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Research article

The L_1-induced norm analysis for linear multivariable differential equations

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Abstract: In this paper, we consider the L_1-induced norm analysis for linear multivariable differential equations. Because such an analysis requires integrating the absolute value of the associated impulse response on the infinite-interval $[0, \infty)$, this interval was divided into $[0, H)$ and $[H, \infty)$, with the truncation parameter H. The former was divided into M subintervals with an equal width, and the kernel function of the relevant input/output operator on each subinterval was approximated by a pth order polynomial with $p = 0, 1, 2, 3$. This derived to an upper bound and a lower bound on the L_1-induced norm for $[0, H)$, with the convergence rate of $1/M^{p+1}$. An upper bound on the L_1-induced norm for $[H, \infty)$ was also derived, with an exponential order of H. Combining these bounds led to an upper bound and a lower bound on the original L_1-induced norm on $[0, \infty)$, within the order of $1/M^{p+1}$. Furthermore, the l_1-induced norm of difference equations was tackled in a parallel fashion. Finally, numerical studies were given to demonstrate the overall arguments.

Keywords: linear multivariable differential equations; L_1-induced norm; l_1-induced norm; convergence analysis; operator approximations
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1. Introduction

Quantitative evaluation of the input/output relationship in differential equations has been regarded as one of the important issues in applied mathematics and control engineering. Depending on the characteristics of the differential equations considered, various systems norms can be taken. For example, the L_2-induced (or the l_2-induced) norm is used in [1, 2] to address energy-bounded disturbances, and the L_∞-induced (or the l_∞-induced) norm is taken in [3–5] to deal with peak-bounded disturbances. Subsequently, the induced norm is considered in a mixed fashion from L_2 to L_∞ as in [6–8].

However, these three norms do not reflect some real-world problems of the maximizing fuel efficiency [9], the population management [10,11], and so on. These problems can be tackled by taking the L_1-induced (or the l_1-induced) norm since it corresponds to the ratio of the total sums between the input and the output. In this line, this induced norm is widely used for practical systems such as switched systems [12, 13], Markov jump systems [14], positive systems [15, 16], and so on. Here, it should be remarked that the L_1-induced norm of positive systems is analytically obtained as in [16], this induced norm cannot be exactly computed even for general linear time-invariant (LTI) differential equations since it involves integrating the absolute value of a matrix exponential function on the infinite interval $[0, \infty)$.

With respect to computing the L_1-induced norm, an adaptive algorithm is introduced in [17], but arguments are confined to single variable, strictly proper differential equations, and no clear extension to the case of multivariable proper differential equations is provided in that study. More importantly, the associated convergence order in [17] is limited to $1/M^2$ in terms of the approximation parameter M. For hybrid continuous/discrete-time differential/difference equations (i.e., sampled-data systems), a method for computing the L_1-induced norm is recently developed in [18], but the convergence rate is $1/M$.

Motivated by the above facts, we develop methods for computing both the L_1-induced norm and the l_1-induced norm of linear multivariable differential and difference equations, with the convergence orders higher than $1/M^2$. We first derive a closed-form expression of the L_1-induced norm and clarify that the 1-norm of a matrix exponential function should be integrated on the infinite interval $[0, \infty)$. This interval is then divided into $[0, H)$ and $[H, \infty)$ for a sufficiently large H. The L_1-induced norm on $[0, H)$ is treated in a relatively rigorous fashion, while that on $[H, \infty)$ is considered in a rough fashion. More precisely, an upper bound and a lower bound on the former are obtained by dividing the interval $[0, H)$ into M subintervals with an equal width together with applying a pth order Taylor expansion to the kernel function of the matrix exponential function with $p = 0, 1, 2, 3$, while an upper bound on the latter is only derived. Here, we show that the gap between the upper and lower bounds on the L_1-induced norm on $[0, H)$ tends to 0 at the rate of $1/M^{p+1}$ and the upper bound on the L_1-induced norm on $[H, \infty)$ converges to 0 in an exponential order of H. Combining these bounds leads to an upper bound and a lower bound on the original L_1-induced norm, and their gap is ensured to converge to 0 within the order of $1/M^{p+1}$. These arguments are extended to the l_1 analysis of discrete-time difference equations in a parallel fashion. The contributions of this paper over the existing studies [16–18] are also summarized in Table 1.

Table 1. Comparsion between studies on the L_1-induced analysis.

	Considered system	Convergence
[16]	Positive continuous-time systems	Exact result
[17]	Almost strictly proper continuous-time systems	$1/M^2$
[18]	Hybrid continuous/discrete-time systems	$1/M$
This paper	Multivariable proper continuous-time and discrete-time systems	$1/M^4$

This paper is organized as follows. In Section 2, we derive a tractable representation, i.e., a closed-form expression, of the L_1-induced norm of linear multivariable differential equations, as a preliminary to computing this induced norm. The major results of this paper, i.e., the methods for computing an upper bound and a lower bound on the L_1-induced norm and the corresponding convergence proof, are provided in Section 3. The method for computing the l_1-induced norm of linear multivariable difference equations is introduced in Section 4. A numerical example is provided in Section 5 to demonstrate the theoretical validity and the practical effectiveness of the overall arguments developed in this paper. Finally, the notations used in this paper are shown in Table 2.

Table 2. Notations used in the paper.

Notations	Definitions
$\mathbb{R}_1^{\nu}$	The Banach space of ν-dimensional real vectors equipped with the 1-norm
$\mathbb{R}_1^{\nu_1 \times \nu_2}$	The Banach space of $\nu_1 \times \nu_2$-dimensional real matrices equipped with the 1-norm
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$ with the set of positive integers $\mathbb{N}$
$\mathrm{row_i}(\cdot)$	ith row of a matrix $(\cdot)$
$\mathrm{col_i}(\cdot)$	ith column of a matrix $(\cdot)$
$\lvert\cdot\rvert_1$	The 1-norm of a matrix, i.e., $\max_{1\le j\le n} \sum_{i=1}^{m} \lvert T_{ij}\rvert$, for a $T \in \mathbb{R}_1^{m\times n}$
$\lVert\cdot\rVert_{L_1}$	The $L_1[0, T)$ norm of a function with $T = H, H/M$, or ∞, i.e., $\int_{t=0}^{T} \lvert f(t)\rvert_1 dt$, for a real vector-valued function $f(\cdot)$ with $\lVert f\rVert_{L_1} < \infty$
$\lVert\cdot\rVert_{l_1}$	The l_1 norm of a sequence, i.e., $\sum_{k=0}^{\infty} \lvert v[k]\rvert_1$, for a real vector-valued sequence $v[k]$ $(k \in \mathbb{N}_0)$
$\lVert\cdot\rVert_{X/X}$	The induced norm of an operator defined as $\lVert\mathcal{T}\rVert_{X/X} := \sup_{f\neq 0} \lVert\mathcal{T} f\rVert_X / \lVert f\rVert_X$

2. Tractable description of the L_1-induced norm

Let us consider the continuous-time (CT) linear time-invariant (LTI) system Σ_C with differential and algebraic equations given by

$$\Sigma_C : \begin{cases} \dot{x} = Ax + Bw \\ y = Cx + Dw \end{cases}, \tag{2.1}$$

where $x(t) \in \mathbb{R}_1^n$ is the state, $w(t) \in \mathbb{R}_1^{n_w}$ is the input and $y(t) \in \mathbb{R}_1^{n_y}$ is the output. With assuming the zero initial condition of x, i.e., $x(0) = 0$, the output of Σ_C is described by

$$y(t) = (\mathbf{G}_C w)(t) := \int_0^t Ce^{A(t-s)} Bw(s)\, ds + Dw(t), \tag{2.2}$$

where $\mathbf{G}_C$ corresponds to the operator describing the input/output relation of Σ_C.

On the basis of the above operator-based representation, we denote the L_1-induced norm of Σ_C by $\lVert\mathbf{G}_C\rVert_{L_1/L_1}$, and it is defined as

$$\lVert\mathbf{G}_C\rVert_{L_1/L_1} := \sup_{w\neq 0} \frac{\lVert y\rVert_{L_1}}{\lVert w\rVert_{L_1}} = \sup_{\lVert w\rVert_{L_1}\le 1} \lVert y\rVert_{L_1}. \tag{2.3}$$

The matrix A is assumed to be Hurwitz stable (i.e., all the eigenvalues of A are located in the open left half-plane) for the L_1-induced norm $\lVert\mathbf{G}_C\rVert_{L_1/L_1}$ to be bounded and well-defined, throughout the paper.

On the other hand, it is a non-trivial task to compute $\|\mathbf{G}\|_{L1/L_1}$ in terms of (2.3) (and (2.2)) because it should be required to consider every w with $\|w\|_{L_1} \leq 1$. To alleviate such a difficulty, we provide the following lemma associated with a more tractable expression of $\|\mathbf{G}_C\|_{L_1/L_1}$ by extending the arguments in [19] without considering the feed through term D.

Lemma 1. *The L_1-induced norm $\|\mathbf{G}_C\|_{L_1/L_1}$ can be described by*

$$\|\mathbf{G}_C\|_{L_1/L_1} = \max_{1\leq j\leq n_w} \sum_{i=1}^{n_y} f_{ij}, \tag{2.4}$$

where

$$f_{ij} := \int_0^\infty \left|(Ce^{At}B)_{ij}\right| dt + \left|D_{ij}\right|, \tag{2.5}$$

and $(\cdot)_{ij}$ means the (i, j)th element of $(\cdot)$.

Proof. Let us denote the vector ∞-norm by $|\cdot|_\infty$ and denote the L_∞ norm of a function equipped with the vector ∞-norm for the spatial space by $\|\cdot\|_{L_\infty}$. We then see from [19, 20] that

$$\begin{aligned}
\|y\|_{L_1} &= \sup_{\|g\|_{L_\infty}=1} \int_0^\infty y^T(t)g(t)dt \\
&= \sup_{\|g\|_{L_\infty}=1} \int_0^\infty \left(\int_0^t w^T(s)\left(Ce^{A(t-s)}B + \delta(t-s)D\right)^T ds\right) g(t)dt \\
&= \sup_{\|g\|_{L_\infty}=1} \int_0^\infty w^T(s)\left(\int_s^\infty \left(Ce^{A(t-s)}B + \delta(t-s)D\right)^T g(t)dt\right) ds \\
&\leq \sup_{\|g\|_{L_\infty}=1} \int_0^\infty \left|w^T(s)\right|_\infty \cdot \left|\left(\int_s^\infty \left(Ce^{A(t-s)}B + \delta(t-s)D\right)^T g(t)dt\right)\right|_\infty ds \\
&\leq \int_0^\infty \left|w^T(s)\right|_\infty ds \cdot \sup_{\|g\|_{L_\infty}=1} \sup_{0\leq s<\infty} \left|\int_s^\infty (Ce^{A(t-s)}B + \delta(t-s)D)^T g(t)dt\right|_\infty \\
&= \|w\|_{L_1} \cdot \sup_{\|g\|_{L_\infty}=1} \|v\|_{L_\infty},
\end{aligned} \tag{2.6}$$

where $\delta(t)$ is the Dirac-delta function and

$$v(s) := \int_s^\infty \left(Ce^{A(t-s)}B + \delta(t-s)D\right)^T g(t)dt. \tag{2.7}$$

From [21, 22], we note that

$$\begin{aligned}
\sup_{\|g\|_{L_\infty}=1} \|v\|_{L_\infty} &= \max_{1\leq i\leq n_w} \sum_{j=1}^{n_y} \int_0^\infty \left(|(Ce^{At}B)^T_{ij}| + |\delta(t)D^T_{ij}|\right) dt \\
&= \max_{1\leq i\leq n_w} \sum_{j=1}^{n_y} \int_0^\infty \left(|(Ce^{At}B)_{ji}| + |\delta(t)D_{ji}|\right) dt = \max_{1\leq j\leq n_w} \sum_{i=1}^{n_y} f_{ij}.
\end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8) clearly implies that

$$\|\mathbf{G}_C\|_{L_1/L_1} \leq \max_{1\leq j\leq n_w} \sum_{i=1}^{n_y} f_{ij}. \tag{2.9}$$

For $j = 1, \ldots, n_w$, let us next take $\hat{w}_j$ by

$$\hat{w}_j(t) := \delta(t)e_j, \tag{2.10}$$

where e_j is the jth standard basis in $\mathbb{R}_1^{n_w}$. If we denote the corresponding output by $\hat{y}_j$, then we see that

$$\begin{aligned}\|\hat{y}_j\|_{L_1} &= \int_0^\infty \left|Ce^{At}Be_j + \delta(t)De_j\right|_1 dt = \int_0^\infty |Ce^{At}Be_j|_1 dt + |De_j|_1 \\ &= \sum_{i=1}^{n_y}\left(\int_0^\infty |(Ce^{At}B)_{ij}|dt + |D_{ij}|\right) = \sum_{i=1}^{n_y} f_{ij}.\end{aligned} \tag{2.11}$$

This together with the fact that $\|\hat{w}_j\|_{L_1} = 1$ for $j = 1, \ldots, n_w$ leads to that

$$\max_{1\leq j\leq n_w} \sum_{i=1}^{n_y} f_{ij} \leq \|\mathbf{G}_C\|_{L_1/L_1}. \tag{2.12}$$

Combining (2.9) and (2.12) completes the proof. □

It would be worthwhile to note that f_{ij} given by (2.5) coincides with the L_1 norm of the impulse response for the single-input/single-output (SISO) system obtained from replacing C, B and D in (2.1) with $\mathrm{row}_i(C)$, $\mathrm{col}_j(B)$ and D_{ij}, respectively. To put it another way, the assertions in Lemma 1 can also be interpreted as showing that the worst input w^* for achieving the L_1-induced norm defined as (2.3) corresponds to the impulse signal (i.e., the Dirac-delta function $\delta(t)$). Thus, it is not required to rigorously consider the input w for computing the L_1-induced norm $\|\mathbf{G}_C\|_{L_1/L_1}$ when we employ the arguments in Lemma 1.

However, it is difficult to directly treat the first term of the right-hand side (RHS) in (2.5), although we can obtain a tractable representation of $\|\mathbf{G}_C\|_{L_1/L_1}$ in Lemma 1. In connection with this, we introduce a truncation idea by which the interval $[0, \infty)$ taken in f_{ij} is divided into $[0, H)$ and $[H, \infty)$ with a sufficiently large parameter H. More precisely, let us introduce $f_{ij}^{[H-]}$ and $f_{ij}^{[H+]}$ defined respectively as

$$f_{ij}^{[H-]} := \int_0^H \left|(Ce^{At}B)_{ij}\right| dt + \left|D_{ij}\right|, \qquad f_{ij}^{[H+]} := \int_H^\infty \left|(Ce^{At}B)_{ij}\right| dt. \tag{2.13}$$

Then, it readily follows from $f_{ij} = f_{ij}^{[H-]} + f_{ij}^{[H+]}$ that

$$f_{ij}^{[H-]} \leq f_{ij} \leq f_{ij}^{[H-]} + f_{ij}^{[H+]}. \tag{2.14}$$

From the point of view of (2.14), we can lead to the following lemma associated with deriving an upper bound and a lower bound on the L_1-induced norm $\|\mathbf{G}_C\|_{L_1/L_1}$ given by (2.4).

Lemma 2. *The inequality*

$$|F^{[H-]}|_1 \le \|\mathbf{G}_C\|_{L_1/L_1} \le \left|F^{[H-]} + F^{[H+]}\right|_1 \tag{2.15}$$

holds, where

$$F^{[H-]} := \begin{bmatrix} f_{11}^{[H-]} & \cdots & f_{1n_w}^{[H-]} \\ \vdots & \ddots & \vdots \\ f_{n_y 1}^{[H-]} & \cdots & f_{n_y n_w}^{[H-]} \end{bmatrix}, \quad F^{[H+]} := \begin{bmatrix} f_{11}^{[H+]} & \cdots & f_{1n_w}^{[H+]} \\ \vdots & \ddots & \vdots \\ f_{n_y 1}^{[H+]} & \cdots & f_{n_y n_w}^{[H+]} \end{bmatrix}. \tag{2.16}$$

From the stability assumption on A, it could be expected that $F^{[H+]}$ converges to 0 as the truncation parameter H becomes larger. Hence, it would be reasonable to take a sufficiently large H for computing $\|\mathbf{G}_C\|_{L_1/L_1}$ if we can explicitly compute $F^{[H-]}$. In this sense, the following section provides two methods for computing an upper bound on each entry of $F^{[H+]}$ and an upper bound and a lower bound on each entry of $F^{[H-]}$, respectively, and derives relevant convergence rates.

3. Computing upper and lower bounds on $\|\mathbf{G}_C\|_{L_1/L_1}$

As mentioned at the end of the preceding section, $F^{[H+]}$ converges to 0 by taking H larger. Thus, it might be useful to compute $F^{[H-]}$ as accurately as possible but $F^{[H+]}$ is treated in a relatively simple fashion when we take a sufficiently large H. With respect to this, we derive an upper bound and a lower bound on each entry of $F^{[H-]}$ while an upper bound on each entry of $F^{[H+]}$ is only obtained.

3.1. Upper bound on each entry of $F^{[H+]}$

Even though $F^{[H+]}$ is very close to 0 when H is large enough, its direct computation is also a non-trivial task. Thus, this subsection aims at obtaining an upper bound on $f_{ij}^{[H+]}$ and showing a relevant convergence property. To this end, we note from the stability assumption on A that there should exist an $q \in \mathbb{N}_0$ ensuring $\left|e^{Aq}\right|_1 < 1$ since $e^{At} \to 0$ $(t \to \infty)$. Using the q, we can derive the following Lemma.

Lemma 3. *For an q with $|e^{Aq}|_1 < 1$, the following inequality holds.*

$$f_{ij}^{[H+]} \le \frac{\left|\mathrm{row}_i(C)e^{AH}\right|_1}{1 - |e^{Aq}|_1} \frac{(e^{|A|_1 q} - 1)}{|A|_1} |\mathrm{col}_j(B)|_1 := f_{ij,\mathrm{U}}^{[H+]}. \tag{3.1}$$

Furthermore, $f_{ij,\mathrm{U}}^{[H+]}$ converges to 0 regardless of the choice of q with an exponential order of H.

Proof. Note that

$$\begin{aligned} f_{ij}^{[H+]} &= \int_H^\infty \left|\mathrm{row}_i(C)e^{At}\mathrm{col}_j(B)\right|_1 dt \\ &= \int_0^\infty \left|\mathrm{row}_i(C)e^{AH} \cdot e^{At}\mathrm{col}_j(B)\right|_1 dt \\ &= \int_0^q \left(\left|\mathrm{row}_i(C)e^{AH} \cdot e^{At}\mathrm{col}_j(B)\right|_1 + \left|\mathrm{row}_i(C)e^{AH} \cdot e^{Aq} \cdot e^{At}\mathrm{col}_j(B)\right|_1 + \cdots \right) dt \\ &\le \left|\mathrm{row}_i(C)e^{AH}\right|_1 \left(1 + |e^{Aq}|_1 + |e^{2Aq}|_1 + \cdots\right) \int_0^q \left|e^{At}\mathrm{col}_j(B)\right|_1 dt \end{aligned}$$

$$\leq \frac{\left|\mathrm{row}_i(C)e^{AH}\right|_1}{1-|e^{Aq}|_1}\frac{(e^{|A|_1 q}-1)}{|A|_1}|\mathrm{col}_j(B)|_1 := f_{ij,\mathrm{U}}^{[H+]}. \tag{3.2}$$

The second assertion follows by the fact that $|\mathrm{row}_i(C)e^{AH}|_1$ tends to 0 with an exponential order of H. This completes the proof. □

From Lemma 3, an upper bound on $f_{ij}^{[H+]}$ can be computed when H is fixed and an appropriate q with $|e^{Aq}|_1 < 1$ is selected. We next consider $F_{\mathrm{U}}^{[H+]} \in \mathbb{R}_1^{n_y \times n_w}$ defined as

$$F_{\mathrm{U}}^{[H+]} := \begin{bmatrix} f_{11,\mathrm{U}}^{[H+]} & \cdots & f_{1n_w,\mathrm{U}}^{[H+]} \\ \vdots & \ddots & \vdots \\ f_{n_y 1,\mathrm{U}}^{[H+]} & \cdots & f_{n_y n_w,\mathrm{U}}^{[H+]} \end{bmatrix}, \tag{3.3}$$

where the subscript 'U' stands for the upper bound, and this will be used for computing $\|\mathbf{G}_C\|_{L_1/L_1}$ instead of $F^{[H+]}$ in Subsection 3.3.

3.2. *Upper and lower bounds on each entry of* $F^{[H-]}$

To determine each entry of $F^{[H-]}$ as accurately as possible through its upper and lower bounds, we consider dividing the interval $[0, H)$ into M subintervals with an equal width. To put it another way, we note that

$$\begin{aligned} f_{ij}^{[H-]} &= \int_0^h \left|(Ce^{At}B)_{ij}\right| dt + \int_h^{2h} \left|(Ce^{At}B)_{ij}\right| dt + \cdots + \int_{(M-1)h}^{H} \left|(Ce^{At}B)_{ij}\right| dt + \left|D_{ij}\right| \\ &= \left\|\mathrm{row_i(C)_M}\mathbf{B}_{\mathrm{j}}(\mathrm{t})\right\|_{L_1} + \left|D_{ij}\right| \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} &h := H/M, \quad A_d := e^{Ah}, \quad \mathbf{B}_j(t) := e^{At}\mathrm{col}_j(B) \\ &\mathrm{row}_i(C)_M := \left[\mathrm{row}_i(C)^T \quad (\mathrm{row}_i(C)A_d)^T \quad \cdots \quad (\mathrm{row}_i(C)A_d^{M-1})^T\right]^T. \end{aligned} \tag{3.5}$$

Because this allows us to deal with the interval $[0, h)(= [0, H/M)$ smaller than the original interval $[0, H)$, an approximate scheme on the former interval could be expected to lead to an approximation error smaller than that on the latter interval.

With this in mind, we develop a kernel approximation approach to $\mathbf{B}_j(t)$ in (3.4) by introducing $\mathbf{B}_{j,p}^{[\alpha]}(t)$ defined as

$$\mathbf{B}_{j,p}^{[\alpha]}(t) = \sum_{i=0}^{p} e^{A\alpha h} \cdot \frac{A^i (t-\alpha h)^i}{i!}\mathrm{col_j(B)} \tag{3.6}$$

for $p \in \{0, 1, 2, 3\}$ and $\alpha \in [0, 1]$. This $\mathbf{B}_{j,p}^{[\alpha]}(t)$ corresponds to the pth order Taylor expansion of e^{At} in $\mathbf{B}_j(t)$ around $t = \alpha h$. Replacing the $\mathbf{B}_j(t)$ with $\mathbf{B}_{j,p}^{[\alpha]}$ in (3.4) derives the approximate treatment of $f_{ij}^{[H-]}$ given by

$$f_{ij,M,p}^{[H-,\alpha]} := \left\|\mathrm{row}_i(C)_M \cdot \mathbf{B}_{j,p}^{[\alpha]}(t)\right\|_{L_1} + |D_{ij}|. \tag{3.7}$$

Regarding an approximation error in terms of $f_{ij,M,p}^{[H-,\alpha]}$ defined as (3.7), we provide the following lemma.

Lemma 4. *For $p \in \{0,1,2,3\}$, the inequality*

$$f_{ij,M,p}^{[H-,\alpha]} - \frac{k_{ij,M,p}^{[H-,\alpha]}}{M^{p+1}} \leq f_{ij}^{[H-]} \leq f_{ij,M.p}^{[H-,\alpha]} + \frac{k_{ij,M,p}^{[H-,\alpha]}}{M^{p+1}} \tag{3.8}$$

holds, where

$$k_{ij,M,p}^{[H-,\alpha]} := \frac{H^{p+2}}{M} \cdot c_\alpha \cdot \left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1}\right|_1 \cdot e^{|A|_1 h} \cdot \left|\mathrm{col}_j(B)\right|_1 \tag{3.9}$$

with c_α given by

$$c_\alpha := \frac{\alpha^{(p+2)} + (1-\alpha)^{(p+2)}}{(p+2)!}. \tag{3.10}$$

Furthermore, $k_{ij,M,p}^{[H-,\alpha]}$ has a uniform upper bound $\hat{k}_{ij,p}^{[H-,\alpha]}$ with respect to M given by

$$\hat{k}_{ij,p}^{[H-,\alpha]} := H^{p+2} \cdot c_\alpha \cdot \left|\mathrm{row}_i(C) e^{A\alpha h} A^{p+1}\right|_1 \cdot e^{|A|_1 H} \cdot \left|\mathrm{col}_j(B)\right|_1. \tag{3.11}$$

Proof. In terms of the triangular inequality, we first note that

$$\begin{aligned} \left|f_{ij}^{[H-]} - f_{ij,M,p}^{[H-,\alpha]}\right| &= \left| \left\|\mathrm{row}_i(C)_M \mathbf{B}_j(t)\right\|_{L_1} - \left\|\mathrm{row}_i(C)_M \mathbf{B}_{j,p}^{[\alpha]}(t)\right\|_{L_1} \right| \\ &\leq \left\|\mathrm{row}_i(C)_M \mathbf{B}_j(t) - \mathrm{row}_i(C)_M \mathbf{B}_{j,p}^{[\alpha]}(t)\right\|_{L_1}. \end{aligned} \tag{3.12}$$

Next, it readily follows from the pth order Taylor expansion of $\exp(At)$ around $t = \alpha h$ that

$$\begin{aligned} &\left\|\mathrm{row}_i(C)_M \mathbf{B}_j(t) - \mathrm{row}_i(C)_M \mathbf{B}_{j,p}^{[\alpha]}(t)\right\|_{L_1} \\ &= \int_0^h \left|\mathrm{row}_i(C)_M \cdot \left(\sum_{i=0}^{\infty} e^{A\alpha h} \frac{A^i (t-\alpha h)^i}{i!} - \sum_{i=0}^{p} e^{A\alpha h} \frac{A^i (t-\alpha h)^i}{i!}\right) \mathrm{col}_j(B)\right|_1 dt \\ &= \int_0^h \left|\mathrm{row}_i(C)_M \left(\sum_{i=p+1}^{\infty} e^{A\alpha h} \frac{A^i (t-\alpha h)^i}{i!}\right) \mathrm{col}_j(B)\right|_1 dt \\ &= \int_0^h \left|\mathrm{row}_i(C)_M \left(\sum_{i=0}^{\infty} e^{A\alpha h} \frac{A^{i+p+1} (t-\alpha h)^{i+p+1}}{(i+p+1)!}\right) \mathrm{col}_j(B)\right|_1 dt \\ &= \int_0^h \left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1} \left(\sum_{i=0}^{\infty} \frac{A^i (t-\alpha h)^{i+p+1}}{(i+p+1)!}\right) \mathrm{col}_j(B)\right|_1 dt \\ &\leq \left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1}\right|_1 \cdot \int_0^h \left(\sum_{i=0}^{\infty} \frac{|A|_1^i |t-\alpha h|^{i+p+1}}{(i+p+1)!}\right) dt \left|\mathrm{col}_j(B)\right|_1 \\ &= \left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1}\right|_1 \cdot \left(\int_0^{\alpha h} \sum_{i=0}^{\infty} \frac{|A|_1^i (\alpha h - t)^{i+p+1}}{(i+p+1)!} dt + \int_{\alpha h}^h \sum_{i=0}^{\infty} \frac{|A|_1^i (t-\alpha h)^{i+p+1}}{(i+p+1)!} dt\right) \cdot \left|\mathrm{col}_j(B)\right|_1 \\ &= \left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1}\right|_1 \cdot \left(\sum_{i=0}^{\infty} \frac{|A|_1^i (\alpha h)^{i+p+2}}{(i+p+2)!} + \sum_{i=0}^{\infty} \frac{|A|_1^i ((1-\alpha)h)^{i+p+2}}{(i+p+2)!}\right) \cdot \left|\mathrm{col}_j(B)\right|_1 \end{aligned}$$

$$\leq \left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1}\right|_1 \cdot h^{p+2} \cdot e^{|A|_1 h} \cdot \frac{\alpha^{p+2} + (1-\alpha)^{p+2}}{(p+2)!} \cdot \left|\mathrm{col}_j(B)\right|_1 = \frac{k_{ij,M,p}^{[H-,\alpha]}}{M^{p+1}}. \tag{3.13}$$

Combining (3.12) and (3.13) establishes the first assertion. The second assertion follows by

$$\frac{1}{M}\left|\mathrm{row}_i(C)_M e^{A\alpha h} A^{p+1}\right|_1 \leq \frac{1}{M}\sum_{k=0}^{M-1}\left|\mathrm{row}_i(C)(A_d)^k e^{A\alpha h} A^{p+1}\right|_1$$

$$\leq \left|\mathrm{row}_i(C) e^{A\alpha h} A^{p+1}\right|_1 \cdot \frac{1}{M}\sum_{k=0}^{M-1} e^{|A|_1 kh} \leq \left|\mathrm{row}_i(C) e^{A\alpha h} A^{p+1}\right|_1 \cdot e^{|A|_1 (M-1)h}. \tag{3.14}$$

This completes the proof. □

Remark 1. *The rationale behind taking p as* $p \in \{0, 1, 2, 3\}$ *is related with explictly computing* $f_{ij,M,p}^{[H-,\alpha]}$ *in* (3.7)*, although the arguments in Lemma 4 are equivalently extended for the case of* $p \geq 4$*. To put it another way, it is well-known that no exact solution can be obtained for a general pth order polynomial with* $p \geq 4$*, and thus it is quite difficult to exactly compute* $f_{ij,M,p}^{[H-,\alpha]}$ *for* $p \geq 4$*. In contrast, the exact solution formulae exist for general pth order polynomials with* $p = 0, 1, 2, 3$*, and thus* $f_{ij,M,p}^{[H-,\alpha]}$ *considered in* (3.7) *can be exactly obtained.*

We can easily obtain from Lemma 4 an upper bound and a lower bound on each entry of $F^{[H-]}$. With these values in mind, let us consider the matrices $F_{M,p,\mathrm{U}}^{[H-,\alpha]}$ and $F_{M,p,\mathrm{L}}^{[H-,\alpha]}$ whose $(i, j)$the elements are given by $f_{ij,M,p}^{[H-,\alpha]} + \frac{k_{ij,M,p}^{[H-,\alpha]}}{M^{p+1}}$ and $f_{ij,M,p}^{[H-,\alpha]} - \frac{k_{ij,M,p}^{[H-,\alpha]}}{M^{p+1}}$, respectively, i.e.,

$$F_{M,p,\mathrm{U}}^{[H-,\alpha]} := \begin{bmatrix} f_{11,M,p}^{[H-,\alpha]} + \frac{k_{11,M,p}^{[H-,\alpha]}}{M^{p+1}} & \cdots & f_{1n_w,M,p}^{[H-,\alpha]} + \frac{k_{1n_w,M,p}^{[H-,\alpha]}}{M^{p+1}} \\ \vdots & \ddots & \vdots \\ f_{n_y1,M,p}^{[H-,\alpha]} + \frac{k_{n_y1,M,p}^{[H-,\alpha]}}{M^{p+1}} & \cdots & f_{n_yn_w,M,p}^{[H-,\alpha]} + \frac{k_{n_yn_w,M,p}^{[H-,\alpha]}}{M^{p+1}} \end{bmatrix}, \tag{3.15}$$

$$F_{M,p,\mathrm{L}}^{[H-,\alpha]} := \begin{bmatrix} f_{11,M,p}^{[H-,\alpha]} - \frac{k_{11,M,p}^{[H-,\alpha]}}{M^{p+1}} & \cdots & f_{1n_w,M,p}^{[H-,\alpha]} - \frac{k_{1n_w,M,p}^{[H-,\alpha]}}{M^{p+1}} \\ \vdots & \ddots & \vdots \\ f_{n_y1,M,p}^{[H-,\alpha]} - \frac{k_{n_y1,M,p}^{[H-,\alpha]}}{M^{p+1}} & \cdots & f_{n_yn_w,M,p}^{[H-,\alpha]} - \frac{k_{n_yn_w,M,p}^{[H-,\alpha]}}{M^{p+1}} \end{bmatrix}. \tag{3.16}$$

Here, the subscripts 'U' and 'L' stand for the upper bound and the lower bound, respectively, and $F_{M,p,\mathrm{U}}^{[H-,\alpha]}$ and $F_{M,p,\mathrm{L}}^{[H-,\alpha]}$ will be used for computing $\|\mathbf{G}_C\|_{L_1/L_1}$ instead of $F^{[H-]}$ in Subsection 3.3.

3.3. Upper and lower bounds on $\|\mathbf{G}_C\|_{L_1/L_1}$

The preceding subsections are devoted to providing approximate but asymptotically exact values of $F^{[H+]}$ and $F^{[H-]}$.In terms of $F_{\mathrm{U}}^{[H+]}$, $F_{M,p,\mathrm{U}}^{[H-,\alpha]}$ and $F_{M,p,\mathrm{L}}^{[H-,\alpha]}$ given by (3.3), (3.15) and (3.16), respectively, combining Lemmas 2–4 leads to the following theorem.

Theorem 1. *For a* $p \in \{0, 1, 2, 3\}$ *and an* $\alpha \in [0, 1]$, *assume that we take sufficiently large* $q \in \mathbb{N}$ *and* M *such that* $\left|e^{Aq}\right|_1 < 1$ *and every element of* $F_{M,p,\mathrm{L}}^{[H-,\alpha]}$ *is not smaller than* 0. *Then, the following inequality holds.*

$$\left|F_{M,p,\mathrm{L}}^{[H-,\alpha]}\right|_1 \leq \|\mathbf{G}_C\|_{L_1/L_1} \leq \left|F_{M,p,\mathrm{U}}^{[H-,\alpha]} + F_{\mathrm{U}}^{[H+]}\right|_1 . \tag{3.17}$$

Furthermore, the gap between the upper and lower bounds in (3.17) *tends to* 0 *as* M *and* H *become larger, with the convergence rate proportional to* $1/M^{p+1}$ *regardless of* q *and* α.

Proof. The first assertion readily follows from substituting Lemmas 3 and 4 into Lemma 2. The second assertion is easily established by noting from Lemmas 3 and 4 that $f_{ij,\mathrm{U}}^{[H+]}$ converges to 0 at an exponential order of H regardless of the choice of q and $\hat{k}_{ij,M,p}^{[H-,\alpha]}$ has a uniform upper bound with respect to $\alpha \in [0, 1]$ because c_α achieves its maximum at $\alpha = 1$ and $\left|\mathrm{row}_i(C)e^{A\alpha h}A^{p+1}\right|_1$ is bounded by $\left|\mathrm{row}_i(C)A^{p+1}\right|_1 \cdot e^h$. □

This theorem clearly implies that we can compute an upper bound and a lower bound on the L_1-induced norm $\|\mathbf{G}_C\|_{L_1/L_1}$, and their gap converges to 0 by taking the approximation parameter M and the truncation parameter H larger, within the order of $1/M^{p+1}$. Here, it would be worthwhile to note that taking H larger to reduce $F_{\mathrm{U}}^{[H+]}$ leads to increasing $k_{ij,M,p}^{[H-,\alpha]}$ in (3.15) and (3.16), by which the gap between the corresponding upper and lower bounds becomes larger. Thus, to achieve the desired accuracy of the L_1-induced norm based on Theorem 1, it is crucial to select appropriate values of the parameters H, q, M and α. In connection with this, we provide a pseudo-code based guideline for determining those parameters as follows.

Algorithm 1 Guideline for determining the parameters H, q, M, and α.

```
Require: System matrices A ∈ R_1^{n_x×n_x}, B ∈ R_1^{n_x×n_w}, C ∈ R_1^{n_y×n_x}, D ∈ R_1^{n_y×n_w}, approximation order p
Define the acceptable tolerance for the L_1 induced norm as ϵ
Define variables ϵ_1, ϵ_2(≤ ϵ), δq, δH(> 0)
Initialize q, H
while |e^{Aq}|_1 < 1 do
    Update q ← q + δq
while 0 < ((e^{|A|_1 q} − 1)|B|_1 |e^{AH}|_1 |C|_1) / (|A|_1 (1 − |e^{Aq}|_1)) ≤ ϵ_1 do
    H ← H + δH or q ← q + δq
Initialize α ← 0.5
Initialize M ← 0
while |F_{M,p,U}^{[H−,α]} + F_U^{[H+]}|_1 − |F_{M,p,L}^{[H−,α]}|_1 ≤ ϵ_2 do
    Update M ← M + 1
Initialize x ← |F_{M,p,U}^{[H−,α]} + F_U^{[H+]}|_1 − |F_{M,p,L}^{[H−,α]}|_1
Initialize y ← 0
for 0 ≤ α ≤ 1 do
    y ← |F_{M,p,U}^{[H−,α]} + F_U^{[H+]}|_1 − |F_{M,p,L}^{[H−,α]}|_1
    if y < x then
        x ← y
    Update α
```

4. Computing the l_1-induced norm of LTI system

Stimulated by the success of computing the L_1-induced norm of linear multivariable differential equations developed in the preceding section, in this section, we establish parallel results on computing the l_1-induced norm of linear multivariable difference equations.

Let us consider the discrete-time (DT) LTI system Σ_D with difference and algebraic equations described by

$$\Sigma_D : \begin{cases} x[k+1] = A_d x[k] + B_d w[k] \\ y[k] = C_d x[k] + D_d w[k] \end{cases}, \tag{4.1}$$

where $x[k] \in \mathbb{R}_1^n$ is the state, $w[k] \in \mathbb{R}_1^{n_w}$ is the input and $y[k] \in \mathbb{R}_1^{n_y}$ is the output. The output of Σ_D is given by

$$y[k] = (\mathbf{G}_D w)[k] := \sum_{i=1}^{k} C_d A_d^{k-i} B_d w[k] + D_d w[k] \tag{4.2}$$

assuming the zero initial condition of x similar to the continuous-time case. For the l_1-induced norm of Σ_D to be well-defined and bounded, we assume that A_d is Schur stable; all the eigenvalues of A_d are located in the open unit disc.

In an equivalent fashion to the arguments in Section 2, we denote the l_1-induced norm of Σ_D by $\|\mathbf{G}_D\|_{l_1/l_1}$ and can derive that

$$\|\mathbf{G}_D\|_{l_1/l_1} = \max_{1\le j\le n_w} \sum_{i=1}^{n_y} g_{ij}, \tag{4.3}$$

where

$$g_{ij} := \sum_{l=0}^{\infty} \left|(C_d A_d^l B_d)_{ij}\right| + \left|(D_d)_{ij}\right|. \tag{4.4}$$

To alleviate difficulties in treating an infinite number of g_{ij} in (4.3), we propose the truncation idea with a sufficiently large parameter N as follows.

$$g_{ij}^{[N-]} := \sum_{l=0}^{N} \left|(C_d A_d^l B_d)_{ij}\right| + |(D_d)ij|, \quad g_{ij}^{[N+]} := \sum_{l=N+1}^{\infty} \left|(C_d A_d^l B_d)_{ij}\right|. \tag{4.5}$$

On the basis of (4.5), it immediately follows from (4.3) that

$$|G^{[N-]}|_1 \le \|\mathbf{G}_D\|_{l_1/l_1} \le |G^{[N-]} + G^{[N+]}|_1 \tag{4.6}$$

holds, where

$$G^{[N-]} := \begin{bmatrix} g_{11}^{[N-]} & \cdots & g_{1n_w}^{[N-]} \\ \vdots & \ddots & \vdots \\ g_{n_y1}^{[N-]} & \cdots & g_{n_yn_w}^{[N-]} \end{bmatrix}, \quad G^{[N+]} := \begin{bmatrix} g_{11}^{[N+]} & \cdots & g_{1n_w}^{[N+]} \\ \vdots & \ddots & \vdots \\ g_{n_y1}^{[N+]} & \cdots & g_{n_yn_w}^{[N+]} \end{bmatrix}. \tag{4.7}$$

From the stability assumption on A_d, $G^{[N+]}$ is expected to converge to 0 as the truncation parameter N becomes larger, and thus taking a sufficiently large N is reasonable for computing $\|G_D\|_{l_1/l_1}$. In connection with this, we note from the stability assumption on A_d that there should exist a $q \in \mathbb{N}_0$ ensuring $\left|A_d^q\right|_1 < 1$, and can derive the following Lemma.

Lemma 5. *For a $q \in \mathbb{N}_0$ with $\left|A_d^q\right|_1 < 1$, the following inequality holds.*

$$g_{ij}^{[N+]} \leq |\mathrm{row_i(C_d)}|_1 \cdot \frac{\left|\left[(A_d^{N+1})^T \quad \cdots \quad (A_d^{N+q})^T\right]^T\right|_1}{1 - \left|A_d^q\right|_1} \cdot \left|\mathrm{col_j(B_d)}\right|_1 := g_{ij,\mathrm{U}}^{[N+]}. \tag{4.8}$$

Furthermore, $g_{ij,\mathrm{U}}^{[N+]}$ converges to 0 *regardless of the choice of q with an exponential order of N*

Proof. Note that

$$g_{ij}^{[N+]} = \sum_{l=N+1}^{\infty} \left|(C_d A_d^l B_d)_{ij}\right|_1 = \sum_{l=N+1}^{\infty} \left|\mathrm{row}_i(C_d) A_d^l \mathrm{col}_j(B_d)\right| = \left\| \begin{bmatrix} \mathrm{row}_i(C_d) & & \\ & \mathrm{row}_i(C_d) & \\ & & \ddots \end{bmatrix} \begin{bmatrix} A_d^{N+1} \\ A_d^{N+2} \\ \vdots \end{bmatrix} \mathrm{col}_j(B_d) \right\|_1$$

$$\leq \left\| \begin{bmatrix} \mathrm{row}_i(C_d) & & \\ & \mathrm{row}_i(C_d) & \\ & & \ddots \end{bmatrix} \right\|_1 \cdot \left\| \begin{bmatrix} A_d^{N+1} \\ A_d^{N+2} \\ \vdots \end{bmatrix} \right\|_1 \cdot \left|\mathrm{col}_j(B_d)\right|_1 = |\mathrm{row}_i(C_d)|_1 \cdot \left\| \begin{bmatrix} A_d^{N+1} \\ A_d^{N+2} \\ \vdots \end{bmatrix} \right\|_1 \cdot \left|\mathrm{col}_j(B_d)\right|_1. \tag{4.9}$$

On the other hand,

$$\left\| \begin{bmatrix} A_d^{N+1} \\ A_d^{N+2} \\ \vdots \end{bmatrix} \right\|_1 \leq \sum_{l=0}^{\infty} \left\| \begin{bmatrix} A_d^{N+1+lq} \\ \vdots \\ A_d^{N+(l+1)q} \end{bmatrix} \right\|_1 \leq \frac{\left|\left[(A_d^{N+1})^T \quad \cdots \quad (A_d^{N+q})^T\right]^T\right|_1}{1 - \left|A_d^q\right|_1}. \tag{4.10}$$

Substituting (4.10) into (4.9) completes the proof of the first assertion. The second assertion follows the fact that the RHS of (4.10) tends to 0 with an exponential order of N. □

From Lemma 5, an upper bound on $g_{ij}^{[N+]}$ can be computed when N is fixed and an appropriate q with $|A_d^q|_1 < 1$ is selected. We next consider $G_{\mathrm{U}}^{[N+]} \in \mathbb{R}_1^{n_y \times n_w}$ defined as

$$G_{\mathrm{U}}^{[N+]} := \begin{bmatrix} g_{11,\mathrm{U}}^{[N+]} & \cdots & g_{1n_w,\mathrm{U}}^{[N+]} \\ \vdots & \ddots & \vdots \\ g_{n_y 1,\mathrm{U}}^{[N+]} & \cdots & g_{n_y n_w,\mathrm{U}}^{[N+]} \end{bmatrix}, \tag{4.11}$$

where the subscript 'U' stands for the upper bound, and this will be used for computing $\|\mathbf{G}_D\|_{l_1/l_1}$ instead of $G^{[N+]}$ in the following arguments. The remaining part, determining each entry of $G^{[N-]}$, can be immediately obtained since only finite numbers of summations are required. Substituting $G^{[N-]}$ and $G_{\mathrm{U}}^{[N+]}$ given respectively by (4.7) and (4.11) into (4.6) leads to the following result.

Theorem 2. *For a sufficiently large $q \in \mathbb{N}$ with $|A_d^q|_1 < 1$, the following inequality holds.*

$$\left|G^{[N-]}\right|_1 \leq \|\mathbf{G}_D\|_{l_1/l_1} \leq \left|G^{[N-]} + G_{\mathrm{U}}^{[N+]}\right|_1. \tag{4.12}$$

Furthermore, the gap between the upper and lower bounds in (4.12) *converges to 0 regardless of the choice of q with an exponential order of N.*

5. Numerical example

In this section, we provide a numerical example for verifying the effectiveness of the methods for computing L_1-induced and l_1-induced norms proposed in this paper. With respect to this, let us consider the twin-rotor MIMO system as shown in Figure 1.

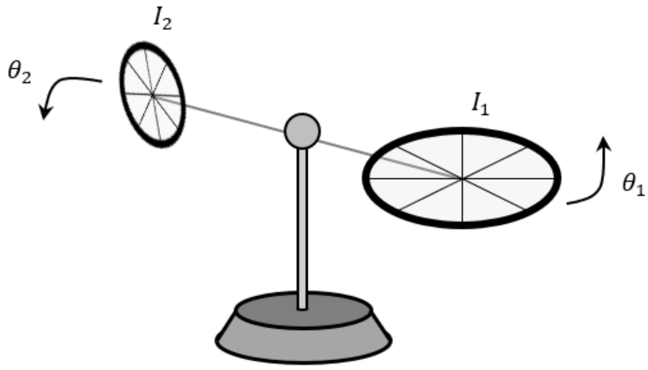

Figure 1. Twin-rotor MIMO system.

5.1. *L_1-induced norm computation*

The dynamic behavior of the twin-rotor mimo system is described by the state-space equation [23]

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{M_g}{I_1} & -\frac{B_{11}}{I_1} & 0 & 0 & \frac{b_1}{I_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{B_{12}}{I_2} & -\frac{a_1 k_c}{I_2} & \frac{b_2}{I_2} \\ 0 & 0 & 0 & 0 & -\frac{1}{T_{11}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{k_1}{T_{11}} & 0 \\ 0 & k_2 \end{bmatrix} v + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{k_1}{T_{11}} & 0 \\ 0 & k_2 \end{bmatrix} w \\ y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x \end{cases}, \tag{5.1}$$

where $x := \begin{bmatrix} \theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 & \tau_1 & \tau_2 \end{bmatrix}^T$ and θ_1, θ_2, τ_1 and τ_2 denote the pitch angle, the yaw angle, the momentum of the rotor 1 and the momentum of the rotor 2, respectively. The control input $v := \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$ contains the input voltages V_1 and V_2 supplied to the rotor 1 and the rotor 2, respectively, w denotes the disturbance affecting the control input and y is the regulated output. We take the corresponding system parameters as shown in Table 3.

Table 3. System parameters.

I_1	Moment of inertia of vertical rotor	$6.8 \times 10^{-1}\, kgm^2$
I_2	Moment of inertia of horizontal rotor	$2 \times 10^{-1}\, kgm^2$
M_g	Gravity momentum	$0.96\, Nm$
B_{11}	Friction momentum function parameter for pitch term	$5.998 \times 10^{-1}\, Nms^2/rad$
B_{12}	Friction momentum function parameter for yaw term	$1 \times 10^{-1}\, Nms^2/rad$
k_c	Cross reaction momentum gain	-1.848×10^{-2}
a_1	Static characteristic parameter	1.75
b_1	Static characteristic parameter	0.2772
b_2	Static characteristic parameter	0.09
T_{11}	Motor denominator parameter	1.1
k_1	Motor 1 gain	1.1
k_2	Motor 2 gain	0.8

Because the system given by (5.1) is unstable, we consider to employ the full-state stabilizing feedback controller proposed in [24], i.e.,

$$v = -\begin{bmatrix} -1.43159 & 1.63792 & 0.11873 & 0.37713 & 0.86570 & 0.21950 \\ 0.33447 & -0.06749 & 0.57959 & 1.34767 & 0.16206 & 0.79287 \end{bmatrix} x. \tag{5.2}$$

Computing the L_1-induced norm of the feedback system consisting of (5.1) and (5.2) is practically meaningful since it corresponds to evaluating the effect of the disturbances occurring in the input voltages on the pitch and yaw angles. In other words, we can clarify how much the angles would rotate by the disturbances in the twin-rotor MIMO system. Based on the fact that c_α in (3.9) attains its minimum at $\alpha = 0.5$, we take $\alpha = 0.5$ for this example. With letting $\epsilon_1 = 10^{-6}$, we can obtain $(H, q) = (150, 6)$ by following Algorithm 1. These parameter values lead to $\|F_{\mathrm{U}}^{[H+]}\|_1 = 3.165 \times 10^{-7} < 10^{-6} = \epsilon_1$. After the pair $(H, q) = (150, 6)$ is fixed, we take the approximation parameter M ranging from 800 to 2000 with the approximation order $p = 0, 1, 2, 3$. The computation results are shown in Table 4.

We can observe from Table 4 that both the upper and lower bounds on the L_1-induced norm converge to 0.6462 by making M larger, and the gaps between these bounds are decreasing at a rate no smaller than $1/M^{p+1}$ for all the approximation order $p = 0, 1, 2, 3$. To make the practical effectiveness of the arguments in Theorem 1 and Algorithm 1 clearer, the results of computing the L_1-induced norm (i.e., an upper bound, a lower bound and their gap) obtained through the conventional arguments in [17] are also shown in Table 5. In a comparison between the results in Table 4 for $p = 2, 3$ to those in Table 5, we can observe that the gaps in the former table are quite smaller than that in the latter table under the same parameter M. Furthermore, the convergence speeds observed from Table 4 for $p = 2, 3$ are much faster than that from Table 5. From these observations, the arguments developed in this paper (i.e., Theorem 1 and Algorithm 1) are demonstrated in both the theoretical and practical aspects and shown to outperform the conventional arguments in [17] for computing the L_1-induced norm of linear multivariable differential equations.

Table 4. Computation results of the L_1-induced norm.

M	800	1200	1500	2000
Approximation order $p = 0$				
Upper bound	1.22448	0.96317	0.88058	0.80886
Lower bound	0.07056	0.33046	0.41230	0.48397
Gap	1.15392	0.63272	0.46828	0.32488
Approximation order $p = 1$				
Upper bound	0.69246	0.66324	0.65631	0.65147
Lower bound	0.60350	0.63072	0.63706	0.64145
Gap	0.08895	0.03252	0.01925	0.01002
Approximation order $p = 2$				
Upper bound	0.65096	0.64729	0.64653	0.64636
Lower bound	0.64041	0.64472	0.64531	0.64588
Gap	0.01055	0.00257	0.00121	0.00048
Approximation order $p = 3$				
Upper bound	0.64626	0.64610	0.64596	0.64613
Lower bound	0.64512	0.64591	0.64589	0.64611
Gap	0.00114	0.00019	0.00007	0.00002

Table 5. Computation results of the L_1-induced norm via the conventional method [17].

M	800	1200	1500	2000
Conventional method [17]				
Upper bound	0.66107	0.65046	0.64835	0.64709
Lower bound	0.62396	0.63864	0.64191	0.64408
Gap	0.03711	0.01182	0.00643	0.00301

5.2. l_1-induced norm computation of the DT LTI system

With respect to verifying the arguments in Theorem 2 relevant to discrete-time systems, we consider a discretization of (5.1) through the zero-order-hold(ZOH) method [25] with the sampling time $T = 0.1$ [s]. Based on such a discretized model, we also consider the discrete-time stabilizing controller given by

$$v[k] = -\begin{bmatrix} -3.23972 & 12.88198 & -4.27361 & -4.81018 & 2.99291 & -0.79350 \\ 7.65089 & 3.80181 & 8.04928 & 12.08854 & 0.84949 & 3.56504 \end{bmatrix} x[k], \tag{5.3}$$

and we take the truncation parameter N ranging from 60 to 180 under the condition $q = 28$ that results in $|A_d^q|_1 = 0.9076(< 1)$. The corresponding computation results are shown in Table 6.

We can observe from Table 6 that the upper and lower bounds on the l_1-induced norm converge to 1.3951 as the truncation parameter N becomes larger, and the gaps between these bounds tend to 0 in an exponential order of N. Both observations demonstrate the theoretical validity and the practical effectiveness of the approximation method developed in this paper for computing the discrete-time l_1-induced norm.

Table 6. Computation results of the l_1-induced norm.

N	60	90	120	180
Upper bound	6.06242	1.64220	1.40619	1.39511
Lower bound	1.39165	1.39477	1.39508	1.39509
Gap	4.67077	0.24743	0.01111	0.00002

6. Conclusions

In this paper, we developed methods for computing the L_1-induced norm of multivariable linear differential equations. We first derived a closed-form representation of the L_1-induced norm and clarified that it should be required to integrate the absolute value of a matrix exponential function on the infinite time interval $[0, \infty)$. To alleviate this difficulty, we aimed to compute an upper bound and a lower bound on the L_1-induced norm within any degree of accuracy. To this end, the time interval $[0, \infty)$ is divided into $[0, H)$ and $[H, \infty)$, for a sufficiently large but finite H. An upper bound on the L_1-induced norm relevant to the infinite interval $[H, \infty)$ was derived, and we showed that the upper bound is decreasing with an exponential order of H. An upper bound and a lower bound on the L_1-induced norm with respect to the finite interval $[0, H)$ were also obtained by dividing the interval into M subintervals with equal width and applying a pth order Taylor approximation with $p = 0, 1, 2, 3$. More precisely, the gap between the upper and lower bounds was ensured to converge to 0 at the rate of $1/M^{p+1}$. Combining these bounds for both the intervals $[0, H)$ and $[H, \infty)$ allowed us to compute the L_1-induced norm on the original interval $[0, \infty)$, within any degree of accuracy. Parallel results on computing the l_1-induced norm of discrete-time differential equations were further established. Some numerical examples were given to demonstrate the theoretical validity and the practical effectiveness of the overall arguments developed in this paper.

Finally, it would also be worthwhile to note that the computation method developed in this paper is straightforward and thus it might be extended to more involved problems. Regarding practical applications of the present study, for example, a new stability criterion for biped walking robots and a new quantitative performance measure could be constructed by modifying the relevant arguments in [26,27], respectively. With respect to theoretical improvements, the quasi-finite-rank approximation of compression operators in [28] can be reformulated in the L_1-induced norm sense. Furthermore, an optimal controller minimizing the L_1-induced norm can be also studied for involved systems such as neutral differential-algebraic equations [29], piecewise continuous systems [30–32], and so on. However, the aforementioned extensions are non-trivial tasks because it is unclear how an optimal controller for minimizing the L_1-induced norm for multivariable linear differential equations can be obtained, and thus they are left for meaningful but quite difficult future works.

Author contributions

Junghoon Kim: Methodology;software;writing original draft; Jung Hoon Kim: Conceptualization; supervision, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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