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Research article

Intuitionistic fuzzy variational inequalities and their applications

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Abstract: In this paper, a new class of generalized convex (concave) fuzzy mappings are introduced, which is called intuitionistic convex (concave) fuzzy mappings from the convex set $K \subseteq \mathbb{R}^n$ to the set of intuitionistic fuzzy numbers. By using the concept of epigraph, the characterization of intuitionistic convex fuzzy mappings is also discussed. Different types of intuitionistic convex (concave) fuzzy mappings are defined and their properties are investigated. Then, we discuss some applications of intuitionistic fuzzy convex mappings in fuzzy optimization. Additionally, some variational inequalities, known as intuitionistic fuzzy variational inequality and intuitionistic fuzzy variational mixed inequalities, are introduced. The results obtained in this paper can be regarded as refinements and extensions of previously established results.

Keywords: intuitionistic fuzzy numbers**;** intuitionistic convex fuzzy mappings; intuitionistic quasi-convex fuzzy mappings; global minimum, intuitionistic fuzzy variational inequalities
Mathematics Subject Classification: 62G10, 62G20, 62N05

1. Introduction

The extensive body of research on work fuzzy sets and systems has contributed to the development of fields. Fuzzy sets play an important role in wide range of problems in pure mathematics and applied sciences, including operation research, computer science, managements sciences, artificial intelligence, control engineering, and decision sciences. Convex analysis is the foundation of optimization theory. Similarly, convex fuzzy analysis serves as a fundamental theory in fuzzy optimization. Fuzzy convex sets have been widely discussed. For example, Liu [1] investigated some properties of convex fuzzy sets and modified the concept of Zadeh [2] of the shadow of fuzzy sets. Lowen [3], compiled foundational results about convex sets and proved a separation theorem for convex fuzzy sets. Ammar and Metz [4] studied different types of convexity and defined the generalized convexity of fuzzy sets. Furthermore, they formulated a general fuzzy nonlinear programing problem with the application of the concept of convexity. Additionally, the properties of convex fuzzy sets have attracted a wide range of interest ([5–11], and references therein). Fuzzy numbers, a generalized form of an interval (in crisp set theory), were defined by Zadeh [2]. Dubois and Prade [12] extended this work introducing new conditions for fuzzy numbers. Goetschel and Voxman [13] polished this idea and modified different conditions on fuzzy numbers to make them adaptable. For instance, in [12], a fuzzy number should be a continuous function; in [13], fuzzy number can be upper semi continuous. Such relaxation of conditions on fuzzy number allow us to easily define a metrics for fuzzy numbers facilitating the study of their basic properties in topological space. Nanda and Kar [14], Syau [15], and Furukawa [16] introduced the concept of fuzzy convex mappings from $\mathbb{R}^n$ to the set of fuzzy numbers. They defined different types of convex fuzzy mappings such as fuzzy logarithmic convex mappings and fuzzy quasi-convex mappings, and studied Lipschitz continuity in fuzzy-valued mappings. Based on the concept of ordering introduced by Goetschel and Voxman [17], Yan and Xu [18] presented the concepts of epigraphs and convexity of fuzzy mappings and characterized convex fuzzy mappings and quasi-convex fuzzy mappings. Syau [19], introduced notions of (ϕ_1, ϕ_2) -convexity, ϕ_1 -B-vexity, and ϕ_1 -convexity fuzzy mappings through the so-called fuzzy max-order among fuzzy numbers. These classes B-vexity, convexity, and preinvexity, were shown to be interrelated subclasses, [20]. Khan et al. [21–26] discussed continuity and convexity through linear ordering and metrics defined on fuzzy numbers. They also extended the Weirstrass theorem from real-valued functions to fuzzy mappings. For recent applications, see [27–33] and the references therein.

Atanassov [34,35], introduced intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets, which are a generalized form of fuzzy sets. Fuzzy sets consist of two functions: the membership function (or degree of acceptance), which defined the truth values of elements and the non-membership function (or degree of rejection). The question raised is why do we need to introduce intuitionistic fuzzy sets. To explain this, we take an optimization problem. In fuzzy optimization problems relationships like the objective function and constraints are expressed using fuzzy sets. For example in “Min” fuzzy optimization if " $\preceq$ " represents the fuzzy inequality, then the problem is

$$\text{Min}: f_j\,(u), \;\; j = 1, 2, 3 \ldots \ldots \ldots \ldots \ldots \ldots k$$

such that

$$tg_j(u) \preceq 0, \;\; j = 1, 2, 3 \ldots \ldots \ldots \ldots \ldots \ldots n.$$

Similarly, in the case when the degree of non-membership $\phi: \widetilde{X} \to [0,1])$ is discussed simultaneously with the degree of membership $\psi: \widetilde{X} \to [0, 1]$ and if these degrees are not supportive of each other, then we apply intuitionistic fuzzy sets to formulate an intuitionistic fuzzy problem. $\psi_j(u)$ and $\phi_j(u)$ denote the degree of satisfaction and degree of rejection of u from the jth intuitionistic fuzzy set:

$$\text{Max}: \psi_j\,(u), \;\; j = 1, 2, 3 \ldots \ldots \ldots \ldots \ldots \ldots k + n$$

$$\text{Min: } \phi_j(u), \;\; j = 1, 2, 3 \ldots \ldots \ldots \ldots \ldots \ldots k + n$$

such that

$$\phi_j(u) \leq 0, \;\; j = 1, 2, 3 \ldots \ldots \ldots \ldots \ldots \ldots k + n$$

$$\phi_j(u) \leq \psi_j(u), \text{and } \phi_j(u) + \psi_j(u) \geq 1, \;\; j = 1, 2, 3 \ldots \ldots \ldots \ldots \ldots \ldots k + n.$$

From this method, it can easily be seen that intuitionistic fuzzy sets contribute to minimize or maximize optimization problems.

Fuzzy numbers are not capable of dealing a lack of knowledge about the membership degree. When the degree of non-membership is discussed simultaneously with the degree of membership, and if these degrees are not supportive of each other, then intuitionistic fuzzy sets are used. Similarly, intuitionistic fuzzy numbers are helpful when the degree of non-membership is discussed simultaneously with the degree of membership. Burillo and Mohendano [36], generalized fuzzy numbers to intuitionistic fuzzy numbers, incorporating both the Membership and non-membership functions. Intuitionistic fuzzy number depended on the behaving like fuzzy numbers. Shen et al. [37], introduced the concept of intuitionistic fuzzy mappings from $\mathbb{R}^n$ to the set of intuitionistic fuzzy sets proving some properties. Additionally, they proved that all of intuitionistic fuzzy mappings constitute a soft algebra, [38,39]. Furthermore, they also studied decomposition and representation theorems of intuitionistic fuzzy mappings. For intuitionistic fuzzy numbers and generalized fuzzy mappings, see [40–44]. Dong and Wan [45], Wan et al. [46], and Dong et al. [47] discussed the interval-valued intuitionistic fuzzy best-worst method with additive consistency proposing the applications using intuitionistic fuzzy preference relations, and multiplicative preference relations based on bounded confidence. Additionally, Dong and Wan [48] introduced the concept of Type-2 interval-valued intuitionistic fuzzy matrix game and applyied this idea to energy vehicle industry development.

Inspired by ongoing research work and the importance of convexity of mappings, in Section 2, we review basic results and introduce the concept of intuitionistic convex (concave) and quasi-convex (quasi-concave) fuzzy mappings. The main results are considered and discussed in Section 3. Section 4 discusses the intuitionistic fuzzy variational inequality and intuitionistic fuzzy variational mixed inequality. In Section 5, some applications of intuitionistic convex fuzzy mappings are discussed in optimization theory.

2. Preliminaries

Let A be a non-empty set. Then, $A = \{< u, \psi_A(u) > : u \in \mathbb{R}\}$ is called a fuzzy set, where ψ_A is a membership function that maps each element of $\mathbb{R}$ in $[0, 1]$ and $A \subseteq \mathbb{R}$, see [2]. In other words, a fuzzy set of $\mathbb{R}$ is a fuzzy mapping $\psi: \mathbb{R} \to [0,1]$. For each fuzzy set and $\tilde{\alpha} \in (0, 1]$, then $\tilde{\alpha}$-level sets of ψ are defined as follows: $\psi_{\tilde{\alpha}} = \{u \in \mathbb{R} | \; \psi(u) \geq \tilde{\alpha}\}$. If $\tilde{\alpha} = 0$, then $\psi_0 = \{u \in \mathbb{R} | \; \psi(u) > 0\}$ is called the support of ψ, and we define the closure of $\sup(\psi)$. A fuzzy set is normal if there exists $u \in \mathbb{R}$ such that $\psi(u) = 1$. A fuzzy set is convex and concave if $\psi\big((1-\tau)u + \tau v\big) \geq \min\big(\psi(u), \psi(v)\big)$ and $\psi((1-\tau)u + \tau v) \leq \max\big(\psi(u), \psi(v)\big)$ for $u, v \in \mathbb{R}$, $\tau \in [0, 1]$, respectively. A fuzzy convex set is a generalization of classical convex, see [12].

A fuzzy set is said to be a fuzzy number if it satisfies following properties: (a) ψ is normal. (b) ψ is a convex fuzzy set. (c) $\sup(\psi)$ is compact. $FN(\mathbb{R})$ denotes the set of all fuzzy numbers.

A mapping $\tilde{F}: K \to IFN(\mathbb{R})$ is called a fuzzy mapping. For each $\tilde{\alpha}, \in [0,1]$, associated with $\tilde{F}$, we define the family of interval valued functions $\tilde{F}_{\tilde{\alpha}}: K \to Ƙ_{FN}$ defined by $\tilde{F}_{\tilde{\alpha}}(u) = [\tilde{F}(u)]^{\tilde{\alpha}}$ or $\tilde{F}_{\tilde{\alpha}}(u) =_{\psi} [\tilde{F}(u)]^{\tilde{\alpha}}$ and denoted by $\tilde{F}_{\tilde{\alpha}}(u) =_{\psi} [\tilde{F}_*(\tilde{\alpha}), \tilde{F}^*(\tilde{\alpha})]$. Now, for any $\tilde{\alpha} \in [0,1]$, the end point functions $\tilde{F}_*(\tilde{\alpha})$, $\tilde{F}_*(\tilde{\beta}): K \to \mathbb{R}$ are called lower and upper functions, respectively.

Let a set $\mathbb{R}$ be fixed. An intuitionistic fuzzy set A in $\mathbb{R}$ is an object having the form, $A = \{\langle u, \psi(u), \phi(u)\rangle | \, u \in E:\}$, where $\psi: \mathbb{R} \to [0,1]$ is a membership function, $\phi: \mathbb{R} \to [0,1]$ is a non-membership function, and $0 \leq \psi(u) + \phi(u) \leq 1$. $IFS(\mathbb{R})$ denotes the set of all intuitionistic fuzzy sets on $\mathbb{R}$, see [42].

In other words, we may say that an intuitionistic fuzzy set is a pair of mappings (ψ, ϕ) such that $\psi: \mathbb{R} \to [0,1]$ (for a grade of membership), $\phi: \mathbb{R} \to [0,1]$ (for a grade of non-membership), $0 \leq \psi(u) + \phi(u) \leq 1$. For each intuitionistic fuzzy set and $\tilde{\alpha}, \tilde{\beta} \in (0,1]$, then $(\tilde{\alpha}, \tilde{\beta})$-level sets of (ψ, ϕ) are denoted and defined as $\psi_{\tilde{\alpha}} = \{u \in \mathbb{R} | \, \psi(u) \geq \tilde{\alpha}\}$ and $\phi_{\tilde{\beta}} = \{u \in \mathbb{R} | \, \phi(u) \leq \tilde{\beta}\}$ or $(\psi, \phi)_{(\tilde{\alpha}, \tilde{\beta})} = \psi_{\tilde{\alpha}} \cap \phi_{\tilde{\beta}} = \{u \in \mathbb{R} | \, \psi(u) \geq \tilde{\alpha} \text{ and } \phi(u) \leq \tilde{\beta}\}$, where $\tilde{\alpha} + \tilde{\beta} \leq 1$. An intuitionistic fuzzy set (ψ, ϕ) is normal if there exists $u, v \in \mathbb{R}$ such that $\psi(u) = 1$ and $\phi(v) = 1$. An intuitionistic fuzzy set (ψ, ϕ) of $\mathbb{R}$ is a convex mapping if $\psi((1-\tau)u + \tau v) \geq \min(\psi(u), \psi(v))$ and $\psi((1-\tau)u + \tau v) \leq \max(\psi(u), \psi(v))$ for $u, v \in \mathbb{R}$, $\tau \in [0,1]$ or ψ is convex and ϕ is concave. An intuitionistic fuzzy convex set is the generalization of fuzzy convex set; see [37].

An intuitionistic fuzzy number is a pair of mappings (ψ, ϕ) defined by $\psi: \mathbb{R} \to [0,1]$ and $\phi: \mathbb{R} \to [0,1]$ with the following properties: (a)(ψ, ϕ) is normal. (b) ψ is fuzzy convex and ϕ is fuzzy concave. (c) ψ is upper semi-continuous and ϕ is lower semicontinuous. (d) $\text{supp}(\psi, \phi) = \{u \in \mathbb{R} | \phi(u) < 1\}$ is bounded. $IFN(\mathbb{R})$ is denoted by the set of all intuitionistic fuzzy numbers. It is worth noting that each intuitionistic fuzzy number is the conjunction of two fuzzy numbers: ψ with membership function and ϕ with non-membership function. For an intuitionistic fuzzy number, it is convenient to distinguish the followings $(\tilde{\alpha}, \tilde{\beta})$-levels:

$$\psi_{\tilde{\alpha}} = \{u \in \mathbb{R} | \, \psi(u) \geq \tilde{\alpha}\}, \qquad \phi_{\tilde{\beta}} = \{u \in \mathbb{R} | \, \phi(u) \leq \tilde{\beta}\}.$$

From these definitions, we have

$$\psi_{\tilde{\alpha}} = [\psi_*(\tilde{\alpha}), \psi^*(\tilde{\alpha})] \text{ and } \phi_{\tilde{\beta}} = [\phi_*(\tilde{\beta}), \phi^*(\tilde{\beta})],$$

where

$$\psi_*(\tilde{\alpha}) = \inf\{u \in \mathbb{R} | \, \psi(u) \geq \tilde{\alpha}\}, \psi^*(\tilde{\alpha}) = \sup\{u \in \mathbb{R} | \, \psi(u) \geq \tilde{\alpha}\},$$

$$\phi_*(\tilde{\beta}) = \inf\{u \in \mathbb{R} | \, \phi(u) \leq \tilde{\beta}\}, \phi^*(\tilde{\beta}) = \sup\{u \in \mathbb{R} | \, \phi(u) \leq \tilde{\beta}\}.$$

From the definitions, we can also define the intuitionistic fuzzy number in this way:

$$(\psi_{\tilde{\alpha}}, \phi_{\tilde{\beta}}) = ([\psi_*(\tilde{\alpha}), \psi^*(\tilde{\alpha})], [\phi_*(\tilde{\beta}), \phi^*(\tilde{\beta})]).$$

It can be easily seen that the $(\tilde{\alpha}, \tilde{\beta})$-levels of the intuitionistic fuzzy number are a pair of enclosed and bonded intervals. Since each $r \in \mathbb{R}$ is also a intuitionistic fuzzy number, defined as

$$\tilde{r}(u) = \begin{cases} 1 \text{ if } u = r \\ 0 \text{ if } u \neq r \end{cases} \text{ and } \breve{r}(u) = \begin{cases} 0 \text{ if } u = r \\ 1 \text{ if } u \neq r \end{cases}.$$

We discuss some properties of intuitionistic fuzzy numbers under addition and scaler multiplication, if $(\psi_1, \psi_2), (\phi_1, \phi_2) \in \text{IFN}(\mathbb{R})$ and $\gamma \in \mathbb{R}$, then $(\psi_1, \psi_2)\tilde{\mp}(\phi_1, \phi_2)$ and $\gamma(\psi_1, \psi_2)$ are defined

$$(\psi_1\tilde{\mp}\phi_1)(w) = \sup_{u+v=w}\min\{(\psi_1)(w), (\phi_1)(w)\}$$

$$(\psi_2\tilde{\mp}\phi_2)(w) = \min_{u+v=w}\sup\{(\psi_2)(w), (\phi_2)(w)\}.$$

It is also well-known that

$$(\psi_1\tilde{\mp}\phi_1)^*(\tilde{\alpha}) = {\psi_1}^*(\tilde{\alpha}) + {\phi_1}^*(\tilde{\alpha}), (\psi_1\tilde{\mp}\phi_1)_*(\tilde{\alpha}) = {\psi_1}_*(\tilde{\alpha}) + {\phi_1}_*(\tilde{\alpha}),$$

$$(\psi_2\tilde{\mp}\phi_2)^*(\tilde{\beta}) = {\psi_2}^*(\tilde{\beta}) + {\phi_2}^*(\tilde{\beta}), (\psi_2\tilde{\mp}\phi_2)_*(\tilde{\beta}) = {\psi_2}_*(\tilde{\beta}) + {\phi_2}_*(\tilde{\beta}),$$

$$(\gamma\psi_1)_*(\tilde{\alpha}) = \begin{cases} \gamma{\psi_1}_*(\tilde{\alpha}) \text{ if } \gamma \geq 0 \\ \gamma{\psi_1}^*(\tilde{\alpha}) \text{ if } \gamma < 0 \end{cases}, (\gamma\psi_1)^*(\tilde{\alpha}) = \begin{cases} \gamma{\psi_1}^*(\tilde{\alpha}) \text{ if } \gamma \geq 0 \\ \gamma{\psi_1}_*(\tilde{\alpha}) \text{ if } \gamma < 0 \end{cases},$$

$$(\gamma\psi_2)_*(\tilde{\beta}) = \begin{cases} \gamma{\psi_2}_*(\tilde{\beta}) \text{ if } \gamma \geq 0 \\ \gamma{\psi_2}^*(\tilde{\beta}) \text{if } \gamma < 0 \end{cases}, (\gamma\psi_2)^*(\tilde{\alpha}) = \begin{cases} \gamma{\psi_2}^*(\tilde{\beta}) \text{ if } \gamma \geq 0 \\ \gamma{\psi_2}_*(\tilde{\beta}) \text{ if } \gamma < 0 \end{cases},$$

for each $\tilde{\alpha}, \tilde{\beta} \in [0, 1]$ with $\tilde{\alpha} + \tilde{\beta} \leq 1$.

For any $(\psi_1, \psi_2), (\phi_1, \phi_2) \in \text{IFN}(\mathbb{R})$, we say that $(\psi_1, \psi_2) \preceq (\phi_1, \phi_2)$ if for all $\tilde{\alpha} \in (0, 1]$, ${\psi_1}^*(\tilde{\alpha}) \leq {\phi_1}^*(\tilde{\alpha})$, ${\psi_1}_*(\tilde{\alpha}) \leq {\phi_1}_*(\tilde{\alpha})$ and ${\psi_2}^*(\tilde{\beta}) \leq {\phi_2}^*(\tilde{\beta})$, ${\psi_2}_*(\tilde{\beta}) \leq {\phi_2}_*(\tilde{\beta})$. If $(\psi_1, \psi_2) \leq (\phi_1, \phi_2)$, then there exists $\tilde{\alpha} \in (0, 1]$ such that ${\psi_1}^*(\tilde{\alpha}) < {\phi_1}^*(\tilde{\alpha})$ or ${\psi_1}_*(\tilde{\alpha}) \leq {\phi_1}_*(\tilde{\alpha})$ or ${\psi_2}^*(\tilde{\beta}) < {\phi_2}^*(\tilde{\beta})$ or ${\psi_2}_*(\tilde{\beta}) < {\phi_2}_*(\tilde{\beta})$. We say they are comparable if for any $(\psi_1, \psi_2), (\phi_1, \phi_2) \in \text{IFN}(\mathbb{R})$, we have $(\psi_1, \psi_2) \preceq (\phi_1, \phi_2)$ or $(\psi_1, \psi_2) \succeq (\phi_1, \phi_2)$; otherwise, they are non-comparable. Occasionally, we may write $(\psi_1, \psi_2) \preceq (\phi_1, \phi_2)$ instead of $(\phi_1, \phi_2) \succeq (\psi_1, \psi_2)$ and we may say that $\text{IFN}(\mathbb{R})$ is a partial ordered set under the relation $\preceq$.

If $(\psi_1, \psi_2), (\phi_1, \phi_2) \in \text{IFN}(\mathbb{R})$, there exists $(\omega_1, \omega_2) \in \text{IFN}(\mathbb{R})$ such that $(\psi_1, \psi_2) = (\phi_1, \phi_2)\tilde{\mp}(\omega_1, \omega_2)$, then by this result we have the existence of Hukuhara difference of (ψ_1, ψ_2) and (ϕ_1, ϕ_2), and we say that (ω_1, ω_2) is the H-difference of (ψ_1, ψ_2) and (ϕ_1, ϕ_2), and denoted by $(\psi_1, \psi_2)\tilde{-}(\phi_1, \phi_2)$; see [36]. If H-difference exists, then

$$(\psi_1\tilde{-}\phi_1)^*(\tilde{\alpha}) = {\psi_1}^*(\tilde{\alpha}) - {\phi_1}^*(\tilde{\alpha}), (\psi_1\tilde{-}\phi_1)_*(\tilde{\alpha}) = {\psi_1}_*(\tilde{\alpha}) - {\phi_1}_*(\tilde{\alpha}),$$

$$(\psi_2\tilde{-}\phi_2)^*(\tilde{\beta}) = {\psi_2}^*(\tilde{\beta}) - {\phi_2}^*(\tilde{\beta}), (\psi_2\tilde{-}\phi_2)_*(\tilde{\beta}) = {\psi_2}_*(\tilde{\beta}) - {\phi_2}_*(\tilde{\beta}).$$

From now onward, we will use " $\leq_\psi$ " and " $\geq_\phi$ " inequalities when mappings are membership and non-membership, respectively, unless otherwise specified.

Definition 1. [36] Let K be a convex set in $\mathbb{R}^n$. Then, the fuzzy mapping $\tilde{F}: K \to FN(R)$ is said to be convex at $u \in K$ if

$$\tilde{F}((1-\tau)u+\tau v) \leq_{\psi} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v)\ \forall\ v \in K, \tau \in [0,1].$$

$\tilde{F}$ is said to be convex on K if it is convex at each $u \in K$ and a strictly convex fuzzy mapping if strict inequality holds for $\tilde{F}(u) \neq_{\psi} \tilde{F}(v)$. $\tilde{F}: K \to FN(R)$ is said to be a concave fuzzy mapping at $u \in K$, if $\tilde{F}((1-\tau)u+\tau v) \geq_{\psi} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v)\ \forall\ v \in K, \tau \in [0,1]$ and $\tilde{F}$ is said to be concave on K if it is concave at each $u \in K$. Strictly concave fuzzy mapping depends on inequality holding for $\tilde{F}(u) \neq_{\psi} \tilde{F}(v)$.

Definition 2. [36] The fuzzy mapping $\tilde{F}: K \to FN(R)$ is said to be quasi-convex at $u \in K$ if

$$\tilde{F}((1-\tau)u+\tau v) \leq_{\psi} \max\left(\tilde{F}(u), \tilde{F}(v)\right)\ \forall\ v \in K, \tau \in [0,1],$$

and $\tilde{F}$ is said to be quasi-convex on K if it is quasi-convex at each $u \in K$. Strictly quasi-convex fuzzy mapping if depend on inequality holding for $\tilde{F}(u) \neq_{\psi} \tilde{F}(v)$. $\tilde{F}: K \to FN(R)$ is said to be quasi-concave fuzzy mapping at $u \in K$, if $\tilde{F}((1-\tau)u+\tau v) \geq_{\psi} \min\left(\tilde{F}(u), \tilde{F}(v)\right)\ \forall\ v \in K,\ \tau \in [0,1]$ and $\tilde{F}$ is said to be quasi-concave on K if it is quasi-concave at each $u \in K$. Strictly quasi-concave fuzzy mapping depends on strict inequality holding for $\tilde{F}(u) \neq_{\psi} \tilde{F}(v)$.

Definition 3. [37] Let K be a non-empty set and $\mathbb{R}$ be a set of real numbers. Then, the mapping $\tilde{F}: K \to IFS(\mathbb{R})$ is called an intuitionistic fuzzy mappings from K to $\mathbb{R}$ and defined as,

$$\tilde{F}(u) = \left\{< w, \tilde{F}_{\psi_u}(w), \tilde{F}_{\phi_u}(w) > \middle| w \in \mathbb{R}\right\}, \text{ or } \tilde{F}(u) = \left(\tilde{F}_{\psi},\ \tilde{F}_{\phi}\right),$$

where $\tilde{F}_{\psi}: \mathbb{R} \to [0,1]$ and $\tilde{F}_{\phi}: \mathbb{R} \to [0,1]$ are called a positive membership function and a negative membership function, respectively. Also $\tilde{F}_{\psi}$ and $\tilde{F}_{\phi}$ satisfies the following condition:

$$0 \leq \tilde{F}_{\psi}(w) + \tilde{F}_{\phi}(w) \leq 1 \text{ for all } w \in \mathbb{R}, \text{ and for } u \in K.$$

And $IFS(\mathbb{R})^K$ denotes the set of all intuitionistic fuzzy mappings from K to $\mathbb{R}$. If we take $IFN(\mathbb{R})$ instead of $IFS(\mathbb{R})$, then $IFN(\mathbb{R})^K$ also denotes the set of all intuitionistic fuzzy mappings from K to $\mathbb{R}$.

By the support of $(\tilde{\alpha}, \tilde{\beta})$-cut where $\tilde{\alpha}, \tilde{\beta} \in [0,1]$ and $\tilde{\alpha} + \tilde{\beta} \leq 1$, we can define intuitionistic fuzzy mappings such that

Definition 4. A mapping $\tilde{F}: K \to IFN(\mathbb{R})$ is called an intuitionistic fuzzy mapping. For each $\tilde{\alpha}, \tilde{\beta} \in [0,1]$ with $\tilde{\alpha} + \tilde{\beta} \leq 1$, associated to $\tilde{F}$, we define the family of interval valued functions $\tilde{F}_{(\tilde{\alpha},\tilde{\beta})}: K \to \mathcal{K}_{IFN}$ (pair of compact intervals) defined by $\tilde{F}_{(\tilde{\alpha},\tilde{\beta})}(u) =_{(\psi,\phi)} \left(\tilde{F}_*\left(u, (\tilde{\alpha}, \tilde{\beta})\right),\ \tilde{F}^*\left(u, (\tilde{\alpha}, \tilde{\beta})\right)\right)$. Now, for any $\tilde{\alpha}, \tilde{\beta} \in [0,1]$, the end point functions $\tilde{F}_*\left(., (\tilde{\alpha}, \tilde{\beta})\right): K \to \mathbb{R}$ and $\tilde{F}^*\left(., (\tilde{\alpha}, \tilde{\beta})\right): K \to \mathbb{R}$ are called lower and upper functions, respectively.

Definition 5. Let $\tilde{F}: K \to IFN(\mathbb{R})$ be a intuitionistic fuzzy mapping. Then

(i) $\tilde{F}$ is non-decreasing on K if for every $u < v$ in K if $\tilde{F}(u) \leq_{(\psi,\phi)} \tilde{F}(v)$.

(ii) $\tilde{F}$ is non-increasing on K if for every $u < v$ in K if $\tilde{F}(u) \geq_{(\psi,\phi)} \tilde{F}(v)$.

(iii) $\tilde{F}$ is strictly non-decreasing on K if for every $u < v$ in K if $\tilde{F}(u) <_{(\psi,\phi)} \tilde{F}(v)$.

(iv) $\tilde{F}$ is strictly non-increasing on K if for every $u < v$ in K if $\tilde{F}(u) >_{(\psi,\phi)} \tilde{F}(v)$.

If $\tilde{F}$ is either increasing or decreasing on the entire K, then $\tilde{F}$ is called a monotone.

Definition 6. Let $K \subseteq \mathbb{R}^n$ be a convex set. Then, the intuitionistic fuzzy mapping $\tilde{F}: K \to IFN(\mathbb{R})$ is said to be convex at $u \in K$ if

$$\tilde{F}((1-\tau)u + \tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u) \tilde{+} \tau\tilde{F}(v)\ \forall\ v \in K, \tau \in [0,1],$$

or the end point mappings $\tilde{F}_*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ and $\tilde{F}^*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ are convex and concave at $u \in K$, respectively. $\tilde{F}$ is said to be convex on K if it is convex at each $u \in K$. Strictly intuitionistic convex fuzzy mapping depends on strict inequality holding for $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$. $\tilde{F}: K \to IFN(\mathbb{R})$ is said to be intuitionistic concave fuzzy mapping at $u \in K$ if $-\tilde{F}$ is convex, and $\tilde{F}$ is said to be concave on K if it is concave at each $u \in K$. Strictly intuitionistic concave fuzzy mapping depends on inequality holding for $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$.

Remark 1. The intuitionistic convex fuzzy mappings have some very nice properties:
(1) If $\tilde{F}$ is an intuitionistic convex fuzzy mapping, then $\Upsilon\tilde{F}$ is also convex for $\Upsilon \geq 0$.
(2) If $\tilde{F}$ and $\tilde{G}$ are both intuitionistic convex fuzzy mappings, then $\tilde{F}\tilde{+}\tilde{G}$ and $\max\left(\tilde{F}(u), \tilde{G}(u)\right)$ is also a convex fuzzy mapping.

Example 1. We consider the intuitionistic fuzzy mapping $\tilde{F}: [1,\infty) \to IFN(\mathbb{R})$ defined by,

$$\tilde{F}(u)(\sigma) =_{\psi} \begin{cases} \frac{\sigma-u}{u}\omega, & \sigma \in [u,2u] \\ \frac{4u-\sigma}{2u}\omega, & \sigma \in [2u,4u] \\ 0, & \sigma \notin [u,4u] \end{cases},$$

$$\tilde{F}(u)(\sigma) =_{\phi} \begin{cases} \frac{2u-\sigma+\mu(\sigma-\log(u))}{2u-\log(u)}, & \sigma \in [\log(u), 2u] \\ \frac{\sigma-2u+\mu(110\log(u)-\sigma)}{110\log(u)-2u}, & \sigma \in [2u, 110\log(u)] \\ 1, & \sigma \notin [\log(u), 110\log(u)] \end{cases},$$

where the values ω and μ represent the maximum membership degree and minimum degree of non-membership, respectively. Then, for each $\tilde{\alpha}, \tilde{\beta} \in [0,1]$, we have

$$\tilde{F}_{(\tilde{\alpha},\tilde{\beta})}(u) = \left(\left[\frac{\tilde{\alpha}+\omega}{\omega}u, \frac{4\omega-2\tilde{\alpha}}{\omega}u\right], \left[\frac{(\tilde{\beta}-\mu)\log(u)+2(1-\tilde{\beta})u}{1-\mu}, \frac{110(\tilde{\beta}-\mu)\log(u)+2(1-\tilde{\beta})u}{1-\mu}\right]\right).$$

Further, these values satisfy the following conditions: $0 \leq \tilde{\alpha} \leq \omega, \mu \leq \tilde{\beta} \leq 1$ and $0 \leq \tilde{\alpha} + \tilde{\beta} \leq 1$. Since the end point functions $\tilde{F}_*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ and $\tilde{F}^*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ are convex and concave, respectively, then $\tilde{F}$ is an intuitionistic convex fuzzy mapping.

Definition 7. Let $K \subseteq \mathbb{R}^n$ be a convex set. Then, the intuitionistic fuzzy mapping $\tilde{F}: K \to IFN(\mathbb{R})$ is said to be trivial convex at $u \in K$ if

$$\tilde{F}((1-\tau)u+\tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v) \forall\, v \in K, \tau \in [0,1],$$

or the end point functions $\tilde{F}_*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ and $\tilde{F}^*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ are convex at $u \in K$. $\tilde{F}$ is said to be trivial convex on K if it is convex at each $u \in K$. Strictly intuitionistic trivial convex fuzzy mapping depend son the inequality holding for $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$ and $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$. $\tilde{F}: K \to IFN(\mathbb{R})$ is said to be intuitionistic fuzzy trivial concave mapping at $u \in K$ if $-\tilde{F}$ is simply convex at $u \in K$, and $\tilde{F}$ is said to be trivial concave on K if it is concave at each $u \in K$. Strictly intuitionistic trivial concave fuzzy mapping depends on inequality holding for $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$ and $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$.

Example 2. We consider the intuitionistic fuzzy mappings $\tilde{F}: (-1,1) \to IFN(\mathbb{R})$ defined by,

$$\tilde{F}(u)(\sigma) =_{(\psi,\phi)} \begin{cases} \frac{\sigma-u^2}{1-u^2}\omega, \sigma \in [u^2,1) \\ 0 \qquad , \text{otherwise} \end{cases}, \tilde{F}(u)(\sigma) =_{(\psi,\phi)} \begin{cases} \frac{(1-\sigma)+\mu(\sigma-u^2)}{1-u^2}, \sigma \in [u^2,1) \\ 1 \qquad , \text{otherwise}, \end{cases}$$

where the values ω and u represent, the maximum membership degree and the minimum degree of non-membership, respectively. Further, these values satisfy the following conditions: $\omega, u \in [0,1]$ and $0 \leq \omega + u \leq 1$. Then, for each $\tilde{\alpha}, \tilde{\beta} \in [0,1]$, and we have

$$\tilde{F}_{(\tilde{\alpha},\tilde{\beta})}(u) = \left(\left[\frac{\tilde{\alpha}+(\omega-\tilde{\alpha})u^2}{\omega}, 1\right], \left[\frac{1-\tilde{\beta}+(\tilde{\beta}-\mu)u^2}{1-\mu}, u^2\right]\right).$$

Further, these values satisfy the following conditions: $0 \leq \tilde{\alpha} \leq \omega, \mu \leq \tilde{\beta} \leq 1$ and $0 \leq \tilde{\alpha} + \tilde{\beta} \leq 1$. Since end point functions $\tilde{F}_*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ and $\tilde{F}^*\left(.,\left(\tilde{\alpha},\tilde{\beta}\right)\right)$ are convex, then $\tilde{F}$ is an intuitionistic trivial convex fuzzy mapping.

Note that if $\omega = 1, u = 0$, then an intuitionistic trivial convex fuzzy mapping $\tilde{F}$ becomes a convex fuzzy mapping.

Definition 8. The intuitionistic fuzzy mapping $\tilde{F}: K \subseteq \mathbb{R}^n \to IFN(\mathbb{R})$ is said to be quasi-convex at $u \in K$ if

$$\tilde{F}((1-\tau)u+\tau v) \leq_{(\psi,\phi)} \max\left(\tilde{F}(u), \tilde{F}(v)\right) \forall\, v \in K, \tau \in [0,1],$$

or the end point functions (ψ,ϕ) are quasi-convex and quasi-concave at $u \in K$, respectively. $\tilde{F}$ is said to be quasi-convex on K if it is quasi-convex at each $u \in K$. Strictly intuitionistic quasi-convex fuzzy mapping depends on strict inequality holding $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$. $\tilde{F}: K \to IFN(\mathbb{R})$ is said to be intuitionistic quasi-concave fuzzy mapping at $u \in K$ if $-\tilde{F}$ is quasi-convex, and $\tilde{F}$ is said to be quasi-concave on K if it is quasi-concave at each $u \in K$. Strictly intuitionistic quasi-concave fuzzy mapping depends on inequality holding $\tilde{F}(u) \neq_{(\psi,\phi)} \tilde{F}(v)$.

Theorem 1. Let $\tilde{F}: K \subseteq \mathbb{R}^n \to IFN(\mathbb{R})$ be an intuitionistic fuzzy mapping. Then

Intuitionistic fuzzy convexity ⇒ Intuitionistic fuzzy quasi-convexity.

Proof. Let $\tilde{F}: K \to IFN(\mathbb{R})$ be an intuitionistic fuzzy convexity. Then, for all $\tau \in (0,1)$ and $u, v \in K$ such that

$$((1-\tau)u+\tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v) \leq_{(\psi,\phi)} \max\left(\tilde{F}(u), \tilde{F}(v)\right),$$

hence, the required result follows.

Similarly, we can see that "Intuitionistic fuzzy concavity ⇒ Intuitionistic fuzzy quasi-concavity".

Example 3. We consider the intuitionistic fuzzy mapping $\tilde{F}: \mathbb{R} \to \mathrm{IFN}(\mathbb{R})$ defined by

$$[\tilde{F}(u)]^{(\tilde{\alpha},\tilde{\beta})} = \begin{cases} ([1, \quad 2]u^3, [-2,-1]u^3) \text{ if } u \geq 0 \\ ([2, \quad 1]x^3, [-1,-2]x^3) \text{ if } u < 0' \end{cases}$$

for each $\tilde{\alpha}, \tilde{\beta} \in [0,1]$, $\tilde{\alpha} + \tilde{\beta} \leq 1$. Then, $\tilde{F}$ is obviously an intuitionistic fuzzy quasi-convex mapping but not convex.

Definition 9. Let $\tilde{F}: K \subseteq \mathbb{R}^n \to \mathrm{IFN}(\mathbb{R})$ be an intuitionistic fuzzy mapping. Then, the level set of $\tilde{F}$ is denoted by $\tilde{F}_{\mathfrak{p}}$ and defined as,

$$\tilde{\Lambda}_{\mathfrak{p}} = \{u : u \in K, \tilde{F}(u) \leq_{(\psi,\phi)} \mathfrak{p}, \mathfrak{p} \in \mathrm{IFN}(\mathbb{R})\}.$$

Note that the level set is also called $\mathfrak{p}$-cut of $\tilde{F}$, and the set $\tilde{\Lambda}_{\mathfrak{p}}$ generalizes the standard form of $\mathfrak{p}$-cut of $\tilde{F}$. The set of all level sets of $\tilde{F}$ is represented as $\mathrm{IF}(\tilde{\Lambda}_{\mathfrak{p}})$.

From now onward, we assume that K is a convex set in $\mathbb{R}^n$, unless otherwise specified.

3. Main results

In this section, we discuss some properties of the intuitionistic convex and quasi-convex fuzzy mappings.

Theorem 2. If $\tilde{F}_j: K \to \mathrm{IFN}(\mathbb{R})$ are intuitionistic convex fuzzy mappings and $\omega_j \geq 0$, for $j = 1,2,3, \ldots \ldots \ldots, m$, then $\sum_{j=1}^m \omega_j \tilde{F}_j(u)$ is an intuitionistic convex fuzzy mapping.

Proof. Let $\tilde{F}_j: K \to \mathrm{IFN}(\mathbb{R})$ be intuitionistic convex such that

$$\tilde{F}_j((1-\tau)u + \tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}_j(u) \tilde{+} \tau \tilde{F}_j(v), \; j = 1,2,3, \ldots \ldots \ldots, m.$$

Now, by the convex combination,

$$\tau \sum_{j=1}^{m} \omega_j \tilde{F}_j(u) \tilde{+} (1-\tau) \sum_{j=1}^{m} \omega_j \tilde{F}_j(v) =_{(\psi,\phi)} \sum_{j=1}^{m} \omega_j \{(1-\tau)\tilde{F}_j(u) \tilde{+} \tau \tilde{F}_j(v)\}$$
$$\geq_{(\psi,\phi)} \sum_{j=1}^{m} \omega_j \tilde{F}_j((1-\tau)u + \tau v),$$

which implies that $\sum_{j=1}^m \omega_j \tilde{F}_j(u)$ is an intuitionistic convex fuzzy mapping.

Corollary 1. If $\tilde{F}_j: K \to \mathrm{IFN}(\mathbb{R})$ are intuitionistic concave fuzzy mappings and $\omega_j \geq 0$, for $j = 1,2,3, \ldots \ldots \ldots, m$, then $\sum_{j=1}^m \omega_j \tilde{F}_j(u)$ is an intuitionistic concave fuzzy mapping.

Theorem 3. Let $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ be an intuitionistic convex fuzzy mapping. Then, the level set $\tilde{\Lambda}_{\mathfrak{p}}$ is a convex set.

Proof. Let $u, v \in \widetilde{\Lambda}_{\mathfrak{p}}$. Then, $\tilde{F}(u) \leq_{(\psi,\phi)} \mathfrak{p}$ and $\tilde{F}(v) \leq_{(\psi,\phi)}$. Since $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is an intuitionistic convex fuzzy mapping, we have

$$\tilde{F}((1-\tau)u+\tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v) \leq_{(\psi,\phi)} (1-\tau)\mathfrak{p}\tilde{+}\tau\mathfrak{p} = \mathfrak{p},$$

which shows that $(1-\tau)u+\tau v \in \widetilde{\Lambda}_{\mathfrak{p}}$. Hence, $\widetilde{\Lambda}_{\mathfrak{p}}$ is a convex set.

Corollary 2. Let $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ be an intuitionistic concave fuzzy mapping. Then, the level set $\widetilde{\Lambda}_{\mathfrak{p}} = \{u: u \in K, \tilde{F}(u) \geq_{(\psi,\phi)} \mathfrak{p}, \mathfrak{p} \in \mathrm{IFN}(\mathbb{R})\}$ is a convex set.

Theorem 4. The intuitionistic fuzzy mapping $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is said to be quasi-convex if and only if the level set $\widetilde{\Lambda}_{\mathfrak{p}}$ is a convex set.

Proof. Let $u, v \in \widetilde{\Lambda}_{\mathfrak{p}}$. Then, by definition of $\widetilde{\Lambda}_{\mathfrak{p}}$, we have $\tilde{F}(u) \leq_{(\psi,\phi)} \mathfrak{p}$. As $u, v \in K$ and K is convex so $(1-\tau)u+\tau v \in K$. Since $\tilde{F}$ is quasi-convex, then

$$\begin{aligned}\tilde{F}((1-\tau)u+\tau v) &\leq_{(\psi,\phi)} \max\left(\tilde{F}(u),\ \tilde{F}(v)\right), \forall\, u, v \in \mathbb{R}, \tau \in [0,1]\\ &\leq_{(\psi,\phi)} \max(\mathfrak{p},\ \mathfrak{p}) = \mathfrak{p},\end{aligned}$$

from which we can note that $(1-\tau)u+\tau v \in \widetilde{\Lambda}_{\mathfrak{p}}$ is a convex set.

Conversely, assume that $\widetilde{\Lambda}_{\mathfrak{p}}$ is a convex set to prove $\tilde{F}$ is quasi-convex. As $\widetilde{\Lambda}_{\mathfrak{p}}$ is a convex set, then for any $u, v \in \widetilde{\Lambda}_{\mathfrak{p}}$ such that $(1-\tau)u+\tau v \in \widetilde{\Lambda}_{\mathfrak{p}}$ with $\tau \in [0,1]$. Now, we take $\max\left(\tilde{F}(u), \tilde{F}(v)\right) =_{(\psi,\phi)} \mathfrak{p}$ and $\tilde{F}(u) \geq_{(\psi,\phi)} \tilde{F}(v)$. By definition of convex set $\widetilde{\Lambda}_{\mathfrak{p}}$, we have

$$\tilde{F}((1-\tau)u+\tau v) \leq_{(\psi,\phi)} \max\left(\tilde{F}(u),\ \tilde{F}(v)\right) \leq_{(\psi,\phi)} \mathfrak{p},$$

hence, the intuitionistic fuzzy mapping $\tilde{F}$ is quasi-convex.

Corollary 3. The intuitionistic fuzzy mapping $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is said to be quasi-concave if and only if the level set $\widetilde{\Lambda}_{\mathfrak{p}} = \{u: u \in K, \tilde{F}(u) \geq_{(\psi,\phi)} \mathfrak{p}, \mathfrak{p} \in \mathrm{IFN}(\mathbb{R})\}$ is a convex set.

Theorem 5. The intuitionistic fuzzy mapping $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is convex if and only if the $\mathrm{epi}(\tilde{F}) = \{(u, \mathfrak{p}): u \in K, \tilde{F}(u) \leq_{(\psi,\phi)} \mathfrak{p}, \mathfrak{p} \in \mathrm{IFN}(\mathbb{R})\}$ is a convex set.

Proof. Let $(u, \mathfrak{p})$, $(v, \mathfrak{p}) \in K$ and $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ be convex. As $(u, \mathfrak{p})$, $(v, \mathfrak{p}) \in K$ so $\tilde{F}(u) \leq_{(\psi,\phi)} \mathfrak{p}_1$ and $\tilde{F}(u) \leq_{(\psi,\phi)} \mathfrak{p}_2$.

Since $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is convex for all $u, v \in K$ and $\tau \in (0,1)$, we have

$$\tilde{F}((1-\tau)u+\tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v), \leq_{(\psi,\phi)} (1-\tau)\mathfrak{p}_1\tilde{+}\tau\mathfrak{p}_2,$$

from which it follows that

$$(1-\left(\tau)u+\tau v, \left((1-\tau)\mathfrak{p}_1\tilde{+}\tau\mathfrak{p}_2, (1-\tau)\tilde{\beta}_1\tilde{+}\tau\tilde{\beta}_2\right)\right) \in \mathrm{epi}(\tilde{F})$$

so $\mathrm{epi}(\tilde{F})$ is a convex set.

Conversely, let $\mathrm{epi}(\tilde{F})$ be a convex set to prove $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is a convex mapping. Assume that $v, u \in K$, then, $\left(u, \left(\tilde{F}(u), \tilde{F}(u)\right)\right) \in \mathrm{epi}(\tilde{F})$ and $\left(v, \left(\tilde{F}(v), \tilde{F}(v)\right)\right) \in \mathrm{epi}(\tilde{F})$. As $\mathrm{epi}(\tilde{F})$ is a convex set, then

$$\left((1-\tau)u + \tau v, \left((1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v), (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v)\right)\right) \in \mathrm{epi}\left(\tilde{F}\right),$$

so by definition of $\mathrm{epi}\left(\tilde{F}\right)$,

$$\tilde{F}((1-\tau)u + \tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v),$$

hence, the intuitionistic fuzzy mapping $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is convex.

Corollary 4. The intuitionistic fuzzy mapping $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is concave if and only if, $\mathrm{hyp}(\tilde{F}) = \left\{(u, \mathfrak{p})): u \in K, \tilde{F}(u) \geq_{(\psi,\phi)} \mathfrak{p}, \mathfrak{p} \in \mathrm{IFN}(\mathbb{R})\right\}$ is a convex set.

Remark 1 and Theorem 5 can be used to verify the convexity or concavity of mapping $\tilde{F}$.

Theorem 6. Let K be an any interval in $\mathbb{R}$. If $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ is a monotonic (increasing or decreasing) intuitionistic fuzzy mapping, then $\tilde{F}$ is intuitionistic quasi-convex and quasi-concave fuzzy mapping on K.

Proof. Let $\tilde{F}: K \to \mathrm{IFN}(\mathbb{R})$ be monotonic and $u, v \in K$ such that $u < v$ to prove $\tilde{F}$ is intuitionistic fuzzy quasi-convex and quasi-concave. Now for $\tau \in (0, 1)$, we have $u < (1-\tau)u + \tau v < v$. As $\tilde{F}$ is non-increasing mapping,

$$\tilde{F}(u) \geq_{(\psi,\phi)} \tilde{F}(1-\tau)u + \tau v) \geq_{(\psi,\phi)} \tilde{F}(v).$$

Now

$$\tilde{F}(u) =_{(\psi,\phi)} \max\,(\tilde{F}(u), \tilde{F}(v)), \tilde{F}(v) =_{(\psi,\phi)} \min\,(\tilde{F}(u), \tilde{F}(v)),$$

we have

$$\min\left(\tilde{F}(u), \tilde{F}(v)\right) \leq_{(\psi,\phi)} \tilde{F}(1-\tau)u + \tau v) \leq_{(\psi,\phi)} \max\left(\tilde{F}(u), \tilde{F}(v)\right).$$

Using the same steps, we can be proved that when $\tilde{F}$ is non-decreasing mapping, then $\tilde{F}$ is an intuitionistic quasi-convex and quasi-concave fuzzy mapping on K. This completes the proof.

Theorem 7. The following results hold

i) If $\tilde{F}$ is an intuitionistic fuzzy convex, $\tilde{F}(0) \leq_{(\psi,\phi)} 0$, then $\tilde{F}(\tau v) \leq_{(\psi,\phi)} \tau\tilde{F}(v)$ for $\tau \in [0,1]$.

ii) If $\tilde{F}$ is an intuitionistic fuzzy quasi-convex, $\tilde{F}(0) \leq_{(\psi,\phi)} 0$, then $\tilde{F}(\tau v) \leq_{(\psi,\phi)} \tilde{F}(v)$ for $\tau \in [0,1]$.

Proof.

$$\text{i)}\tilde{F}(\tau v) =_{(\psi,\phi)} \tilde{F}((1-\tau)0 + \tau v) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(0)\tilde{+}\tau\tilde{F}(v) \leq_{(\psi,\phi)} \tau\tilde{F}(v).$$

$$\text{ii)}\tilde{F}(\tau v) =_{(\psi,\phi)} \tilde{F}((1-\tau)0 + \tau v) \leq_{(\psi,\phi)} \max\left(\tilde{F}(0), \tilde{F}(v)\right) \leq_{(\psi,\phi)} \tilde{F}(v).$$

Theorem 8. The following results hold:

i) If $\tilde{F}$ is an intuitionistic fuzzy convex, and $\tilde{F}(0) =_{(\psi,\phi)} 0$, then $\tilde{g}(\tau) \equiv \frac{\tilde{F}(\tau v)}{\tau}$ is non-decreasing for $\tau > 0$.
ii) If $\tilde{F}$ is an intuitionistic fuzzy quasi-convex, and $\tilde{F}(v) \leq_{(\psi,\phi)} \tilde{F}(0)$, then $\tilde{g}(\tau) \equiv \tilde{F}(\tau v)$ is non-decreasing for $\tau > 0$.
Proof. For any $\tau_2 > \tau_1$, it follows that:

$$\text{i)}(\tau_1) \equiv \frac{\tilde{F}(\tau_1 v)}{\tau_1} =_{(\psi,\phi)} \frac{\tilde{F}\left(\left(1-\frac{\tau_1}{\tau_2}\right)0 + \frac{\tau_1}{\tau_2}(\tau_2 v)\right)}{\tau_1}$$

$$\leq_{(\psi,\phi)} \frac{\left(1-\frac{\tau_1}{\tau_2}\right)\tilde{F}(0)\tilde{+}\frac{\tau_1}{\tau_2}\tilde{F}(\tau_2 v)}{\tau_1} =_{(\psi,\phi)} \frac{\tilde{F}(\tau_2 v)}{\tau_2} =_{(\psi,\phi)} \tilde{g}(\tau_2),$$

$$\text{ii)}\tilde{g}(\tau_1) \equiv \tilde{F}(\tau_1 v) =_{(\psi,\phi)} \tilde{F}\left(\left(1-\frac{\tau_1}{\tau_2}\right)0 + \frac{\tau_1}{\tau_2}(\tau_2 v)\right)$$

$$\leq_{(\psi,\phi)} \max\left(\tilde{F}(0), \tilde{F}(\tau_2 v)\right) \leq_{(\psi,\phi)} \tilde{F}(\tau_2 v) =_{(\psi,\phi)} \tilde{g}(\tau_2).$$

Hence, the required result is obtained.

Theorem 9. The following result hold:
i) $\tilde{g}(\tau) \equiv \tilde{F}((1-\tau)u + \tau v)$ is an intuitionistic fuzzy convex over $\tau \in [0,1]$ for any $u, v \in K$ if and if $\tilde{F}$ is an intuitionistic fuzzy convex over K.
ii) $\tilde{g}(\tau) \equiv \tilde{F}((1-\tau)u + \tau v)$ is an intuitionistic fuzzy quasi-convex over $\tau \in [0,1]$ for any $u, v \in K$ if and only if, $\tilde{F}$ is intuitionistic fuzzy quasi-convex over K.

Proof. i) Let $\tilde{F}$ be intuitionistic fuzzy convex over K. Then, for any $\tau_1, \tau_2 \in [0,1]$ and $\lambda \in [0,1]$, we have

$$\tilde{g}((1-\lambda)\tau_1 + \lambda\tau_2) \equiv \tilde{F}\left[\left\{1-\left((1-\lambda)\tau_1 + \lambda\tau_2\right)\right\}u + \left\{\left((1-\lambda)\tau_1 + \lambda\tau_2\right)\right\}v\right],$$

$$=_{(\psi,\phi)} \tilde{F}[\lambda\{(1-\tau_1)u + \tau_1 v\} + (1-\lambda)\{(1-\tau_2)u + \tau_2 v\}],$$

$$\leq_{(\psi,\phi)} \lambda\tilde{F}((1-\tau_1)u + \tau_1 v)\tilde{+}\,(1-\lambda)\tilde{F}\left((1-\tau_2)u + \tau_2 v\right),$$

$$\leq_{(\psi,\phi)} \lambda\tilde{g}(\tau_1)\tilde{+}(1-\lambda)\tilde{g}(\tau_2).$$

Conversely for any $\tau_1, \tau_2 \in [0,1]$ there exists $\lambda \in [0,1]$, we have

$$\tilde{g}((1-\lambda)\tau_1 + \lambda\tau_2) \leq_{(\psi,\phi)} \lambda\tilde{g}(\tau_1)\tilde{+}(1-\lambda)\tilde{g}(\tau_2)$$

that is,

$$\tilde{F}\left[\left\{1-\left((1-\lambda)\tau_1 + \lambda\tau_2\right)\right\}u + \left\{\left((1-\lambda)\tau_1 + \lambda\tau_2\right)\right\}v\right] \leq_{(\psi,\phi)} \lambda\tilde{F}((1-\tau_1)u + \tau_1 v)\tilde{+}\,(1-\lambda)\tilde{F}\left((1-\tau_2)u + \tau_2 v\right),$$

now, taking $\tau_1 = 1, \tau_2 = 0$, we can obtain

$$\tilde{F}((1-\lambda)u+\lambda v) \leq_{(\psi,\phi)} (1-\lambda)\tilde{F}(u)\tilde{+}\lambda\tilde{F}(v).$$

ii) Let $\tilde{F}$ be an intuitionistic fuzzy convex over K to prove $\tilde{g}(\tau)$ is convex. Now, for any $\tau_1, \tau_2 \in [0,1]$ and $\lambda \in [0,1]$, we have

$$\tilde{g}((1-\lambda)\tau_1+\lambda\tau_2) \equiv \tilde{F}\left[\left\{1-\left((1-\lambda)\tau_1+\lambda\tau_2\right)\right\}u+\left\{\left((1-\lambda)\tau_1+\lambda\tau_2\right)\right\}v\right],$$

$$=_{(\psi,\phi)} \tilde{F}[\lambda\{(1-\tau_1)u+\tau_1 v\}\tilde{+}(1-\lambda)\{(1-\tau_2)u+\tau_2 v\}],$$

$$\leq_{(\psi,\phi)} \max\left(\tilde{F}((1-\tau_1)u+\tau_1 v), \tilde{F}\left((1-\tau_2)u+\tau_2 v\right)\right),$$

$$\leq_{(\psi,\phi)} \max\left(\tilde{g}(\tau_1), \tilde{g}(\tau_2)\right).$$

Conversely, for any $\tau_1, \tau_2 \in [0,1]$, there exists $\lambda \in [0,1]$, we have

$$\tilde{g}((1-\lambda)\tau_1+\lambda\tau_2) \leq_{(\psi,\phi)} \max\left(\tilde{g}(\tau_1), \tilde{g}(\tau_2)\right),$$

that is,

$$\tilde{F}\left[\left\{1-\left((1-\lambda)\tau_1+\lambda\tau_2\right)\right\}+\left\{\left((1-\lambda)\tau_1+\lambda\tau_2\right)\right\}v\right]$$

$$\leq_{(\psi,\phi)} \max\left(\tilde{F}((1-\tau_1)u+\tau_1 v), \tilde{F}\left((1-\tau_2)u+\tau_2 v\right)\right).$$

Now, taking $\tau_1 = 1, \tau_2 = 0$, we can obtain

$$\tilde{F}((1-\lambda)u+\lambda v) \leq_{(\psi,\phi)} \max\left(\tilde{F}(u), \tilde{F}(v)\right).$$

By using the same arguments, part (ii) can be proved.

Theorem 10. Let $\tilde{F}: \mathrm{IFN}(\mathbb{R}) \to \mathrm{IFN}(\mathbb{R})$ be an intuitionistic convex fuzzy mapping (resp. concave), and $\tilde{G}: K \to \mathrm{IFN}(\mathbb{R})$ an intuitionistic fuzzy convex and concave. Then, the intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is also a convex (resp. concave).

Proof. Let $\tilde{F}: \mathrm{IFN}(\mathbb{R}) \to \mathrm{IFN}(\mathbb{R})$ be an intuitionistic convex fuzzy mapping, and $\tilde{G}: K \to \mathrm{IFN}(\mathbb{R})$ be a convex and concave. Since $\tilde{G}: K \to \mathrm{IFN}(\mathbb{R})$ is convex and concave, then for each $u, v \in K$ and $t \in (1, 0)$, we have

$$\tilde{G}\left((1-\tau)u+\tau v\right) =_{(\psi,\phi)} (1-\tau)\tilde{G}(u)\tilde{+}\tau\tilde{G}(v),$$

since $\tilde{F}$ is an intuitionistic convex fuzzy mapping, so

$$\tilde{F}\left(\tilde{G}\left((1-\tau)u+\tau v\right)\right) =_{(\psi,\phi)} \tilde{F}\left((1-\tau)\tilde{G}(u)\tilde{+}\tau\tilde{G}(v)\right),$$

$$\leq_{(\psi,\phi)} (1-\tau)\tilde{F}\left(\tilde{G}(u)\right)\tilde{+}\tau\tilde{F}\left(\tilde{G}(v)\right),$$

hence, intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is convex.

By using the same steps, we can prove that if $\tilde{F}: \mathrm{IFN}(\mathbb{R}) \to \mathrm{IFN}(\mathbb{R})$ is intuitionistic fuzzy concave mapping and $\tilde{G}: K \to \mathrm{IFN}(\mathbb{R})$ an intuitionistic fuzzy convex and concave, then intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is also a concave. Hence, the result follows.

Theorem 11. Let $\tilde{F}: IFN(\mathbb{R}) \rightarrow IFN(\mathbb{R})$ and $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ be intuitionistic convex (resp. concave) fuzzy mappings, and $\tilde{F}$ is a non-decreasing (resp. non-inreasing) mapping. Then, the intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is also convex.

Proof. Let $\tilde{F}: IFN(\mathbb{R}) \rightarrow IFN(\mathbb{R})$ and $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ be convex mappings. As $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ is convex, so for each $u, v \in K$ and $t \in (1, 0)$, we have

$$\tilde{G}\big((1-\tau)u + \tau v\big) \leq_{(\psi,\phi)} (1-\tau)\tilde{G}(u) \tilde{+} \tau\tilde{G}(v),$$

since $\tilde{F}$ is a non-decreasing mapping and convex, so

$$\tilde{F}\left(\tilde{G}\big((1-\tau)u + \tau v\big)\right) \leq_{(\psi,\phi)} \tilde{F}\left((1-\tau)\tilde{G}(u) \tilde{+} \tau\tilde{G}(v)\right) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}\left(\tilde{G}(u)\right) \tilde{+} \tau\tilde{F}\left(\tilde{G}(v)\right).$$

Hence, the intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is convex. Similarly, we can show that, if $\tilde{F}: IFN(\mathbb{R}) \rightarrow IFN(\mathbb{R})$ and $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ are intuitionistic concave fuzzy mappings and $\tilde{F}$ is a non-increasing mapping, then the intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is also concave. Hence, the result follows.

Theorem 12. Let $\tilde{F}: IFN(\mathbb{R}) \rightarrow IFN(\mathbb{R})$ be a non-decreasing (resp. non-increasing) mapping and $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ be intuitionistic quasi-convex (resp. quasi-concave) fuzzy mappings. Then intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is also quasi-convex (resp. quasi-concave).

Proof. Let $\tilde{F}: IFN(\mathbb{R}) \rightarrow IFN(\mathbb{R})$ be a non-decreasing mapping and $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ be convex mapping. Since $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ is convex, then for each $u, v \in K$ and $t \in (1, 0)$, we have

$$\tilde{G}\big((1-\tau)u + \tau v\big) \leq_{(\psi,\phi)} \max\left(\tilde{G}(u), \tilde{G}(v)\right),$$

since $\tilde{F}$ is a non-decreasing mapping, it follows that

$$\tilde{F}\left(\tilde{G}\big((1-\tau)u + \tau v\big)\right) \leq_{(\psi,\phi)} \tilde{F}\left(\max\left(\tilde{G}(u), \tilde{G}(v)\right)\right) \leq_{(\psi,\phi)} \max\left(\tilde{F}\left(\tilde{G}(u)\right), \tilde{F}\left(\tilde{G}(u)\right)\right),$$

hence, the intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is convex. Similarly, we can show that, if $\tilde{F}: IFN(\mathbb{R}) \rightarrow IFN(\mathbb{R})$ is a non-decreasing mapping and $\tilde{G}: K \rightarrow IFN(\mathbb{R})$ is an intuitionistic quasi-convex (resp. quasi-concave) fuzzy mappings. Then, the intuitionistic fuzzy mapping $u \mapsto \tilde{F}\left(\tilde{G}(u)\right)$ is also quasi-convex (resp. quasi-concave). Hence, the result follows.

4. Intuitionistic fuzzy variational inequalities

In this section, some variational inequalities are obtained over intuitionistic fuzzy functions, known as intuitionistic fuzzy variational inequalities.

Definition 13. An element $u \in K$ is called the minimum of intuitionistic fuzzy function $\tilde{F}: K \rightarrow \mathfrak{I}\mathcal{F}\mathcal{N}(R)$ if for all $u \in K$ such that

$$\tilde{F}(u) \leq_{(\psi,\phi)} \tilde{F}(v).$$

Theorem 13. A differentiable intuitionistic fuzzy convex function $\tilde{F}: K \rightarrow \mathfrak{I}\mathcal{F}\mathcal{N}(R)$ has minimum $u \in K$ if and only if, $u \in K$ satisfies

$$0 \leq_{(\psi,\phi)} \langle \tilde{F}'(u), \mathcal{v} - u \rangle \text{ for all } \mathcal{v} \in K. \tag{1}$$

Proof. Let $\tilde{F}: K \to \mathfrak{T}\mathcal{F}\mathcal{N}(R)$ be a differentiable intuitionistic fuzzy convex function and $u \in K$ satisfy (1) to prove $u \in K$ is a minimum of $\tilde{F}$.

As $\tilde{F}$ is a intuitionistic fuzzy convex function, then, by hypothesis, for all $u, \mathcal{v} \in K$, we have

$$\tilde{F}(\mathcal{v}) \tilde{-} \tilde{F}(u) \geq_{(\psi,\phi)} \langle \tilde{F}'(u), \mathcal{v} - u \rangle,$$

(1) implies that $\geq_{(\psi,\phi)} 0$, and implies that $\tilde{F}(u) \leq_{(\psi,\phi)} \tilde{F}(\mathcal{v})$, $\forall\, \mathcal{v} \in K$.

Conversely, let $u \in K$ be a minimum of $\tilde{F}$ to prove (7). As $u \in K$ is a minimum of $\tilde{F}$, then by definition and for all $\mathcal{v} \in K$,

$$\tilde{F}(u) \leq_{(\psi,\phi)} \tilde{F}(\mathcal{v}). \tag{2}$$

As K is a convex set, $\mathcal{v}_\tau = u + \tau(\mathcal{v} - u)$, for all $u, \mathcal{v} \in K$ and $\tau \in (0, 1)$. By solving (2) and replacing $\mathcal{v}$ by $\mathcal{v}_\tau$, then we have

$$\tilde{F}(u) \leq_{(\psi,\phi)} \tilde{F}(\mathcal{v}_\tau) \leq_{(\psi,\phi)} \tilde{F}(u + \tau(\mathcal{v} - u)), 0 \leq_{(\psi,\phi)} \tilde{F}(u + \tau(\mathcal{v} - u)) \tilde{-} \tilde{F}(u), \tag{3}$$

now dividing (3) by "τ" and taking $\lim_{\tau \to \infty}$, we have

$$0 \leq_{(\psi,\phi)} \lim_{\tau \to \infty} \frac{\tilde{F}(u + \tau(\mathcal{v} - u)) \tilde{-} \tilde{F}(u)}{\tau},$$

$$0 \leq_{(\psi,\phi)} \langle \tilde{F}'(u), \mathcal{v} - u \rangle,$$

for all $\mathcal{v} \in K$.

Note that (1) is called an intuitionistic fuzzy variational inequality. This theorem shows that, we characterized the minimum of intuitionistic fuzzy convex function with the help of intuitionistic variational inequalities.

In the next result, we obtain the generalized form of intuitionistic fuzzy variational inequalities, known as intuitionistic fuzzy mixed variational inequalities.

Theorem 14. Let $\tilde{F}: K \to \mathfrak{T}\mathcal{F}\mathcal{N}(R)$ be a differentiable intuitionistic fuzzy convex function and $\tilde{\aleph}: K \to \mathfrak{T}\mathcal{F}\mathcal{N}(R)$ be an intuitionistic fuzzy convex function. Then, the function $\tilde{A}$ defined by $\tilde{A}(\mathcal{v}) = \tilde{F}(\mathcal{v}) \tilde{+} \tilde{\aleph}(\mathcal{v})$ has minimum $u \in K$ if and only if $u \in K$ satisfies

$$0 \leq_{(\psi,\phi)} \langle \tilde{F}'(u), \mathcal{v} - u \rangle \tilde{+} \tilde{\aleph}(\mathcal{v}) \tilde{-} \tilde{\aleph}(u), \text{for all } \mathcal{v} \in K. \tag{4}$$

Proof. Let $\tilde{F}: K \to \mathfrak{T}\mathcal{F}\mathcal{N}(R)$ be a differentiable intuitionistic fuzzy convex function, $\tilde{\aleph}: K \to \mathfrak{T}\mathcal{F}\mathcal{N}(R)$ a non-differentiable intuitionistic fuzzy convex function, and $u \in K$ be the minimum of $\tilde{A}$ to prove that $u \in K$ satisfies (11) for all $\mathcal{v} \in K$.

As $u \in K$ is the minimum of $\tilde{A}$, then by definition, for all $\mathcal{v} \in K$, we have

$$\tilde{A}(u) \leq_{(\psi,\phi)} \tilde{A}(\mathcal{v}), \tag{5}$$

as K is a convex set, $\mathcal{v}_\tau = u + \tau(\mathcal{v} - u)$, for all $u, \mathcal{v} \in K$ and $\tau \in (0, 1)$. Replacing, $\mathcal{v}$ by $\mathcal{v}_\tau$ in (5), we get

$$\tilde{A}(u) \leq_{(\psi,\phi)} \tilde{A}(u + \tau(\mathcal{v} - u))$$

implying that

$$\tilde{F}(u)\tilde{+}\tilde{\aleph}(u) \leq_{(\psi,\phi)} \tilde{F}(u+\tau(\mathscr{v}-u))\tilde{+}\tilde{\aleph}(u+\tau(\mathscr{v}-u)),$$

since $\tilde{\aleph}$ is an intuitionistic fuzzy convex function, then we have

$$\tilde{F}(u)\tilde{+}\tilde{\aleph}(u) \leq_{(\psi,\phi)} \tilde{F}(u+\tau(\mathscr{v}-u))\tilde{+}(1-\tau)\tilde{\aleph}(u)\tilde{+}\tau\tilde{\aleph}(\mathscr{v}),$$

implying that

$$0 \leq_{(\psi,\phi)} \tilde{F}(u+\tau(\mathscr{v}-u))\tilde{-}\tilde{F}(u)\tilde{+}\tau\left(\tilde{\aleph}(\mathscr{v})-\tilde{\aleph}(u)\right),$$

now dividing by "τ" and taking $\lim\limits_{\tau\to\infty}$, we have

$$0 \leq_{(\psi,\phi)} \lim_{\tau\to\infty}\frac{\tilde{F}(u+\tau(\mathscr{v}-u))\tilde{-}\tilde{F}(u)}{\tau}\tilde{+}\tilde{\aleph}(\mathscr{v})\tilde{-}\tilde{\aleph}(u),$$

implying that

$$0 \leq_{(\psi,\phi)} \langle \tilde{F}'(u), \mathscr{v}-u\rangle\tilde{+}\tilde{\aleph}(\mathscr{v})\tilde{-}\tilde{\aleph}(u).$$

Conversely, let (4) be satisfied to prove that $u \in K$ is a minimum of Ã. Assume that for all $\mathscr{v} \in K$, we have

$$\begin{aligned}\tilde{A}(u)\tilde{-}\tilde{A}(\mathscr{v}) &=_{(\psi,\phi)} \tilde{F}(u)\tilde{+}\tilde{\aleph}(u)\tilde{-}\tilde{F}(\mathscr{v})\tilde{-}\tilde{\aleph}(\mathscr{v}),\\ &=_{(\psi,\phi)} \tilde{-}\left(\tilde{F}(\mathscr{v})\tilde{-}\tilde{F}(u)\right)\tilde{+}\tilde{\aleph}(u)\tilde{-}\tilde{\aleph}(\mathscr{v}),\\ &\leq_{(\psi,\phi)} \tilde{-}\left[\langle \tilde{F}'(u), \mathscr{v}-u\rangle\tilde{+}\tilde{\aleph}(u)\tilde{-}\tilde{\aleph}(\mathscr{v})\right],\\ &\leq_{(\psi,\phi)} 0,\end{aligned} \tag{6}$$

hence, $\tilde{A}(u) \leq_{(\psi,\phi)} \tilde{A}(\mathscr{v})$.

Note that (4) is called an intuitionistic fuzzy mixed variational inequality. This result shows the characterization of the minimum of an intuitionistic fuzzy convex function with the support of intuitionistic fuzzy mixed variational inequalities. If $\tilde{\aleph}=0$, then (4) becomes an intuitionistic fuzzy variational inequality.

5. Applications

We now discuss some applications of intuitionistic convex fuzzy mappings in optimization theory.

Definition 10. Let K be a non-empty subset of $\mathbb{R}^n$.

(1) An element $\tilde{u} \in K$ is said to be a local minimum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) \leq_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K \cap B_\delta(\tilde{u}).$$

(2) An element $\tilde{u} \in K$ is said to be a strict local minimum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) <_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K \cap B_\delta(\tilde{u}).$$

(3) An element $\tilde{u} \in K$ is said to be a global minimum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) \leq_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K.$$

(4) An element $\tilde{u} \in K$ is said to be a strict global minimum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) <_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K.$$

(5) An element $\tilde{u} \in K$ is said to be a local maximum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) \geq_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K \cap B_\delta(\tilde{u}).$$

(6) An element $\tilde{u} \in K$ is said to be a strict local maximum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) >_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K \cap B_\delta(\tilde{u}).$$

(7) An element $\tilde{u} \in K$ is said to be a global maximum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) \geq_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K.$$

(8) An element $\tilde{u} \in K$ is said to be a global maximum of intuitionistic fuzzy mapping $\tilde{F}$ if there exists $\delta > 0$ such that

$$\tilde{F}(\tilde{u}) >_{(\psi,\phi)} \tilde{F}(v), \text{ for all } v \in K.$$

Theorem 15. Let K be a non-empty convex set and $\tilde{F}: K \to IFN(\mathbb{R})$ be an intuitionistic fuzzy mapping, with $\tilde{\omega} =_{(\psi,\phi)} \inf_{u \in K}\left(\tilde{F}(u)\right)$ existing in $IFN(\mathbb{R})$.

(1) If $\tilde{F}: K \to IFN(\mathbb{R})$ is an intuitionistic convex fuzzy mapping, then set $\$ = \{u : u \in K, \tilde{F}(u) =_{(\psi,\phi)} \tilde{\omega}\}$ is a convex set.

(2) If $\tilde{F}: K \to IFN(\mathbb{R})$ is a strictly intuitionistic fuzzy convex, then \$ is a singleton set or empty. That is, if $\tilde{F}$ is strictly convex, then $\tilde{F}$ has at least one global minimum.

Proof. (1). Let $\tilde{F}$ be an intuitionistic convex fuzzy mapping. Then, by definition of $\tilde{F}$, for all $u, v \in \$$ implies that $u, v \in K$ and $\tau \in (0,1)$, we have

$$\tilde{F}\big((1-\tau)u + \tau v\big) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u) \tilde{+} \tau\tilde{F}(v),$$
$$=_{(\psi,\phi)} (1-\tau)\tilde{\omega} \tilde{+} \tau\tilde{\omega} =_{(\psi,\phi)} \tilde{\omega}.$$

Hence, \$ is a convex set.

(2). Let $\tilde{F}$ be a strictly intuitionistic fuzzy convex to prove \$ is a singleton set. Otherwise, suppose that $\tilde{F}(u) =_{(\psi,\phi)} \tilde{\omega} =_{(\psi,\phi)} \tilde{F}(v)$. As $\tilde{F}$ is strictly intuitionistic fuzzy convex, for all $u, v \in K$ and $\tau \in (0,1)$, we have

$$\tilde{F}\big((1-\tau)u + \tau v\big) <_{(\psi,\phi)} (1-\tau)\tilde{F}(u) \tilde{+} \tau\tilde{F}(v),$$
$$=_{(\psi,\phi)} (1-\tau)\tilde{\omega} \tilde{+} \tau\tilde{\omega} =_{(\psi,\phi)} \tilde{\omega},$$

which implies that $\tilde{\omega} \neq_{(\psi,\phi)} \inf_{u\in K}\left(\tilde{F}(u)\right)$ so this contradicts the fact. Hence, the result follows.

Corollary 5. Let K be a non-empty convex set and $\tilde{F}\colon K \to IFN(\mathbb{R})$ be an intuitionistic fuzzy mapping, with $\tilde{\omega} =_{(\psi,\phi)} \sup_{u\in K}\left(\tilde{F}(u)\right)$ existing in $IFN(\mathbb{R})$.

(1) If $\tilde{F}\colon K \to IFN(\mathbb{R})$ is an intuitionistic concave fuzzy mapping, then set $\$ = \{u\colon u \in K, \tilde{F}(u) =_{(\psi,\phi)} \tilde{\omega}\}$ is a convex set.

(2) If $\tilde{F}\colon K \to IFN(\mathbb{R})$ is a strictly intuitionistic concave, then \$ is a singleton set or empty. That is, if a strictly intuitionistic fuzzy concave, then $\tilde{F}$ has at least one global maximum.

Theorem 16. Let K be a non-empty convex set and $\tilde{F}\colon K \to IFN(\mathbb{R})$ be a intuitionistic convex fuzzy mapping, with $\tilde{\omega} =_{(\psi,\phi)} \inf_{u\in K}\left(\tilde{F}(u)\right)$ existing in $IFN(\mathbb{R})$, and that the set

$$\$ = \{u\colon u \in K, \tilde{F}(u) =_{(\psi,\phi)} \tilde{\omega}\} \neq \emptyset.$$

If $\tilde{u}$ is a local minimum of $\tilde{F}$, then it is also a global minimum of $\tilde{F}$ on K.

Proof. Let $\tilde{F}\colon K \to IFN(\mathbb{R})$ be an intuitionistic fuzzy mapping and $\tilde{u}$ be a local minimum of $\tilde{F}$. If $\tilde{u}$ is not a global minimum of $\tilde{F}$, then $\tilde{u} \notin \$$. By hypothesis, $\$ \neq \emptyset$. If $v \in \$$, then we must have $\tilde{F}(v) <_{(\psi,\phi)} \tilde{F}(\tilde{u})$. As $\tilde{F}$ is convex, by definition, for $\tau \in (1,0)$, we have

$$\tilde{F}\big((1-\tau)u + \tau v\big) \leq_{(\psi,\phi)} (1-\tau)\tilde{F}(u)\tilde{+}\tau\tilde{F}(v) <_{(\psi,\phi)} \tilde{F}(\tilde{u}), \text{ because } \tilde{F}(v) <_{(\psi,\phi)} \tilde{F}(\tilde{u}),$$

for any small positive number τ; this contradiction proves the required result.

Corollary 6. Let K be a non-empty convex set and $\tilde{F}\colon K \to IFN(\mathbb{R})$ be an intuitionistic concave fuzzy mapping, with $\tilde{\omega} =_{(\psi,\phi)} \sup_{u\in K}\left(\tilde{F}(u)\right)$ existing in $IFN(\mathbb{R})$, and that the set

$$\$ = \{u\colon u \in K, \tilde{F}(u) =_{(\psi,\phi)} \tilde{\omega}\} \neq \emptyset.$$

If $\tilde{u}$ is a local maximum of $\tilde{F}$, then it is also a global maximum of $\tilde{F}$ on K.

Theorem 17. Let $\tilde{F}\colon K \to IFN(\mathbb{R})$ be an intuitionistic quasi-convex fuzzy mapping. If $\tilde{u} \in K$ is a strict local minimum of $\tilde{F}$, then it is also a strict global minimum of $\tilde{F}$ in K.

Proof. Let $\tilde{u} \in K$ be a strict local minimum of $\tilde{F}\colon K \to IFN(\mathbb{R})$. Then, by definition of strict local minimum, there exists a $\delta > 0$ such that $\tilde{F}(\tilde{u}) <_{(\psi,\phi)} \tilde{F}(v)$ for all $v \in K \cap B_\delta(\tilde{u})$. On the contrary, that is, for some $v' \in K$, $\tilde{F}(\tilde{u}) \geq_{(\psi,\phi)} \tilde{F}(v')$.

As $\tilde{F}$ is an intuitionistic quasi-convex fuzzy mapping, by definition, for $\tau \in (1,0)$, we have

$$\tilde{F}\big((1-\tau)u + \tau v\big) \leq_{(\psi,\phi)} \max\left(\tilde{F}(u), \tilde{F}(v)\right) \leq_{(\psi,\phi)} \tilde{F}(\tilde{u}), \text{ because } \tilde{F}(\tilde{u}) \geq_{(\psi,\phi)} \tilde{F}(v').$$

This contradicts that $\tilde{u} \in K$ is a strict local minimum of $\tilde{F}$, and hence the results as follows:

Corollary 7. Let $\tilde{F}\colon K \to IFN(\mathbb{R})$ be an intuitionistic quasi-concave fuzzy mapping. If $\tilde{u} \in K$ is a strict local maximum of $\tilde{F}$, then it is also a strict global maximum of $\tilde{F}$ in K.

Theorem 18. Let $\tilde{F}\colon K \to IFN(\mathbb{R})$ be a strictly intuitionistic quasi-convex fuzzy mapping. If $\tilde{u} \in K$ is a local minimum of $\tilde{F}$, then it is also a global minimum of $\tilde{F}$ in K.

Proof. Proof is similar as Theorem 17, so it is omitted here.

Corollary 8. Let $\tilde{\mathrm{F}}: \mathrm{K} \to \mathrm{IFN}(\mathbb{R})$ be a strictly intuitionistic quasi-concave fuzzy mapping. If $\tilde{u} \in \mathrm{K}$ is a local maximum of $\tilde{\mathrm{F}}$, then it is also a global maximum of $\tilde{\mathrm{F}}$ in K.

6. Conclusions

During the last decades, the fuzzy convex analysis has attracted increasing attention. This research work has been dedicated on the development of optimization theory, which has brought the fuzzy convexity into many theoretical and application problems. In this paper, the notions of intuitionistic fuzzy mappings were introduced and some results, including epigraph of intuitionistic fuzzy mapping were discussed; some applications to G non-differentiable optimization were studied. Additionally, some new versions of intuitionistic fuzzy inequalities, known as intuitionistic fuzzy variational and intuitionistic fuzzy mixed variational inequalities over intuitionistic fuzzy mappings, were studied. Therefore, this work will provide duality and optimality results for intuitionistic convex fuzzy programing problems, potentially guiding future research.

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Conflict of interest

The author declares there is no conflict of interest.

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