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Research article

On eigenfunctions corresponding to first non-zero eigenvalue of the sphere $S^n(c)$ on a Riemannian manifold

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Abstract: We recall classical themes such as "on hearing the shape of a drum" or "can one hear the shape of a drum?", and the discovery of Milnor, who constructed two flat tori which are isospectral but not isometric. In this article, we consider the question of finding conditions under which an n-dimensional closed Riemannian manifold (M^n, g) having a non-zero eigenvalue nc for a positive constant c (that is, has same non-zero eigenvalue as first non-zero eigenvalue of the sphere $S^n(c)$), is isometric to $S^n(c)$. We address this issue in two situations. First, we consider the compact (M^n, g) as the hypersurface of the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$ with isometric immersion $f : (M^n, g) \to \left(R^{n+1}, \langle,\rangle\right)$ and a constant unit vector $\overrightarrow{a}$ such that the function $\rho = \left\langle f, \overrightarrow{a}\right\rangle$ satisfying $\Delta\rho = -nc\rho$ for a positive constant c is isometric to $S^n(c)$ if and only if (M^n, g) is isometric to $S^n(c)$ provided the integral of Ricci curvature $Ric(\nabla\rho, \nabla\rho)$ has an appropriate lower bound. In the second situation, we consider that the compact (M^n, g) admits a non-trivial concircular vector field ξ with potential function σ satisfying $\Delta\sigma = -nc\sigma$ for a positive constant c and a specific function f related to ξ (called circular function) is constant along the integral curves of ξ if and only if (M^n, g) is isometric to $S^n(c)$.

Keywords: first non-zero eigenvalue; eigenfunction; Laplace operator; Ricci curvature; isometric to sphere
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1. Introduction

The Laplace operator Δ acting on smooth functions of a Riemannian manifold (M^n, g), is defined by $\Delta f = \operatorname{div}(\nabla f)$, where ∇f is the gradient of f. It is known that Δ is a self adjoint elliptic operator with respect to the inner product $(,)$ defined on the algebra of smooth functions $C^\infty(M^n)$ with compact support by

$$(f, h) = \int_{M^n} fh dV_g,$$

where dV_g is the volume form on M^n with respect to the metric g. If $\Delta f = -\lambda f$, for a constant λ, then λ is said to be an eigenvalue of the Laplace operator Δ, the negative sign in the definition chosen so that for non-zero eigenvalue λ, $\lambda > 0$. The set of eigenvalues λ_i of the Laplace operator Δ on a Riemannian manifold (M^n, g) is called the spectrum of (M^n, g). Spectra of known Riemannian manifolds such as the sphere $S^n(c)$ and the real projective space RP^n are known, and a nice description could be found in [2] (Chapter 2, Section 5). Two Riemannian manifolds of same dimension (M^n, g) and $\left(\overline{M}^n, \overline{g}\right)$ having the same spectra are said to be isospectral manifolds, whereas they are said to be isometric if there exists a diffeomorphism $\phi : M^n \to \overline{M}^n$ that preserves the metric $\phi^*(\overline{g}) = g \circ \phi$, that is, ϕ is an isometry. In the mid Nineteenth century, it was an open question whether isospectral Riemannian manifolds are isometric. We see that physicists followed this question under the topic "On hearing the shape of a drum" or "Can one hear the shape of a drum?" (cf. [5, 11]), while Milnor constructed two flat tori in dimension 16, which are isospectral but not isometric (cf. [12]). In [12], the author constructed two 2-dimensional compact manifolds of constant negative curvature which are isospectral and not isometric. Further, Ejiri constructed two non-flat compact Riemannian manifolds which are isospectral but not isometric (cf. [6]). This initiated an interest in comparing spectra of two Riemannian manifolds, for further results in this direction, we refer to (cf. [1, 7–10, 15]). One of the natural questions is under what conditions two isospectral Riemannian manifolds are isometric? We know that the n-sphere $S^n(c)$ has first non-zero eigenvalue $\lambda_1 = nc$, and, supposing an n-dimensional Riemannian manifold (M^n, g) also has non-zero eigenvalue $\lambda_1 = nc$, we reduce the above general question to the following specific question: Under what condition is an n-dimensional Riemannian manifold (M^n, g) that has non-zero eigenvalue $\lambda_1 = -nc$ for a positive constant c isometric to $S^n(c)$? In order to answer this question, naturally, we need a smooth function ρ on an n-dimensional Riemannian manifold (M^n, g) that satisfies $\Delta\rho = -nc\rho$. In order to address this issue of finding a smooth function, we consider two situations:

(i) Considering (M^n, g) as an isometrically immersed hypersurface in the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$ with isometric immersion $f : (M^n, g) \to \left(R^{n+1}, \langle,\rangle\right)$, though there are several known functions on the hypersurface (M^n, g), namely, the mean curvature function, the scalar curvature function, as well as the support function, but they cannot be eigenfunctions of the Laplace operator corresponding to a non-zero eigenvalue, while (M^n, g) is isometric to the sphere $S^n(c)$. Then, we go for the alternative, namely, for a constant unit vector field $\overrightarrow{a}$ on the Euclidean space R^{n+1}, we get a smooth function $\rho = \left\langle f, \overrightarrow{a} \right\rangle$ on the hypersurface (M^n, g), which we require to satisfy $\Delta\rho = -nc\rho$, $c > 0$.

(ii) We seek the Riemannian manifold (M^n, g), and there exists a non-trivial concircular vector field ξ with potential function ρ and we require that the potential function ρ satisfies $\Delta\rho = -nc\rho$, $c > 0$.

In this article, we explore the above two situations that an n-dimensional Riemannian manifold (M^n, g) has an eigenvalue of the Laplace operator same as the first non-zero eigenvalue of the sphere $S^n(c)$, and find an additional condition so that these two are isometric (See Theorems 1, 2, 3). In the language of physics, "yes, we could hear a node through an apparatus to predict the shape of a drum".

2. Preliminaries

On an n-dimensional Riemannian manifold (M^n, g), denote by ∇ the Riemannian connection. The curvature tensor R of (M^n, g) is given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]\, Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in \Gamma\,(TM^n)\,, \tag{2.1}$$

where $\Gamma\,(TM^n)$ is the space of smooth sections of the tangent bundle TM^n. The Ricci tensor Ric of (M^n, g) is a symmetric tensor given by

$$Ric\,(X,Y) = \sum_{l=1}^{n} g\,(R\,(e_l, X)\,Y, e_l)\,, \qquad X, Y \in \Gamma\,(TM^n)\,, \tag{2.2}$$

where $\{e_1, .., e_n\}$ is a local orthonormal frame (or lof) on (M^n, g). The Ricci operator Q of (M^n, g) is related to the Ricci tensor by

$$Ric\,(X,Y) = g\,(QX, Y)\,, \qquad X, Y \in \Gamma\,(TM^n)$$

and therefore Q is a symmetric $(1, 1)$ tensor field. The scalar curvature τ of the Riemannian manifold (M^n, g) is given by

$$\tau = \sum_{l=1}^{n} Ric\,(e_l, e_l)\,.$$

The following formula is well known (cf. [2]),

$$\frac{1}{2}\nabla\tau = \sum_{l=1}^{n} (\nabla_{e_l} Q)\,(e_l)\,,$$

where $\nabla\tau$ is the gradient of τ, and the covariant derivative is given by

$$(\nabla_X Q)\,(Y) = \nabla_X QY - Q\,(\nabla_X Y)\,.$$

Given a smooth function $f : M^n \to R$ on a Riemannian manifold (M^n, g), the Laplace operator Δ acts on f, given by

$$\Delta f = \operatorname{div}(\nabla f)\,,$$

where ∇f is the gradient of f and

$$\operatorname{div} X = \sum_{l=1}^{n} g\,(\nabla_{e_l} X, e_l)\,.$$

If (M^n, g) is closed, then Stokes's theorem implies

$$\int_{M^n} (\operatorname{div} X)\, dV_g = 0,$$

where dV_g is the volume element of (M^n, g).

Given a symmetric $(1,1)$ tensor field T on an n-dimensional Riemannian manifold (M^n, g) with trace t, that is,

$$t = \sum_{l=1}^{n} g\left(Te_l, e_l\right),$$

then the Cauchy–Schwartz inequality is

$$\|T\|^2 \geq \frac{1}{n} t^2, \tag{2.3}$$

where

$$\|T\|^2 = \sum_{l=1}^{n} g\left(Te_l, Te_l\right).$$

Moreover, the equality in (2.3) holds if and only if

$$T = \frac{t}{n} I,$$

where I is the identity $(1,1)$ tensor.

Suppose on an n-dimensional closed Riemannian manifold (M^n, g) there is a smooth function f that satisfies $\Delta f = -ncf$ for a constant c. Then we have $f\Delta f = -ncf^2$. Then, integrating by parts, the last equation leads to

$$\int_{M^n} \|\nabla f\|^2 \, dV_g = nc \int_{M^n} f^2 dV_g. \tag{2.4}$$

3. Compact hypersurfaces in a Euclidean space with non-zero eigenvalue *nc* of the Laplace operator

In this section, we consider an n-dimensional Riemannian manifold (M^n, g) that admits an isometric immersion $f : (M^n, g) \to \left(R^{n+1}, \langle , \rangle\right)$ into the Euclidean space $\left(R^{n+1}, \langle , \rangle\right)$, where $\langle , \rangle$ is the Euclidean inner product. We denote by ς the unit normal to the hypersurface (M^n, g), and by B the shape operator with respect to the isometric immersion f. Then, denoting the Euclidean connection on $\left(R^{n+1}, \langle , \rangle\right)$ by $\overline{\nabla}$ and the Riemannian connection on the hypersurface (M^n, g) by ∇, we have the following fundamental equations for the hypersurface (cf. [2])

$$\overline{\nabla}_X Y = \nabla_X Y + g\left\langle BX, Y\right\rangle \varsigma, \qquad X, Y \in \Gamma\left(TM^n\right) \tag{3.1}$$

and

$$\overline{\nabla}_X \varsigma = -BX, \qquad X \in \Gamma\left(TM^n\right). \tag{3.2}$$

The mean curvature β of the hypersurface (M^n, g) is given by

$$\beta = \frac{1}{n} \sum_{l=1}^{n} g\left(Be_l, e_l\right), \tag{3.3}$$

where $\{e_1, .., e_n\}$ is a local orthonormal frame on (M^n, g).

The curvature tensor, the Ricci tensor, and the scalar curvature of the hypersurface (M^n, g) are given by (cf. [2])

$$R(X,Y)Z = g(BY,Z)BX - g(BX,Z)BX, \qquad X,Y,Z \in \Gamma(TM^n),$$
$$Ric(X,Y) = n\beta g(BX,Y) - g(BX.BY), \qquad X,Y \in \Gamma(TM^n)$$

and

$$\tau = n^2\beta^2 - \|B\|^2.$$

Also, the shape operator B has the following wonderful property (Codazzi equation of the hypersurfaces in flat spaces)

$$(\nabla_X B)(Y) = (\nabla_Y B)(X), \qquad X,Y \in \Gamma(TM^n). \tag{3.4}$$

Using Eqs (3.3) and (3.4) as well as the symmetry of the shape operator, we have for any $X \in \Gamma(TM^n)$

$$\begin{aligned} nX(\beta) &= \sum_{l=1}^{n} g(\nabla_X Be_l, e_l) + \sum_{l=1}^{n} g(Be_l, \nabla_X e_l) \\ &= \sum_{l=1}^{n} g((\nabla_X B)(e_l) + B(\nabla_X e_l), e_l) + \sum_{l=1}^{n} g(Be_l, \nabla_X e_l) \\ &= \sum_{l=1}^{n} g((\nabla_{e_l} B)(e_l), X) + 2\sum_{l=1}^{n} g(Be_l, \nabla_X e_l). \end{aligned}$$

Note that $Be_l = \sum_j g\left(Be_l, e_j\right) e_j$ and $\nabla_X e_l = \sum_k \omega_l^k(X) e_k$, where $g\left(Be_l, e_j\right)$ is symmetric while the connection forms ω_l^k are skew symmetric. Therefore,

$$\sum_{l=1}^{n} g(Be_l, \nabla_X e_l) = \sum_{jkl} g\left(Be_l, e_j\right) \omega_l^k(X) g\left(e_j, e_k\right) = \sum_{l=1}^{n} g\left(Be_l, e_j\right) \omega_l^j(X) = 0.$$

Thus, above equation becomes

$$nX(\beta) = \sum_{l=1}^{n} g((\nabla_{e_l} B)(e_l), X),$$

that is, we have

$$\sum_{l=1}^{n} (\nabla_{e_l} B)(e_l) = n\nabla\beta. \tag{3.5}$$

Treating the isometric immersion $f : (M^n, g) \to \left(R^{n+1}, \langle , \rangle\right)$ as a position vector of points of M^n in R^{n+1}, and defining $\sigma = \langle f, \varsigma \rangle$, called the support function of the hypersurface (M^n, g), we express f as

$$f = \xi + \sigma\varsigma,$$

where $\xi \in \Gamma(TM^n)$ is tangential to (M^n, g). Differentiating equation (2.4), while using Eqs (3.1) and (3.2), we have upon equating the tangential and normal parts

$$\nabla_X \xi = X + \sigma BX \tag{3.6}$$

and

$$\nabla\sigma = -B\xi.$$

Taking a constant unit vector field $\overrightarrow{a}$ on the Euclidean space R^{n+1} (for instance a coordinate vector field), we define a smooth function h on the hypersurface (M^n, g), by $h = \left\langle \overrightarrow{a}, \varsigma \right\rangle$. Denoting the tangential part of $\overrightarrow{a}$ to the hypersurface (M^n, g) by ζ, we have

$$\overrightarrow{a} = \zeta + h\varsigma. \tag{3.7}$$

Differentiating the above equation with respect to $X \in \Gamma(TM^n)$, while using Eqs (3.1) and (3.2), we have upon equating the tangential and normal parts

$$\nabla_X \zeta = hBX, \qquad X \in \Gamma(TM^n) \tag{3.8}$$

and

$$\nabla h = -B\zeta. \tag{3.9}$$

Now, we prove the main result of this section.

Theorem 1. *An n-dimensional compact and connected isometrically immersed hypersurface* $f : (M^n, g) \to \left(R^{n+1}, \langle , \rangle\right)$ *in the Euclidean space* $\left(R^{n+1}, \langle , \rangle\right)$ *with mean curvature* β *and a constant unit vector* $\overrightarrow{a} = \zeta + h\varsigma$ *on* R^{n+1}*, where the function* $\rho = \left\langle f, \overrightarrow{a} \right\rangle$ *satisfies* $\Delta\rho = -nc\rho$ *for a positive constant c, is isometric to the sphere* $S^n(c)$ *if and only if the Ricci curvature* $Ric(\zeta, \zeta)$ *satisfies*

$$\int_{M^n} Ric(\zeta, \zeta)\, dV_g \geq n(n-1) \int_{M^n} h^2\beta^2 dV_g.$$

Proof. Consider an n-dimensional compact and connected Riemannian manifold (M^n, g) that admits an isometric immersion $f : (M^n, g) \to \left(R^{n+1}, \langle , \rangle\right)$ in the Euclidean space $\left(R^{n+1}, \langle , \rangle\right)$ with shape operator B, mean curvature β, and a constant unit vector $\overrightarrow{a} = \zeta + h\varsigma$ on R^{n+1}, where the function $\rho = \left\langle f, \overrightarrow{a} \right\rangle$ satisfies

$$\Delta\rho = -nc\rho \tag{3.10}$$

for a positive constant c. Also, the Ricci curvature $Ric(\zeta, \zeta)$ satisfies

$$\int_{M^n} Ric(\zeta, \zeta)\, dV_g \geq n(n-1) \int_{M^n} h^2\beta^2 dV_g. \tag{3.11}$$

Now, differentiating $\rho = \left\langle f, \overrightarrow{a} \right\rangle$in the direction of $X \in \Gamma(TM^n)$, we get $X(\rho) = \left\langle X, \overrightarrow{a} \right\rangle = \langle f, \zeta \rangle$. This gives us the gradient of ρ as

$$\nabla\rho = \zeta. \tag{3.12}$$

The Hessian operator $\mathcal{H}_\rho$ of the function ρ is given by $\mathcal{H}_\rho X = \nabla_X \nabla\rho$, $X \in \Gamma(TM^n)$, and using Eqs (3.8) and (3.12), we arrive at

$$\mathcal{H}_\rho X = hBX, \qquad X \in \Gamma(TM^n). \tag{3.13}$$

Taking the trace in the above equation and taking account of Eq (3.10), we get

$$c\rho = -h\beta \tag{3.14}$$

and therefore, through Eq (3.13), we conclude

$$\mathcal{H}_\rho X + c\rho X = hBX - h\beta X, \qquad X \in \Gamma(TM^n).$$

From the above equation, we reach

$$\left\|\mathcal{H}_\rho + c\rho I\right\|^2 = h^2 \|B - \beta I\|^2. \tag{3.15}$$

Next, using Eq (3.8), we have

$$h(BX - \beta X) = \nabla_X \zeta - h\beta X,$$

which yields

$$h^2 \|B - \beta I\|^2 = \|\nabla\zeta\|^2 + nh^2\beta^2 - 2h\beta \operatorname{div}\zeta.$$

Inserting $\operatorname{div}\zeta = nh\beta$ (an outcome of Eq (3.8)), in the above equation, we arrive at

$$h^2 \|B - \beta I\|^2 = \|\nabla\zeta\|^2 - nh^2\beta^2. \tag{3.16}$$

Recalling the following well known integral formula (cf. [16])

$$\int_{M^n} \left(Ric(\zeta,\zeta) + \frac{1}{2}\left|\pounds_\zeta g\right|^2 - \|\nabla\zeta\|^2 - (\operatorname{div}\zeta)^2\right) dV_g = 0,$$

and integrating Eq (3.16) while using the above integral formula, we conclude

$$\int_{M^n} h^2 \|B - \beta I\|^2 \, dV_g = \int_{M^n} \left(Ric(\zeta,\zeta) + \frac{1}{2}\left|\pounds_\zeta g\right|^2 - (\operatorname{div}\zeta)^2 - nh^2\beta^2\right) dV_g. \tag{3.17}$$

Using Eq (3.8), we compute

$$\left(\pounds_\zeta g\right)(X,Y) = 2hg(BX,Y), \qquad X, Y \in \Gamma(TM^n)$$

and consequently, we have

$$\frac{1}{2}\left|\pounds_\zeta g\right|^2 = 2h^2 \|B\|^2.$$

Thus, inserting above equation and $\operatorname{div}\zeta = nh\beta$ in Eq (3.17) confirms

$$\int_{M^n} h^2 \|B - \beta I\|^2 \, dV_g = \int_{M^n} \left(Ric(\zeta,\zeta) + 2h^2 \|B\|^2 - n^2h^2\beta^2 - nh^2\beta^2\right) dV_g,$$

that is,

$$\int_{M^n} h^2 \|B - \beta I\|^2 \, dV_g = \int_{M^n} \left(Ric(\zeta,\zeta) + 2h^2\left(\|B\|^2 - n\beta^2\right) - n(n-1)h^2\beta^2\right) dV_g. \tag{3.18}$$

For a local orthonormal frame $\{e_1, .., e_n\}$, we have

$$\|B - \beta I\|^2 = \sum_k g(Be_k - \beta e_k, Be_k - \beta e_k)$$

$$= \|B\|^2 + n\beta^2 - 2\beta \sum_k g(Be_k, e_k)$$
$$= \|B\|^2 - n\beta^2.$$

Utilizing the above equation in (3.18), we arrive at

$$\int_{M^n} h^2 \|B - \beta I\|^2 dV_g = \int_{M^n} \left(n(n-1)h^2\beta^2 - Ric(\zeta, \zeta)\right) dV_g.$$

Inserting from Eq (3.15) in the above equation, we have

$$\int_{M^n} \left\|\mathcal{H}_\rho + c\rho I\right\|^2 dV_g = \int_{M^n} \left(n(n-1)h^2\beta^2 - Ric(\zeta, \zeta)\right) dV_g$$

and treating it with inequality (3.11) allows us to reach the conclusion

$$\mathcal{H}_\rho = -c\rho I.$$

Note that ρ satisfies Eq (3.10), that is, $\Delta\rho = -nc\rho$ for a non-zero constant c. We claim that ρ can not be a constant, for if it were, Equation (3.10) will imply $\rho = 0$, and then Eq (3.14) will imply $h\beta = 0$. Note that by Eq (3.6) we have $\mathrm{div}\xi = n(1 + \sigma\beta)$, and therefore on the compact hypersurface (M^n, g), we have (cf. [4])

$$\int_{M^n} (1 + \sigma\beta) = 0,$$

which does not allow $\beta = 0$. Hence, in the situation where ρ is a constant, we have $h = 0$, and, also, by Eq (3.12), $\zeta = 0$, and by virtue of Eq (3.7), we will reach the conclusion $\overrightarrow{a} = 0$, contrary to the assumption that $\overrightarrow{a}$ is a unit vector. Thus, ρ is a non-constant function which satisfies Obata's equation (3.8) proving that (M^n, g) is isometric to the sphere $S^n(c)$ (cf. [13, 14]).

Conversely, consider the isometric immersion $f : S^n(c) \to \left(R^{n+1}, \langle , \rangle\right)$ of the sphere $S^n(c)$ in the Euclidean space $\left(R^{n+1}, \langle , \rangle\right)$ given by $f(x) = x$. Then, the unit normal $\varsigma = \sqrt{c}f$, the shape operator $B = -\sqrt{c}I$, and the mean curvature $\beta = -\sqrt{c}$. Consider the unit constant vector $\overrightarrow{a}$ given by the first Euclidean coordinate vector field, that is,

$$\overrightarrow{a} = \frac{\partial}{\partial x^1} = \zeta + h\varsigma,$$

where ζ is tangent to the sphere $S^n(c)$ and $h = \left\langle \overrightarrow{a}, \varsigma \right\rangle = \left\langle \overrightarrow{a}, \sqrt{c}f \right\rangle = \sqrt{c}\left\langle f, \overrightarrow{a} \right\rangle$. Thus, defining $\rho = \left\langle f, \overrightarrow{a} \right\rangle$, we have

$$h = \sqrt{c}\rho. \tag{3.19}$$

Now, differentiating (3.19) in the direction of $X \in \Gamma(TS^n(c))$ and equating the tangential and normal parts, we confirm

$$\nabla_X\zeta = -\sqrt{c}hX, \qquad \nabla h = \sqrt{c}\zeta \tag{3.20}$$

Using Eqs (3.19) and (3.20), we see $\sqrt{c}\zeta = \sqrt{c}\nabla\rho$, that is,

$$\zeta = \nabla\rho, \tag{3.21}$$

which, in view of the first equation in (3.20), provides

$$\Delta\rho = \mathrm{div}\zeta = -n\sqrt{c}h = -nc\rho. \tag{3.22}$$

Finally, using Eq (3.21), the Ricci curvature $Ric(\zeta,\zeta)$ of the sphere $S^n(c)$ is given by

$$Ric(\zeta,\zeta) = (n-1)c\|\zeta\|^2 = (n-1)c\|\nabla\rho\|^2. \tag{3.23}$$

However, Equations (3.19) and (3.22) confirm

$$\int_{S^n(c)} \|\nabla\rho\|^2 dV_g = nc\int_{S^n(c)} \rho^2 dV_g = n\int_{S^n(c)} h^2 dV_g = \frac{n}{c}\int_{S^n(c)} h^2\beta^2 dV_g.$$

Now, integrating Eq (3.23) while using above equation yields

$$\int_{S^n(c)} Ric(\zeta,\zeta)\, dV_g = n(n-1)\int_{S^n(c)} h^2\beta^2 dV_g.$$

Hence, the converse also holds. □

Next, we prove the following result for the complete hypersurface (M^n, g) of the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$.

Theorem 2. *An n-dimensional complete and simply connected isometrically immersed hypersurface $f : (M^n, g) \to \left(R^{n+1}, \langle,\rangle\right)$, $n > 1$, in the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$ with mean curvature β and a constant unit vector $\overrightarrow{a} = \zeta + h\varsigma$ on R^{n+1}, where the function $h = \left\langle\overrightarrow{a}, \varsigma\right\rangle \neq 0$ satisfies $\Delta h = -nch$ for a positive constant c, is isometric to the sphere $S^n(c)$ if and only if the mean curvature β is a constant along the integral curves of ζ and $\beta^2 \geq c$ holds.*

Proof. Consider an n-dimensional complete and simply connected Riemannian manifold (M^n, g) that admits an isometric immersion $f : (M^n, g) \to \left(R^{n+1}, \langle,\rangle\right)$ in the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$ such that the function $h = \left\langle\overrightarrow{a}, \varsigma\right\rangle \neq 0$ satisfies

$$\Delta h = -nch, \tag{3.24}$$

with mean curvature β satisfying

$$\zeta(\beta) = 0$$

and

$$\beta^2 \geq c. \tag{3.25}$$

We use Eqs (3.5) and (3.8), the symmetry of the shape operator B, and a local orthonormal frame $\{e_1, .., e_n\}$ in order to find $\mathrm{div}(B\zeta)$,

$$\begin{aligned}\mathrm{div}(B\zeta) &= \sum_{l=1}^{n} g\left(\nabla_{e_l}B\zeta, e_l\right) = \sum_{l=1}^{n} g\left(\left(\nabla_{e_l}B\right)(\zeta) + B\left(hBe_l\right), e_l\right)\\ &= \sum_{l=1}^{n} g\left(\zeta, \left(\nabla_{e_l}B\right)(e_l)\right) + h\|B\|^2 = n\zeta(\beta) + h\|B\|^2.\end{aligned}$$

Using Eq (3.15), we get

$$\operatorname{div}(B\zeta) = h\,\|B\|^2\,.$$

Now, taking the divergence in Eq (3.9) and using the above equation with Eq (3.24) yields

$$h\,\|B\|^2 = nch,$$

that is,

$$h^2\left(\|B\|^2 - n\beta^2\right) = nh^2\left(c - \beta^2\right). \tag{3.26}$$

Combining above equation with inequality (3.25), while keeping in view Cauchy–Schwartz's inequality $\|B\|^2 \geq n\beta^2$, we get

$$h^2\left(\|B\|^2 - n\beta^2\right) = 0.$$

Since $h \neq 0$, we get

$$\|B\|^2 = n\beta^2,$$

which, being an inequality in Cauchy–Schwartz's inequality $\|B\|^2 \geq n\beta^2$, we must have

$$B = \beta I. \tag{3.27}$$

The above equation implies

$$(\nabla_X B)(Y) = X(\beta)\,Y, \qquad X, Y \in \Gamma(TM^n),$$

which gives

$$\sum_{l=1}^{n} (\nabla_{e_l} B)(e_l) = \nabla\beta.$$

Combining the above equation with Eq (3.5) yields

$$(n-1)\,\nabla\beta = 0$$

and, as $n > 1$, we get that β is a constant, and by virtue of Eqs (3.26) and (3.27), we have

$$\beta^2 = c.$$

Now, using Eq (3.27) in the expression of the curvature tensor of the hypersurface with the above equation gives

$$R(X, Y)Z = c\,\{g(Y, Z)\,X - g(X, Z)\,Y\}, \qquad X, , Y, Z \in \Gamma(TM^n),$$

that is, (M^n, g) is a complete and simply connected space of constant positive curvature c. Hence, (M^n, g) is isometric to $S^n(c)$. The converse is trivial. □

4. Concircular vector field with potential function σ an eigenfunction of Laplace operator with eigenvalue nc

Consider an n-dimensional Riemannian manifold (M^n, g) that possesses a concircular vector field ξ (cf. [3]), that is, the vector field satisfies

$$\nabla_X \xi = \sigma X, \qquad X \in \Gamma(TM^n), \tag{4.1}$$

where σ is a smooth function, called the potential function of the concircular vector field. A concircular vector field is said to be non-trivial if the potential function $\sigma \neq 0$. Using Eq (4.1), we immediately have

$$\text{div}\xi = n\sigma. \tag{4.2}$$

In this section, we are interested in an n-dimensional compact Riemannian manifold (M^n, g) that possesses a non-trivial concircular vector field ξ with potential function σ satisfying

$$\Delta\sigma = -nc\sigma,$$

where $c > 0$ is a constant, that is, σ is an eigenfunction of the Laplace operator with eigenvalue the same as the first non-zero eigenvalue of the sphere $S^n(c)$, and we find a condition under which (M^n, g) is isometric to the sphere $S^n(c)$.

Before we approach this issue, we first prepare an auxiliary result to prove the main result. First, for Eq (4.1), using Eq (2.1) immediately gives the following expression of the curvature tensor, namely

$$R(X, Y)\xi = X(\sigma)Y - Y(\sigma)X, \qquad X, Y \in \Gamma(TM^n).$$

Taking the trace in the above equation and using Eq (2.2), we reach

$$Ric(X, \xi) = -(n-1)X(\sigma) \tag{4.3}$$

and this equation gives the following expression for the Ricci operator Q operating on ξ, namely

$$Q(\xi) = -(n-1)\nabla\sigma,$$

where $\nabla\sigma$ is the gradient of the potential function σ.

In the following paragraph, we show that for each concircular vector field ξ on a connected Riemannian manifold (M^n, g) there corresponds a smooth function f, which we call a concircular function of the concircular vector field ξ. Note that Eq (4.2) implies

$$R(X, \xi)\xi = X(\sigma)\xi - \xi(\sigma)X, \qquad X \in \Gamma(TM^n)$$

and the operator $R(X, \xi)\xi$, $X \in \Gamma(TM^n)$ is symmetric in X, and, therefore, the above equation implies

$$X(\sigma)g(\xi, Y) = Y(\sigma)g(\xi, X), \qquad X \in \Gamma(TM^n).$$

The above equation implies

$$X(\sigma)\xi = g(\xi, X)\nabla\sigma$$

and taking the inner product in the above equation with $\nabla\sigma$ and replacing X by ξ, we conclude

$$(\xi(\sigma))^2 = \|\xi\|^2 \|\nabla\sigma\|^2,$$

that is

$$\|\xi\|^2 \|\nabla\sigma\|^2 = g(\xi, \nabla\sigma)^2.$$

This proves that vector fields $\nabla\sigma$ and ξ are parallel, and, consequently, there exists a smooth function f such that

$$\nabla\sigma = f\xi.$$

We call this function f the *concircular function* of the concircular vector field ξ.

First, we prove the following proposition.

Proposition 1. *Let ξ be a non-trivial concircular vector field on an n-dimensional compact Riemannian manifold (M^n, g) with potential function σ and concircular function f. If the potential function σ satisfies*

$$\Delta\sigma = -nc\sigma$$

for a positive constant c, then

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2\right) dV_g = \frac{n-1}{n} \int_{M^n} (\xi(f))^2 \, dV_g.$$

Proof. Let ξ be a non-trivial concircular vector field on an n-dimensional compact Riemannian manifold (M^n, g) with potential function σ and concircular function f, and the potential function σ satisfies

$$\Delta\sigma = -nc\sigma \tag{4.4}$$

for a positive constant c. Using Eqs (4.2) and (4.3), we have

$$Ric(\xi, \xi) = -(n-1)\xi(\sigma) = -(n-1)\left[\operatorname{div}(\sigma\xi) - n\sigma^2\right]$$

and integrating the above equation, we confirm

$$\int_{M^n} Ric(\xi, \xi) \, dV_g = n(n-1) \int_{M^n} \sigma^2 dV_g.$$

Using the integral formula in [16], we have for a vector field ζ on (M^n, g)

$$\int_{M^n} \left(Ric(\zeta, \zeta) + \frac{1}{2}\left|\pounds_\zeta g\right|^2 - \|\nabla\zeta\|^2 - (\operatorname{div}\zeta)^2\right) dV_g = 0.$$

Replacing ζ in the above equation by $\nabla\sigma$ and noting that

$$\frac{1}{2}\left|\pounds_\zeta g\right|^2 = 2\|\mathcal{H}_\sigma\|^2, \qquad \|\nabla\zeta\|^2 = \|\mathcal{H}_\sigma\|^2,$$

we conclude

$$\int_{M^n} \left(Ric\,(\nabla\sigma, \nabla\sigma) + \|\mathcal{H}_\sigma\|^2 - (\Delta\sigma)^2 \right) dV_g = 0.$$

Thus, we have

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2 \right) dV_g = \int_{M^n} \left(\frac{n-1}{n}(\Delta\sigma)^2 - Ric\,(\nabla\sigma, \nabla\sigma) \right) dV_g$$

and, inserting Eq (4.4), we reach

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2 \right) dV_g = \int_{M^n} \left(n(n-1)c^2\sigma^2 - Ric\,(\nabla\sigma, \nabla\sigma) \right) dV_g.$$

Now, inserting from Eq (4.4) in the above equation takes us to

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2 \right) dV_g = \int_{M^n} \left(n(n-1)c^2\sigma^2 - f^2 Ric\,(\xi, \xi) \right) dV_g. \tag{4.5}$$

Using Eqs (4.3) and (4.4), we have

$$Ric\,(\xi, \xi) = -(n-1)\xi\,(\sigma) = -(n-1)g\,(\xi, \nabla\sigma) = -(n-1)f\,\|\xi\|^2$$

and using this in Eq (4.5) leads to

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2 \right) dV_g = (n-1)\int_{M^n} \left(nc^2\sigma^2 + f^3\,\|\xi\|^2 \right) dV_g. \tag{4.6}$$

Note that taking the divergence on both sides of Eq (4.4) and using Eq (4.2) gives

$$\Delta\sigma = \xi\,(f) + nf\sigma$$

and combining this with Eq (4.4) allows us to conclude

$$\xi\,(f) = -n\sigma\,(c + f)\,. \tag{4.7}$$

Using Eqs (4.2), (4.4), and (4.7), we compute

$$\begin{aligned} \operatorname{div}\left(f^2\sigma\xi\right) &= \xi\left(f^2\sigma\right) + nf^2\sigma^2 = f^2 g\,(\xi, \nabla\sigma) + 2\sigma f\xi\,(f) + nf^2\sigma^2 \\ &= f^3\,\|\xi\|^2 - 2n\sigma^2 f\,(c + f) + nf^2\sigma^2 \end{aligned}$$

and, integrating the above equation, gives

$$\int_{M^n} f^3\,\|\xi\|^2\,dV_g = \int_{M^n} \left(nf^2\sigma^2 + 2ncf\sigma^2 \right) dV_g.$$

Inserting the above equation into Eq (4.6) leads to

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n} (\Delta\sigma)^2 \right) dV_g = n(n-1) \int_{M^n} \sigma^2 (c+f)^2 \, dV_g$$

and combining it with Eq (4.7) yields

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n} (\Delta\sigma)^2 \right) dV_g = \frac{(n-1)}{n} \int_{M^n} (\xi (f))^2 \, dV_g.$$

□

As a straightforward application of the above result, we have the following theorem.

Theorem 3. *An n-dimensional compact and connected Riemannian manifold (M^n, g) admits a non-trivial concircular vector field ξ with potential function σ and concircular function f such that the potential function σ satisfies*

$$\Delta\sigma = -nc\sigma$$

for a positive constant c, and the concircular function f is a constant along the integral curves of ξ if and only if (M^n, g) is isometric to $S^n(c)$.

Proof. Suppose an n-dimensional compact and connected Riemannian manifold (M^n, g) admits a non-trivial concircular vector field ξ with potential function σ and concircular function f such that the potential function σ satisfies

$$\Delta\sigma = -nc\sigma \tag{4.8}$$

for a positive constant c, and the concircular function f is a constant along the integral curves of ξ. Then, by Proposition 1 we have

$$\int_{M^n} \left(\|\mathcal{H}_\sigma\|^2 - \frac{1}{n} (\Delta\sigma)^2 \right) dV_g = 0. \tag{4.9}$$

The Cauchy–Schwartz inequality implies

$$\|\mathcal{H}_\sigma\|^2 \geq \frac{1}{n} (\Delta\sigma)^2 \tag{4.10}$$

and equality holds if and only if

$$\mathcal{H}_\sigma = \frac{\Delta\sigma}{n} I. \tag{4.11}$$

In view of inequality (4.10) and Eq (4.9), we are ready to conclude the equality

$$\|\mathcal{H}_\sigma\|^2 = \frac{1}{n} (\Delta\sigma)^2$$

and, therefore, Eq (4.11) holds. Combining Eqs (4.8) and (4.11), we arrive at

$$\mathcal{H}_\sigma = -c\sigma I. \tag{4.12}$$

Note that the potential function σ can not be a constant, for if it were a constant, the above equation would imply $\sigma = 0$, which is contrary to the assumption that ξ is a non-trivial concircular vector field. Thus, Equation (4.12) is Obata's equation, and therefore (M^n, g) is isometric to the sphere $S^n(c)$ (cf. [13, 14]).

Conversely, take a constant unit vector $\overrightarrow{a}$ on the Euclidean space $\left(R^{n+1}, \langle , \rangle\right)$ while treating $S^n(c)$ as a hypersurface of $\left(R^{n+1}, \langle , \rangle\right)$ with unit normal ς, shape operator $B = -\sqrt{c}I$, and expressing $\overrightarrow{a}$ as

$$\overrightarrow{a} = \xi + h\varsigma, \tag{4.13}$$

where ξ is tangent to $S^n(c)$ and $h = \left\langle \overrightarrow{a}, \varsigma \right\rangle$. Differentiating equation (4.13) with respect to $X \in \Gamma(TS^n(c))$ and equating the tangential and normal parts, we arrive at

$$\nabla_X \xi = -\sqrt{c}hX, \qquad \nabla h = \sqrt{c}\xi. \tag{4.14}$$

This confirms that ξ is a concircular vector field on $S^n(c)$ with potential function $\sigma = -\sqrt{c}h$, and the second equation gives $\nabla\sigma = -c\xi$. This proves that the concircular function $f = -c$. Moreover, if the potential function $\sigma = 0$, we get $h = 0$, and by the second equation in (4.14), we get $\xi = 0$. In this case, Eq (4.13) confirms $\overrightarrow{a} = 0$, a contradiction to the fact that $\overrightarrow{a}$ is a unit vector. Thus, the potential function $\sigma \neq 0$, that is, the concircular vector field ξ on $S^n(c)$ is non-trivial. Note that $\operatorname{div}\xi = -n\sqrt{c}h = n\sigma$, and, combining it with the equation $\nabla\sigma = -c\xi$, we get $\Delta\sigma = -nc\sigma$ with c a positive constant. Hence, the converse holds. □

5. Conclusions

We have initiated the study of an n-dimensional compact Riemannian manifold (M^n, g) that has an eigenvalue nc for a positive constant c of the Laplace operator the same as the first non-zero eigenvalue of the n-sphere $S^n(c)$ of constant curvature c, and searched for an additional condition under which (M^n, g) is isometric to the sphere $S^n(c)$. The main aim was to find an appropriate smooth function on (M^n, g) that will become the eigenfunction of the Laplace operator with eigenvalue nc as seen in Theorems 1 and 3. Naturally, the scope of this study is quite modest, for instance one can consider an n-dimensional compact Riemannian manifold (M^n, g) that admits a torse forming vector field ξ (cf. [17]). Recall that a torse forming vector field ξ on (M^n, g) satisfies

$$\nabla_X \xi = \sigma X + \omega(X)\xi, \qquad X \in \Gamma(TM^n),$$

where σ is a smooth function defined on M^n called the conformal scalar and ω is a smooth 1-form on M^n called the generating form of the a torse forming vector field ξ. It will be an interesting question to consider torse forming vector field ξ on an n-dimensional compact Riemannian manifold (M^n, g) such that its conformal scalar σ satisfies $\Delta\sigma = -nc\sigma$ for a positive constant c, and find conditions under which (M^n, g) is isometric to $S^n(c)$.

We know that the second non-zero eigenvalue of the sphere $S^n(c)$ is given by $\lambda_2 = 2(n+1)c$, and another aspect of our work could be, if there is a smooth function f on an n-dimensional compact Riemannian manifold (M^n, g) such that $\Delta f = -2(n+1)cf$, that is, (M^n, g) has an eigenvalue same as second non-zero eigenvalue of the sphere $S^n(c)$, to find additional condition on (M^n, g) so that (M^n, g) is isometric to $S^n(c)$.

Author contributions

Sharief Deshmukh: Conceptualization, Methodology, Writing-original draft, Writing-review and editing, Supervision; Amira Ishan: Conceptualization, Methodology, Writing-review and editing; Olga Belova: Formal analysis, Writing-original draft, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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