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Research article

Fourth-order differential equations with neutral delay: Investigation of monotonic and oscillatory features

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Abstract: For fourth-order neutral differential equations (NDE) in the canonical case, we present new relationships between the solution and its corresponding function in two casses: $p < 1$ and $p > 1$. Through these relationships, we discover new monotonic properties for this equation of fourth order. Using the new relationships and properties, we derive some oscillation conditions for the equation under study. By using the Comparison and Ricatti technique, the positive solutions are excluded by providing some conditions. Lastly, we provide examples and review previous theorems from the literature to compare our findings.

Keywords: functional differential equation; neutral; oscillation; fourth-order; canonical
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1. Introduction

Differential equations (DEs) are a powerful tool that can be used to model and understand a wide variety of systems. It plays a crucial role in solving real-world problems in many fields; see [1, 2]. During the 20th century, the rapid progress of science resulted in applications across biology, population studies, chemistry, medicine, social sciences, genetic engineering, economics, and more. Many of the phenomena that appear in these fields are modeled using delay differential equations (DDEs). This led to many disciplines being elevated, and significant discoveries were made with this type of mathematical modeling.

The DEs that have the delayed argument in the highest derivative of the state variable are known as neutral differential equations (NDEs). NDEs have an extremely diverse historical background. In reality, they have a wide range of uses in natural science, as in the process of chemical reactions. Time-delayed transitions may be seen in chemical reaction kinetics, especially in complex processes. These kinetics are described by NDDEs; see [3]. The presence of the delay term in NDDs, which expresses the need for historical information, expands the solution space, and complicates numerical methods. This led to studying the qualitative behavior of these equations because finding closed solutions is often impossible due to their complexity. In recent years, there has been significant research focused on the asymptotic behavior of solutions to DEs; see [4–6]. By examining the asymptotic properties, researchers can forecast the future behavior of systems modeled by DEs from simple physical processes to complicated biological and economic systems. This part of the study supports the practical use of theoretical models in a variety of scientific and engineering domains in addition to aiding in their refinement; see [7]. In recent years, one of the most significant branches of qualitative theory has been oscillation theory; it was introduced in a pioneering paper of Fite; see [8–10]. This theory answers a lot of questions regarding the oscillatory behavior and asymptotic properties of DE solutions.

Finding adequate criteria to guarantee that all DE solutions oscillate while eliminating positive solutions is one of the main objectives of oscillation theory; see [11–13]. One of the main characteristics of oscillation theory is the variety of mathematical and analytical approaches it uses; see [14–16]. Over the past decade, there has been significant progress in the study of the oscillatory properties of DEs; see [17–19]. This interest stems from the fact that comprehending mathematical models and the phenomena they describe is made easier by examining the oscillatory and asymptotic behavior of these models; also, see [20–22]. Moreover, oscillation theory is abundant with fascinating theoretical problems that require the tools of mathematical analysis. In recent decades, oscillation theory has attracted the attention of many researchers, resulting in numerous books and hundreds of studies on several kinds of functional DEs; see [23–25]. Due to the critical roles of NDDE in various fields, such as civil engineering and application-oriented research that can support research with the potential to develop the ship-building, airplane, and rocket industries, the study of the oscillatory properties of these equations has advanced significantly. This makes them extremely important practically in addition to their abundance of interesting analytical problems. For more recent results regarding the oscillatory properties of NDDE solutions; see [26, 27].

1.1. Fourth-order NDE

In this paper, we examine the oscillatory behavior of solutions to the neutral equation

$$\left(r(u)\left(\Omega'''(u)\right)\right)' + q(u)\,x(\theta(u)) = 0, \quad u \geq u_0, \tag{1.1}$$

where $\Omega(u) = x(u) + p(u)\,x(\zeta(u))$. Here, we accomplish our important results by considering the next conditions:

(H_1) $p, \zeta \in C^4([u_0, \infty))$, $0 < p(u) < p_0 < \infty$, $\zeta(u) \leq u$, $\lim_{u\to\infty} \zeta(u) = \infty$, and $\zeta(u)$ invertible;
(H_2) $\theta, q \in C([u_0, \infty))$, $q(u) > 0$, $\theta(u) \leq u$, $\lim_{u\to\infty} \theta(u) = \infty$;
(H_3) $r(u) \in C^1([u_0, \infty))$, $r(u) > 0$ and satisfy

$$\pi_0(u) = \int_{u_0}^{u} \frac{1}{r(\xi)}\mathrm{d}\xi \longrightarrow \infty, \text{as } u \longrightarrow \infty. \tag{1.2}$$

By a solution of (1.1), we mean a function $x \in C^3([u_x, \infty))$ for $u_x \geq u_0$, which has the properties $r(\Omega''') \in C^1([u_x, \infty))$, and satisfies (1.1) on $[u_x, \infty)$. We only take into account the solutions x of (1.1) that satisfy $Sup\{|x(u)| : u \geq u_*\} > 0$ for all $u_* \geq u_x$.

Definition 1.1. *[22] A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on* $[u_0, \infty)$*; otherwise, it is called non-oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.*

1.2. Literature review

The majority of studies have focused on establishing a condition that assures excluding increasing positive solutions using a variety of techniques. The oscillations of higher-order NDE have been investigated by many researchers, and many techniques have been presented for establishing oscillatory criteria for these equations. A lot of research has been conducted regarding the canonical condition; see [28–30]. We will now outline some of the results from previous papers that have contributed to an important part in advancing research on fourth-order NDEs, particularly Moaaz et al. in [31] established criteria for oscillation of solutions of NDDE

$$(r(u)((\Omega(u))''')^{\alpha})' + q(u)x^{\beta}(\theta(u)) = 0, \tag{1.3}$$

by applying two Riccati substitutions in each case of the derivatives of the corresponding function Ω. This criteria guarantees that all solutions oscillate under the canonical condition, where $\beta \geq \alpha$ and $0 \leq p(u) < p_0 < \infty$.

In [32], Bazighifan et al. obtained the Philos type the oscillation criteria to ensure oscillation of solutions of the equation

$$(r(u)((\Omega(u))''')^{\alpha})' + \sum_{i=1}^{k} q_i(u)x^{\beta}(\theta_i(u)) = 0,$$

and by employing the well-known Riccati transformation, they established an asymptotic criterion that enhances and supplements previous results, where $0 \leq p(u) < p_0 < 1$.

To comprehend the asymptotic and oscillatory behavior of solutions to NDEs, it is essential to understand the relationship between the solution x and its associated function Ω. Through this relationship, many researchers discovered several criteria that simplified and enhanced their earlier results. Here, we will present some of the relationships identified from previous research. For $p(u) = p_0$, the conventional relationship:

$$x > (1 - p_0)\Omega, \tag{1.4}$$

is generally employed for second-order equations under the condition (1.2), while in [33, 34] they applied the following relationship

$$x > (1 - p_0\frac{\pi(\zeta(u))}{\pi(u)})\Omega(u), \tag{1.5}$$

in the non-canonical case. Moaaz et al. [35] in the canonical condition, they obtained some oscillation criteria for the next form of the equation

$$(r(u)(\Omega'(u))^{\alpha})' + q(u)x^{\beta}(\zeta(u)) = 0, \tag{1.6}$$

by optimizing the relationship (1.4), where $\alpha,\beta \in \mathbb{Q}^{+}_{odd}$. They provided for $p_0 > 1$ the following relationship

$$x(u) > \Omega(u) \sum_{i=1}^{m/2} \frac{1}{p_0^{[2i-1]}} \left(1 - \frac{1}{p_0} \frac{\pi\left(\zeta^{[-2i]}(u)\right)}{\pi\left(\zeta^{-[2i-1]}(u)\right)} \right),$$

for m is even, while for $p_0 < 1$ they provided the relationship

$$x(u) > \Omega(u)(1 - p_0) \sum_{i=0}^{(m-1)/2} p_0^{[2i]} \left(\frac{1}{p_0} \frac{\pi\left(\zeta^{[2i+1]}(u)\right)}{\pi(u)} \right),$$

for m is odd. In [36], Hassan et al. enhanced the relationship (1.5) by the next one

$$x(u) > \Omega(u) \sum_{i=0}^{(m-1)/2} p_0^{2i} \left(1 - p_0 \frac{\pi\left(\zeta^{[2i+1]}(u)\right)}{\pi^{[2i]}(u)} \right),$$

for m an odd integer, when examined, the oscillatory properties of the equation

$$\left(r(u)\left(\Omega'(u)\right)^{\alpha}\right)' + q(u)\, x^{\alpha}(\zeta_i(u)) = 0,$$

where $\alpha \in \mathbb{Q}^{+}_{odd}$. In [37], Moaaz et al. improved the relationship (1.4) to the following

$$x(u) > (1 - p_0)\, \Omega(u) \sum_{i=0}^{(m-1)/2} p_0^{2i} \left(\frac{\zeta^{[2i+1]}(u) - u_1}{u - u_1} \right)^2,$$

and they obtained a criterion to ensure that there are no Kneser solutions of the equation of third-order NDDE

$$\left(r(u)\left(\Omega''(u)\right)^{\alpha}\right)' + q(u)\, x^{\alpha}(\theta(u)) = 0, \tag{1.7}$$

by comparing (1.7) with a first-order DDE (comparison technique).
Moaaz and Alnafisah [38] examined the oscillatory behavior of solutions to DE

$$\left(r_2(u)\left(r_1(u)\left[x(u) + p(u)\, x(\zeta(u))\right]'\right)'\right)' + q(u)\, x(\theta(u)) = 0,$$

and derived inequalities and relationships, by enhancing the relationship (1.4) considering the two cases $p_0 > 1$ and $p_0 < 1$ without restrictions on the delay functions. Then, using an improved approach, they obtained new monotonic properties for the positive solutions.

Recently, Bohner et al. [39], by considering two cases $\zeta \le u$ and $\zeta \ge u$, studied the NDDE

$$\left(r(u)\left(\Omega'(u)\right)^{\alpha}\right)' + q(u)\, x^{\alpha}(\theta(u)) = 0,$$

and improved the relationship (1.5) by getting the next one

$$x(u) > \Omega(u)(1 - p_0)(1 + H_k(u)),$$

where

$$H_k(u) = \begin{cases} 0 & \text{for } k = 0, \\ \sum_{i=1}^{k} \prod_{j=0}^{2i-1} p\left(\theta^{j}(u)\right) & \text{for } \theta(u) \le u, \\ \sum_{i=1}^{k} \frac{\pi\left(\zeta^{[2i]}(u)\right)}{\pi(u)} \prod_{j=0}^{2i-1} p\left(\theta^{j}(u)\right) & \text{for } \theta(u) \ge u, \end{cases}$$

where $k \in \mathbb{N}$.

In addition, recently, for higher-order some research improved the relationship of (1.4). Among these research, Alnafisah et al. [40] presented the following relationship:

$$x(u) > \sum_{i=0}^{k}\left(\prod_{j=0}^{2i} p\left(\zeta^{[\alpha]}(u)\right)\right)\left[\frac{1}{p\left(\zeta^{[2i]}(u)\right)} - 1\right]\left(\frac{\zeta^{[2i]}(u)}{u}\right)^{(n-2)/\epsilon} \Omega(u),$$

by investigating the asymptotic and oscillatory behaviors of solutions to the NDEs

$$\left(r(u)\left(\Omega^{(n-1)}(u)\right)^{\alpha}\right)' + q(u)\, x^{\alpha}(\theta(u)) = 0,$$

where $n \geq 4$ and α is the ratio of two positive odd integers.

The key to our contribution to this work, We categorize the positive solutions to the studied equation based on the signs of its derivatives. After that, we obtain new monotonic properties in certain cases of positive solutions. Based on these properties, we discover the relationship between the solution and its corresponding function Ω of our Eq (1.1) in the two cases $p_0 > 1$ and $p_0 < 1$. Additionally, we use these new relationships to exclude positive solutions by obtaining some oscillation criteria. The results are illustrated by an example. These results obtained extend and improve upon previous findings in the literature, providing a more comprehensive framework for analyzing these equations.

The paper is organized as follows: In Section 2, we present the fundamental notation and definitions that will be used in our proofs. In Section 3, we present a series of lemmas that enhance the monotonicity properties of nonoscillatory solutions. In Section 4, we establish oscillation criteria for (1.1) as our main result. Lastly, we illustrate our results with an example. In conclusion, briefly discuss what we have done in this research and the results we have obtained.

2. Preliminaries

In this section, we will display the following constants and functions that are used in this paper. The class of all positive non-oscillatory solutions to (1.1) is denoted by the symbol S^{+}.

Notation 2.1. *For any integer $k \geq 0$. In order to present the results, we will need the following notation:*

$$Y^{[0]}(u) = u,\ Y^{[i]}(u) = Y\left(Y^{[i-1]}(u)\right) \text{ and } Y^{[-i]}(u) = Y^{-1}\left(Y^{[-i+1]}(u)\right),$$

for $i = 1, 2, \ldots$.

Lemma 2.1. *[41] Let $F \in C^{\kappa}([u_0, \infty), \mathbb{R}^{+})$ and $F^{(\kappa)}$ be of constant sign, eventually. Then there are a $u_x \geq u_0$ and a $j \in \mathbb{Z}$, $0 \leq j \leq \kappa$, with $\kappa + j$ even for $F^{(\kappa)}(u) \geq 0$, or $\kappa + j$ odd for $F^{(\kappa)}(u) \leq 0$ such that*

$$j \geq 0 \text{ implies that } F^{(l)}(u) > 0 \text{ for } u \geq u_x,\ l = 0, 1, ...j-1.$$

And $j \leq \kappa - 1$ implies that $(-1)^{j+l} F^{(l)}(u) > 0$ for $u \geq u_x$, $l = j, j+1, ...\kappa - 1$.

Lemma 2.2. *[42] Assume F is stated in Lemma 2.1. If $F^{(\kappa-1)}(u)F^{(\kappa)}(u) \leq 0$, eventually, and $\lim_{u\to\infty} F(u) \neq 0$, then, there exists $u_k \in [u_1, \infty)$ for every $\epsilon \in (0, 1)$, such that*

$$F(u) \geq \frac{\epsilon}{(\kappa-1)!} u^{\kappa-1} \left|F^{(\kappa-1)}(u)\right|, \quad \text{for } u \in [u_k, \infty).$$

Lemma 2.3. *[43] If the Eq (1.1) has a solution x that is eventually positive, then*

$$x(u) > \sum_{i=0}^{k}\left(\prod_{j=0}^{2i} p\left(\zeta^{[j]}(u)\right)\right)\left\{\frac{\Omega\left(\zeta^{[2i]}(u)\right)}{p\left(\zeta^{[2i]}(u)\right)} - \Omega\left(\zeta^{[2i+1]}(u)\right)\right\}, \tag{2.1}$$

for $k \geq 0$.

Lemma 2.4. *[41] If* $h \in C^{\kappa}([u_0,\infty),\mathbb{R}^{+})$, $h^{(i)}(u) > 0$ *for* $i = 0,1,2,...,\kappa$, *and* $h^{(\kappa+1)}(u) \leq 0$, *then eventually,*

$$h(u) \geq \frac{1}{\kappa} u h'(u).$$

3. Results and discussion

We will introduce the next lemma that describes the behavior of positive solutions.

Lemma 3.1. *Assume that* $x \in S^{+}$. *Then, eventually, we have two cases for* Ω *eventually:*
Case (1)

$$\Omega(u) > 0,\ \Omega'(u) > 0,\ \Omega''(u) > 0,\ \Omega'''(u) > 0,\ \Omega^{(4)}(u) < 0,$$

Case (2)

$$\Omega(u) > 0,\ \Omega'(u) > 0,\ \Omega''(u) < 0,\ \Omega'''(u) > 0.$$

Proof. Suppose that x is a positive solution of (1.1); we obtain $\Omega^{(4)}(u) \leq 0$ from (1.1). From the Lemma 2.1 Cases (1) and (2), and their derivatives, are obtained. □

Notation 3.1. *We will refer to the symbol* $\aleph_1$ *as the class of all eventually positive solutions of Eq (1.1) whose corresponding function satisfies Case* (1) *and* $\aleph_2$ *as the class of all eventually positive solutions of Eq (1.1) whose corresponding function satisfies Case* (2). *Moreover, we will use the following notation throughout the proof of our lemmas.*

Notation 3.2. *For any positive integer k, we define the functions*

$$\pi_i(u) = \int_{u_0}^{u} \pi_{i-1}(\xi)\,\mathrm{d}\xi,\ i = 1,2,$$

and

$$\rho_1(u,k) = \sum_{i=0}^{k}\left(\prod_{j=0}^{2i} p\left(\zeta^{[j]}(u)\right)\right)\left[\frac{1}{p\left(\zeta^{[2i]}(u)\right)} - 1\right]\frac{\pi_2\left(\zeta^{[2i]}(u)\right)}{\pi_2(u)};$$

$$\widehat{\rho_1}(u;k) = \sum_{i=0}^{k}\left(\prod_{j=0}^{2i} p\left(\zeta^{[j]}(u)\right)\right)\left[\frac{1}{p\left(\zeta^{[2i]}(u)\right)} - 1\right]\left(\frac{\zeta^{[2i]}(u)}{u}\right)^{1/\epsilon};$$

$$\rho_2(u,k) = \sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1} \frac{1}{p\left(\zeta^{[-j]}(u)\right)}\right)\left[\frac{\left(\zeta^{[-2i+1]}(u)\right)^3}{\left(\zeta^{[-2i]}(u)\right)^3} - \frac{1}{p\left(\zeta^{[-2i]}(u)\right)}\right];$$

$$\widehat{\rho_2}(u;k)=\sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1}\frac{1}{p\left(\zeta^{[-j]}(u)\right)}\right)\left[\frac{\left(\zeta^{[-2i+1]}(u)\right)^{1/\epsilon}}{\left(\zeta^{[-2i]}(u)\right)^{1/\epsilon}}-\frac{1}{p\left(\zeta^{[-2i]}(u)\right)}\right],$$

and

$$R(u;k)=\begin{cases}\rho_1(u,k) & \text{for } x\in\aleph_1,\ p<1,\\ \widehat{\rho_1}(u;k) & \text{for } x\in\aleph_2,\ p<1.\end{cases}$$

$$\widetilde{R}(u,k)=\begin{cases}\rho_2(u,k) & \text{for } x\in\aleph_1,\ p>\frac{1}{\zeta(u)},\\ \widehat{\rho_2}(u;k) & \text{for } x\in\aleph_2,\ p>\frac{1}{\zeta(u)}.\end{cases}$$

For every $k\in\mathbb{N}_0$, we assume that

$$\beta_k^*=\liminf_{u\longrightarrow\infty}r(u)\,\pi_2(\theta(u))\,\pi_0(u)\,q(u)\,R(\theta(u);k).$$

It is clear that β_k^* is positive. Our reasoning will usually rely on the obvious truth that a $u_1\geq u_0$ is large enough such that for fixed but arbitrary $\beta_k\in\left(0,\beta_k^*\right)$, we have

$$\beta_k\leq r(u)\,\pi_2(\theta(u))\,\pi_0(u)\,q(u)\,R(\theta(u);k),\tag{3.1}$$

on $[u_1,\infty)$.

3.1. Asymptotic and monotonic properties

This section contains several lemmas regarding the asymptotic properties of solutions that are part of the classes $\aleph_1$ and $\aleph_2$.

Lemma 3.2. *Suppose that $x\in S^+$. If $\Omega''(u)>0$ eventually, then,*
(I) $\Omega(u)\geq\frac{1}{3}u\Omega'(u)$.
However, if $\Omega''(u)<0$, eventually, then
(II) $\Omega(u)\geq\epsilon u\Omega'(u)$, for $\epsilon\in(0,1)$.

Proof. Suppose that x is a positive solution of (1.1) and for $u\geq u_1$, $\Omega''(u)>0$. By applying Lemma 2.4 with $F=\Omega$ and $\kappa\geq 3$, we obtain

$$\Omega(u)\geq\frac{1}{3}u\Omega'(u),$$

which gives (I). Next, for $u\geq u_1$, $\Omega''(u)<0$. Then, $u_2>u_1$ exists, such that

$$\Omega(u)\geq\int_{u_1}^{u}\Omega'(s)\,\mathrm{d}s\geq(u-u_1)\,\Omega'(u)\geq\epsilon u\Omega'(u),$$

which gives (II), for all $\epsilon\in(0,1)$ and $u\geq u_2$. □

3.1.1. Properties in the case where $p(u)\leq p_0<1$

Lemma 3.3. *Let $\beta_0^*>0$ and $x\in\aleph_1$. Then, for u large enough,*
($\boldsymbol{A}_1$) $\lim_{u\to\infty}r(u)\,\Omega'''(u)=\lim_{u\to\infty}\Omega^{(k)}(u)/\pi_{2-k}(u)=0$, $k=0,1,2$;
($\boldsymbol{A}_2$) $\Omega''(u)/\pi_0(u)$ is decreasing;
($\boldsymbol{A}_3$) $\Omega'(u)/\pi_1(u)$ is decreasing;
($\boldsymbol{A}_4$) $\Omega(u)/\pi_2(u)$ is decreasing.

Proof. Assume that $x \in \aleph_1$. From the definition of Ω

$$\Omega(u) = x(u) + p(u)\,x(\zeta(u)),$$

we have

$$x(u) = \Omega(u) - p(u)\,x(\zeta(u)).$$

Since $\Omega(u) > x(u)$, $\Omega'(u) > 0$, and $\zeta(u) \leq u$, we have

$$\begin{aligned} x(u) &\geq \Omega(u) - p(u)\,\Omega(\zeta(u)) \\ &\geq (1 - p(u))\,\Omega(u), \end{aligned} \tag{3.2}$$

which with (1.1) we have

$$(r(u)\,\Omega'''(u))' + q(u)\,(1 - p(\theta(u)))\,\Omega(\theta(u)) \leq 0. \tag{3.3}$$

(A_1): Since we have $\Omega'''(u)$ as a non-increasing and positive function, then

$$\lim_{u \to \infty} r(u)\,\Omega'''(u) = \ell \geq 0.$$

Assume $\ell > 0$; then $r(u)\,\Omega'''(u) \geq \ell > 0$, by integrating three times

$$\Omega(u) \geq \ell \pi_2(u), \quad u \geq u_2 \geq u_1. \tag{3.4}$$

From (3.1) with $R(u;0) = (1 - p(u))$ and (3.3) we obtain

$$(r(u)\,\Omega'''(u))' + \frac{\beta_0}{r(u)\,\pi_2(\theta(u))\,\pi_0(u)}\Omega(\theta(u)) \leq 0. \tag{3.5}$$

From (3.4) into (3.5) we obtain

$$-(r(u)\,\Omega'''(u))' \geq \frac{\ell\beta_0}{r(u)\,\pi_0(u)}. \tag{3.6}$$

By integrating the above inequality from u_3 to u, we obtain

$$r(u_3)\,\Omega'''(u_3) \geq r(u)\,\Omega'''(u) + \beta_0 \ell \ln \frac{\pi_0(u)}{\pi_0(u_3)}, \tag{3.7}$$

which is

$$r(u_3)\,\Omega'''(u_3) \geq \ell + \beta_0 \ell \ln \frac{\pi_0(u)}{\pi_0(u_3)} \to \infty \text{ as } u \longrightarrow \infty,$$

we find that there is a contradiction; therefore, $\ell = 0$. When $x \in \aleph_1$, we have $\Omega(u) \longrightarrow \infty$, $\Omega'(u) \longrightarrow \infty$ as $u \longrightarrow \infty$ and $\Omega''(u)/\pi_0(u) \longrightarrow 0$ as $u \longrightarrow \infty$ such that $\Omega''(u) > 0$ for $k = 2$ is increasing, then by l'Hôpital's rule we find that (A_1) satisfied.
(A_2): As $r(u)\,\Omega'''(u)$ is nonincreasing in $\aleph_1$, we are able to say that

$$\Omega''(u) = \Omega''(u_1) + \int_{u_1}^{u} r(\xi)\,\Omega'''(\xi)\,\frac{1}{r(\xi)}\mathrm{d}\xi$$

$$\geq \quad \Omega''(u_1) + r(u)\,\Omega'''(u)\left(\int_{u_0}^{u} \frac{1}{r(\xi)}\mathrm{d}\xi - \int_{u_0}^{u_1} \frac{1}{r(\xi)}\mathrm{d}\xi\right),$$

that is

$$\begin{aligned}\Omega''(u) &\geq \Omega''(u_1) + r(u)\,\Omega'''(u)\left(\pi_0(u) - \int_{u_0}^{u_1} \frac{1}{r(\xi)}\mathrm{d}\xi\right), \\ &> r(u)\,\Omega'''(u)\,\pi_0(u) + \Omega''(u_1) - r(u)\,\Omega'''(u)\int_{u_0}^{u_1} \frac{1}{r(\xi)}\mathrm{d}\xi.\end{aligned}$$

Since $\Omega''(u) > 0$, and $r(u)\,\Omega'''(u)$ converges to zero by (A_1) , there exists $u_4 > u_3$ such that

$$\Omega''(u_1) - r(u)\,\Omega'''(u)\int_{u_0}^{u_1} \frac{1}{r(\xi)}\mathrm{d}\xi > 0,$$

so we obtain

$$\Omega''(u) > r(u)\,\Omega'''(u)\,\pi_0(u).$$

Therefore,

$$\left(\frac{\Omega''(u)}{\pi_0(u)}\right)' = \frac{r(u)\,\Omega'''(u)\,\pi_0(u) - \Omega''(u)}{r(u)\,\pi_0^2(u)} < 0, \quad u \geq u_4,$$

then $\Omega''(u)/\pi_0$ is decreasing; that proves (A_2).

(A_3): From (A_1) and (A_2), we have $\Omega''(u)/\pi_0$ decreasing and tending to zero, then we find

$$\begin{aligned}\Omega'(u) &= \Omega'(u_4) + \int_{u_4}^{u} \frac{\Omega''(\xi)}{\pi_0(\xi)}\pi_0(\xi)\,\mathrm{d}\xi \\ &\geq \Omega'(u_4) + \frac{\Omega''(u)}{\pi_0(u)}\left(\int_{u_0}^{u} \pi_0(\xi) - \int_{u_0}^{u_4} \pi_0(\xi)\right)\mathrm{d}\xi,\end{aligned}$$

then we obtain

$$\Omega'(u) \geq \frac{\Omega''(u)\,\pi_1(u)}{\pi_0(u)} + \Omega'(u_4) - \frac{\Omega''(u)}{\pi_0(u)}\int_{u_0}^{u_4} \pi_0(\xi)\,\mathrm{d}\xi > \frac{\Omega''(u)\,\pi_1(u)}{\pi_0(u)},\ u \geq u_5,$$

for $u_5 > u_4$. Hence

$$\left(\frac{\Omega'(u)}{\pi_1(u)}\right)' = \frac{\Omega''(u)\,\pi_1(u) - \pi_0(u)\,\Omega'(u)}{\pi_1^2(u)} < 0, \quad u \geq u_5,$$

from that we arrive at (A_3).

(A_4): Likewise, since $\Omega'(u)/\pi_1(u)$ is decreasing and tends to zero, we obtain

$$\begin{aligned}\Omega(u) &= \Omega(u_5) + \int_{u_5}^{u} \frac{\Omega'(\xi)}{\pi_1(\xi)}\pi_1(\xi)\,\mathrm{d}\xi \\ &\geq \Omega(u_5) + \frac{\Omega'(u)}{\pi_1(u)}\left(\int_{u_0}^{u} \pi_1(\xi) - \int_{u_0}^{u_5} \pi_1(\xi)\right),\end{aligned}$$

then we arrive at

$$\Omega(u) \geq \frac{\Omega'(u)}{\pi_1(u)}\pi_2(u) + \Omega(u_5) - \frac{\Omega'(u)}{\pi_1(u)}\int_{u_0}^{u_5} \pi_1(\xi)\,\mathrm{d}\xi > \frac{\Omega'(u)\,\pi_2(u)}{\pi_1(u)},\ u \geq u_6,$$

for $u_6 > u_5$, so
$$\left(\frac{\Omega(u)}{\pi_2(u)}\right)' = \frac{\Omega'(u)\pi_2(u) - \pi_1(u)\Omega(u)}{\pi_2^2(u)} < 0,\ u \geq u_6,$$
that proves (A_4). □

Lemma 3.4. *Let $\beta_0^* > 0$ and $x \in \aleph_1$. Then, the corresponding function Ω eventually satisfies*
$$(r(u)\Omega'''(u))' + q(u)\rho_1(\theta(u);k)\Omega(\theta(u)) \leq 0. \tag{3.8}$$

Proof. By using the facts that $\zeta^{[2i+1]}(u) \leq \zeta^{[2i]}(u) < u$ and $\Omega'(u) > 0$, we find that
$$\Omega\left(\zeta^{[2i]}(u)\right) \geq \Omega\left(\zeta^{[2i+1]}(u)\right).$$
By utilizing $(\Omega(u)/\pi_2(u))' < 0$, we arrive at
$$\frac{\Omega\left(\zeta^{[2i]}(u)\right)}{\pi_2\left(\zeta^{[2i]}(u)\right)} \geq \frac{\Omega(u)}{\pi_2(u)},$$
this leads to
$$\Omega\left(\zeta^{[2i]}(u)\right) \geq \frac{\pi_2\left(\zeta^{[2i]}(u)\right)}{\pi_2(u)}\Omega(u).$$
From this inequality in (2.1), we obtain
$$x(u) \geq \Omega(u)\sum_{i=0}^{k}\left(\prod_{j=0}^{2i} p\left(\zeta^{[j]}(u)\right)\right)\left\{\frac{1}{p\left(\zeta^{[2i]}(u)\right)} - 1\right\}\frac{\pi_2\left(\zeta^{[2i]}(u)\right)}{\pi_2(u)} > \Omega(u)\rho_1(u;k).$$
From this and (1.1) we obtain
$$(r(u)\Omega'''(u))' + q(u)\rho_1(\theta(u);k)\Omega(\theta(u)) \leq 0.$$
This is completes the proof. □

Lemma 3.5. *Let $\beta_k^* > 0$ for some $k \in N$, and $x \in \aleph_1$. Then, for u large enough, (A_1)–(A_4) (in Lemma 3.3) hold.*

Proof. Replacing inequality (3.5) in Lemma 3.2 (II) and proceeding in the same manner, we obtain properties in (A_1)–(A_4). □

Lemma 3.6. *Assume that $\beta_k^* > 0$ and $x \in \aleph_2$. If $\Omega''(u) < 0$, eventually, then*
$$(r(u)\Omega'''(u))' + q(u)\widehat{\rho_1}(\theta(u),k)\Omega(\theta(u)) \leq 0. \tag{3.9}$$

Proof. Suppose that $x \in \aleph_2$. For $u \geq u_1$, $\Omega''(u) < 0$. From the facts $\Omega'(u) > 0$ and (II), we obtain
$$\Omega\left(\zeta^{[2i]}(u)\right) \geq \Omega\left(\zeta^{[2i+1]}(u)\right),$$
and
$$\Omega\left(\zeta^{[2i]}(u)\right) \geq \frac{\left(\zeta^{[2i]}(u)\right)^{1/\epsilon}}{u^{1/\epsilon}}\Omega(u).$$

Then, Eq (2.1) becomes

$$x(u) > \Omega(u) \sum_{i=0}^{k} \left(\prod_{j=0}^{2i} p\left(\zeta^{[j]}(u)\right) \right) \left\{ \frac{1}{p\left(\zeta^{[2i]}(u)\right)} - 1 \right\} \left(\frac{\zeta^{[2i]}(u)}{u} \right)^{1/\epsilon} > \widehat{\rho_1}(u;k)\Omega(u), \quad for\ k \geq 0,$$

which together with (1.1) gives (3.9). □

Lemma 3.7. *Let $\beta_k^* > 0$ for some $k \in \mathbb{N}$ and $x \in \aleph_1$. Then,*

$$(r(u)\Omega'''(u))' + q(u)R(\theta(u);k)\Omega(\theta(u)) \leq 0. \tag{3.10}$$

Proof. Follows from Lemmas 3.3, 3.6, and 3.4 with (1.1) gives (3.10). □

3.1.2. Properties in the case where $p(u) \geq p_0 > 1$

Lemma 3.8. *Let $x \in \aleph_1 \cup \aleph_2$. Then there exists k such that*

$$x(u) > \sum_{i=1}^{k} \left(\prod_{j=0}^{2i-1} \frac{1}{p\left(\zeta^{[-j]}(u)\right)} \right) \left[\Omega\left(\zeta^{[-2i+1]}(u)\right) - \frac{1}{p\left(\zeta^{[-2i]}(u)\right)} \Omega\left(\zeta^{[-2i]}(u)\right) \right]. \tag{3.11}$$

Proof. From the definition $\Omega(u)$, we find that

$$\begin{aligned} p\left(\zeta^{-1}(u)\right)x(u) &= \Omega\left(\zeta^{-1}(u)\right) - x\left(\zeta^{-1}(u)\right) \\ &= \Omega\left(\zeta^{-1}(u)\right) - \frac{1}{p\left(\zeta^{[-2]}(u)\right)} \left[\Omega\left(\zeta^{[-2]}(u)\right) - x\left(\zeta^{[-2]}(u)\right)\right], \end{aligned}$$

hence, we obtain

$$\begin{aligned} p\left(\zeta^{-1}(u)\right)x(u) &= \Omega\left(\zeta^{-1}(u)\right) - \Omega\left(\zeta^{[-2]}(u)\right) \prod_{i=2}^{2} \frac{1}{p\left(\zeta^{[-i]}(u)\right)} \\ &\quad + \prod_{i=2}^{3} \frac{1}{p\left(\zeta^{[-i]}(u)\right)} \left[\Omega\left(\zeta^{[-3]}(u)\right) - x\left(\zeta^{[-3]}(u)\right)\right]. \end{aligned}$$

After k steps we arrive at there exists a k such that

$$x(u) > \sum_{i=1}^{k} \left(\prod_{j=0}^{2i-1} \frac{1}{p\left(\zeta^{[-j]}(u)\right)} \right) \left[\Omega\left(\zeta^{[-2i+1]}(u)\right) - \frac{1}{p\left(\zeta^{[-2i]}(u)\right)} \Omega\left(\zeta^{[-2i]}(u)\right) \right]. \tag{3.12}$$

This concludes the proof. □

Lemma 3.9. *Let $x \in \aleph_1 \cup \aleph_2$. Then there exists k such that (1.1) implies,*

$$(r(u)\Omega'''(u))' + q(u)\widetilde{R}(\theta(u),k)\Omega(\theta(u)) \leq 0. \tag{3.13}$$

When

$$x(u) > \widetilde{R}(u,k)\Omega(u).$$

Proof. Suppose that $x \in \aleph_1$. From the fact that $\Omega'(u) > 0$ and from (I) in Lemma 3.2, we obtain

$$\zeta^{[-2i]}(u) \geq \zeta^{[-2i+1]}(u),$$

and

$$\Omega\left(\zeta^{[-2i+1]}(u)\right) \geq \frac{\left(\zeta^{[-2i+1]}(u)\right)^3}{\left(\zeta^{[-2i]}(u)\right)^3}\Omega\left(\zeta^{[-2i]}(u)\right).$$

Then, from Lemma (3.8), Eq (3.11) becomes

$$x(u) > \sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1}\frac{1}{p\left(\zeta^{[-j]}(u)\right)}\right)\left[\frac{\left(\zeta^{[-2i+1]}(u)\right)^3}{\left(\zeta^{[-2i]}(u)\right)^3}\Omega\left(\zeta^{[-2i]}(u)\right) - \frac{1}{p\left(\zeta^{[-2i]}(u)\right)}\Omega\left(\zeta^{[-2i]}(u)\right)\right],$$

in view of the fact that $\Omega\left(\zeta^{[-2i]}(u)\right) \geq \Omega(u)$, the above inequality becomes

$$x(u) > \Omega(u)\sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1}\frac{1}{p\left(\zeta^{[-j]}(u)\right)}\right)\left[\frac{\left(\zeta^{[-2i+1]}(u)\right)^3}{\left(\zeta^{[-2i]}(u)\right)^3} - \frac{1}{p\left(\zeta^{[-2i]}(u)\right)}\right] > \Omega(u)\rho_2(u,k),$$

which is together with (1.1), we obtain

$$\left(r(u)\Omega'''(u)\right)' + q(u)\Omega(\theta(u))\rho_2(\theta(u),k) \leq 0. \tag{3.14}$$

Suppose that $x \in \aleph_2$. For $u \geq u_1$, $\Omega''(u) < 0$. From the facts $\Omega'(u) > 0$ and (II) in Lemma 3.2, we obtain

$$\zeta^{[-2i]}(u) \geq \zeta^{[-2i+1]}(u),$$

and

$$\Omega\left(\zeta^{[-2i+1]}(u)\right) \geq \frac{\left(\zeta^{[-2i+1]}(u)\right)^{1/\epsilon}}{\left(\zeta^{[-2i]}(u)\right)^{1/\epsilon}}\Omega\left(\zeta^{[-2i]}(u)\right).$$

Then, from Lemma (3.8), Eq (3.11) becomes

$$x(u) > \Omega(u)\sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1}\frac{1}{p\left(\zeta^{[-j]}(u)\right)}\right)\left[\frac{\left(\zeta^{[-2i+1]}(u)\right)^{1/\epsilon}}{\left(\zeta^{[-2i]}(u)\right)^{1/\epsilon}} - \frac{1}{p\left(\zeta^{[-2i]}(u)\right)}\right] > \Omega(u)\hat{\rho}_2(u;k),$$

which is together with (1.1), we obtain

$$\left(r(u)\Omega'''(u)\right)' + q(u)\hat{\rho}_2(\theta(u);k)\Omega(\theta(u)) \leq 0. \tag{3.15}$$

Followed by (3.14) and (3.15) gives (3.13). □

4. Oscillation theorems

In the following theorem, we will obtain oscillation criteria for Eq (1.1) in the case where $p_0 < 1$. For clarity, we will define that:

$$M_1(u) = q(u)\rho_1(\theta(u); k)\left(\frac{\theta(u)}{u}\right)^3;$$

$$D(u) = \frac{\epsilon}{2}\frac{u^2}{r(u)};$$

$$M_2(u) = \int_u^\infty \left(\frac{1}{r(\xi)}\int_\xi^\infty q(s)\widehat{\rho_1}(\theta(s), k)\left(\frac{\theta(s)}{s}\right)^{1/\epsilon} ds\right) d\xi.$$

Theorem 4.1. *Assume that there is $\epsilon \in (0, 1)$ and $p_0 < 1$ such that, if $x \in \aleph_1$, then*

$$\liminf_{u\longrightarrow\infty} \frac{1}{\widetilde{M}_1(u)} \int_u^\infty D(s)\widetilde{M}_1^2(s)\, ds \geq \frac{1}{4}, \tag{4.1}$$

and, if $x \in \aleph_2$, then

$$\liminf_{u\longrightarrow\infty} \frac{1}{\widetilde{M}_2(u)} \int_u^\infty \widetilde{M}_2^2(u)\, ds \geq \frac{1}{4}, \tag{4.2}$$

where

$$\widetilde{M}_1(u) = \int_u^\infty M_1(s)\, ds,\ \widetilde{M}_2(u) = \int_u^\infty M_2(s)\, ds.$$

Then, Eq (1.1) is oscillatory.

Proof. Assume the contrary, that (1.1) has a non-oscillatory solution x. Then, there exists a $u_1 \geq u_0$ such that $x(u) > 0$, $x(\theta(u)) > 0$, and $x(\zeta(u)) > 0$ for $u \geq u_1$. There are two possible classes from Lemma 3.1: $\aleph_1$ and $\aleph_2$. Assume $\aleph_1$ holds. From (3.8) in Lemma 3.4 we have

$$(r(u)\Omega'''(u))' + q(u)\Omega(\theta(u))\rho_1(\theta(u); k) \leq 0. \tag{4.3}$$

Introduce Riccati substitutions

$$\omega(u) = \frac{r(u)\Omega'''(u)}{\Omega(u)}, \quad u \geq u_1. \tag{4.4}$$

We observe that $\omega(u) > 0$ for $u \geq u_1$; by differentiating (4.4), we obtain

$$\omega'(u) = \frac{(r(u)(\Omega'''(u)))'}{\Omega(u)} - \frac{r(u)\Omega'''(u)\Omega(u)'}{\Omega^2(u)},$$

from (4.3) we obtain

$$\omega'(u) \leq -q(u)\rho_1(\theta(u); k)\frac{\Omega(\theta(u))}{\Omega(u)} - \frac{r(u)\Omega'''(u)\Omega(u)'}{\Omega^2(u)}. \tag{4.5}$$

From (I) in Lemma 3.2, we have

$$\Omega(u) \geq \frac{1}{3}u\Omega'(u),$$

then we obtain

$$\frac{\Omega(\theta(u))}{\Omega(u)} \geq \frac{\theta^3(u)}{u^3}, \tag{4.6}$$

by applying Lemma 2.2 for every $\epsilon \in (0, 1)$, we obtain

$$\Omega'(u) \geq \frac{\epsilon}{2}u^2\Omega'''(u). \tag{4.7}$$

Therefore, from (4.5)–(4.7), we obtain

$$\omega'(u) \leq -q(u)\rho_1(\theta(u);k)\left(\frac{\theta(u)}{u}\right)^3 - \frac{\epsilon}{2}\frac{r^2(u)u^2(\Omega'''(u))^2}{r(u)\Omega^2(u)},$$

which is

$$\omega'(u) + M_1(u) + D(u)\omega^2(u) \leq 0.$$

By integrating the above inequality, from u to∞, we obtain

$$\omega(u) \geq \int_u^\infty M_1(s)\,\mathrm{d}s + \int_u^\infty D(s)\omega^2(s)\,\mathrm{d}s,$$

since $\omega > 0$ and $\omega' < 0$, we have

$$\omega(u) \geq \widetilde{M}_1(u) + \int_u^\infty D(s)\omega^2(s)\,\mathrm{d}s,$$

which is

$$\frac{\omega(u)}{\widetilde{M}_1(u)} \geq 1 + \frac{1}{\widetilde{M}_1(u)}\int_u^\infty \widetilde{M}_1^2(s)D(s)\left(\frac{\omega(s)}{\widetilde{M}_1(s)}\right)^2\mathrm{d}s. \tag{4.8}$$

Let $\lambda = \inf_{u\geq u_*}\omega(u)/\widetilde{M}_1(u)$, then from (4.8) we notice

$$\lambda \geq 1 + (\lambda)^2,$$

which contradicts that$\lambda \geq 1$ in (4.8).

Assume $\aleph_2$ holds. From (3.9) in Lemma 3.6 we have

$$(r(u)\Omega'''(u))' + q(u)\widehat{\rho}_1(\theta(u),k)\Omega(\theta(u)) \leq 0. \tag{4.9}$$

By introducing Riccati substitutions

$$\varpi(u) = \frac{\Omega'(u)}{\Omega(u)}, \quad u \geq u_1. \tag{4.10}$$

Integrating (4.9) from u to ∞, we obtain

$$r(u)\Omega'''(u) \geq \int_u^\infty q(s)\widehat{\rho_1}(\theta(s),k)\Omega(\theta(s)). \tag{4.11}$$

From (II) in Lemma 3.2, we have

$$\Omega(u) \geq \epsilon u \Omega'(u),$$

hence

$$\Omega(\theta(u)) \geq \frac{\theta^{1/\epsilon}(u)}{u^{1/\epsilon}}\Omega(u). \tag{4.12}$$

By using (4.12) in (4.11), we obtain

$$r(u)\Omega'''(u) \geq \Omega(u)\int_u^\infty q(s)\widehat{\rho_1}(\theta(s),k)\left(\frac{\theta(u)}{u}\right)^{1/\epsilon} ds,$$

by integrating again this inequality from u to ∞, we obtain

$$\Omega''(u) \leq -\Omega(u)\int_u^\infty \left(\frac{1}{r(\xi)}\int_\xi^\infty q(s)\widehat{\rho_1}(\theta(s),k)\left(\frac{\theta(s)}{s}\right)^{1/\epsilon} \mathrm{d}s\right)\mathrm{d}\xi, \tag{4.13}$$

by differentiating $\varpi(u)$ in (4.10), we obtain

$$\begin{aligned}\varpi'(u) &= \frac{\Omega''(u)}{\Omega(u)} - \left(\frac{\Omega'(u)}{\Omega(u)}\right)^2 \\ &\leq -\varpi^2(u) - \int_u^\infty \left(\frac{1}{r(\xi)}\int_\xi^\infty q(s)\widehat{\rho_1}(\theta(s),k)\left(\frac{\theta(s)}{s}\right)^{1/\epsilon} \mathrm{d}s\right)\mathrm{d}\xi\end{aligned}$$

for $u \geq u_2$.

Then, from (4.13) and (4.10), we obtain

$$\varpi'(u) + M_2(u) + \varpi^2(u) \leq 0,$$

by integrating the above inequality from u to ∞

$$\varpi(u) \geq \int_u^\infty M_2(s)\,\mathrm{d}s + \int_u^\infty \varpi^2(s)\,\mathrm{d}s,$$

from this we obtain

$$\frac{\varpi(u)}{\widetilde{M}_2(u)} \geq 1 + \frac{1}{\widetilde{M}_2(u)}\int_u^\infty \widetilde{M}_2^2(s)\left(\frac{\varpi(s)}{\widetilde{M}_2(s)}\right)^2 \mathrm{d}s.$$

The rest of the proof is done as in the case $\aleph_1$. As a result, the theorem is established. □

The next theorem provides two oscillation conditions for Eq (1.1), which require that $p_0 > 1$. These conditions are established by using a comparison method with a first-order equation, under the next constraints: $g(u) \leq \theta(u)$.

Theorem 4.2. *Let $p_0 > 1$ hold, if there exists $\epsilon \in (0,1)$ and k such that ρ_2 and $\hat{\rho}_2$ are defined, such that the two first-order DDEs*

$$\psi'(u) + \epsilon\theta^3(u)\frac{q(u)\rho_2(\theta(u);k)}{3!r(\theta(u))}\psi(\theta(u)) = 0, \tag{4.14}$$

and when $g(u) \leq \theta(u)$

$$\nu'(u) + \epsilon g(u)\int_u^\infty \frac{E(\xi)}{r(\xi)}\nu(g(u))\,\mathrm{d}\xi = 0, \tag{4.15}$$

where

$$E(u) = \int_u^\infty q(\xi)\hat{\rho}_2(\theta(\xi);k)\mathrm{d}\xi,$$

are oscillatory, then (1.1) is oscillatory.

Proof. Assume the contrary, that (1.1) has a non-oscillatory solution x. Then, there exists a $u_1 \geq u_0$ such that $x(u) > 0$, $x(\theta(u)) > 0$, and $x(\zeta(u)) > 0$ for $u \geq u_1$. There are two possible classes from Lemma 3.1: $\aleph_1$ and $\aleph_2$. Assume $\aleph_1$ holds. Then from Eq (3.14) in Lemma 3.9, we have

$$(r(u)\Omega'''(u))' + q(u)\Omega(\theta(u))\rho_2(\theta(u),k) \leq 0. \tag{4.16}$$

From (4.7) and (4.16) we notice

$$\Omega(\theta(u)) \geq \frac{\epsilon\theta^3(u)}{3!}\Omega'''(\theta(u)),$$

$$(r(u)\Omega'''(u))' + \epsilon\theta^3(u)\frac{r(\theta(u))q(u)\rho_2(\theta(u),k)}{3!r(\theta(u))}\Omega'''(\theta(u)) \leq 0.$$

If we set $\psi(u) = r(u)\Omega'''(u)$, then $\psi(u)$ is a positive solution of the first-order delay differential inequality

$$\psi'(u) + \epsilon\theta^3(u)\frac{q(u)\rho_2(\theta(u);k)}{3!r(\theta(u))}\psi(\theta(u)) \leq 0. \tag{4.17}$$

By [44, Theorem 1], the DDE (4.14) also has a positive solution; this leads to a contradiction.
Assume $\aleph_2$ holds. Then from Eq (3.15) in Lemma 3.9, we have

$$(r(u)\Omega'''(u))' + q(u)\hat{\rho}_2(\theta(u),k)\Omega(\theta(u)) \leq 0.$$

Since $g(u) \leq \theta(u)$, we find

$$(r(u)\Omega'''(u))' + q(u)\hat{\rho}_2(\theta(u),k)\Omega(g(u)) \leq 0, \tag{4.18}$$

by integrating (4.18) from u to ∞ we obtain

$$-r(u)\Omega'''(u) + \Omega(g(u))\int_u^\infty q(\xi)\hat{\rho}_2(\theta(\xi);k)\mathrm{d}\xi \leq 0, \tag{4.19}$$

which is

$$-\Omega'''(u)+\frac{E(u)\,\Omega(g(u))}{r(u)}\leq 0. \tag{4.20}$$

Integrating (4.20) again from u to ∞, we obtain

$$\Omega''(u)+\Omega(g(u))\int_{u}^{\infty}\frac{E(\xi)}{r(\xi)}\mathrm{d}\xi\leq 0. \tag{4.21}$$

From (II) in Lemma 3.2, we have

$$\Omega(u)\geq \epsilon u\Omega'(u). \tag{4.22}$$

Let $\nu(u)=\Omega'(u)$ and by using (4.22) in (4.21), we find that $\nu(u)$ is a positive solution of the inequality

$$\nu'(u)+\epsilon g(u)\int_{u}^{\infty}\frac{E(\xi)}{r(\xi)}\nu(g(u))\,\mathrm{d}\xi\leq 0.$$

But according to [44, Theorem 1], the condition (4.15) also has a positive solution $\nu(u)$; this leads to a contradiction. □

Corollary 4.1. *If* $p_0>1$ *such that*

$$\liminf_{u\to\infty}\int_{\theta(u)}^{u}q(\xi)\frac{\epsilon\theta^3(\xi)\,\rho_2(\theta(\xi);k)}{3!r(\theta(\xi))}\mathrm{d}\xi\geq\frac{1}{\mathrm{e}}, \tag{4.23}$$

and

$$\liminf_{u\to\infty}\int_{g(u)}^{u}\epsilon g(\vartheta)\left(\int_{\vartheta}^{\infty}\frac{E(\xi)}{r(\xi)}\mathrm{d}\xi\right)\mathrm{d}\vartheta\geq\frac{1}{\mathrm{e}}. \tag{4.24}$$

Then (1.1) is oscillatory.

5. Applications and discussion

Example 5.1. *Let the fourth-order NDE*

$$\left([x(u)+p_0x(\delta u)]'''\right)'+\frac{q_0}{u^4}x(\beta u)=0,\quad u>1, \tag{5.1}$$

where p_0, q_0 *are positive and* $\delta,\beta\in(0,1)$. *We note that* $\zeta^{-1}(\theta(u))=\frac{\beta u}{\delta}$, $r(u)=1$. *As a result, it is clear that* $\zeta^{[-2i]}(u)=\delta^{-2i}u$, $\zeta^{[-2i+1]}(u)=\delta^{-2i+1}u$. *Thus, for* $p_0>1$, *we define*

$$\rho_2(u;k)=\left[\delta^3-\frac{1}{p_0}\right]\sum_{i=1}^{k}p_0^{-2i},\ \hat{\rho}_2(u;k)=\left[\delta^{1/\epsilon}-\frac{1}{p_0}\right]\sum_{i=1}^{k}p_0^{-2i},$$

for $k>0$. *Then, the condition (4.23) in Corollary 4.1, becomes*

$$\liminf_{u\to\infty}\frac{\epsilon q_0\beta^3}{6}\left(\delta^3-\frac{1}{p_0}\right)\sum_{i=1}^{k}p_0^{-2i}\int_{\theta(u)}^{u}\frac{1}{u}\mathrm{d}\xi\geq\frac{1}{\mathrm{e}},$$

$$q_0 \geq \frac{6p_0}{\epsilon \mathrm{e}\beta^3 \left(p_0\delta^3 - 1\right) \ln \frac{1}{\beta} \sum\limits_{i=1}^{k} p_0^{-2i}}. \tag{5.2}$$

Also, we have

$$E(u) = q_0 \left[\delta^{1/\epsilon} - \frac{1}{p_0}\right] \left(\frac{1}{3u^3}\right) \sum_{i=1}^{k} p_0^{-2i}.$$

Then, the condition (4.24) simplifies to

$$\frac{\epsilon\beta q_0}{6} \left(\delta^{1/\epsilon} - \frac{1}{p_0}\right) \sum_{i=1}^{k} p_0^{-2i} \ln \frac{1}{\beta} \geq \frac{1}{\mathrm{e}},$$

which is

$$q_0 > \frac{6}{\epsilon\beta\mathrm{e} \left(\delta^{1/\epsilon} - \frac{1}{p_0}\right) \sum\limits_{i=1}^{k} p_0^{-2i} \ln \frac{1}{\beta}}. \tag{5.3}$$

Using Corollary 4.1, Eq (5.1) is oscillatory if

$$q_0 > Max \left\{ \frac{6p_0}{\epsilon \mathrm{e}\beta^3 \left(p_0\delta^3 - 1\right) \ln \frac{1}{\beta} \sum\limits_{i=1}^{k} p_0^{-2i}}, \frac{6}{\epsilon\beta\mathrm{e} \left(\delta^{1/\epsilon} - \frac{1}{p_0}\right) \sum\limits_{i=1}^{k} p_0^{-2i} \ln \frac{1}{\beta}} \right\}. \tag{5.4}$$

Example 5.2. *Now, consider the fourth-order NDE*

$$\left(\left[x(u) + p_0 x(\lambda u)\right]'''\right)' + \frac{q_0}{u^4} x(\mu u) = 0, \quad u > 1, \tag{5.5}$$

where p_0, q_0 *are positive,* $\lambda, \mu \in (0,1)$, $r(u) = 1$, $\theta(u) = \mu u$, $q(u) = \frac{q_0}{u^4}$, $p(u) = p_0$ *and* $\zeta(u) = \lambda u$. *As a result, it is clear that* $\pi_0(u) = u$, $\pi_1(u) = \frac{u^2}{2}$, $\pi_2(u) = \frac{u^3}{6}$ *and* $\pi_2\left(\zeta^{[2i]}(u)\right) = \frac{\left(\lambda^{2i}u\right)^3}{6}$, $\pi_0\left(\zeta^{[2i]}(u)\right) = \lambda^{2i}u$, *then for* $p_0 < 1$, *we define*

$$\rho_1(u;k) = \left(\frac{1}{p_0} - 1\right) \sum_{i=0}^{k} p_0^{2i}\lambda^{6i}, \ \widehat{\rho_1}(u;k) = \left(\frac{1}{p_0} - 1\right) \sum_{i=0}^{k} \left(p_0^{2i}\right) \left(\lambda^{2i}\right)^{1/\epsilon};$$

$$M_1(u) = \frac{q_0}{u^4}\mu^3 \left(\frac{1}{p_0} - 1\right) \sum_{i=0}^{k} p_0^{2i}\lambda^{6i};$$

$$M_2(u) = q_0 (\mu)^{1/\epsilon} \left(\frac{1}{6u^2}\right) \left(\frac{1}{p_0} - 1\right) \sum_{i=0}^{k} \left(p_0^{2i}\right) \left(\lambda^{2i}\right)^{1/\epsilon};$$

$$D(u) = \frac{\epsilon u^2}{2};$$

$$\widetilde{M}_1(u) = q_0\mu^3 \left(\frac{1}{3u^3}\right) \left(\frac{1}{p_0} - 1\right) \sum_{i=0}^{k} p_0^{2i}\lambda^{6i};$$

and

$$\widetilde{M}_2(u) = \mu^{1/\epsilon} q_0 \left(\frac{1}{6u}\right)\left(\frac{1}{p_0} - 1\right)\sum_{i=0}^{k}\left(p_0^{2i}\right)\left(\lambda^{2i}\right)^{1/\epsilon}.$$

By applying condition (4.1) in Theorem 4.1 when $x \in \aleph_1$, *we see*

$$\liminf_{u \longrightarrow \infty} \frac{\epsilon 3u^3}{18}\left(q_0 \mu^3 \left(\frac{1}{p_0} - 1\right)\sum_{i=0}^{k} p_0^{2i}\lambda^{6i}\right)\int_u^\infty \left(\frac{1}{s^4}\right) \mathrm{d}s \geq \frac{1}{4},$$

this implies

$$q_0 \frac{\epsilon}{18}\mu^3 \left(\frac{1}{p_0} - 1\right)\sum_{i=0}^{k} p_0^{2i}\lambda^{6i} \geq \frac{1}{4},$$

which get all solutions of (5.5) are oscillatory if

$$q_0 > \frac{18}{4\epsilon\mu^3\left(\frac{1}{p_0} - 1\right)\sum\limits_{i=0}^{k} p_0^{2i}\lambda^{6i}}. \tag{5.6}$$

By applying condition (4.2) in Theorem 4.1 when $x \in \aleph_2$, *we see*

$$q_0 \frac{\mu^{1/\epsilon}}{6}\left(\frac{1}{p_0} - 1\right)\sum_{i=0}^{k}\left(p_0^{2i}\right)\left(\lambda^{2i}\right)^{1/\epsilon} \geq \frac{1}{4},$$

which obtain all solutions of (5.5) are oscillatory if

$$q_0 > \frac{6}{4\mu^{1/\epsilon}\left(\frac{1}{p_0} - 1\right)\sum\limits_{i=0}^{k}\left(p_0^{2i}\right)\left(\lambda^{2i}\right)^{1/\epsilon}}. \tag{5.7}$$

Remark 5.1. *Consider a special case of Eq (5.5) in the form*

$$([x(u) + 0.5x(0.9u)]''')' + \frac{q_0}{u^4}x(\mu u) = 0, \quad u > 1, \tag{5.8}$$

when talking $\epsilon = p_0 = 0.5$. *By using our conditions (5.6) and (5.7), then Eq (5.8) is oscillatory if*

$$q_0 > \frac{9}{(\mu)^3 \sum\limits_{i=0}^{3}(0.5)^{2i}(0.9)^{6i}}. \tag{5.9}$$

By applying [32, Corollary 1], we see that

$$S_1(t) = kq_0(1 - p_0)\mu^3, \; S_2(t) = \frac{kq_0(1 - p_0)\mu}{4\mu t}.$$

Then, by choosing $\theta(t) = t^4$ *and* $\phi(t) = t^2$, *we find that (5.8) is oscillatory if*

$$q_0 > Max\left\{\frac{32}{3\mu^3}, \frac{4}{\mu}\right\}. \tag{5.10}$$

While [4, Theorem 2.1] ensures the oscillation of Eq (5.8) if

$$q_0 > \frac{9}{\mu^3}, \tag{5.11}$$

Figures 1 illustrates the efficiency of the conditions (5.9) in studying the oscillation of the solutions of (5.8) for values of $\mu \in (0,1)$ *. Thus, our results present a better criterion for oscillation.*

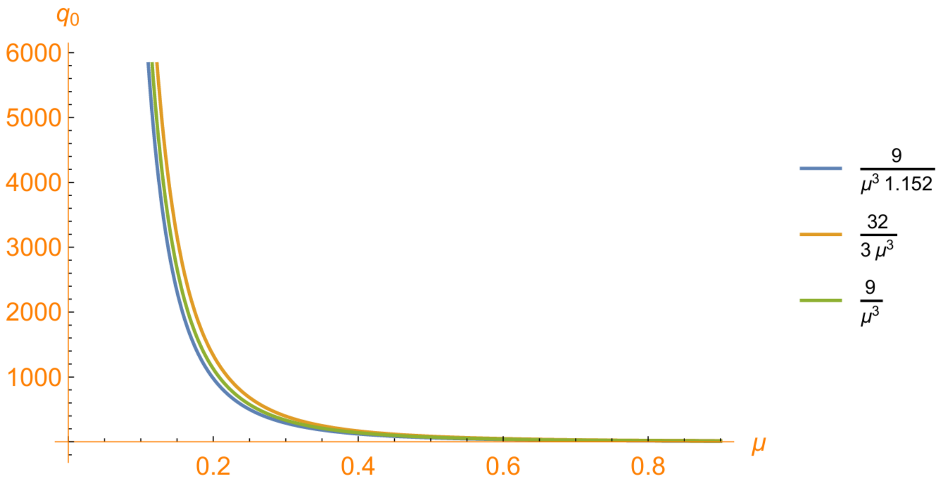

Figure 1. Comparison of the oscillation conditions of (5.8).

6. Conclusions

Finding conditions that exclude each of the cases of the derivatives of the positive solution is often the foundation of the idea of establishing oscillation criteria for differential equations. This study examines the oscillatory behavior of fourth-order NDEs in the canonical case. The relationship between the solution and the corresponding function is vital to the oscillation theory of NDEs. Therefore, we improve these relationships by applying the modified monotonic properties of positive solutions. The conditions that we obtained using these relationships subsequently proved that there are no positive solutions in categories $\aleph_1$ and $\aleph_2$. Then, using the newly deduced relationships and properties, we employed a number of approaches by using different techniques, including recatti and comparison techniques, to develop a set of oscillation criteria. Additionally, we provided examples that illustrated and clarified the importance of our results; they were compared with some previous results in the literature. In the future, we can try to develop new conditions that ensure that every solution of (1.1) is oscillatory in the noncanonical case.

Author contributions

H. Salah, C. Cesarano, and E. M. Elabbasy: Conceptualization, methodology, writing-original draft; M. Anis, S. S. Askar, and S. A. M. Alshamrani: Formal analysis, investigation, writing-review and editing. All authors have read and approved the final version of the manuscript for publication

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Conflict of interest

The authors declare that there is no conflict of interest.

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