Re$, we first sort $\Re^{(k)}$ for $k = \mathfrak{B} + 1, \ldots, \mathfrak{M}$. Then, the 100%$(1 - \alpha)$ BCI is

$$\left\{\Re^{(\alpha(\mathfrak{M}-\mathfrak{B})/2)}, \Re^{((1-\alpha/2)(\mathfrak{M}-\mathfrak{B}))}\right\}.$$

For the HPD credible interval, specify $k^*, k^* = \mathfrak{B} + 1, \ldots, \mathfrak{M}$, where

$$\Re^{(k^*+[(1-\alpha)(\mathfrak{M}-\mathfrak{B})])} - \Re^{(k^*)} = \min_{1 \leqslant k \leqslant \alpha(\mathfrak{M}-\mathfrak{B})} \left\{\Re^{(k+[(1-\alpha)(\mathfrak{M}-\mathfrak{B})])} - \Re^{(k)}\right\},$$

where $[v]$ refers to the largest integer that is less than or equal to v. Then, the 100%$(1-\alpha)$ HPD credible interval for $\Re$ is

$$\left\{\Re^{(k^*)}, \Re^{(k^*+(1-\alpha)(\mathfrak{M}-\mathfrak{B}))}\right\}.$$

4. Monte Carlo comparisons

This part presents extensive Monte Carlo simulations to evaluate the validity of the offered estimations of all new theoretical results reported in the preceding sections.

4.1. Comparison scenarios

Each acquired point (or interval) estimator of $a_i,\ i = 1, 2,\ b$, and $\Re$ is assessed based on 1,000 APTIIC samples gathered from two different groups of $IC(a_1, a_2, b)$ population, namely: Set-I:(0.4,0.8,0.5) and Set-II:(1.5,1.2,1). At the same time, the plausible value of $\Re$ using Set-I and Set-II is considered to be 0.3333 and 0.5555, respectively. To achieve this objective, besides different fashions of progressive censoring (PC) $\underline{R}$ and $\underline{S}$, several comparison designs of $M_i,\ m_i$, and T_i for $i = 1, 2$ are presented in Table 1. For distinction, in Table 1, we present four PC designs, called PC[A], PC[B], PC[C], and PC[D] as uniformly, left-, center-, and right-censored patterns, respectively. For brevity, in Table 1, the progressive pattern PC[A]:{(1⋆20),(1⋆15)} (as an example) means that one survival element will be removed for twenty times in $\underline{R}$ and for fifteen times in $\underline{S}$.

To collect an APTIIC sample of sizes $m_i,\ i = 1, 2$, after assigning the level of $M_i,\ T_i,\ i = 1, 2$, and PC:$\{\underline{R}, \underline{S}\}$, do the following procedure:

Step 1: Fix the actual values of $IC(a_1, b)$ population.

Step 2: Draw a traditional PTIIC sample as:

(a) Simulate ζ independent items (say $\zeta_1, \zeta_2, \ldots, \zeta_{m_1}$) from uniform $U(0, 1)$ distribution.
(b) Set $\Lambda_i = \zeta_i^{\left(i+\sum_{j=m_1-i+1}^{m_1} R_j\right)^{-1}}$, for $i = 1, 2, \ldots, m_1$.
(c) Set $U_i = 1 - \Lambda_{m_1}\Lambda_{m_1-1}\cdots\Lambda_{m_1-i+1}$ for $i = 1, 2, \ldots, m_1$.
(d) Collect a PTIIC sample (with size m_1) from $IC(a_1, b)$ distribution by putting:

$$X_i = \left[\log\left(1 - \frac{\log(u_i)}{a_1}\right)\right]^{-\frac{1}{b}},\ i = 1, 2, \ldots, m_1.$$

Step 3: Determine d_1 at predetermined T_1.

Step 4: Discard the remaining sample $(X_{d_1+2}, \ldots, X_{m_1})$ when $T < X_{m_1}$.

Step 5: Use the truncated $f(x; a_1, b)[1 - F(x_{d_1+1}; a_1, b)]^{-1}$ distribution to get $(X_{d_1+2}, \ldots, X_{m_1})$ order statistics with size $M_1 - d_1 - \sum_{j=1}^{d_1} R_j - 1$.

Step 6: Redo Steps 1–5 for $IC(a_2, b)$ population.

Once 1,000 APTIIC samples are gathered, we install two recommended packages in R software, namely:

- A 'maxLik' package (by Henningsen and Toomet [31]) to calculate the frequentist results.
- A 'coda' package (by Plummer et al. [35]) to calculate the Bayes' results.

The primary challenge in a Bayesian setting is determining the hyper-parameter values. To do this, we will select values for the hyper-parameters θ_i and β_i for $i = 1, 2$, of $a_i,\ i = 1, 2$, in the joint gamma density prior using the past-sample data approach. In this case, we create 10,000 past-complete samples when $M_1 = M_2 = 50$. As a result, we assign the values of $(\theta_1,\beta_1,\theta_2,\beta_2)$ as (32.3375,80.4758,43.5030,53.1782) and (34.6828,22.0867,39.8016,31.9652) for Set-I and Set-II, respectively. Additionally, without loss of generality, we shall fix (θ_3,β_3) of b as (1,2) and (1,1) for Set-I and Set-II, respectively. To run the MCMC technique proposed in Section 3, to ignore the effect of initial guessing, the first 2000 iterations (out of a total of 12000 MCMC iterations) are produced. Following that, the Bayes estimates and BCI/HPD interval estimates of a_i, $i = 1, 2$, b, and $\Re$ are obtained.

Table 1. Eight scenarios for comparison in Monte Carlo simulation.

Test	(M_1, m_1)	(M_2, m_2)	(T_1, T_2)	PC:$\{\underline{R}, \underline{S}\}$
1	(40,20)	(30,15)	(0.5,1.5)	PC[A]:{(1⋆20),(1⋆15)}
				PC[B]:{(5⋆4,0⋆16),(5⋆3,0⋆12)}
				PC[C]:{(0⋆8,5⋆4,0⋆8),(0⋆6,5⋆3,0⋆6)}
				PC[D]:{(0⋆16,5⋆4),(0⋆12,5⋆3)}
2	(40,20)	(30,15)	(2.5,2.0)	PC[A]:{(1⋆20),(1⋆15)}
				PC[B]:{(5⋆4,0⋆16),(5⋆3,0⋆12)}
				PC[C]:{(0⋆8,5⋆4,0⋆8),(0⋆6,5⋆3,0⋆6)}
				PC[D]:{(0⋆16,5⋆4),(0⋆12,5⋆3)}
3	(40,30)	(30,25)	(0.5,1.5)	PC[A]:{(1⋆10,0⋆20),(1⋆5,0⋆20)}
				PC[B]:{(5⋆2,0⋆28),(5⋆1,0⋆24)}
				PC[C]:{(0⋆14,5⋆2,0⋆14),(0⋆12,5⋆1,0⋆12)}
				PC[D]:{(0⋆28,5⋆2),(0⋆24,5⋆1)}
4	(40,30)	(30,25)	(2.5,2.0)	PC[A]:{(1⋆10,0⋆20),(1⋆5,0⋆20)}
				PC[B]:{(5⋆2,0⋆28),(5⋆1,0⋆24)}
				PC[C]:{(0⋆14,5⋆2,0⋆14),(0⋆12,5⋆1,0⋆12)}
				PC[D]:{(0⋆28,5⋆2),(0⋆24,5⋆1)}
5	(60,30)	(80,40)	(0.5,1.5)	PC[A]:{(1⋆30),(1⋆40)}
				PC[B]:{(5⋆6,0⋆24),(5⋆8,0⋆32)}
				PC[C]:{(0⋆12,5⋆6,0⋆12),(0⋆16,5⋆8,0⋆16)}
				PC[D]:{(0⋆24,5⋆6),(0⋆32,5⋆8)}
6	(60,30)	(80,40)	(2.5,2.0)	PC[A]:{(1⋆30),(1⋆40)}
				PC[B]:{(5⋆6,0⋆24),(5⋆8,0⋆32)}
				PC[C]:{(0⋆12,5⋆6,0⋆12),(0⋆16,5⋆8,0⋆16)}
				PC[D]:{(0⋆24,5⋆6),(0⋆32,5⋆8)}
7	(60,50)	(80,60)	(0.5,1.5)	PC[A]:{(1⋆10,0⋆40),(1⋆20,0⋆40)}
				PC[B]:{(5⋆2,0⋆48),(5⋆4,0⋆56)}
				PC[C]:{(0⋆24,5⋆2,0⋆24),(0⋆28,5⋆4,0⋆28)}
				PC[D]:{(0⋆48,5⋆2),(0⋆56,5⋆4)}
8	(60,50)	(80,60)	(2.5,2.0)	PC[A]:{(1⋆10,0⋆40),(1⋆20,0⋆40)}
				PC[B]:{(5⋆2,0⋆48),(5⋆4,0⋆56)}
				PC[C]:{(0⋆24,5⋆2,0⋆24),(0⋆28,5⋆4,0⋆28)}
				PC[D]:{(0⋆48,5⋆2),(0⋆56,5⋆4)}

All estimates of $\Re$ (as an example) are assessed based on their root mean squared-errors (RMSEs), average relative absolute biases (ARABs), average interval lengths (AILs), and coverage percentages

(CPs) as

$$\mathrm{RMSE}(\breve{\Re}) = \sqrt{\frac{1}{1000}\sum\nolimits_{j=1}^{1000}\left(\breve{\Re}^{[j]} - \Re\right)^2},$$
$$\mathrm{ARAB}(\breve{\Re}) = \frac{1}{1000}\sum\nolimits_{j=1}^{1000}\Re^{-1}\left|\breve{\Re}^{[j]} - \Re\right|,$$
$$\mathrm{AIL}_{(1-\alpha)\%}(\Re) = \frac{1}{1000}\sum\nolimits_{j=1}^{1000}(\mathcal{U}_{\breve{\Re}^{[j]}} - \mathcal{L}_{\breve{\Re}^{[j]}}),$$

and

$$\mathrm{CP}_{(1-\alpha)\%}(\Re) = \frac{1}{1000}\sum_{i=1}^{1000}\mathbf{J}^{\bullet}{}_{(\mathbb{L}_{\Re^{[j]}};\mathbb{U}_{\Re^{[j]}})}(\Re),$$

respectively, where $\mathbf{J}^{\bullet}(\cdot)$ is the indicator function and $(\mathbb{L}(\cdot), \mathbb{U}(\cdot))$ denotes the (lower,upper) sides of the $(1-\alpha)\%$ frequentist (or Bayesian) interval estimate of $\Re$. Similarly, the RMSE, ARAB, AIL, and CP values of $a_i,\ i = 1, 2$, and b can be easily computed.

4.2. Comparison results

In Tables 2–9, the RMSEs (reported in first-column) and ARABs (reported in second-column) for the maximum likelihood and Bayes estimates (from SSE and LINEX($\tau(= -3, +3)$)) of a_1, a_2, b, and $\Re$ are listed, respectively. In Tables 10–17, the AILs (reported in first-column) and CPs (reported in second-column) for the ACI-NA/ACI-NT and BCI/HPD interval estimates (at $\alpha = 5\%$) of the same parameters are listed, respectively.

From Tables 2–17, in terms of the lowest level of simulated RMSE, ARAB, and AIL values, we report the following facts:

- A general note of this study is that all offered estimates of $a_i,\ i = 1, 2$, b, and $\Re$ derived by the suggested estimation methods perform well and produce.
- As M_i(or m_i) for $i = 1, 2$, increases, the associated accuracy of all unknown quantities becomes even better. A similar comment is also noted when $\sum_{i=1}^{m_1} R_i$ (or $\sum_{i=1}^{m_2} S_i$) narrowed down.
- As $T_i,\ i = 1, 2$, grows, the simulated RMSE, ARAB, and AIL values of all estimates of $a_i,\ i = 1, 2$, b, or $\Re$ decrease while those CPs increase; thus, their estimates behaved satisfactory.
- To acquire high-quality assessments of any unknown lifespan factor in the presence of the suggested censored data, the investigator ought to maximize the total test length while accounting for the test's cost.
- Due to prior gamma knowledge, all Bayes' MCMC results of $a_i,\ i = 1, 2$, b, or $\Re$ perform better than other competitor estimates, as predicted. A similar comment is also reached when comparing the asymptotic (ACI-NA/ACI-NT) with credible (BCI/HPD) methods.
- Comparing the proposed point and interval procedures, it is clear that:
  - All results developed from the Bayes' MCMC and HPD interval methods outperformed frequentist estimates. It is the best we expected.
  - The 95% ACI-NA results of $a_i,\ i = 1, 2$, and $\Re$ performed superior compared to those developed from 95% ACI-NT, whereas the 95% ACI-NT results of b performed superior compared to 95% ACI-NA results.

  - The 95% HPD results of $a_i,\ i = 1, 2, b$, or $\Re$ performed superior compared to those developed from 95% BCI results.
  - The estimates obtained using the LINEX loss function for $a_i,\ i = 1, 2, b$, or $\Re$ outperform those acquired using the classical approach or the Bayes estimates obtained using the SSE loss function.

- As a_1, a_2, and b increase, it is observed that:
  - The RMSE and ARAB values of $a_i,\ i = 1, 2$ and $\Re$ decrease; while those of b increase.
  - The AIL values of $a_i,\ i = 1, 2, b$, or $\Re$ increase, while the associated CPs decrease.

- Comparing the proposed schemes PC[A], PC[B], PC[C], and PC[D], for Set-I or Set-II, it is observed that the offered estimates of $a_i,\ i = 1, 2$, or $\Re$ perform well using PC[B] 'left-censoring' whereas those of b perform well using PC[A] 'uniform-censoring' than others.
- As a summary, in the case of data generated using the proposed sampling mechanism, the Bayesian method via the MH procedure is recommended to analyze the IC parameters or the stress–strength index.

Table 2. Point assessments of a_1 from Set-I.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.459	0.778	0.253	0.623	0.241	0.568	0.209	0.513
	PC[B]	0.332	0.630	0.238	0.590	0.229	0.547	0.193	0.476
	PC[C]	0.364	0.718	0.247	0.614	0.238	0.551	0.201	0.494
	PC[D]	0.495	0.883	0.263	0.635	0.253	0.606	0.218	0.533
2	PC[A]	0.289	0.613	0.224	0.580	0.216	0.520	0.193	0.473
	PC[B]	0.252	0.588	0.208	0.551	0.200	0.476	0.180	0.443
	PC[C]	0.271	0.603	0.214	0.556	0.202	0.480	0.180	0.446
	PC[D]	0.282	0.604	0.224	0.557	0.215	0.515	0.192	0.466
3	PC[A]	0.247	0.580	0.206	0.547	0.198	0.465	0.174	0.430
	PC[B]	0.238	0.571	0.190	0.454	0.182	0.440	0.167	0.414
	PC[C]	0.244	0.573	0.195	0.473	0.188	0.452	0.168	0.415
	PC[D]	0.245	0.574	0.196	0.500	0.188	0.453	0.172	0.427
4	PC[A]	0.235	0.450	0.181	0.438	0.172	0.412	0.167	0.402
	PC[B]	0.221	0.415	0.167	0.404	0.158	0.388	0.156	0.345
	PC[C]	0.228	0.446	0.179	0.431	0.168	0.408	0.165	0.346
	PC[D]	0.230	0.448	0.180	0.434	0.170	0.409	0.167	0.399
5	PC[A]	0.221	0.411	0.165	0.400	0.155	0.384	0.145	0.299
	PC[B]	0.211	0.320	0.131	0.299	0.113	0.277	0.091	0.168
	PC[C]	0.214	0.333	0.134	0.327	0.121	0.299	0.092	0.179
	PC[D]	0.216	0.399	0.160	0.388	0.150	0.373	0.134	0.269
6	PC[A]	0.208	0.302	0.123	0.283	0.107	0.265	0.085	0.165
	PC[B]	0.187	0.259	0.106	0.244	0.094	0.230	0.075	0.145
	PC[C]	0.191	0.286	0.115	0.269	0.102	0.252	0.076	0.148
	PC[D]	0.193	0.288	0.117	0.280	0.107	0.261	0.083	0.162
7	PC[A]	0.187	0.249	0.102	0.234	0.089	0.220	0.073	0.143
	PC[B]	0.178	0.178	0.072	0.175	0.068	0.169	0.067	0.130
	PC[C]	0.180	0.214	0.088	0.203	0.078	0.192	0.067	0.132
	PC[D]	0.186	0.224	0.090	0.219	0.085	0.213	0.070	0.136
8	PC[A]	0.173	0.141	0.063	0.138	0.057	0.135	0.054	0.127
	PC[B]	0.142	0.118	0.047	0.103	0.035	0.100	0.029	0.092
	PC[C]	0.149	0.121	0.050	0.110	0.044	0.107	0.042	0.105
	PC[D]	0.154	0.124	0.056	0.122	0.050	0.119	0.048	0.116

Table 3. Point assessments of a_1 from Set-II.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.316	0.166	0.210	0.088	0.176	0.085	0.116	0.076
	PC[B]	0.308	0.152	0.164	0.085	0.134	0.083	0.116	0.076
	PC[C]	0.309	0.153	0.167	0.085	0.134	0.083	0.116	0.076
	PC[D]	0.310	0.153	0.176	0.088	0.134	0.083	0.116	0.076
2	PC[A]	0.280	0.149	0.163	0.082	0.124	0.075	0.113	0.071
	PC[B]	0.238	0.142	0.150	0.077	0.113	0.070	0.107	0.065
	PC[C]	0.254	0.145	0.159	0.080	0.114	0.072	0.109	0.071
	PC[D]	0.273	0.149	0.159	0.082	0.116	0.072	0.109	0.071
3	PC[A]	0.227	0.135	0.136	0.075	0.112	0.069	0.105	0.057
	PC[B]	0.199	0.125	0.131	0.070	0.104	0.067	0.102	0.055
	PC[C]	0.207	0.128	0.133	0.075	0.105	0.067	0.102	0.055
	PC[D]	0.209	0.135	0.135	0.075	0.110	0.068	0.103	0.055
4	PC[A]	0.189	0.123	0.130	0.067	0.093	0.061	0.092	0.054
	PC[B]	0.173	0.117	0.121	0.046	0.090	0.037	0.044	0.018
	PC[C]	0.186	0.120	0.122	0.047	0.090	0.041	0.048	0.031
	PC[D]	0.186	0.121	0.127	0.050	0.090	0.041	0.049	0.032
5	PC[A]	0.171	0.116	0.120	0.043	0.081	0.034	0.031	0.017
	PC[B]	0.143	0.111	0.090	0.033	0.050	0.029	0.023	0.012
	PC[C]	0.144	0.112	0.099	0.035	0.062	0.029	0.025	0.012
	PC[D]	0.166	0.114	0.117	0.038	0.078	0.029	0.029	0.012
6	PC[A]	0.136	0.106	0.087	0.028	0.047	0.026	0.022	0.012
	PC[B]	0.117	0.101	0.082	0.024	0.044	0.019	0.019	0.011
	PC[C]	0.123	0.104	0.085	0.026	0.045	0.019	0.020	0.011
	PC[D]	0.130	0.105	0.087	0.028	0.047	0.021	0.021	0.011
7	PC[A]	0.114	0.099	0.080	0.021	0.042	0.019	0.017	0.008
	PC[B]	0.085	0.097	0.075	0.018	0.033	0.016	0.014	0.007
	PC[C]	0.096	0.098	0.076	0.019	0.037	0.017	0.015	0.007
	PC[D]	0.099	0.098	0.076	0.020	0.038	0.018	0.016	0.007
8	PC[A]	0.082	0.095	0.064	0.018	0.032	0.015	0.013	0.006
	PC[B]	0.072	0.082	0.061	0.016	0.027	0.013	0.008	0.004
	PC[C]	0.076	0.082	0.061	0.016	0.029	0.014	0.011	0.005
	PC[D]	0.080	0.084	0.062	0.017	0.031	0.014	0.012	0.006

Table 4. Point assessments of a_2 from Set-I.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.392	0.477	0.153	0.173	0.139	0.172	0.124	0.150
	PC[B]	0.266	0.290	0.140	0.158	0.126	0.156	0.113	0.136
	PC[C]	0.272	0.325	0.147	0.166	0.132	0.163	0.116	0.139
	PC[D]	0.352	0.424	0.152	0.171	0.138	0.170	0.122	0.145
2	PC[A]	0.264	0.260	0.136	0.151	0.118	0.145	0.104	0.122
	PC[B]	0.213	0.239	0.098	0.106	0.075	0.092	0.065	0.079
	PC[C]	0.226	0.243	0.098	0.112	0.090	0.108	0.084	0.103
	PC[D]	0.231	0.246	0.120	0.131	0.100	0.124	0.089	0.104
3	PC[A]	0.210	0.210	0.098	0.106	0.072	0.089	0.063	0.078
	PC[B]	0.173	0.178	0.097	0.104	0.071	0.086	0.057	0.069
	PC[C]	0.185	0.179	0.097	0.105	0.072	0.088	0.060	0.073
	PC[D]	0.206	0.193	0.098	0.105	0.072	0.088	0.063	0.077
4	PC[A]	0.168	0.173	0.096	0.104	0.070	0.086	0.057	0.068
	PC[B]	0.161	0.161	0.092	0.092	0.070	0.085	0.055	0.065
	PC[C]	0.165	0.162	0.092	0.101	0.070	0.085	0.056	0.068
	PC[D]	0.166	0.168	0.093	0.101	0.070	0.085	0.056	0.068
5	PC[A]	0.156	0.158	0.089	0.089	0.068	0.084	0.055	0.065
	PC[B]	0.152	0.146	0.078	0.076	0.060	0.073	0.052	0.064
	PC[C]	0.153	0.150	0.079	0.081	0.064	0.077	0.054	0.064
	PC[D]	0.153	0.152	0.088	0.088	0.064	0.078	0.054	0.065
6	PC[A]	0.149	0.145	0.078	0.076	0.059	0.073	0.051	0.061
	PC[B]	0.142	0.141	0.076	0.075	0.057	0.070	0.050	0.058
	PC[C]	0.144	0.143	0.077	0.076	0.058	0.071	0.050	0.060
	PC[D]	0.146	0.145	0.078	0.076	0.059	0.072	0.050	0.061
7	PC[A]	0.137	0.136	0.072	0.075	0.057	0.070	0.049	0.057
	PC[B]	0.099	0.093	0.064	0.068	0.053	0.065	0.049	0.057
	PC[C]	0.106	0.102	0.067	0.071	0.055	0.068	0.049	0.057
	PC[D]	0.137	0.135	0.067	0.074	0.057	0.069	0.049	0.057
8	PC[A]	0.094	0.091	0.062	0.065	0.050	0.062	0.047	0.057
	PC[B]	0.086	0.084	0.056	0.039	0.025	0.027	0.012	0.014
	PC[C]	0.087	0.086	0.060	0.054	0.043	0.047	0.028	0.030
	PC[D]	0.091	0.088	0.060	0.061	0.049	0.060	0.046	0.056

Table 5. Point assessments of a_2 from Set-II.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.273	0.172	0.108	0.083	0.099	0.080	0.092	0.076
	PC[B]	0.257	0.157	0.098	0.075	0.090	0.073	0.084	0.070
	PC[C]	0.260	0.162	0.101	0.077	0.093	0.074	0.086	0.071
	PC[D]	0.264	0.166	0.107	0.081	0.098	0.078	0.090	0.075
2	PC[A]	0.256	0.155	0.098	0.074	0.089	0.072	0.082	0.068
	PC[B]	0.221	0.138	0.092	0.070	0.084	0.068	0.079	0.065
	PC[C]	0.229	0.142	0.094	0.071	0.085	0.069	0.080	0.066
	PC[D]	0.248	0.149	0.094	0.072	0.086	0.070	0.081	0.067
3	PC[A]	0.211	0.128	0.083	0.048	0.078	0.027	0.071	0.021
	PC[B]	0.152	0.101	0.081	0.046	0.068	0.025	0.060	0.016
	PC[C]	0.185	0.123	0.082	0.047	0.070	0.025	0.063	0.019
	PC[D]	0.202	0.128	0.083	0.048	0.074	0.027	0.069	0.020
4	PC[A]	0.151	0.098	0.075	0.038	0.065	0.025	0.057	0.016
	PC[B]	0.143	0.093	0.066	0.032	0.058	0.022	0.048	0.010
	PC[C]	0.144	0.094	0.068	0.032	0.060	0.024	0.051	0.010
	PC[D]	0.148	0.096	0.072	0.037	0.061	0.025	0.053	0.015
5	PC[A]	0.142	0.093	0.064	0.030	0.056	0.020	0.043	0.008
	PC[B]	0.137	0.090	0.061	0.029	0.048	0.019	0.036	0.008
	PC[C]	0.138	0.091	0.062	0.029	0.051	0.020	0.039	0.008
	PC[D]	0.139	0.091	0.063	0.030	0.053	0.020	0.040	0.008
6	PC[A]	0.134	0.089	0.059	0.029	0.045	0.018	0.033	0.008
	PC[B]	0.122	0.082	0.053	0.028	0.035	0.018	0.028	0.006
	PC[C]	0.129	0.085	0.055	0.028	0.040	0.018	0.029	0.008
	PC[D]	0.131	0.088	0.056	0.029	0.043	0.018	0.031	0.008
7	PC[A]	0.116	0.077	0.052	0.019	0.033	0.016	0.026	0.006
	PC[B]	0.092	0.063	0.048	0.017	0.027	0.013	0.023	0.005
	PC[C]	0.101	0.065	0.049	0.017	0.028	0.014	0.024	0.005
	PC[D]	0.102	0.068	0.051	0.018	0.029	0.016	0.025	0.006
8	PC[A]	0.086	0.057	0.046	0.017	0.025	0.013	0.021	0.004
	PC[B]	0.068	0.044	0.041	0.017	0.023	0.012	0.013	0.004
	PC[C]	0.075	0.049	0.043	0.017	0.024	0.013	0.015	0.004
	PC[D]	0.076	0.052	0.045	0.017	0.024	0.013	0.019	0.004

Table 6. Point assessments of b from Set-I.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.342	0.663	0.335	0.617	0.333	0.596	0.293	0.559
	PC[B]	0.296	0.577	0.246	0.460	0.231	0.446	0.199	0.375
	PC[C]	0.264	0.502	0.230	0.440	0.216	0.430	0.192	0.363
	PC[D]	0.253	0.496	0.222	0.424	0.205	0.409	0.181	0.334
2	PC[A]	0.221	0.429	0.180	0.325	0.160	0.316	0.137	0.252
	PC[B]	0.218	0.422	0.167	0.305	0.148	0.292	0.129	0.238
	PC[C]	0.216	0.403	0.161	0.291	0.141	0.279	0.123	0.226
	PC[D]	0.195	0.378	0.154	0.281	0.132	0.263	0.117	0.215
3	PC[A]	0.194	0.372	0.143	0.245	0.118	0.232	0.102	0.193
	PC[B]	0.192	0.337	0.127	0.218	0.100	0.198	0.096	0.185
	PC[C]	0.179	0.336	0.124	0.215	0.100	0.197	0.089	0.164
	PC[D]	0.174	0.327	0.118	0.191	0.094	0.187	0.089	0.162
4	PC[A]	0.167	0.279	0.108	0.190	0.090	0.175	0.078	0.143
	PC[B]	0.161	0.272	0.100	0.185	0.085	0.166	0.077	0.140
	PC[C]	0.143	0.257	0.095	0.154	0.069	0.135	0.066	0.132
	PC[D]	0.135	0.217	0.094	0.136	0.063	0.125	0.063	0.116
5	PC[A]	0.123	0.210	0.092	0.129	0.058	0.113	0.056	0.106
	PC[B]	0.108	0.176	0.087	0.129	0.058	0.111	0.053	0.099
	PC[C]	0.084	0.128	0.075	0.114	0.056	0.109	0.052	0.096
	PC[D]	0.078	0.106	0.066	0.104	0.046	0.086	0.044	0.083
6	PC[A]	0.078	0.098	0.064	0.091	0.040	0.070	0.036	0.065
	PC[B]	0.076	0.094	0.063	0.090	0.037	0.070	0.035	0.064
	PC[C]	0.076	0.088	0.056	0.087	0.037	0.069	0.034	0.063
	PC[D]	0.072	0.086	0.053	0.082	0.036	0.066	0.032	0.058
7	PC[A]	0.072	0.081	0.052	0.078	0.035	0.066	0.031	0.056
	PC[B]	0.069	0.080	0.051	0.076	0.034	0.064	0.028	0.052
	PC[C]	0.068	0.079	0.051	0.065	0.033	0.056	0.024	0.043
	PC[D]	0.067	0.074	0.047	0.064	0.031	0.055	0.022	0.040
8	PC[A]	0.066	0.068	0.044	0.062	0.029	0.048	0.019	0.034
	PC[B]	0.065	0.064	0.040	0.058	0.028	0.048	0.017	0.031
	PC[C]	0.064	0.058	0.033	0.053	0.028	0.044	0.014	0.026
	PC[D]	0.062	0.055	0.030	0.052	0.027	0.039	0.010	0.019

Table 7. Point assessments of *b* from Set-II.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.715	0.712	0.566	0.564	0.554	0.511	0.431	0.420
	PC[B]	0.653	0.646	0.305	0.294	0.296	0.282	0.247	0.239
	PC[C]	0.577	0.572	0.298	0.287	0.290	0.272	0.241	0.235
	PC[D]	0.525	0.514	0.275	0.262	0.264	0.252	0.221	0.214
2	PC[A]	0.489	0.473	0.273	0.260	0.262	0.251	0.220	0.212
	PC[B]	0.479	0.461	0.256	0.243	0.245	0.234	0.209	0.203
	PC[C]	0.474	0.444	0.256	0.243	0.245	0.234	0.209	0.203
	PC[D]	0.459	0.441	0.245	0.243	0.209	0.213	0.181	0.166
3	PC[A]	0.452	0.438	0.209	0.206	0.179	0.176	0.167	0.164
	PC[B]	0.450	0.426	0.148	0.144	0.146	0.132	0.128	0.123
	PC[C]	0.435	0.413	0.146	0.124	0.128	0.123	0.107	0.102
	PC[D]	0.419	0.407	0.114	0.102	0.107	0.100	0.099	0.097
4	PC[A]	0.402	0.377	0.104	0.085	0.089	0.082	0.084	0.080
	PC[B]	0.392	0.367	0.079	0.065	0.071	0.058	0.071	0.052
	PC[C]	0.387	0.345	0.067	0.061	0.060	0.047	0.056	0.042
	PC[D]	0.333	0.296	0.064	0.057	0.059	0.046	0.053	0.039
5	PC[A]	0.184	0.141	0.064	0.055	0.056	0.045	0.051	0.038
	PC[B]	0.174	0.137	0.064	0.053	0.054	0.044	0.049	0.037
	PC[C]	0.148	0.114	0.063	0.043	0.049	0.042	0.044	0.037
	PC[D]	0.148	0.113	0.062	0.043	0.047	0.040	0.042	0.037
6	PC[A]	0.143	0.112	0.060	0.042	0.045	0.037	0.041	0.034
	PC[B]	0.139	0.109	0.057	0.040	0.042	0.035	0.039	0.032
	PC[C]	0.124	0.097	0.056	0.039	0.040	0.032	0.038	0.031
	PC[D]	0.116	0.091	0.056	0.037	0.037	0.029	0.035	0.028
7	PC[A]	0.111	0.088	0.052	0.037	0.034	0.028	0.034	0.021
	PC[B]	0.107	0.085	0.048	0.029	0.032	0.028	0.031	0.013
	PC[C]	0.101	0.079	0.046	0.025	0.031	0.018	0.031	0.013
	PC[D]	0.095	0.076	0.044	0.024	0.027	0.017	0.030	0.010
8	PC[A]	0.085	0.067	0.040	0.022	0.024	0.012	0.020	0.007
	PC[B]	0.079	0.064	0.038	0.022	0.021	0.009	0.018	0.004
	PC[C]	0.075	0.059	0.035	0.018	0.019	0.005	0.013	0.004
	PC[D]	0.074	0.059	0.031	0.017	0.016	0.004	0.012	0.003

Table 8. Point assessments of $\Re$ from Set-I.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.126	0.368	0.095	0.274	0.089	0.266	0.086	0.253
	PC[B]	0.097	0.263	0.085	0.232	0.078	0.230	0.076	0.228
	PC[C]	0.101	0.290	0.086	0.245	0.081	0.242	0.079	0.235
	PC[D]	0.126	0.357	0.090	0.257	0.085	0.255	0.083	0.246
2	PC[A]	0.095	0.247	0.083	0.231	0.076	0.227	0.074	0.218
	PC[B]	0.088	0.221	0.076	0.213	0.071	0.211	0.069	0.202
	PC[C]	0.089	0.233	0.077	0.220	0.074	0.219	0.069	0.207
	PC[D]	0.091	0.237	0.081	0.229	0.075	0.222	0.073	0.218
3	PC[A]	0.075	0.210	0.070	0.208	0.069	0.201	0.068	0.193
	PC[B]	0.071	0.204	0.068	0.201	0.066	0.197	0.064	0.154
	PC[C]	0.072	0.207	0.069	0.205	0.067	0.199	0.066	0.164
	PC[D]	0.074	0.208	0.069	0.206	0.068	0.200	0.067	0.187
4	PC[A]	0.069	0.202	0.067	0.200	0.065	0.195	0.059	0.147
	PC[B]	0.068	0.191	0.064	0.187	0.062	0.184	0.058	0.138
	PC[C]	0.069	0.194	0.065	0.190	0.063	0.187	0.059	0.139
	PC[D]	0.069	0.199	0.066	0.195	0.064	0.189	0.059	0.142
5	PC[A]	0.067	0.188	0.063	0.185	0.062	0.180	0.058	0.138
	PC[B]	0.056	0.133	0.047	0.120	0.041	0.117	0.039	0.114
	PC[C]	0.056	0.134	0.053	0.134	0.046	0.131	0.044	0.125
	PC[D]	0.065	0.183	0.061	0.179	0.060	0.174	0.057	0.135
6	PC[A]	0.053	0.132	0.044	0.113	0.038	0.111	0.037	0.108
	PC[B]	0.051	0.123	0.041	0.090	0.031	0.085	0.027	0.081
	PC[C]	0.051	0.124	0.042	0.098	0.033	0.096	0.032	0.093
	PC[D]	0.052	0.126	0.043	0.105	0.036	0.103	0.034	0.101
7	PC[A]	0.048	0.121	0.039	0.088	0.029	0.083	0.025	0.081
	PC[B]	0.041	0.097	0.028	0.045	0.020	0.042	0.018	0.037
	PC[C]	0.044	0.105	0.031	0.072	0.024	0.071	0.020	0.070
	PC[D]	0.046	0.114	0.035	0.086	0.028	0.082	0.023	0.080
8	PC[A]	0.040	0.096	0.027	0.043	0.017	0.040	0.015	0.027
	PC[B]	0.036	0.085	0.018	0.029	0.009	0.011	0.005	0.009
	PC[C]	0.036	0.088	0.022	0.033	0.011	0.016	0.008	0.012
	PC[D]	0.040	0.093	0.025	0.034	0.013	0.017	0.010	0.012

Table 9. Point assessments of $\Re$ from Set-II.

Test	PC	MLE		SSE		LINEX			
$\tau \rightarrow$						-3		+3	
1	PC[A]	0.065	0.093	0.054	0.043	0.032	0.040	0.029	0.038
	PC[B]	0.061	0.087	0.042	0.040	0.030	0.038	0.026	0.036
	PC[C]	0.061	0.088	0.043	0.041	0.031	0.039	0.027	0.037
	PC[D]	0.062	0.088	0.043	0.042	0.032	0.039	0.028	0.038
2	PC[A]	0.060	0.085	0.041	0.039	0.028	0.038	0.025	0.035
	PC[B]	0.056	0.081	0.039	0.037	0.025	0.035	0.022	0.033
	PC[C]	0.058	0.083	0.040	0.038	0.026	0.035	0.023	0.034
	PC[D]	0.059	0.084	0.040	0.038	0.028	0.036	0.024	0.035
3	PC[A]	0.055	0.080	0.038	0.037	0.025	0.028	0.021	0.024
	PC[B]	0.051	0.073	0.035	0.035	0.023	0.027	0.019	0.024
	PC[C]	0.054	0.078	0.037	0.035	0.023	0.028	0.020	0.024
	PC[D]	0.055	0.079	0.037	0.036	0.024	0.028	0.020	0.024
4	PC[A]	0.045	0.067	0.034	0.034	0.022	0.026	0.019	0.024
	PC[B]	0.042	0.064	0.031	0.032	0.019	0.023	0.016	0.022
	PC[C]	0.043	0.065	0.032	0.033	0.020	0.024	0.017	0.024
	PC[D]	0.044	0.066	0.033	0.034	0.021	0.025	0.018	0.024
5	PC[A]	0.041	0.064	0.030	0.030	0.018	0.022	0.015	0.021
	PC[B]	0.037	0.060	0.027	0.026	0.016	0.021	0.012	0.019
	PC[C]	0.039	0.060	0.028	0.027	0.017	0.022	0.013	0.020
	PC[D]	0.040	0.061	0.029	0.029	0.017	0.022	0.014	0.020
6	PC[A]	0.035	0.060	0.026	0.025	0.015	0.021	0.011	0.018
	PC[B]	0.031	0.058	0.022	0.019	0.012	0.014	0.008	0.012
	PC[C]	0.032	0.059	0.023	0.021	0.013	0.020	0.009	0.015
	PC[D]	0.034	0.060	0.025	0.022	0.014	0.020	0.010	0.017
7	PC[A]	0.030	0.058	0.021	0.018	0.011	0.013	0.008	0.011
	PC[B]	0.025	0.055	0.018	0.015	0.008	0.011	0.005	0.008
	PC[C]	0.027	0.055	0.019	0.015	0.010	0.012	0.006	0.008
	PC[D]	0.028	0.056	0.020	0.017	0.010	0.012	0.007	0.010
8	PC[A]	0.024	0.054	0.017	0.015	0.008	0.011	0.005	0.008
	PC[B]	0.022	0.052	0.015	0.015	0.007	0.010	0.004	0.007
	PC[C]	0.021	0.048	0.015	0.014	0.007	0.010	0.004	0.007
	PC[D]	0.020	0.045	0.014	0.014	0.006	0.009	0.003	0.007

Table 10. Interval assessments of a_1 from Set-I.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	0.538	0.926	0.548	0.924	0.473	0.946	0.434	0.949
	PC[B]	0.432	0.933	0.442	0.931	0.374	0.953	0.326	0.956
	PC[C]	0.455	0.932	0.460	0.930	0.404	0.952	0.357	0.955
	PC[D]	0.493	0.929	0.503	0.927	0.423	0.949	0.365	0.952
2	PC[A]	0.422	0.934	0.428	0.932	0.353	0.954	0.316	0.957
	PC[B]	0.401	0.935	0.410	0.933	0.290	0.955	0.278	0.958
	PC[C]	0.409	0.935	0.416	0.932	0.291	0.955	0.289	0.958
	PC[D]	0.415	0.934	0.423	0.932	0.323	0.954	0.305	0.957
3	PC[A]	0.384	0.936	0.398	0.934	0.263	0.957	0.254	0.959
	PC[B]	0.364	0.938	0.375	0.936	0.251	0.959	0.218	0.961
	PC[C]	0.373	0.937	0.384	0.935	0.255	0.958	0.224	0.960
	PC[D]	0.374	0.937	0.387	0.935	0.255	0.958	0.249	0.960
4	PC[A]	0.363	0.938	0.371	0.936	0.248	0.959	0.213	0.961
	PC[B]	0.331	0.939	0.340	0.937	0.237	0.960	0.201	0.962
	PC[C]	0.339	0.939	0.348	0.937	0.240	0.960	0.207	0.962
	PC[D]	0.354	0.937	0.365	0.935	0.246	0.958	0.211	0.960
5	PC[A]	0.323	0.940	0.332	0.938	0.233	0.961	0.183	0.964
	PC[B]	0.313	0.941	0.321	0.939	0.221	0.962	0.164	0.965
	PC[C]	0.318	0.941	0.322	0.938	0.225	0.962	0.169	0.964
	PC[D]	0.321	0.940	0.325	0.938	0.230	0.961	0.171	0.964
6	PC[A]	0.312	0.941	0.318	0.939	0.218	0.962	0.153	0.965
	PC[B]	0.297	0.943	0.304	0.941	0.208	0.964	0.137	0.967
	PC[C]	0.309	0.942	0.313	0.940	0.211	0.963	0.142	0.966
	PC[D]	0.311	0.941	0.316	0.939	0.213	0.962	0.146	0.965
7	PC[A]	0.293	0.943	0.298	0.942	0.189	0.964	0.132	0.967
	PC[B]	0.251	0.947	0.255	0.945	0.126	0.968	0.116	0.971
	PC[C]	0.264	0.946	0.269	0.944	0.146	0.967	0.125	0.970
	PC[D]	0.291	0.944	0.296	0.942	0.169	0.965	0.129	0.968
8	PC[A]	0.247	0.948	0.251	0.946	0.118	0.969	0.110	0.972
	PC[B]	0.224	0.950	0.230	0.948	0.091	0.971	0.082	0.974
	PC[C]	0.232	0.949	0.237	0.947	0.099	0.970	0.090	0.973
	PC[D]	0.242	0.948	0.245	0.946	0.108	0.969	0.098	0.972

Table 11. Interval assessments of a_1 from Set-II.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	1.168	0.896	1.195	0.894	0.517	0.908	0.480	0.911
	PC[B]	1.119	0.902	1.143	0.900	0.408	0.918	0.353	0.921
	PC[C]	1.138	0.900	1.162	0.898	0.438	0.916	0.374	0.919
	PC[D]	1.158	0.899	1.184	0.897	0.460	0.912	0.414	0.915
2	PC[A]	1.079	0.903	1.101	0.901	0.391	0.919	0.320	0.923
	PC[B]	1.023	0.908	1.044	0.906	0.372	0.924	0.302	0.928
	PC[C]	1.053	0.907	1.074	0.905	0.378	0.923	0.311	0.927
	PC[D]	1.075	0.905	1.096	0.903	0.386	0.921	0.317	0.925
3	PC[A]	1.015	0.909	1.025	0.907	0.354	0.927	0.294	0.931
	PC[B]	0.966	0.913	0.974	0.911	0.308	0.931	0.259	0.935
	PC[C]	0.997	0.911	1.017	0.909	0.313	0.929	0.270	0.933
	PC[D]	1.005	0.910	1.024	0.908	0.329	0.928	0.277	0.932
4	PC[A]	0.955	0.915	0.964	0.913	0.307	0.932	0.257	0.936
	PC[B]	0.933	0.917	0.947	0.915	0.298	0.933	0.248	0.937
	PC[C]	0.936	0.917	0.953	0.915	0.302	0.932	0.252	0.936
	PC[D]	0.941	0.916	0.958	0.914	0.305	0.932	0.255	0.936
5	PC[A]	0.925	0.918	0.939	0.916	0.296	0.934	0.246	0.938
	PC[B]	0.871	0.922	0.882	0.920	0.286	0.936	0.233	0.939
	PC[C]	0.884	0.921	0.897	0.919	0.291	0.935	0.238	0.939
	PC[D]	0.895	0.920	0.911	0.918	0.294	0.934	0.239	0.938
6	PC[A]	0.864	0.924	0.875	0.922	0.286	0.938	0.232	0.940
	PC[B]	0.829	0.928	0.849	0.926	0.261	0.942	0.217	0.944
	PC[C]	0.840	0.926	0.850	0.924	0.275	0.940	0.220	0.942
	PC[D]	0.854	0.925	0.865	0.923	0.282	0.939	0.225	0.941
7	PC[A]	0.802	0.930	0.810	0.928	0.226	0.945	0.185	0.946
	PC[B]	0.765	0.934	0.775	0.932	0.206	0.949	0.161	0.950
	PC[C]	0.785	0.932	0.795	0.930	0.207	0.947	0.176	0.948
	PC[D]	0.798	0.931	0.807	0.929	0.225	0.946	0.185	0.947
8	PC[A]	0.717	0.936	0.726	0.934	0.199	0.951	0.155	0.953
	PC[B]	0.656	0.940	0.668	0.938	0.179	0.955	0.145	0.957
	PC[C]	0.676	0.938	0.684	0.936	0.183	0.953	0.147	0.955
	PC[D]	0.693	0.937	0.701	0.935	0.188	0.952	0.148	0.954

Table 12. Interval assessments of a_2 from Set-I.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	0.774	0.918	0.793	0.917	0.264	0.942	0.237	0.943
	PC[B]	0.719	0.922	0.741	0.921	0.251	0.944	0.228	0.945
	PC[C]	0.725	0.922	0.764	0.920	0.256	0.944	0.230	0.945
	PC[D]	0.765	0.919	0.784	0.918	0.261	0.943	0.233	0.944
2	PC[A]	0.710	0.923	0.732	0.922	0.245	0.945	0.226	0.946
	PC[B]	0.685	0.925	0.711	0.924	0.235	0.949	0.216	0.950
	PC[C]	0.693	0.924	0.722	0.923	0.237	0.947	0.218	0.948
	PC[D]	0.707	0.923	0.729	0.922	0.240	0.946	0.221	0.947
3	PC[A]	0.679	0.926	0.701	0.925	0.232	0.950	0.213	0.952
	PC[B]	0.646	0.930	0.668	0.929	0.214	0.954	0.200	0.956
	PC[C]	0.655	0.929	0.672	0.928	0.226	0.953	0.206	0.955
	PC[D]	0.665	0.927	0.686	0.926	0.229	0.951	0.209	0.953
4	PC[A]	0.643	0.930	0.659	0.929	0.211	0.954	0.196	0.956
	PC[B]	0.627	0.933	0.634	0.932	0.201	0.956	0.181	0.958
	PC[C]	0.631	0.932	0.642	0.931	0.206	0.956	0.186	0.958
	PC[D]	0.635	0.931	0.650	0.930	0.207	0.955	0.192	0.957
5	PC[A]	0.578	0.937	0.582	0.936	0.185	0.958	0.169	0.959
	PC[B]	0.476	0.942	0.484	0.941	0.169	0.961	0.160	0.962
	PC[C]	0.516	0.941	0.524	0.940	0.176	0.960	0.163	0.961
	PC[D]	0.546	0.939	0.565	0.938	0.179	0.959	0.166	0.960
6	PC[A]	0.453	0.942	0.464	0.941	0.165	0.961	0.158	0.962
	PC[B]	0.414	0.945	0.418	0.944	0.157	0.964	0.146	0.965
	PC[C]	0.419	0.944	0.423	0.943	0.160	0.963	0.150	0.964
	PC[D]	0.428	0.943	0.435	0.942	0.162	0.962	0.155	0.963
7	PC[A]	0.408	0.945	0.411	0.944	0.153	0.964	0.122	0.965
	PC[B]	0.395	0.946	0.399	0.945	0.138	0.966	0.111	0.967
	PC[C]	0.397	0.946	0.401	0.945	0.140	0.965	0.118	0.966
	PC[D]	0.398	0.946	0.407	0.945	0.145	0.965	0.121	0.966
8	PC[A]	0.384	0.947	0.388	0.946	0.135	0.966	0.108	0.967
	PC[B]	0.348	0.952	0.357	0.951	0.127	0.968	0.088	0.969
	PC[C]	0.358	0.950	0.365	0.949	0.130	0.967	0.092	0.968
	PC[D]	0.371	0.948	0.374	0.947	0.133	0.967	0.101	0.968

Table 13. Interval assessments of a_2 from Set-II.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	1.087	0.906	1.103	0.904	0.286	0.939	0.263	0.942
	PC[B]	0.995	0.908	1.015	0.907	0.271	0.943	0.241	0.944
	PC[C]	1.010	0.907	1.041	0.906	0.276	0.943	0.243	0.944
	PC[D]	1.062	0.907	1.094	0.905	0.281	0.940	0.258	0.943
2	PC[A]	0.989	0.908	1.013	0.907	0.266	0.944	0.235	0.946
	PC[B]	0.960	0.911	0.983	0.909	0.239	0.947	0.218	0.950
	PC[C]	0.965	0.910	0.989	0.909	0.246	0.945	0.220	0.949
	PC[D]	0.980	0.908	1.006	0.908	0.256	0.944	0.227	0.947
3	PC[A]	0.958	0.912	0.978	0.910	0.232	0.948	0.213	0.951
	PC[B]	0.924	0.917	0.945	0.915	0.224	0.952	0.207	0.955
	PC[C]	0.938	0.915	0.959	0.913	0.225	0.951	0.211	0.954
	PC[D]	0.943	0.913	0.966	0.911	0.228	0.949	0.212	0.952
4	PC[A]	0.913	0.918	0.934	0.916	0.222	0.952	0.206	0.955
	PC[B]	0.846	0.922	0.865	0.921	0.208	0.955	0.187	0.958
	PC[C]	0.849	0.922	0.867	0.920	0.213	0.954	0.192	0.957
	PC[D]	0.856	0.921	0.875	0.919	0.218	0.953	0.205	0.956
5	PC[A]	0.690	0.926	0.714	0.924	0.194	0.956	0.184	0.959
	PC[B]	0.579	0.932	0.598	0.930	0.182	0.958	0.172	0.961
	PC[C]	0.593	0.931	0.606	0.929	0.190	0.957	0.175	0.960
	PC[D]	0.628	0.929	0.635	0.927	0.191	0.956	0.176	0.959
6	PC[A]	0.571	0.933	0.584	0.931	0.181	0.958	0.169	0.961
	PC[B]	0.554	0.935	0.572	0.933	0.164	0.961	0.152	0.964
	PC[C]	0.562	0.934	0.575	0.932	0.169	0.960	0.157	0.963
	PC[D]	0.568	0.933	0.578	0.931	0.180	0.958	0.163	0.962
7	PC[A]	0.547	0.936	0.570	0.933	0.155	0.962	0.147	0.965
	PC[B]	0.537	0.937	0.552	0.935	0.147	0.963	0.123	0.967
	PC[C]	0.541	0.937	0.559	0.934	0.149	0.963	0.124	0.967
	PC[D]	0.543	0.936	0.560	0.934	0.151	0.962	0.131	0.966
8	PC[A]	0.535	0.937	0.548	0.935	0.145	0.963	0.118	0.968
	PC[B]	0.518	0.938	0.523	0.937	0.136	0.965	0.098	0.969
	PC[C]	0.529	0.938	0.534	0.936	0.138	0.965	0.107	0.968
	PC[D]	0.533	0.938	0.537	0.936	0.142	0.964	0.112	0.968

Table 14. Interval assessments of b from Set-I.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	0.401	0.932	0.344	0.936	0.238	0.941	0.231	0.942
	PC[B]	0.388	0.933	0.334	0.937	0.230	0.942	0.228	0.943
	PC[C]	0.354	0.935	0.317	0.939	0.229	0.944	0.227	0.945
	PC[D]	0.349	0.936	0.308	0.940	0.214	0.945	0.211	0.946
2	PC[A]	0.332	0.938	0.288	0.942	0.210	0.947	0.196	0.948
	PC[B]	0.321	0.940	0.270	0.943	0.195	0.949	0.194	0.949
	PC[C]	0.311	0.941	0.265	0.945	0.194	0.950	0.193	0.951
	PC[D]	0.306	0.942	0.264	0.946	0.193	0.951	0.191	0.951
3	PC[A]	0.292	0.944	0.244	0.948	0.192	0.951	0.190	0.951
	PC[B]	0.285	0.945	0.239	0.949	0.191	0.952	0.188	0.953
	PC[C]	0.275	0.946	0.228	0.950	0.184	0.953	0.182	0.954
	PC[D]	0.264	0.948	0.224	0.952	0.174	0.955	0.172	0.956
4	PC[A]	0.257	0.950	0.214	0.954	0.169	0.957	0.163	0.958
	PC[B]	0.248	0.951	0.211	0.955	0.166	0.958	0.156	0.959
	PC[C]	0.246	0.952	0.208	0.956	0.163	0.958	0.152	0.960
	PC[D]	0.245	0.952	0.206	0.956	0.160	0.959	0.147	0.960
5	PC[A]	0.241	0.953	0.204	0.956	0.151	0.960	0.144	0.960
	PC[B]	0.239	0.953	0.202	0.957	0.145	0.960	0.141	0.961
	PC[C]	0.237	0.953	0.200	0.957	0.142	0.960	0.139	0.961
	PC[D]	0.232	0.954	0.193	0.958	0.141	0.960	0.136	0.962
6	PC[A]	0.220	0.955	0.178	0.959	0.136	0.961	0.133	0.963
	PC[B]	0.216	0.955	0.178	0.959	0.135	0.962	0.131	0.963
	PC[C]	0.209	0.956	0.171	0.959	0.132	0.963	0.126	0.963
	PC[D]	0.206	0.956	0.169	0.960	0.131	0.963	0.125	0.964
7	PC[A]	0.204	0.956	0.167	0.960	0.128	0.963	0.123	0.964
	PC[B]	0.202	0.957	0.165	0.961	0.122	0.964	0.119	0.965
	PC[C]	0.201	0.957	0.164	0.961	0.117	0.964	0.122	0.965
	PC[D]	0.198	0.958	0.163	0.962	0.111	0.965	0.106	0.966
8	PC[A]	0.196	0.958	0.162	0.962	0.109	0.965	0.103	0.966
	PC[B]	0.189	0.959	0.146	0.964	0.103	0.966	0.101	0.968
	PC[C]	0.182	0.960	0.139	0.965	0.099	0.967	0.095	0.969
	PC[D]	0.178	0.961	0.136	0.965	0.095	0.968	0.091	0.969

Table 15. Interval assessments of b from Set-II.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	0.579	0.917	0.561	0.919	0.538	0.920	0.510	0.922
	PC[B]	0.567	0.919	0.560	0.921	0.482	0.922	0.447	0.924
	PC[C]	0.550	0.920	0.543	0.922	0.474	0.923	0.446	0.925
	PC[D]	0.477	0.924	0.461	0.926	0.425	0.928	0.418	0.929
2	PC[A]	0.461	0.925	0.457	0.927	0.421	0.929	0.417	0.930
	PC[B]	0.457	0.926	0.453	0.928	0.414	0.930	0.388	0.932
	PC[C]	0.435	0.928	0.431	0.930	0.391	0.932	0.379	0.933
	PC[D]	0.429	0.929	0.425	0.931	0.348	0.933	0.330	0.935
3	PC[A]	0.426	0.929	0.420	0.931	0.314	0.935	0.297	0.937
	PC[B]	0.411	0.931	0.405	0.933	0.302	0.937	0.283	0.938
	PC[C]	0.390	0.933	0.387	0.935	0.286	0.938	0.281	0.938
	PC[D]	0.386	0.934	0.383	0.936	0.247	0.938	0.240	0.939
4	PC[A]	0.380	0.934	0.377	0.936	0.235	0.939	0.223	0.940
	PC[B]	0.342	0.936	0.337	0.938	0.216	0.941	0.176	0.942
	PC[C]	0.326	0.937	0.318	0.939	0.201	0.941	0.171	0.943
	PC[D]	0.320	0.937	0.315	0.939	0.185	0.942	0.167	0.943
5	PC[A]	0.318	0.938	0.314	0.940	0.184	0.942	0.165	0.944
	PC[B]	0.312	0.938	0.310	0.940	0.178	0.943	0.162	0.944
	PC[C]	0.306	0.939	0.303	0.941	0.174	0.943	0.159	0.945
	PC[D]	0.303	0.939	0.300	0.941	0.170	0.944	0.158	0.945
6	PC[A]	0.293	0.940	0.289	0.942	0.160	0.946	0.141	0.946
	PC[B]	0.290	0.940	0.287	0.942	0.159	0.946	0.139	0.946
	PC[C]	0.281	0.941	0.278	0.943	0.156	0.947	0.138	0.947
	PC[D]	0.266	0.943	0.265	0.945	0.153	0.948	0.135	0.948
7	PC[A]	0.253	0.944	0.251	0.946	0.150	0.949	0.133	0.950
	PC[B]	0.228	0.946	0.226	0.948	0.144	0.951	0.130	0.952
	PC[C]	0.217	0.947	0.214	0.949	0.141	0.952	0.128	0.953
	PC[D]	0.198	0.948	0.193	0.950	0.137	0.953	0.126	0.954
8	PC[A]	0.196	0.948	0.192	0.950	0.131	0.953	0.120	0.955
	PC[B]	0.188	0.949	0.182	0.951	0.124	0.954	0.119	0.956
	PC[C]	0.185	0.949	0.180	0.951	0.120	0.954	0.105	0.957
	PC[D]	0.178	0.951	0.168	0.953	0.115	0.956	0.101	0.957

Table 16. Interval assessments of $\Re$ from Set-I.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	0.257	0.942	0.264	0.940	0.143	0.950	0.138	0.952
	PC[B]	0.246	0.945	0.251	0.942	0.129	0.953	0.127	0.955
	PC[C]	0.251	0.943	0.255	0.941	0.133	0.952	0.130	0.954
	PC[D]	0.255	0.942	0.262	0.940	0.137	0.951	0.135	0.952
2	PC[A]	0.244	0.945	0.247	0.943	0.130	0.953	0.125	0.956
	PC[B]	0.234	0.946	0.239	0.944	0.119	0.955	0.115	0.957
	PC[C]	0.236	0.946	0.241	0.944	0.122	0.955	0.119	0.957
	PC[D]	0.237	0.946	0.242	0.944	0.125	0.954	0.120	0.957
3	PC[A]	0.231	0.947	0.238	0.945	0.112	0.956	0.110	0.958
	PC[B]	0.223	0.948	0.232	0.946	0.101	0.957	0.092	0.959
	PC[C]	0.227	0.948	0.235	0.946	0.108	0.956	0.105	0.959
	PC[D]	0.231	0.947	0.236	0.945	0.110	0.956	0.108	0.958
4	PC[A]	0.221	0.948	0.227	0.946	0.097	0.957	0.075	0.961
	PC[B]	0.182	0.951	0.185	0.949	0.088	0.960	0.072	0.962
	PC[C]	0.183	0.951	0.194	0.949	0.090	0.960	0.073	0.962
	PC[D]	0.192	0.950	0.213	0.948	0.094	0.959	0.074	0.962
5	PC[A]	0.177	0.952	0.184	0.950	0.085	0.961	0.071	0.963
	PC[B]	0.163	0.954	0.174	0.951	0.077	0.963	0.062	0.965
	PC[C]	0.169	0.953	0.175	0.951	0.079	0.962	0.068	0.964
	PC[D]	0.170	0.953	0.178	0.951	0.082	0.962	0.068	0.964
6	PC[A]	0.158	0.956	0.170	0.952	0.074	0.965	0.060	0.966
	PC[B]	0.147	0.957	0.162	0.954	0.068	0.966	0.056	0.968
	PC[C]	0.149	0.957	0.168	0.953	0.069	0.966	0.057	0.967
	PC[D]	0.153	0.956	0.169	0.953	0.072	0.965	0.058	0.966
7	PC[A]	0.141	0.958	0.156	0.955	0.067	0.966	0.055	0.969
	PC[B]	0.130	0.961	0.141	0.958	0.061	0.970	0.051	0.971
	PC[C]	0.133	0.960	0.147	0.957	0.062	0.969	0.053	0.970
	PC[D]	0.136	0.959	0.151	0.956	0.064	0.967	0.054	0.969
8	PC[A]	0.128	0.961	0.137	0.958	0.060	0.970	0.050	0.971
	PC[B]	0.121	0.963	0.127	0.960	0.053	0.972	0.041	0.974
	PC[C]	0.124	0.962	0.129	0.959	0.058	0.971	0.046	0.973
	PC[D]	0.126	0.962	0.133	0.959	0.059	0.970	0.049	0.972

Table 17. Interval assessments of $\Re$ from Set-II.

Test	PC	ACI-NA		ACI-NT		BCI		HPD	
1	PC[A]	0.273	0.939	0.276	0.938	0.157	0.948	0.146	0.949
	PC[B]	0.257	0.941	0.260	0.939	0.116	0.951	0.110	0.953
	PC[C]	0.262	0.940	0.264	0.939	0.127	0.949	0.119	0.952
	PC[D]	0.271	0.939	0.274	0.938	0.137	0.948	0.127	0.951
2	PC[A]	0.255	0.941	0.258	0.940	0.113	0.951	0.104	0.953
	PC[B]	0.248	0.942	0.251	0.941	0.099	0.953	0.092	0.955
	PC[C]	0.251	0.942	0.253	0.941	0.101	0.952	0.094	0.954
	PC[D]	0.254	0.941	0.256	0.940	0.105	0.952	0.099	0.954
3	PC[A]	0.246	0.942	0.246	0.941	0.098	0.953	0.091	0.955
	PC[B]	0.238	0.944	0.242	0.943	0.091	0.954	0.088	0.956
	PC[C]	0.241	0.943	0.244	0.942	0.092	0.954	0.089	0.956
	PC[D]	0.243	0.943	0.245	0.942	0.094	0.953	0.090	0.955
4	PC[A]	0.232	0.944	0.238	0.943	0.090	0.954	0.087	0.956
	PC[B]	0.217	0.947	0.225	0.946	0.086	0.956	0.084	0.958
	PC[C]	0.220	0.946	0.231	0.945	0.088	0.955	0.085	0.957
	PC[D]	0.224	0.946	0.237	0.944	0.089	0.955	0.086	0.956
5	PC[A]	0.205	0.948	0.220	0.947	0.085	0.957	0.081	0.959
	PC[B]	0.182	0.950	0.192	0.949	0.081	0.960	0.077	0.962
	PC[C]	0.184	0.949	0.202	0.948	0.082	0.959	0.078	0.961
	PC[D]	0.193	0.949	0.213	0.948	0.083	0.958	0.080	0.960
6	PC[A]	0.180	0.950	0.182	0.949	0.078	0.962	0.075	0.963
	PC[B]	0.170	0.952	0.177	0.951	0.075	0.963	0.071	0.965
	PC[C]	0.174	0.951	0.178	0.950	0.076	0.962	0.072	0.964
	PC[D]	0.176	0.951	0.179	0.950	0.077	0.962	0.074	0.964
7	PC[A]	0.168	0.952	0.174	0.951	0.074	0.964	0.070	0.966
	PC[B]	0.162	0.954	0.169	0.953	0.065	0.967	0.061	0.968
	PC[C]	0.165	0.953	0.171	0.952	0.072	0.966	0.068	0.967
	PC[D]	0.166	0.952	0.173	0.951	0.073	0.965	0.069	0.967
8	PC[A]	0.160	0.955	0.168	0.954	0.063	0.967	0.059	0.968
	PC[B]	0.154	0.957	0.164	0.956	0.055	0.970	0.050	0.971
	PC[C]	0.157	0.956	0.165	0.955	0.058	0.969	0.053	0.970
	PC[D]	0.158	0.956	0.166	0.955	0.060	0.968	0.057	0.969

5. Real-world data analysis

This part offers two examples that demonstrate how the proposed methodologies can be utilized in a practical environment, using various actual data sets from the engineering field.

5.1. Light-emitting diode

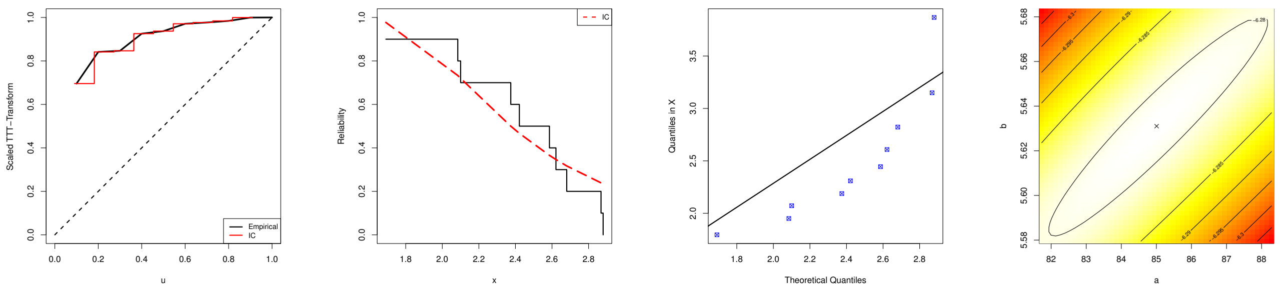
A light-emitting diode (LED) is a semiconductor device that emits infrared or visible light when charged with an electric current. Recently, organic white LEDs (WOLEDs) have emerged as a promising technology for low-power lighting applications because they combine the benefits of high efficiency and low manufacturing costs with the attractive qualities of large surfaces that radiate high-quality white light; see Farinola and Ragni [36]. This application illustrates the applicability of the proposed methodologies for evaluating the lifetime of the M00071 WOLEDs mixed with red, green, and blue colors under two stress levels, namely: 9.64mA (with $M_1 = 10$) and 17.09mA (with $M_2 = 10$). In Table 18, each data point has been divided by 1000 for computational purposes. Also, in Table 18, we assume the data set at 9.64mA by X and the other by Y. This data set was originally reported by Zhang et al. [37] and rediscussed by Nassar et al. [38]. To determine if the IC distribution is suitable for the WOLED data sets, the MLEs (with their standard errors (Std.Errs)) of IC parameters a and b are derived first to create the Kolmogorov-Smirnov (KS) statistic with its P-value; see Table 18. Since the computed P-value is far from the desired significance percentage (5%), the IC distribution matches the WOLED data sets satisfactorily.

Table 18. Newly transformed WOLED data sets.

Group	Data	MLE(Std.Err)		KS(P-Value)
		a	b	
X	1.6915, 2.0847, 2.1003, 2.3745, 2.4215, 2.5860, 2.6215, 2.6805, 2.8680, 2.8795	85.007(81.938)	5.6310(1.2882)	0.2351(0.5614)
Y	0.6015, 0.6897, 0.6973, 0.7165, 0.7855, 0.8545, 0.8895, 1.1157, 1.1313, 1.2515	0.1594(0.0806)	2.2851(0.3753)	0.1287(0.9887)

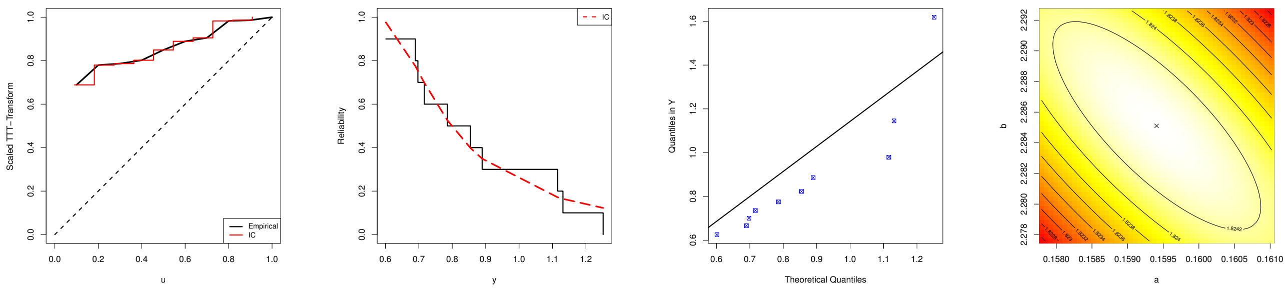
Additionally, in Figure 3, four fitting diagrams are also investigated to examine the goodness-of-fit results, namely: (1) scaled total-time-on-test (TTT) transform, (2) fitted IC reliability, (3) quantile-quantile, and (4) contour for the log-likelihood of a and b. For both groups X and Y, Figure 3 offers that the fitted TTT-transforms indicate that the WOLED data sets provide an increasing failure rate, the fitted reliability (or quantile-quantile) line captured its empirical-line adequately, and that the acquired values of $\hat{a}$ and $\hat{b}$ used in KS calculations existed and are unique. As a result, all subplots in Figure 3 support our appropriate conclusion.

From Table 18, for each group of WOLED data, several artificial APTIIC samples (symbolized by $\mathbb{S}[i]$, $i = 1, 2, 3,$) are generated based on $m_1 = m_2 = 5$ and various selects of $T_i, i = 1, 2$, $\underline{R}$ and $\underline{S}$; see Table 19. For each sample in Table 19, the point estimations (with their Std.Errs) and 95% interval estimates (with their interval widths (IWs)) developed by the likelihood and Bayes approaches of a_i, $i = 1, 2$, b, and $\Re$ are evaluated; see Table 20. Due to our lack of prior information about the IC parameters from the WOLED data, all Bayes calculations will be created using the joint improper gamma prior against (from SSE and LINEX(for $\tau(= -5, -0.05, +5)$). Without loss of generality, we assign $\mathfrak{M} = 50,000$ and $\mathfrak{B} = 10,000$ to develop all acquired Bayes' inferences of a_i, $i = 1, 2$, b, and $\Re$. Here, the estimated values of $\hat{a}$ and $\hat{b}$ (in Table 20) are used as starting points in MCMC iterations. It is noted, from Table 20, that the estimated values of a_i, $i = 1, 2$, b, and $\Re$ obtained from the MCMC method behaved well compared to those obtained from the MCMC method in terms of the smallest estimated Std.Err values. Similar behavior is also noted, from Table 21, if one compares the credible (from BCI/HPD) intervals with the asymptotic (from ACI-NA/ACI-NT) intervals in terms of the smallest estimated IW values.



(a) (b) (c) (d)

(i) Group-X



(a) (b) (c) (d)

(ii) Group-Y

Figure 3. Scaled TTT-transform (a), Fitted IC reliability (b), Fitted quantile-quantile (c), Contour (d) from WOLED data sets.

Table 19. Artificial APTIIC samples from WOLED data.

Sample	$\underline{R}$ / $\underline{S}$	$T_1(d_1)$ / $T_2(d_2)$	R^* / S^*	Data
$\mathbb{S}[1]$	(1*5)	2.1(2)	3	1.6915, 2.1003, 2.3745, 2.4215, 2.6215
	(1*5)	0.7(3)	2	0.6015, 0.6897, 0.7165, 0.8545, 1.1157
$\mathbb{S}[2]$	(2,3,0*3)	2.2(1)	3	1.6915, 2.0847, 2.3745, 2.4215, 2.5860
	(3,2,0*3)	0.8(1)	2	0.6015, 0.6897, 0.7165, 0.8545, 0.8895
$\mathbb{S}[3]$	(0*3,2,3)	2.5(4)	3	1.6915, 2.0847, 2.1003, 2.3745, 2.6215
	(0*3,3,2)	0.8(4)	2	0.6015, 0.6897, 0.6973, 0.7165, 1.1157

Table 20. Point estimates of a_1, a_2, b, and $\Re$ from WOLED data.

Sample	Par.	MLE		SSE		LINEX					
$\tau \rightarrow$						−5		−0.05		+5	
$\mathbb{S}[1]$	a_1	6.7877	3.2279	6.7621	0.0558	7.9319	0.0556	6.6830	0.0553	6.5795	0.0555
	a_2	0.2091	0.1154	0.1902	0.0438	0.2232	0.0436	0.1880	0.0434	0.1851	0.0436
	b	2.1221	0.4063	2.0989	0.0541	2.4620	0.0539	2.0743	0.0536	2.0422	0.0539
	$\Re$	0.9701	0.0264	0.9727	0.0061	1.1409	0.0061	0.9613	0.0060	0.9464	0.0061
$\mathbb{S}[2]$	a_1	8.4977	3.9352	8.4884	0.0259	9.5325	0.0257	8.4290	0.0256	8.3382	0.0257
	a_2	0.1230	0.0788	0.1150	0.0225	0.1292	0.0223	0.1142	0.0222	0.1130	0.0223
	b	2.4420	0.3978	2.4332	0.0257	2.7325	0.0256	2.4162	0.0255	2.3902	0.0255
	$\Re$	0.9857	0.0138	0.9866	0.0026	1.1080	0.0026	0.9797	0.0025	0.9692	0.0026
$\mathbb{S}[3]$	a_1	6.8853	3.2264	6.8760	0.0258	7.9280	0.0256	6.8437	0.0257	6.7061	0.0256
	a_2	0.1942	0.1079	0.1843	0.0244	0.2125	0.0242	0.1835	0.0243	0.1798	0.0242
	b	2.1437	0.4180	2.1350	0.0258	2.4617	0.0256	2.1250	0.0257	2.0823	0.0255
	$\Re$	0.9726	0.0244	0.9739	0.0034	1.1229	0.0033	0.9693	0.0033	0.9498	0.0033

Table 21. Interval estimates of a_1, a_2, b, and $\Re$ from WOLED data.

Sample	Par.	ACI-NA ACI-NT			BCI HPD		
		Low.	Upp.	IW	Low.	Upp.	IW
$\mathbb{S}[1]$	a_1	0.4611	13.114	12.653	6.6657	6.8599	0.1942
		2.6726	17.239	14.566	6.6662	6.8601	0.1939
	a_2	0.0000	0.4353	0.4353	0.1166	0.2710	0.1543
		0.0709	0.6170	0.5461	0.1138	0.2674	0.1535
	b	1.3258	2.9184	1.5926	2.0052	2.1967	0.1915
		1.4581	3.0884	1.6303	2.0034	2.1935	0.1901
	$\Re$	0.9183	1.0219	0.1036	0.9614	0.9830	0.0216
		0.9197	1.0233	0.1036	0.9619	0.9834	0.0215
$\mathbb{S}[2]$	a_1	0.7849	16.210	15.426	8.4420	8.5403	0.0983
		3.4286	21.061	17.632	8.4429	8.5411	0.0982
	a_2	0.0000	0.2775	0.2775	0.0758	0.1590	0.0832
		0.0350	0.4321	0.3971	0.0742	0.1572	0.0830
	b	1.6624	3.2217	1.5594	2.3873	2.4853	0.0979
		1.7746	3.3606	1.5860	2.3872	2.4851	0.0978
	$\Re$	0.9586	1.0128	0.0542	0.9816	0.9912	0.0095
		0.9590	1.0132	0.0542	0.9817	0.9913	0.0095
$\mathbb{S}[3]$	a_1	0.5617	13.209	12.647	6.8297	6.9279	0.0982
		2.7483	17.250	14.502	6.8295	6.9276	0.0981
	a_2	0.0000	0.4057	0.4057	0.1420	0.2324	0.0904
		0.0654	0.5770	0.5116	0.1426	0.2329	0.0903
	b	1.3244	2.9631	1.6387	2.0885	2.1866	0.0982
		1.4628	3.1417	1.6789	2.0889	2.1870	0.0980
	$\Re$	0.9247	1.0204	0.0957	0.9673	0.9798	0.0125
		0.9259	1.0216	0.0957	0.9672	0.9797	0.0125

To see the existence and uniqueness of the reported likelihood estimates of a_1, a_2, and b in Table 20, for $\mathbb{S}[i]$, $i = 1, 2, 3$, the profile log-likelihoods of a_1, a_2, and b are plotted; see Figure 4. It shows that all values of $\hat{a}_1$, $\hat{a}_2$, and $\hat{b}$ (in Table 20) existed and are unique. Using Sample $\mathbb{S}[1]$ (as an example), Figure 5 displays the trace and density plots of a_1, a_2, b, and $\Re$ using their staying 40,000 draws after burn-in. To distinguish, in Figure 5, the sample mean (or Bayes' estimate from SSE) and the two HPD bounds of a_1, a_2, b, and $\Re$ are defined by red-soled and red-dashed lines, respectively. It shows that the simulated MCMC variates for all unknown parameters are appropriately mixed and that the associated conditional PDFs are approximately symmetric.

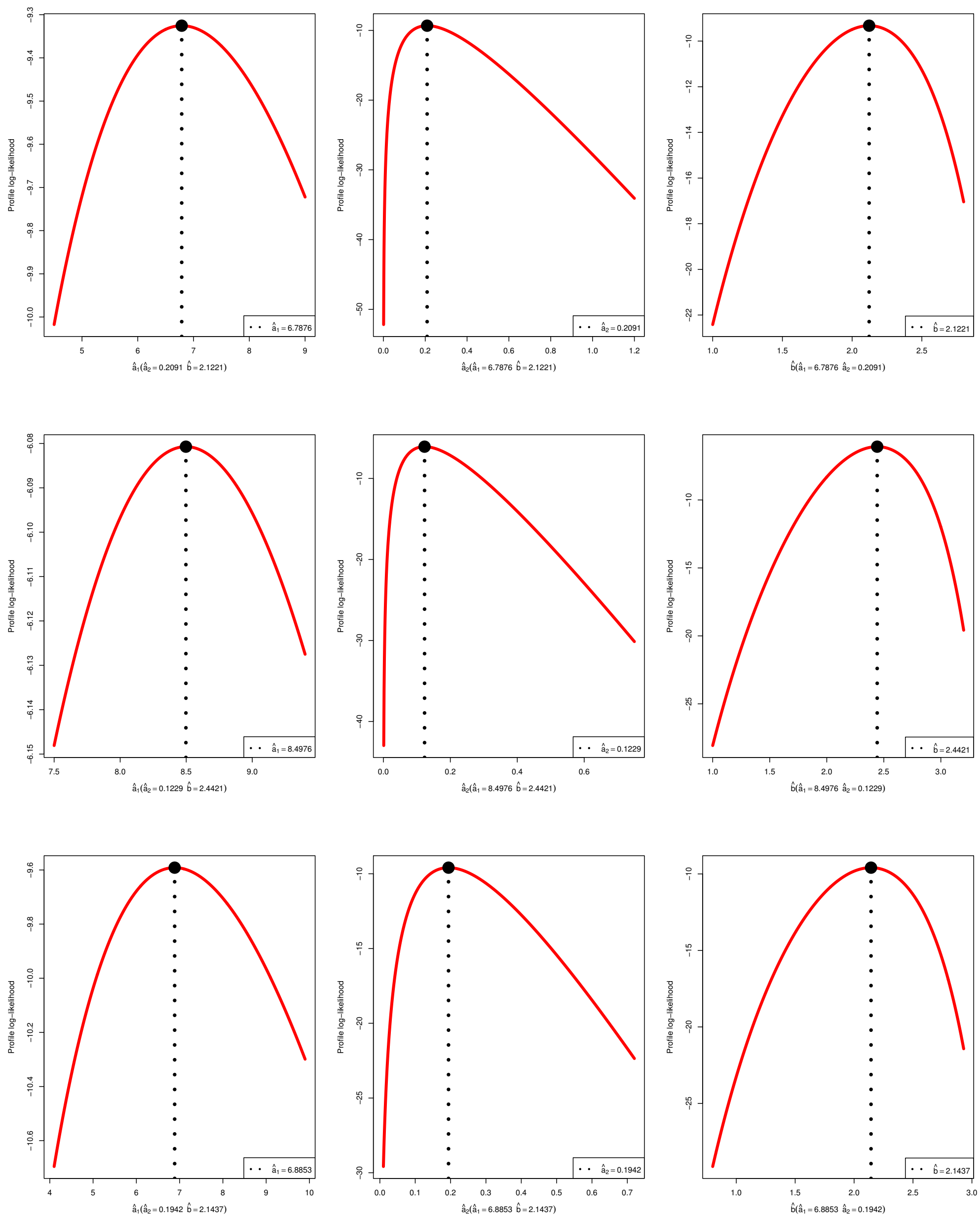


Figure 4. The profile log-likelihoods of a_1, a_2, and b from WOLED data.

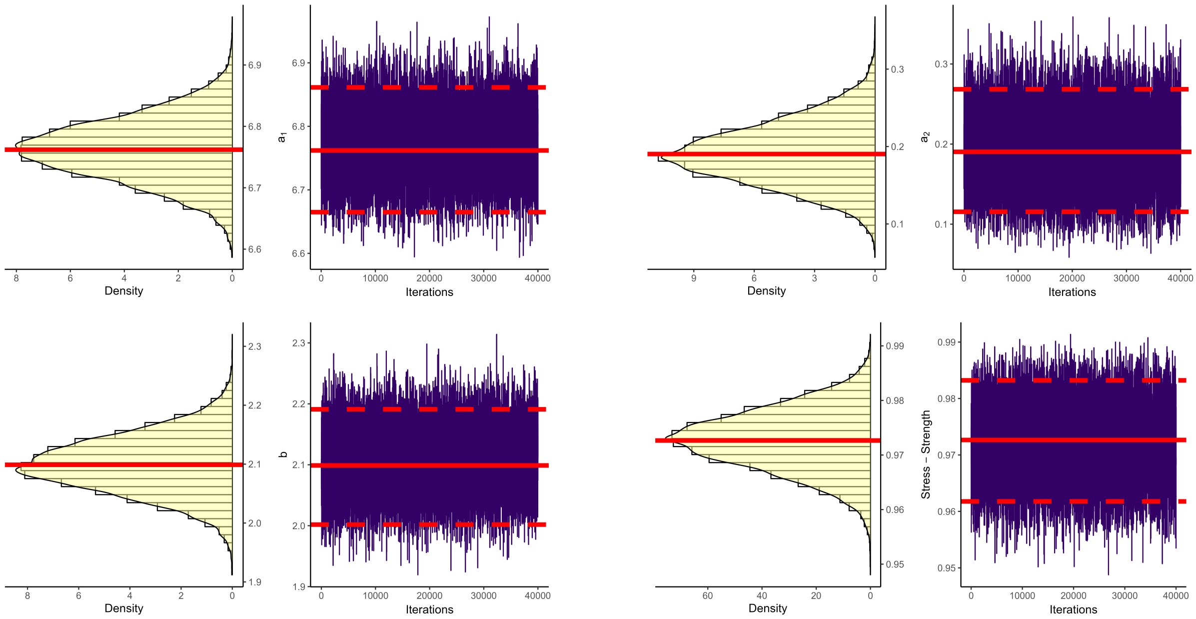


Figure 5. The density (left) and trace (right) plots of a_1, a_2, b, and $\Re$ from WOLED data.

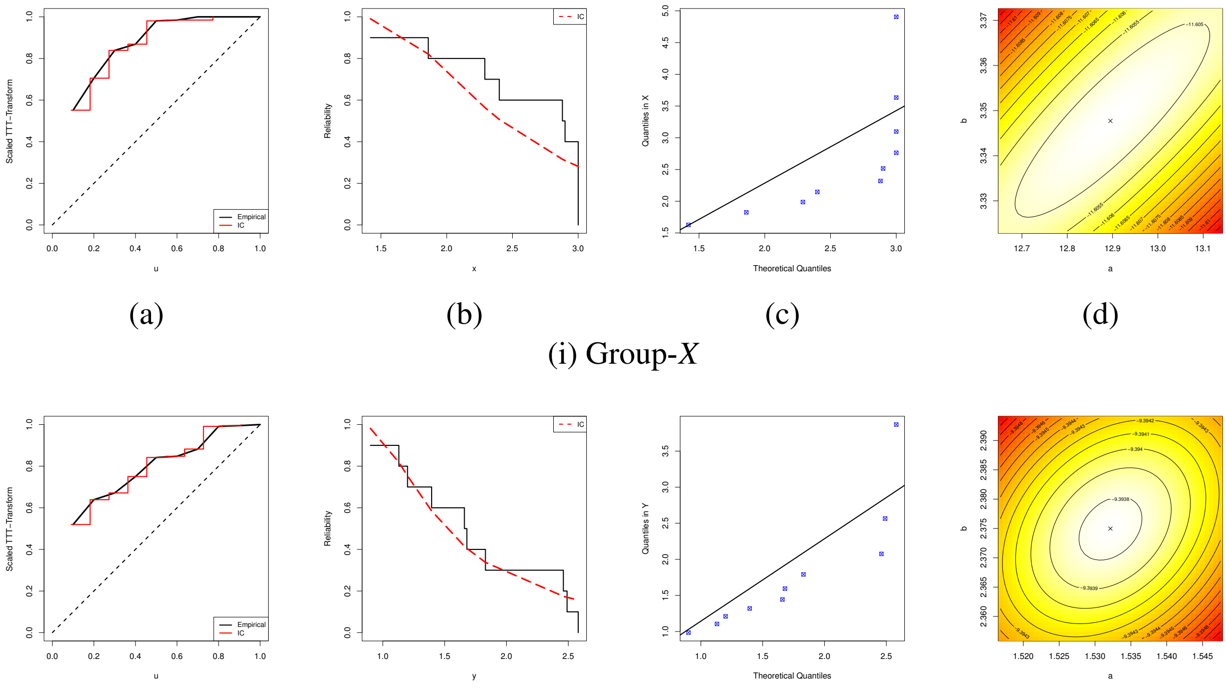
5.2. *Pump motor*

A pump motor is an electrical induction device that transfers electrical energy into mechanical energy. In centrifugal pumps, the motor drives the rotating movement of the pump shaft, which is linked to the pump's impeller. This application analyzes a real-data set that consists of the failure times (in hours) of a motor for a pump that can pump multiple liquids at various rates. The liquid density was used to determine stress and failure times under two stress levels, namely: 1.0g/ml (with $M_1 = 10$) and 1.4g/ml (with $M_2 = 10$); see Abdel Ghaly et al. [39]. In Table 22, each failure time point has been divided by 1000 for computational requirements. Also, in Table 22, we assume the data set at 1.0g/ml by X and the other by Y.

Similar to the same fitting scenarios discussed in Application 6, from the entire pump's data sets, Table 22 shows that the IC lifetime distribution fits the pump's data sets well. Figure 6(a), for Group-X, emphasizes that the Group-X data set has an increasing hazard rate. Figure 6(d), for Group-X, shows that the estimates $\hat{a} \cong 12.895$ and $\hat{b} \cong 3.3477$ (provided in Table 22) existed and are unique. Figure 6(b) and (c) support the same fact. A similar fitting information from Figure 6 is also observed for Group-Y.

Table 22. Newly transformed pump's data sets.

Group	Data	MLE(Std.Err)		KS(P-Value)
		a	b	
X	1.42, 1.86, 2.29, 2.40, 2.88, 2.90, 3.00, 3.00, 3.00, 3.00	12.895(8.0893)	3.3477(0.8132)	0.2845(0.3931)
Y	0.90, 1.13, 1.20, 1.395, 1.66, 1.68, 1.83, 2.46, 2.49, 2.58	1.5321(0.5110)	2.3749(0.6258)	0.1849(0.8241)



(a) (b) (c) (d)
(i) Group-X

(a) (b) (c) (d)
(ii) Group-Y

Figure 6. Scaled TTT-transform (a), Fitted IC reliability (b), Fitted quantile-quantile (c), Contour (d) from pump's data sets.

Now, from the complete pump's data sets, different APTIIC samples with $m_1 = m_2 = 5$ are generated; see Table 23. For $\mathbb{S}[i]$ for $i = 1, 2, 3$, the MLE and Bayesian estimations of $a_i,\ i = 1, 2, b$, and $\Re$ (with their Std.Errs) are obtained and reported in Table 24. Further, 95% ACI-NA/ACI-NT and 95% BCI/HPD interval estimates (with their IWs) of the same subjects are also presented in Table 25. In this application, all Bayes settings are set to be the same as those described in Subsection 5.1. Regarding the lowest Std.Errs, the results (in Table 24) showed that the offered Bayes' estimations of $a_i,\ i = 1, 2, b$, and $\Re$ (created from SSE or LINEX) perform better than those obtained from the competitive likelihood approach. In terms of minimum IWs, from Table 25, the same observation is also reached when comparing asymptotic intervals with credible intervals.

Table 23. Artificial APTIIC samples from pump's data.

Sample	$\underline{R}$ $\underline{S}$	$T_1(d_1)$ $T_2(d_2)$	R^* S^*	Data
$\mathbb{S}[1]$	(1*5) (1*5)	2.89(3) 1.25(2)	2 3	1.42, 2.29, 2.88, 2.90, 3.00 0.90, 1.20, 1.66, 1.68, 1.83
$\mathbb{S}[2]$	(2,3,0*3) (3,2,0*3)	1.55(1) 0.95(1)	3 2	1.42, 1.86, 2.40, 2.88, 2.90 0.90, 1.13, 1.66, 1.83, 2.46
$\mathbb{S}[3]$	(0*3,2,3) (0*3,3,2)	2.45(4) 1.45(4)	3 2	1.42, 1.86, 2.29, 2.40, 3.00 0.90, 1.13, 1.20, 1.395, 1.83

Table 24. Point estimates of a_1, a_2, b, and $\Re$ from pump's data.

Sample	Par.	MLE		SSE		LINEX					
$\tau \rightarrow$						−5		−0.05		+5	
$\mathbb{S}[1]$	a_1	5.6955	2.9694	5.1426	0.5791	5.7237	0.0556	5.0824	0.5774	5.0551	0.5779
	a_2	1.6501	0.5670	1.1131	0.5719	1.2388	0.0436	1.1003	0.5702	1.0941	0.5708
	b	1.9922	0.5089	1.4950	0.5390	1.6639	0.0539	1.4775	0.5374	1.4696	0.5380
	$\Re$	0.7754	0.0993	0.8224	0.0551	0.9154	0.0061	0.8128	0.0550	0.8085	0.0550
$\mathbb{S}[2]$	a_1	4.6630	2.3167	4.6369	0.0560	5.2397	0.0257	4.6230	0.0557	4.6045	0.0555
	a_2	1.5042	0.5353	1.4780	0.0563	1.6702	0.0223	1.4736	0.0560	1.4677	0.0558
	b	1.8537	0.4961	1.8276	0.0560	2.0651	0.0256	1.8221	0.0558	1.8148	0.0556
	$\Re$	0.7561	0.1058	0.7583	0.0068	0.8569	0.0026	0.7561	0.0068	0.7530	0.0068
$\mathbb{S}[3]$	a_1	5.5315	2.8295	5.5055	0.0559	6.0560	0.0256	5.4945	0.0555	5.4796	0.0554
	a_2	1.4685	0.4872	1.4423	0.0562	1.5865	0.0242	1.4394	0.0559	1.4355	0.0557
	b	2.0007	0.5378	1.9746	0.0560	2.1721	0.0256	1.9707	0.0556	1.9654	0.0555
	$\Re$	0.7902	0.0913	0.7924	0.0063	0.8717	0.0033	0.7909	0.0062	0.7887	0.0062

Table 25. Interval estimates of a_1, a_2, b, and $\Re$ from pump's data.

Sample	Par.	ACI-NA ACI-NT			BCI HPD		
		Low.	Upp.	IW	Low.	Upp.	IW
$\mathbb{S}[1]$	a_1	0.0000	11.515	11.515	4.8272	5.4789	0.6518
		2.0500	15.824	13.774	4.8255	5.4576	0.6321
	a_2	0.5388	2.7615	2.2227	0.7709	1.5609	0.7900
		0.8414	3.2361	2.3946	0.7191	1.4409	0.7218
	b	0.9948	2.9896	1.9947	1.1934	1.9094	0.7159
		1.2076	3.2867	2.0791	1.1934	1.9014	0.7080
	$\Re$	0.5808	0.9699	0.3892	0.7607	0.8684	0.1076
		0.6033	0.9965	0.3932	0.7787	0.8823	0.1035
$\mathbb{S}[2]$	a_1	0.1224	9.2037	9.0813	4.5391	4.7345	0.1954
		1.7611	12.347	10.586	4.5426	4.7369	0.1943
	a_2	0.4551	2.5533	2.0982	1.3819	1.5764	0.1945
		0.7489	3.0214	2.2725	1.3826	1.5770	0.1944
	b	0.8813	2.8260	1.9447	1.7334	1.9258	0.1924
		1.0970	3.1321	2.0351	1.7320	1.9236	0.1916
	$\Re$	0.5488	0.9634	0.4147	0.7457	0.7707	0.0250
		0.5748	0.9947	0.4199	0.7457	0.7706	0.0250
$\mathbb{S}[3]$	a_1	0.0000	11.077	11.077	5.4076	5.6030	0.1954
		2.0297	15.075	13.045	5.4110	5.6054	0.1943
	a_2	0.5135	2.4234	1.9099	1.3462	1.5406	0.1944
		0.7664	2.8138	2.0474	1.3469	1.5412	0.1942
	b	0.9466	3.0548	2.1082	1.8808	2.0729	0.1921
		1.1813	3.3884	2.2071	1.8790	2.0706	0.1916
	$\Re$	0.6112	0.9692	0.3580	0.7810	0.8037	0.0227
		0.6300	0.9911	0.3611	0.7810	0.8037	0.0227

The log-likelihoods of a_1, a_2, and b for $\mathbb{S}[i]$, $i = 1, 2, 3$, are displayed in Figure 7 to demonstrate the presence and uniqueness of $\hat{a}_1$, $\hat{a}_2$, and $\hat{b}$ in Table 24. It demonstrates that all values for the MLEs of a_1, a_2, or b (in Table 25) existed and are distinct. Using Sample $\mathbb{S}[1]$ (as an instance), Figure 8 depicts the trace and density diagrams of a_1, a_2, b, and $\Re$ after 40,000 draws following burn-in. It demonstrates that the created MCMC variates of a_1, a_2, b, and $\Re$ are effectively mixed, and that the corresponding conditional PDFs are roughly symmetric.

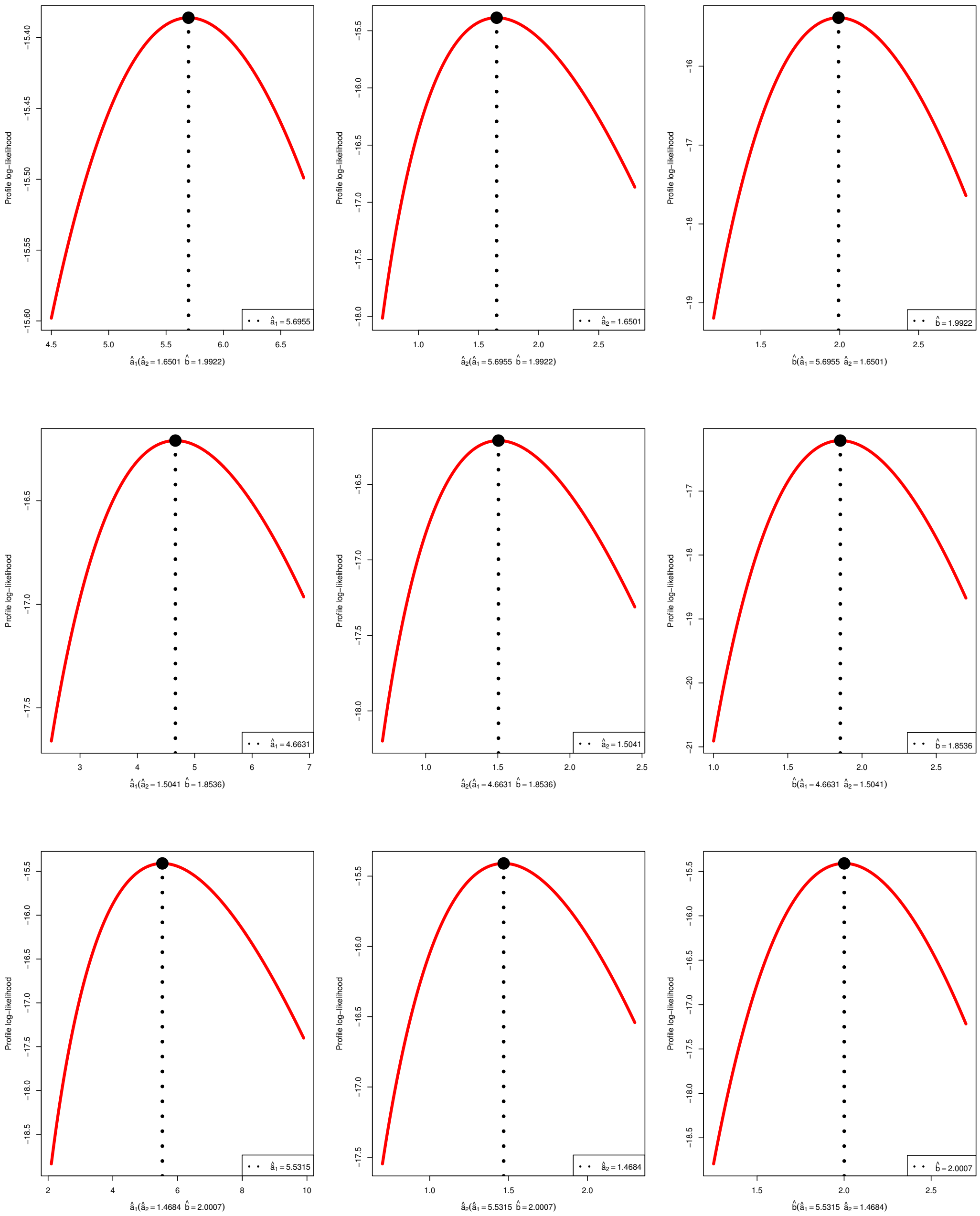


Figure 7. The profile log-likelihoods of a_1, a_2, and b from pump's data.

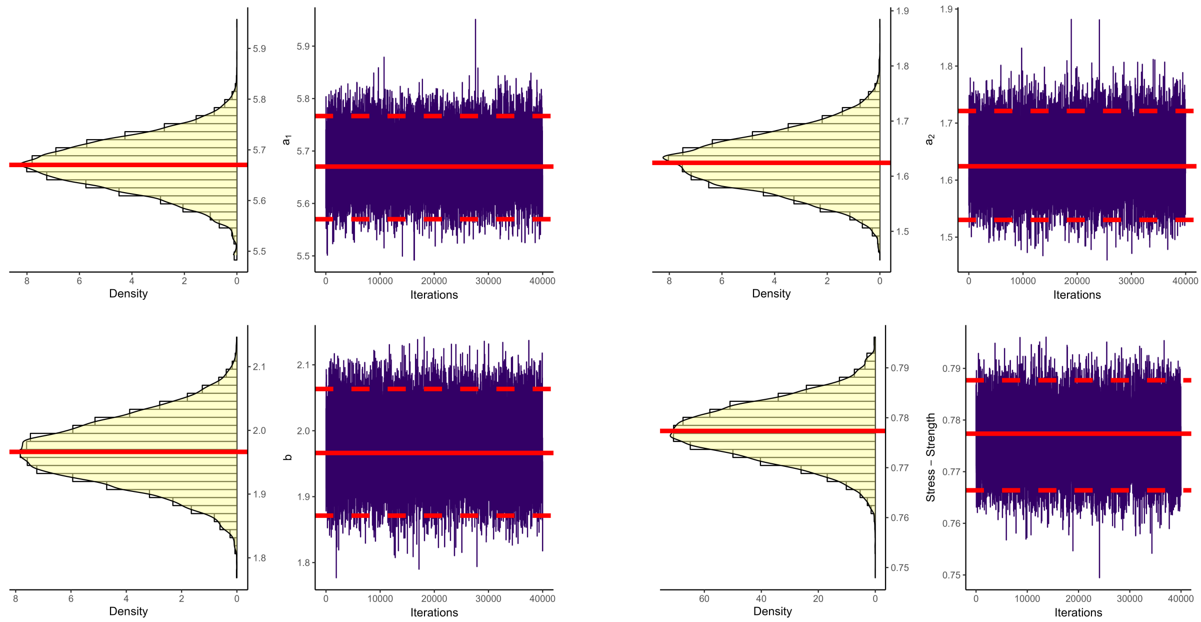


Figure 8. The density (left) and trace (right) plots of a_1, a_2, b, and $\Re$ from pump's data.

6. Conclusions

In this research, we explored different methods for estimating the stress–strength index. We considered the strength and stress components as independent random variables, following inverted Chen distributions. These distributions had similar shape parameters but separate scale parameters. The estimations are constructed using adaptive progressive type-II censoring samples. We used the maximum likelihood method, a traditional approach, to estimate the model parameters, as well as the stress–strength index and two types of approximate confidence intervals. The delta method is used to approximate the variance of the stress–strength index. The Bayes estimates of the model parameters and the stress–strength index are estimated using gamma priors under squared error and LINEX loss functions. Given the complexity of the posterior distribution, the Markov Chain Monte Carlo technique is employed to generate samples from the conditional distributions and then compute the Bayes estimates. Bayesian credible intervals and the highest posterior density credible intervals are also calculated. We perform a comprehensive numerical analysis to assess the provided estimates using various testing plans, considering factors such as root mean squared error, interval lengths, and coverage probabilities. Two illustrative examples are investigated to provide a better understanding of the suggested approaches. The numerical results demonstrate that Bayes estimates using the LINEX loss function are superior to other Bayes and classical estimates for estimating model parameters and the stress–strength index. Furthermore, the highest posterior density credible intervals exhibit a minimum interval length compared to Bayes credible intervals and the two classical confidence intervals for various parameters. The real data analysis shows the effectiveness of the inverted Chen distribution in fitting two engineering data sets as well as in modelling the stress–strength index derived from these data. Although we have not come across any related work developing the asymptotic (or credible) intervals for the inverted Chen parameters or its stress–strength index in the presence of datasets gathered from the proposed sampling strategy, we advise to extend the new results presented in this work to other sampling strategies. Finally, it is important to note that this study assumes the stress and strength random variables are independent. However, in cases where these variables may be dependent, it is crucial to incorporate that relationship into the model. This idea requires further investigation in future work. See for more detail, James and Chandra [40], Chandra et al. [41], and Shang and Yan [42].

Author contributions

Refah Alotaibi: Conceptualization, methodology, investigation, funding acquisition, writing – original draft; Mazen Nassar: Conceptualization, methodology, writing – review and editing, writing – original draft; Zareen A. Khan: Conceptualization, investigation, writing – review and editing; Ahmed Elshahhat: Methodology, software, witing – original draft. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

There is no conflict of interest.

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