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Research article

A characterization of b-generalized skew derivations on a Lie ideal in a prime ring

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Abstract: This paper investigates the analysis of b-generalized skew derivations, denoted as Δ_1 and Δ_2, within a prime ring $\mathcal{R}$ with characteristic different from 2. Here, Q_r represents the right Martindale quotient ring of $\mathcal{R}$, and C denoted its extended centroid. Additionally, $\mathcal{L}$ is a noncentral Lie ideal of $\mathcal{R}$. Assuming Δ_1 and Δ_2 are nontrivial b-generalized skew derivations associated with the same automorphism α, the paper aims to explore the detailed structure of these generalized derivations that satisfy the specific equation:

$$pu\Delta_1(u) + \Delta_1(u)uq = \Delta_2(u^2), \text{ with } p + q \notin C, \quad \text{for all } u \in \mathcal{L}.$$

The above-studied result generalized the already existing results [1, 2] in the literature.

Keywords: prime rings; Lie ideals; right Martindale quotient ring; b-generalized skew derivation; extended centroid
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1. Introduction

A ring $\mathcal{R}$ is considered prime if, for any elements π and ξ in $\mathcal{R}$, the condition $\pi\mathcal{R}\xi = 0$ implies that either $\pi = 0$ or $\xi = 0$. In our discussion, unless otherwise stated, $\mathcal{R}$ refers to a prime ring with its center denoted by $\mathcal{Z}(\mathcal{R})$, and Q_r refers to its right Martindale quotient ring. Notably, Q_r retains the prime property of $\mathcal{R}$. Additionally, the center of Q_r, known as the extended centroid of $\mathcal{R}$, is a field.

To simplify the notation, we use $[\pi, \xi] = \pi\xi - \xi\pi$ for all $\pi, \xi \in \mathcal{R}$. A subset $\mathcal{L}$ of $\mathcal{R}$ is called a Lie ideal of $\mathcal{R}$ if it forms an additive subgroup and satisfies the condition that the commutator of $\mathcal{L}$ with any element of $\mathcal{R}$ remains within $\mathcal{L}$, i.e., $[\mathcal{L}, \mathcal{R}] \subseteq \mathcal{L}$.

Definition 1.1. [3] A mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation if it is additive and

$$d(\pi\xi) = d(\pi)\xi + \pi d(\xi), \quad \text{for all} \quad \pi, \xi \in \mathcal{R}.$$

For a fixed $v \in \mathcal{R}$, the mapping $d_v : \mathcal{R} \to \mathcal{R}$ defined as $d_v(\pi) = [v, \pi]$ for all $\pi \in \mathcal{R}$ is a derivation termed as an inner derivation induced by the element v. A derivation that is not inner is referred to as an outer derivation. In 1957, Posner [4] showed that if R is a prime ring and d is a nontrivial derivation of $\mathcal{R}$ such that $[d(\pi), \pi] \in \mathcal{Z}(\mathcal{R})$ for all $\pi \in \mathcal{R}$, then $\mathcal{R}$ is commutative. Posner's results were later extended in various ways by other mathematicians.

In 1991, M. Brešar [5] proposed a new kind of derivation, known as a generalized derivation.

Definition 1.2. [5] A mapping $\Delta : \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation if Δ is additive and there exists a derivation δ on $\mathcal{R}$ such that

$$\Delta(\pi\xi) = \Delta(\pi)\xi + \pi\delta(\xi), \text{ for all } \pi, \xi \in \mathcal{R}.$$

For fixed elements $v_1, v_2 \in \mathcal{R}$, the mapping $\Delta_{(v_1,v_2)} : \mathcal{R} \to \mathcal{R}$ defined by $\Delta_{(v_1,v_2)}(\pi) = v_1\pi + \pi v_2$ is a generalized derivation on $\mathcal{R}$, often referred to as a generalized inner derivation.

Definition 1.3. [3] A mapping $\mathcal{D} : \mathcal{R} \to \mathcal{R}$ is called a skew derivation associated with the automorphism $\alpha \in \text{Aut}(\mathcal{R})$ if it is additive and it satisfies

$$\mathcal{D}(\pi\xi) = \mathcal{D}(\pi)\xi + \alpha(\pi)\mathcal{D}(\xi), \quad \text{for all } \pi, \xi \in \mathcal{R}.$$

A skew derivation that is associated with the identity automorphism reduces to a derivation. For example, given a fixed element b in Q_r, the mapping defined by $\pi \mapsto b\pi - \alpha(\pi)b$ is a notable example of a skew derivation, commonly known as an inner skew derivation. If a skew derivation does not fit this structure, it is termed an outer skew derivation.

Definition 1.4. [3] A mapping $\phi : \mathcal{R} \to \mathcal{R}$ is called a generalized skew derivation associated with the automorphism $\alpha \in \text{Aut}(\mathcal{R})$ if it is additive and there exists a skew derivation δ on $\mathcal{R}$ such that

$$\phi(\pi\xi) = \phi(\pi)\xi + \alpha(\pi)\delta(\xi), \quad \text{for all } \pi, \xi \in \mathcal{R}.$$

In 2021, De Filippis [6] studied the identity $\Delta_1(\Delta_2(\pi)) = 0$ for all $\pi \in \mathcal{L}$, where Δ_1 and Δ_2 are generalized skew derivations on a prime ring $\mathcal{R}$, with $\mathcal{L}$ being a Lie ideal of $\mathcal{R}$. This identity was examined within the framework of generalized derivations.

In 2018, De Filippis and Wei [7] developed the notion of b-generalized skew derivation, which broadens the concept of derivations and investigates different kinds of linear mappings in noncommutative algebras.

Definition 1.5. [7] Let b be a fixed element in the right Martindale quotient ring Q_r. The mapping $\Delta_1 : \mathcal{R} \to Q_r$ is called a b-generalized skew derivation of $\mathcal{R}$ associated with the triplet $(\mathrm{b}, \alpha, \mathrm{d})$ if it is additive and it satisfies the condition

$$\Delta_1(\pi\xi) = \Delta_1(\pi)\xi + \mathrm{b}\alpha(\pi)\mathrm{d}(\xi)$$

for all $\pi, \xi \in \mathcal{R}$, where $\mathrm{d} : \mathcal{R} \to Q_r$ is an additive mapping and α is an automorphism of $\mathcal{R}$.

Furthermore, the authors showed that when b $\neq 0$, the corresponding additive map d, as defined earlier, acts as a skew derivation. Additionally, it has been established that the additive mapping Δ_1 can be extended to the right Martindale quotient ring Q_r, taking the form $\Delta_1(\pi) = \mathrm{a}\pi + \mathrm{bd}(\pi)$, where $\mathrm{a} \in Q_r$. The concept of b-generalized skew derivation, characterized by the triplet $(\mathrm{b}, \alpha, \mathrm{d})$, includes skew derivations, generalized derivations, and left multipliers, among other concepts. For instance, setting b $= 1$ yields a skew derivation, while choosing b $= 1$ and $\alpha = I_{\mathcal{R}}$ results in a generalized derivation, with $I_{\mathcal{R}}$ representing the identity map on $\mathcal{R}$. Additionally, if b $= 0$ in Definition 1.5, then Δ_1 reduces to a left multiplier map. The mapping $\Delta_1 : \mathcal{R} \to Q_r$, given by $\pi \mapsto a\pi + \mathrm{b}\alpha(\pi)c$ is a notable example of b-generalized skew derivation of $\mathcal{R}$ associated with the triplet $(\mathrm{b}, v_1, \mathrm{d})$, where a, $b, c \in Q_r$, and $\mathrm{d}(\pi) = \alpha(\pi)c - c\pi$ for all $\pi \in \mathcal{R}$. This type of b-generalized skew derivation is known as an inner b-generalized skew derivation. Therefore, the study of b-generalized skew derivations of a ring $\mathcal{R}$ provides insights into the study of other types of derivations.

These broad results concerning b-generalized skew derivations lead to significant corollaries related to derivations, generalized derivations, and generalized skew derivations. Such findings offer valuable insights for applications and further advancements in the study of these related concepts.

It is quite natural to examine the implications of substituting derivations with b-generalized skew derivations in the results originally obtained by Posner and Brešar. In 2021, Filippis et al. [8] made progress in extending Brešar's result by investigating the identity $\Delta_1(\pi)\pi - \pi\Delta_2(\pi) = 0$ involving b-generalized skew derivations Δ_1 and Δ_2 in a prime ring $\mathcal{R}$. Here, π represents elements of the form $\phi(\pi_1, \ldots, \pi_n)$, where $\pi = (\pi_1, \ldots, \pi_n) \in \mathcal{R}^n$, and $\phi(\pi)$ is a multi-linear polynomial over C. Relevant generalizations related to b-generalized skew derivations can be found in [3, 7–12].

Continuing the investigation of above cited results, we focus to study the following identity $p\pi\Delta_1(\pi) + \Delta_1(\pi)\pi q = \Delta_2(\pi^2)$, where $p + q \notin C$ for all $\pi \in \mathcal{L}$. The primary motivation for this identity comes from the articles [1] and [2]. In [1], the authors examined the identity $p\pi\mathcal{F}(\pi)+\mathcal{F}(\pi)\pi q = \mathcal{G}(\pi^2)$, where $\mathcal{F}$, $\mathcal{G}$ are derivations and $\pi \in \mathcal{S}$, a particular subset of $\mathcal{R}$. In [2], the same identity was explored with $\mathcal{F}$ and $\mathcal{G}$ considered as generalized derivations. Naturally, it is of interest to investigate this identity further by taking $\mathcal{F}$ and $\mathcal{G}$ as b-generalized skew derivations. The following theorem establishes our result:

Theorem 1.6. *Let $\mathcal{R}$ be a prime ring with characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, and $\mathcal{L}$ a noncentral Lie ideal of $\mathcal{R}$. Suppose Δ_1 and Δ_2 are non-zero* b*-generalized skew derivations of $\mathcal{R}$ with associated triples* (b, α, d) *and* (b, α, h)*, respectively, satisfying the identity:*

$$p\pi\Delta_1(\pi) + \Delta_1(\pi)\pi q = \Delta_2(\pi^2) \textit{ for some } p, q \in \mathcal{R} \textit{ with } p + q \notin C,\ \forall \pi \in \mathcal{L}.$$

Then, for all $\pi \in \mathcal{R}$, one of the following holds:

1) There exist $a \in C$ and $c \in Q_r$ such that $\Delta_1(\pi) = a\pi$, $\Delta_2(\pi) = \pi c$ with $pa = c - aq \in C$.
2) There exist $a \in C$ and $c, c' \in Q_r$ such that $\Delta_1(\pi) = a\pi$, $\Delta_2(\pi) = c\pi + \pi c'$ with $pa - c \in C$ and $(p + q)a = c + c'$.
3) There exist $a, q \in C$ and $c \in Q_r$ such that $\Delta_1(\pi) = a\pi$, $\Delta_2(\pi) = c\pi$ with $(p + q)a = c$.
4) There exist $a, b, u, v, t \in Q_r$, and $\lambda, \eta \in C$ such that $\Delta_1(\pi) = (a + b\pi)\pi$, $\Delta_2(\pi) = c\pi + bt\pi t^{-1}v$ with $t^{-1}v + \eta q = \lambda \in C$, $(a + b\pi) + \eta bt = 0$, and $p(a + b\pi) - c = \lambda bt$.
5) $\mathcal{R}$ satisfies s_4.

The standard polynomial identity s_4 in four variables is defined as follows:

$$s_4(\pi_1, \pi_2, \pi_3, \pi_4) = \sum_{\sigma \in \mathrm{Sym}(4)} (-1)^\sigma \pi_{\sigma(1)}\pi_{\sigma(2)}\pi_{\sigma(3)}\pi_{\sigma(4)},$$

where $(-1)^\sigma$ is $+1$ or -1 depending on whether σ represents an even or odd permutation in the symmetric group Sym(4).

The general approach for proving the main theorem can be extended to demonstrate a broader result for multi-linear polynomials. Consequently, from a theoretical standpoint, there is no distinction between cases involving Lie ideals and those involving multi-linear polynomials. Applying the proof method suitable for multi-linear polynomials streamlines the process by minimizing excessive calculations. The paper is structured as follows: Section 2 provides a review of fundamental concepts regarding prime rings. Section 3 explores the case where the b-generalized skew derivations Δ_1 and Δ_2 are inner. In Section 4, we establish our main theorem by carefully examining each case.

2. Preliminaries and notations

We frequently utilize the following facts to establish our results:

Fact 2.1. [13] Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a two-sided ideal of $\mathcal{R}$. Then, $\mathcal{R}$, $\mathcal{I}$, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r.

Fact 2.2. [14] Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a two-sided ideal of $\mathcal{R}$. Then, $\mathcal{R}, \mathcal{I}$, and Q_r satisfy the same differential identities.

Fact 2.3. [13] Let $\mathcal{R}$ be a prime ring. Then, every derivation d of $\mathcal{R}$ can be uniquely extended to a derivation of Q_r.

Fact 2.4. [15, Chuang] Let $\mathcal{R}$ be a prime ring, d be a nonzero skew derivation on $\mathcal{R}$, and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\mathcal{I}$ satisfies the differential identity.

$$f(\zeta_1, \zeta_2, \ldots, \zeta_n, d(\zeta_1), d(\zeta_2) \ldots, d(\zeta_n)) = 0$$

for any $\zeta_1, \ldots, zeta_n \in \mathcal{I}$, then either

- $\mathcal{I}$ satisfies the generalized polynomial identity,

$$f(\zeta_1, \zeta_2 \ldots, \zeta_n, \xi_1, xi_2 \ldots, xi_n) = 0$$

for all $\xi_1, \ldots, \xi_n \in \mathcal{R}$.
or

- d is Q_r-inner,

$$f(\zeta_1, \zeta_2, \ldots, \zeta_n, [p, \zeta_1], [p, \zeta_2] \ldots, [p, \zeta_n]) = 0.$$

Fact 2.5. [16] Let $\mathcal{K}$ be an infinite field and $m \geq 2$ an integer. If $P_1, \ldots, P_k$ are non-scalar matrices in $\mathcal{M}_m(\mathcal{K})$, then there exists some invertible matrix $P \in \mathcal{M}_m(\mathcal{K})$ such that each matrix $PP_1P^{-1}, \ldots, PP_kP^{-1}$ has all nonzero entries.

Fact 2.6. [17] Let $\mathcal{R}$ be a noncommutative prime ring of characteristic not equal to 2 with right Martindale quotient ring Q_r and extended centroid C, and let $f(\zeta_1, \ldots, \zeta_n)$ be a multi-linear polynomial over C, which is not central valued on $\mathcal{R}$. Suppose that there exists $a, b, c \in Q_r$ such that $f(\chi)af(\chi) + f(\chi)^2 b - cf(\chi)^2 = 0$ for all $\chi = (\zeta_1, \ldots, \zeta_n) \in \mathcal{R}$. Then, one of the following holds:

1) $b, c \in C, C - b = a = \alpha \in C$.
2) $f(\zeta_1, \ldots, \zeta_n)^2$ is central valued and there exists $\alpha \in C$ such that $c - b = a = \alpha$.

Fact 2.7. [7] If d is a nonzero skew derivation on a prime ring $\mathcal{R}$, then associated automorphism α is unique.

Fact 2.8. [6] Let $\mathcal{R}$ be a prime ring, ϕ, γ be two automorphisms of Q_r and d, g be two skew derivations on $\mathcal{R}$ associated with the same automorphism ϕ. If there exist a nonzero central element ν and $v \in Q_r$ such that

$$\mathcal{G}(\zeta) = (v\zeta - \gamma(\zeta)v) + \nu d(\zeta), \text{ for all } \zeta \in \mathcal{R}.$$

then, $\mathcal{G}(\zeta) = \nu d(\zeta)$ and one of the following holds:

1) $\phi = \gamma$.
2) $v = 0$.

Fact 2.9. [6] Let $\mathcal{R}$ be a prime ring, ϕ, γ be two automorphisms of Q_r, and d, g be two skew derivations on $\mathcal{R}$ associated with the same automorphism ϕ. If there exist a nonzero central element ν and $v \in Q_r$ such that

$$\mathcal{G}(\zeta) = (v\zeta - \gamma(\zeta)v) + \nu d(\zeta), \text{ for all } \zeta \in \mathcal{R}.$$

If d is inner skew derivation, then so is $\mathcal{G}$.

In this paper, $\mathcal{R}$ will consistently refer to a nontrivial, associative prime ring (unless specified otherwise). Additionally, the term "GPI" will be used as a shorthand for generalized polynomial identity.

3. Δ_1 and Δ_2 are inner b-generalized skew derivations

In this section, we focus on the case where Δ_1 and Δ_2 are inner b-generalized skew derivations of $\mathcal{R}$ associated with the pair (b, α). More specifically, we investigate Theorem 1.6 under the conditions $\Delta_1(\pi) = a\pi + \mathrm{b}\alpha(\pi)u$ and $\Delta_2(\pi) = c\pi + \mathrm{b}\alpha(\pi)v$ for all $\pi \in \mathcal{R}$, where $a, \mathrm{b}, c, u, v \in Q_r$. To establish the main result, we first present the following lemmas:

Lemma 3.1. *Let $\mathcal{R}$ be a prime ring of char$(\mathcal{R}) \neq 2$ and a_1, a_2, a_3, a_4, and $a_5 \in \mathcal{R}$ such that*

$$a_1u^2 + a_2u^2a_3 + a_4u^2a_5 = 0, \ \forall\ u = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]. \tag{3.1}$$

Then, one of the following holds:

1) $\mathcal{R}$ satisfies s_4.

2) $a_3, a_4 \in C$, and $a_1 + a_2a_3 = -a_5a_4 \in C$.
3) $a_3, a_5 \in C$, and $a_1 + a_2a_3 + a_5a_4 = 0$.
4) $a_1, a_2, a_4 \in C$, and $a_1 + a_2a_3 + a_5a_4 = 0$.
5) $a_2, a_5 \in C$, and $a_1 + a_4a_5 = -a_2a_3 \in C$.
6) There exist $\lambda, \eta, \mu \in C$ such that $a_5 + \eta a_3 = \lambda$, $a_2 - \eta a_4 = \mu$, and $a_1 + \lambda a_4 = -\mu a_3 \in C$

Proof. If u^2 is a centrally valued element in $\mathcal{R}$, then $\mathcal{R}$ satisfies the identity s_4, which leads to our first conclusion. Now, suppose u^2 is not central. Let S be the additive subgroup of $\mathcal{R}$ generated by the set $\{u^2 : u \in [\mathcal{R}, \mathcal{R}]\}$. Clearly, $S \neq 0$, and we have the relation:

$$a_1\pi + a_2\pi a_3 + a_4\pi a_5 = 0$$

for all $\pi \in S$.

According to [18], either $S \subseteq Z(\mathcal{R})$, or $\mathrm{char}(\mathcal{R}) = 2$ and $\mathcal{R}$ satisfies s_4, unless S contains a noncentral ideal $\mathcal{L}'$ of $\mathcal{R}$. Since u^2 is not centrally valued in $\mathcal{R}$, the first possibility is excluded. Additionally, since $\mathrm{char}(\mathcal{R}) \neq 2$, it follows that S contains a noncentral Lie ideal $\mathcal{L}'$ of $\mathcal{R}$. By [19], there exists a noncentral two-sided ideal $\mathcal{I}$ of $\mathcal{R}$ such that $[\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}'$. Under the given hypothesis, we have

$$a_1[\pi_1, \pi_2] + a_2[\pi_1, \pi_2]a_3 + a_4[\pi_1, \pi_2]a_5 = 0$$

for all $\pi_1, \pi_2 \in \mathcal{I}$. From Fact 2.1, since Q_r, $\mathcal{I}$, and $\mathcal{R}$ satisfy the same GPI, it follows that

$$a_1[\pi_1, \pi_2] + a_2[\pi_1, \pi_2]a_3 + a_4[\pi_1, \pi_2]a_5 = 0$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Therefore, by [[20], Proposition 2.13], we obtain the desired conclusions. □

Lemma 3.2. *Let $\mathcal{R} = \mathcal{M}_m(C)$, where $m \geq 2$, be the ring of all $m \times m$ matrices over an infinite field C with characteristic not equal to 2. Suppose $a, b, c, u, v, p, q \in \mathcal{R}$ satisfy:*

$$p\Pi a\Pi + p\Pi b\Pi u + a\Pi^2 q + b\Pi u\Pi q - c\Pi^2 - b\Pi^2 v = 0$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, either $b \in C$, $u \in C$, or $p + q \in C$.

Proof. Assume the field C is infinite. From the hypothesis:

$$p\Pi a\Pi + p\Pi b\Pi u + a\Pi^2 q + b\Pi u\Pi q - c\Pi^2 - b\Pi^2 v = 0 \tag{3.2}$$

for all $\Pi \in [\mathcal{R}, \mathcal{R}]$. If we assume that $p + q$, b, and u are not central elements, and since Eq (3.2) holds invariantly under any automorphism of $\mathcal{R}$ (as stated in Fact 2.5), it implies that all entries of $p + q$, b, and u are nonzero. By selecting $\Pi = e_{ij}$ in Eq (3.2), we obtain:

$$pe_{ij}ae_{ij} + pe_{ij}be_{ij}u + be_{ij}ue_{ij}q = 0. \tag{3.3}$$

Next, multiplying Eq (3.3) both on the right and the left by e_{ij} gives:

$$(p + q)_{ji}u_{ji}b_{ji}e_{ij} = 0,$$

which implies that either $(p + q)_{ji} = 0$, $u_{ji} = 0$, or $b_{ji} = 0$. Each of these scenarios leads to a contradiction. Therefore, it follows that either $p + q \in C$, or $b \in C$, or $u \in C$. □

Lemma 3.3. *Let $\mathcal{R} = \mathcal{M}_m(C)$, where $m \geq 2$, be the ring of all $m \times m$ matrices over a field C with characteristic not equal to 2. Suppose $a, b, c, u, v, p, q \in \mathcal{R}$ satisfy:*

$$p\Pi a\Pi + p\Pi b\Pi u + a\Pi^2 q + b\Pi u\Pi q - c\Pi^2 - b\Pi^2 v = 0$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, either $b \in C$, $u \in C$, or $p + q \in C$.

Proof. If C is an infinite field, the conclusion follows directly from Lemma 3.2. Now, let's consider the case where the field C is finite. Let $\mathcal{K}$ be an infinite extension field of C, and set $\bar{\mathcal{R}} = \mathcal{M}_m(\mathcal{K}) \cong \mathcal{R} \otimes_C \mathcal{K}$. It is important to note that a multi-linear polynomial is central-valued on $\mathcal{R}$ if and only if it is central-valued on $\bar{\mathcal{R}}$.

Consider the generalized polynomial identity for $\mathcal{R}$ given by

$$\begin{aligned} Q(\pi_1, \pi_2) = p[\pi_1, \pi_2]a[\pi_1, \pi_2] + p[\pi_1, \pi_2]b[\pi_1, \pi_2]u + a[\pi_1, \pi_2]^2 q \\ + b[\pi_1, \pi_2]u[\pi_1, \pi_2]q - c[\pi_1, \pi_2]^2 - b[\pi_1, \pi_2]^2 v. \end{aligned} \tag{3.4}$$

This polynomial has a multi-degree of $(2, 2)$ with respect to the indeterminates π_1 and π_2. Therefore, the complete linearization of $Q(\pi_1, \pi_2)$ results in a multi-linear generalized polynomial $\Theta(\pi_1, \pi_2, \xi_1, \xi_2)$ involving four indeterminates. Additionally, we have the relation $\Theta(\pi_1, \pi_2, \pi_1, \pi_2) = 4Q(\pi_1, \pi_2)$.

It is clear that the multi-linear polynomial $\Theta(\pi_1, \pi_2, \xi_1, \xi_2)$ serves as a generalized polynomial identity for both $\mathcal{R}$ and $\bar{\mathcal{R}}$. Given that the characteristic of $\mathcal{R}$ is not equal to 2, as per the assumption, we conclude that $Q(\pi_1, \pi_2) = 0$ for all $\pi_1, \pi_2 \in \bar{\mathcal{R}}$. Hence, the result follows from Lemma 3.2. □

Lemma 3.4. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2, with Martindale quotient ring Q_r and extended centroid C. Suppose that for some $a, b, c, u, v, p, q \in \mathcal{R}$, the following holds:*

$$p\Pi a\Pi + p\Pi b\Pi u + a\Pi^2 q + b\Pi u\Pi q - c\Pi^2 - b\Pi^2 v = 0$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, either $b \in C$, $u \in C$, or $p + q \in C$.

Proof. **Case 1:** Suppose none of b, u, or $p + q$ is central. Given the hypothesis, we have

$$\begin{aligned} h(\pi_1, \pi_2) = p[\pi_1, \pi_2]a[\pi_1, \pi_2] + p[\pi_1, \pi_2]b[\pi_1, \pi_2]u \\ + a[\pi_1, \pi_2]^2 q + b[\pi_1, \pi_2]u[\pi_1, \pi_2]q - c[\pi_1, \pi_2]^2 - b[\pi_1, \pi_2]^2 v \end{aligned} \tag{3.5}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Define $\mathcal{D} = Q_r \star_C C\{\pi_1, \pi_2\}$, the free product of Q_r and the free C-algebra $C\{\pi_1, \pi_2\}$ in non-commuting indeterminates π_1 and π_2. Since both $\mathcal{R}$ and Q_r satisfy the same GPI (from Facts 2.1 and 2.2), Q_r satisfies $h(\pi_1, \pi_2) = 0$ in $\mathcal{D}$.

Now, let's treat $h(\pi_1, \pi_2)$ as a trivial GPI for $\mathcal{R}$. Thus, $h(\pi_1, \pi_2)$ is a zero element in $\mathcal{D}$. However, since b, u, and $p + q$ are assumed not to be central, it must be that either $b[\pi_1, \pi_2]u[\pi_1, \pi_2]q$ or $p[\pi_1, \pi_2]b[\pi_1, \pi_2]u$ appears nontrivially in $h(\pi_1, \pi_2)$, leading to a contradiction.

Hence, at least one of b, u, or $p + q$ belongs to C.

Case 2: Now, suppose that $h(\pi_1, \pi_2)$ is a nontrivial GPI for Q_r. If C is infinite, then $h(\pi_1, \pi_2) = 0$ for all $\pi_1, \pi_2 \in Q_r \otimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since Q_r and $Q_r \otimes_C \bar{C}$ are both prime and centrally closed (refer to Theorems 2.5 and 3.5 in [21]), we can replace $\mathcal{R}$ by either Q_r or $Q_r \otimes_C \bar{C}$, depending on whether C is finite or infinite. Thus, $\mathcal{R}$ is centrally closed over C, and $h(\pi_1, \pi_2) = 0$ for all $\pi_1, \pi_2 \in \mathcal{R}$.

By Martindale's theorem [22], $\mathcal{R}$ is a primitive ring with a nonzero socle, $soc(\mathcal{R})$, and C as its associated division ring. By Jacobson's theorem (see p.75 in [23]), $\mathcal{R}$ is isomorphic to a dense ring of linear transformations on a vector space V over C.

Assuming first that V is finite-dimensional over C, i.e., $\dim_C V = m$, the density of $\mathcal{R}$ implies $\mathcal{R} \cong \mathcal{M}_m(C)$. Since $\mathcal{R}$ is noncommutative, therefore, $m \geq 2$. In this case, the result follows from Lemma 3.2.

Next, suppose V is infinite-dimensional over C. For any $e^2 = e \in \mathrm{soc}(\mathcal{R})$, we have $e\mathcal{R}e \cong \mathcal{M}_t(C)$ where $t = \dim_C Ve$. Since none of b, u, or $p+q$ is central, there exist $h_1, h_2, h_3 \in \mathrm{soc}(\mathcal{R})$ such that $[b, h_1] \neq 0$, $[u, h_2] \neq 0$, and $[p+q, h_3] \neq 0$. By Litoff's theorem [24], there is an idempotent $e \in \mathrm{soc}(\mathcal{R})$ such that $bh_1, h_1b, uh_2, h_2u, (p+q)h_3, h_3(p+q), h_1, h_2, h_3 \in e\mathcal{R}e$. Then, from Eq (3.5), we have:

$$\begin{aligned} e\{p[e\pi_1 e, e\pi_2 e]a[e\pi_1 e, e\pi_2 e] + p[e\pi_1 e, e\pi_2 e]b[e\pi_1 e, e\pi_2 e]u \\ +a[e\pi_1 e, e\pi_2 e]^2 q + b[e\pi_1 e, e\pi_2 e]u[e\pi_1 e, e\pi_2 e]q \\ -c[e\pi_1 e, e\pi_2 e]^2 - b[e\pi_1 e, e\pi_2 e]^2 v\}e = 0 \end{aligned} \tag{3.6}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. The subring $e\mathcal{R}e$ satisfies:

$$\begin{aligned} epe[\pi_1, \pi_2]eae[\pi_1, \pi_2] + epe[\pi_1, \pi_2]ebe[\pi_1, \pi_2]eue \\ +eae[\pi_1, \pi_2]^2 eqe + ebe[\pi_1, \pi_2]eue[\pi_1, \pi_2]eqe \\ -ece[\pi_1, \pi_2]^2 - ebe[\pi_1, \pi_2]^2 eve = 0 \end{aligned} \tag{3.7}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. By the finite-dimensional case above, either ebe, or eue, or $e(p+q)e$ is a central element of $e\mathcal{R}e$. Thus, one of the following must hold: $bh_1 = (ebe)h_1 = h_1 ebe = h_1 b$, or $uh_2 = (eue)h_2 = h_2(eue) = h_2 u$, or $(p+q)h_3 = e(p+q)eh_3 = h_3(e(p+q)e) = h_3(p+q)$, which contradicts the initial assumption.

Therefore, we conclude that either $b \in C$, or $u \in C$, or $p+q \in C$. □

From the previous arguments, we can prove the following lemmas:

Lemma 3.5. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2 with Martindale quotient ring Q_r and extended centroid C. Suppose that for some $a, p, q \in \mathcal{R}$,*

$$p\Pi a\Pi + a\Pi^2 q = 0$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, either $a \in C$ or both p and $pa \in C$.

Lemma 3.6. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2 with Martindale quotient ring Q_r and extended centroid C. Suppose that for some $p, q \in \mathcal{R}$,*

$$p\Pi^2 + \Pi^2 q = 0$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, either $\mathcal{R}$ satisfies s_4 or $p + q = 0$.

Proposition 3.7. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2, Q_r be its Martindale ring of quotients with extended centroid C, and $\mathcal{L} = [\mathcal{R}, \mathcal{R}]$ be a Lie ideal of $\mathcal{R}$. Let Δ_1, Δ_2 be two b-generalized skew inner derivations of $\mathcal{R}$ with associated pair (b, α). Suppose there exist elements $p, q \in \mathcal{R}$ such that*

$$p\Pi\Delta_1(\Pi) + \Delta_1(\Pi)\Pi q = \Delta_2(\Pi^2), \text{ with } p + q \notin C, \ \forall \Pi \in \mathcal{L}.$$

Then, for all $\pi \in \mathcal{R}$, one of the following holds:

1) There exist $a \in C$ and $c \in Q_r$ such that $\Delta_1(\pi) = a\pi, \Delta_2(\pi) = \pi c$ with $pa = c - aq \in C$.
2) There exist $a \in C$ and $c, c' \in Q_r$ such that $\Delta_1(\pi) = a\pi$, $\Delta_2(\pi) = c\pi + \pi c'$ with $pa - c \in C$ and $(p+q)a = c + c'$.
3) There exist $a, q \in C$ and $c \in Q_r$ such that $\Delta_1(\pi) = a\pi$, $\Delta_2(\pi) = c\pi$ with $(p+q)a = c$.
4) There exist $a, b, u, v, t \in Q_r$ and $\lambda, \eta \in C$ such that $\Delta_1(\pi) = (a+bu)\pi$, $\Delta_2(\pi) = c\pi + bt\pi t^{-1}v$ with $t^{-1}v + \eta q = \lambda \in C$, $(a+bu) + \eta bt = 0$, and $p(a+bu) - c = \lambda bt$.
5) $\mathcal{R}$ satisfies s_4.

Proof. From the hypothesis, we have:

$$p\Pi a\Pi + p\Pi b\alpha(\Pi)u + a\Pi^2 q + b\alpha(\Pi)u\Pi q - c\Pi^2 - b\alpha(\Pi^2)v = 0 \tag{3.8}$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$.
Case 1: Suppose the associated automorphism α is inner, then there exists an invertible element $t \in \mathcal{R}$ such that $\alpha(\pi) = t\pi t^{-1}$ for all $\pi \in \mathcal{R}$. Thus, Eq (3.8) becomes:

$$p\Pi a\Pi + p\Pi bt\Pi t^{-1}u + a\Pi^2 q + bt\Pi t^{-1}u\Pi q - c\Pi^2 - bt\Pi^2 t^{-1}v = 0 \tag{3.9}$$

for all $\Pi \in [\mathcal{R}, \mathcal{R}]$. Then from Lemma 3.4, either $bt \in C$ or $t^{-1}u \in C$.
Sub-case (a): If $bt \in C$, then Eq (3.9) reduces to:

$$p\Pi a\Pi + p\Pi^2 bu + a\Pi^2 q + \Pi bu\Pi q - c\Pi^2 - \Pi^2 bv = 0 \tag{3.10}$$

for all $\Pi \in [\mathcal{R}, \mathcal{R}]$. Again, by previous arguments, one of the following holds:

1) $p, pa, bu \in C$.
2) $a, bu \in C$.
3) $a, q, buq \in C$.

Now, we will discuss each of the above cases in detail.

1) Suppose $pa, p, bu \in C$, then Eq (3.10) reduces to:

$$(pa - c)\Pi^2 + a\Pi^2 q + \Pi^2(pbu + buq - bv) = 0 \tag{3.11}$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, from Lemma 3.1, one of the following holds:

- $\mathcal{R}$ satisfies s_4, which is our Conclusion (5).
- $q \in C$, which implies that $p + q \in C$, a contradiction.
- $q, (pbu + buq - bv) \in C$, which implies that $p + q \in C$, a contradiction.
- $(pa - c), a \in C$, which implies that $c \in C$. Thus, from Eq (3.11), we get $(a + bu)p = (c + bv) - (a + bu)q$. Hence, in this case, we get $\Delta_1(\pi) = (a + bu)\pi$ and $\Delta_2(\pi) = \pi(c + bv)$ for all $\pi \in \mathcal{R}$ with $(a + bu)p = (c + bv) - (a + bu)q$, which is our Conclusion (1).
- $q, (pbu + buq - bv) \in C$, which gives that $p + q \in C$, a contradiction.
- There exist $\eta, \lambda, \mu \in C$ such that $(pbu+buq-bv)+\eta q = \lambda$, $a-\eta = \mu$ and $(pa-c)+\lambda = -\mu q \in C$. If $\mu \neq 0$, then $q \in C$, which implies $p + q \in C$, a contradiction. If $\mu = 0$, then $pa - c, a \in C$; then by previous arguments, we get our Conclusion (1).

2) Suppose $a, bu \in C$. Then, Eq (3.10) reduces to:

$$(p(a+bu)-c)\Pi^2 + a\Pi^2 q + \Pi^2(buq - bv) = 0 \tag{3.12}$$

for all $\Pi = [\pi_1, \pi_2] \in [\mathcal{R}, \mathcal{R}]$. Then, from Lemma 3.1, one of the following holds:

- $\mathcal{R}$ satisfies s_4, which is our Conclusion 5.
- $q \in C$ and $p(a+bu) - c + aq = -buq + bv \in C$. Thus, in this case, we get our Conclusion (2).
- $q, (buq - bv) \in C$, which implies $bv \in C$. Thus, from Eq (3.12), we get $(pa + pbu - c + aq + buq - bv)\Pi^2 = 0$, which implies $(a+bu)q = (c+bv)-(a+bu)p \in C$. Hence, in this case, we get $\Delta_1(\pi) = (a+bu)\pi$ and $\Delta_2(\pi) = (c+bv)\pi$ for all $\pi \in \mathcal{R}$ with $(a+bu)q = (c+bv)-(a+bu)p \in C$, which is our Conclusion (3).
- $(p(a+bu) - c) \in C$. Then, from Eq (3.12), we get $\Pi^2(pa + pbu - c + aq + buq - bv) = 0$, which implies $(p+q)(a+bu) = (c+bv)$. Thus, in this case, we get $\Delta_1(\pi) = (a+bu)\pi$, $\Delta_2(\pi) = c\pi + \pi bv$ for all $\pi \in \mathcal{R}$ with $(p+q)(a+bu) = (c+bv)$, which is our Conclusion (2).
- $(buq - bv), a \in C$, and $p(a+bu) - c + buq - bv = -aq \in C$. Since $buq - bv \in C$, we have $p(a+bu) - c \in C$. Thus, in this case, we get our Conclusion (2).
- There exist $\eta, \lambda, \mu \in C$ such that $(buq-bv)+\eta q = \lambda$, $a-\eta = \mu$, and $(p(a+bu)-c)+\lambda = -\mu q \in C$. If $\mu \neq 0$, then $q \in C$, which implies $bv \in C$. Thus, by previous arguments, we get our Conclusion (3). If $\mu = 0$, then $p(a+bu) - c \in C$, and by previous arguments, we get our Conclusion (2).

3) Suppose $a, q, buq \in C$, then Eq (3.10) reduces to:

$$(ap + aq + buq - c)\Pi^2 + p\Pi^2 bu - \Pi^2 bv = 0 \tag{3.13}$$

for all $\Pi \in [\mathcal{R}, \mathcal{R}]$. Then, from Lemma 3.1, one of the following holds:

- $\mathcal{R}$ satisfies s_4, which is our Conclusion (5).
- $bu \in C$ and $ap + aq + buq - c + pbu = bv \in C$. Therefore, we have $p(a+bu) - c \in C$. Also, we have $(a+bu)q \in C$. If $a + bu = 0$, then $c + bv = 0$, and we get our conclusion (1). If $a + bu \neq 0$, then we get $q \in C$; thus, in this situation, we get Conclusion (3).
- $bu, bv \in C$, and our functions take the form $\Delta_1(\pi) = (a+bu)\pi$, $\Delta_2(\pi) = (c+bv)\pi$ for all $\pi \in \mathcal{R}$ with $(a+bu)q = (c+bv) - (a+bu)p$, which is our Conclusion (3).
- $(ap + aq + buq - c), p \in C$; this gives $p + q \in C$, a contradiction.
- $p, bv \in C$; this gives $p + q \in C$, a contradiction.
- There exist $\eta, \lambda, \mu \in C$ such that $-bv + \eta bu = \lambda$, $p - \eta = \mu$, and $(ap + aq + buq - c) + \lambda = -\mu bu \in C$. This gives that $p = \eta + \mu \in C$, and, hence, $p + q \in C$, a contradiction.

Sub-case (b): If $t^{-1}u \in C$, then Eq (3.9) reduces to:

$$p\Pi a\Pi + p\Pi bu\Pi + a\Pi^2 q + bu\Pi^2 q - c\Pi^2 - bt\Pi^2 t^{-1} v = 0 \tag{3.14}$$

for all $\Pi \in [\mathcal{R}, \mathcal{R}]$. Again, by previous arguments, one of the following holds:

a) $p, p(a+bu) \in C$,
b) $a + bu \in C$.

a) Now, if $p, p(a + bu) \in C$, then Eq (3.14) reduces to the following:

$$(p(a + bu) - c)\Pi^2 + (a + bu)\Pi^2 q - bt\Pi^2 t^{-1} v = 0. \tag{3.15}$$

Now, by Lemma 3.1, one of the following holds:

- $\mathcal{R}$ satisfies s_4, which is our Conclusion (5).
- $q, bt \in C$, which implies $p + q \in C$, a contradiction.
- $t^{-1}v, q \in C$, which implies $p + q \in C$, a contradiction.
- $p(a + bu) - c, (a + bu), bt \in C$ with $p(a + bu) - c + (a + bu)q = bv$, which implies $c \in C$ and $p(a + bu) = -q(a + bu) + (c + bv) \in C$. Thus, in this situation, we get $\Delta_1(\pi) = (a + bu)\pi$ and $\Delta_2(\pi) = \pi(c + bv)$ for all $\pi \in \mathcal{R}$, which is our Conclusion (1).
- $(a + bu), t^{-1}v \in C$ with $p(a + bu) - c - bv = -(a + bu)q \in C$. If $a + bu = 0$, then $c + bv = 0$ and, thus, we get our Conclusion (1). Now, if $a + bu \neq 0$, then $q \in C$. Therefore, $p + q \in C$, which is a contradiction.
- There exist $\eta, \lambda, \mu \in C$ such that $t^{-1}v + \eta q = \lambda \in C$, $(a + bu) + \eta bt = \mu \in C$, and $p(a + bu) - c - \lambda bt = -\mu q \in C$. If $\mu \neq 0$, then $q \in C$ and, thus, $p + q \in C$, a contradiction.
  Again, if $\mu = 0$, then $t^{-1}v + \eta q = \lambda \in C$, $(a + bu) + \eta bt = 0$, and $p(a + bu) - c = \lambda bt$. Thus, in this situation, we get our Conclusion (4).

b) Now, if $(a + bu) \in C$, then Eq (3.14) transforms into the following:

$$(p(a + bu) - c)\Pi^2 - bt\Pi^2 t^{-1} v + \Pi^2(a + bu)q = 0 \tag{3.16}$$

for all $\Pi \in [\mathcal{R}, \mathcal{R}]$. By Lemma 3.1, one of the following holds:

- $\mathcal{R}$ satisfies s_4, which is our Conclusion (5).
- $t^{-1}v \in C$ and $(p(a+bu)-c)-bv = -(a+bu)q \in C$. In this situation, we get $\Delta_1(\pi) = (a+bu)\pi$ and $\Delta_2(\pi) = (c + bv)\pi$ for all $\pi \in \mathcal{R}$ with $q(a + bu) = (c + bv) - p(a + bu)$. This is our Conclusion (2).
- $t^{-1}v, (a + bu)q \in C$, and $p(a + bu) + (a + bu)q - c - bv = 0$. In this situation, we get our conclusion from previous arguments.
- $(p(a + bu) - c), bt \in C$, and $p(a + bu) + (a + bu)q - c - bv = 0$. The functions Δ_1 and Δ_2 take the form $\Delta_1(\pi) = (a + bu)\pi$, $\Delta_2(\pi) = c\pi + \pi bv$ for all $\pi \in \mathcal{R}$ with $(p + q)(a + bu) = (c + bv)$. This is our Conclusion (2).
- $bt, (a+bu)q, t^{-1}v \in C$, and $p(a+bu)-c+(a+bu)q = bv \in C$. Thus, the functions Δ_1 and Δ_2 take the form $\Delta_1(\pi) = (a+bu)\pi$, $\Delta_2(\pi) = (c+bv)\pi$ for all $\pi \in \mathcal{R}$ with $q(a+bu) = (c+bv)-(a+bu)p$. This is our Conclusion (2).
- There exist $\eta, \lambda, \mu \in C$ such that $(a + bu)q + \eta t^{-1}v = \lambda$, $-bt - \eta = \mu$, and $(p(a + bu) - c) + \lambda = -\mu t^{-1}v \in C$. If $\mu \neq 0$, then $q, t^{-1}v \in C$. Thus, from Eq (3.16), we get $(p + q)(a + bu) = (c + bv)$. Hence, in this situation, the functions Δ_1 and Δ_2 take the form $\Delta_1(\pi) = (a + bu)\pi$, $\Delta_2(\pi) = (c + bv)\pi$ for all $\pi \in \mathcal{R}$. This is our Conclusion (3).
  Now, if $\mu = 0$, then $bt, p(a+bu)-c \in C$. Then, by previous arguments, we get our Conclusion (2).

Case 2: Since $\mathcal{R}$ and Q_r satisfy the same differential polynomial identities with coefficients in Q_r (see Fact 2.2), it follows from Eq (3.8) that:

$$p\Pi a\Pi + p\Pi b\alpha(\Pi)u + a\Pi^2 q + b\alpha(\Pi)u\Pi q - c\Pi^2 - b\alpha(\Pi^2)v = 0, \tag{3.17}$$

for all $\Pi = [\pi_1, \pi_2] \in [Q_r, Q_r]$. If α is an outer derivation, then by Fact 2.4, we have:

$$p[\pi_1,\pi_2]a[\pi_1,\pi_2] + p[\pi_1,\pi_2]b[\xi_1,\xi_2]u + a[\pi_1,\pi_2]^2q + b[\xi_1,\xi_2]u[\pi_1,\pi_2]q - c[\pi_1,\pi_2]^2 - b[\xi_1,\xi_2]^2v = 0, \tag{3.18}$$

for all $\pi_1, \pi_2, \xi_1, \xi_2 \in \mathcal{R}$. In particular, Q_r satisfies $b[\xi_1, \xi_2]^2 v = 0$, which implies either $b = 0$ or $v = 0$.

If $b = 0$, then Eq (3.18) simplifies to:

$$p[\pi_1,\pi_2]a[\pi_1,\pi_2] + a[\pi_1,\pi_2]^2q - c[\pi_1,\pi_2]^2 = 0, \tag{3.19}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Thus, from Sub-case (b) of Case 1, we reach our conclusions. Now, if $v = 0$, then Eq (3.18) reduces to:

$$p[\pi_1,\pi_2]a[\pi_1,\pi_2] + p[\pi_1,\pi_2]b[\xi_1,\xi_2]u + a[\pi_1,\pi_2]^2q + b[\xi_1,\xi_2]u[\pi_1,\pi_2]q - c[\pi_1,\pi_2]^2 = 0, \tag{3.20}$$

for all $\pi_1, \pi_2, \xi_1, \xi_2 \in \mathcal{R}$. Specifically, Q_r satisfies:

$$p[\pi_1,\pi_2]b[\xi_1,\xi_2]u + b[\xi_1,\xi_2]u[\pi_1,\pi_2]q = 0, \tag{3.21}$$

for all $\pi_1, \pi_2, \xi_1, \xi_2 \in \mathcal{R}$. Setting $\xi_1 = \pi_1$ and $\xi_2 = \pi_2$ in Eq (3.21), we get:

$$p[\pi_1,\pi_2]b[\pi_1,\pi_2]u + b[\pi_1,\pi_2]u[\pi_1,\pi_2]q = 0, \tag{3.22}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Now, by Lemma (3.4), either $b \in C$ or $u \in C$.

Sub-case 1: First, we assume that $b \in C$. Then, Eq (3.22) reduces to:

$$p[\pi_1,\pi_2]^2bu + [\pi_1,\pi_2]bu[\pi_1,\pi_2]q = 0, \tag{3.23}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Similarly, by parallel arguments, we obtain $bu \in C$ or $q, buq \in C$.

Assume that $bu \in C$. If $bu \neq 0$, then from Eq (3.23), we have:

$$p[\pi_1,\pi_2]^2 + [\pi_1,\pi_2]^2q = 0, \tag{3.24}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Then, by Lemma 3.6, $\mathcal{R}$ either satisfies s_4, which is our conclusion, or $p + q = 0$, a contradiction.

If $bu = 0$, then either $b = 0$ or $u = 0$. If $b = 0$, we conclude as before. Assuming $u = 0$, then from Eq (3.20), Q_r satisfies:

$$p[\pi_1,\pi_2]a[\pi_1,\pi_2] + a[\pi_1,\pi_2]^2q - c[\pi_1,\pi_2]^2 = 0, \tag{3.25}$$

for all $\pi_1, \pi_2 \in \mathcal{R}$. Eq (3.25) is analogous to Eq (3.14), and thus, we reach the required conclusion by previous arguments.

Now, assume that $q, buq \in C$. If $q \neq 0$, then $bu \in C$. Thus, the conclusion follows from the previous argument. If $q = 0$, then from Eq (3.23), we get $p[\pi_1, \pi_2]^2 bu = 0$. It follows from [25] that either $p = 0$ or $bu = 0$. If $p = 0$, then $p + q = 0 \in C$, a contradiction. Again, if $bu = 0$, then we get our conclusion from previous arguments.

Suba-case 2: If $u \in C$, then we get our conclusion by previous arguments. □

4. Proof of Theorem 1.6

In this final section, we aim to prove the main result, Theorem 1.6. Throughout the proof, we assume that $\mathcal{R}$ does not satisfies s_4. According to [26], there exist elements $a, c \in Q_r$ and skew derivations d and g associated with the automorphism α, such that $\Delta_1(\pi) = a\pi + bd(\pi)$ and $\Delta_2(x) = c\pi + bg(\pi)$ for all $\pi \in \mathcal{R}$. Given that $\mathcal{L}$ is noncentral and the characteristic of $\mathcal{R}$ is not 2, there is a nonzero ideal $\mathcal{J}$ of $\mathcal{R}$ such that $0 \neq [\mathcal{J}, \mathcal{R}] \subseteq \mathcal{L}$ (see [27], p.45; [28], Lemma 2 and Proposition 1; [29], Theorem 4). Consequently, we have:

$$p\Pi\Delta_1(\Pi) + \Delta_1(\Pi)\Pi q = \Delta_2(\Pi^2), \quad \text{for all } \Pi \in [\mathcal{J}, \mathcal{J}].$$

Since $\mathcal{R}$, Q_r, and $\mathcal{J}$ satisfy the same generalized differential identities, the following holds for all $X \in [\mathcal{R}, \mathcal{R}]$:

$$p\Pi\Delta_1(\Pi) + \Delta_(\Pi)\Pi q = \Delta_2(\Pi^2).$$

Thus, Q_r satisfies:

$$\begin{aligned} p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]bd([\pi,\xi]) + a[\pi,\xi]^2q + bd([\pi,\xi])[\pi,\xi]q \\ -c[\pi,\xi]^2 - bg([\pi,\xi]^2) = 0, \end{aligned} \tag{4.1}$$

for all $\pi, \xi \in \mathcal{R}$.

d is a skew inner derivation and g is a skew outer derivation.

Since d is a skew inner derivation of $\mathcal{R}$, there exists an element $b' \in Q_r$ such that $d(\pi) = b'\pi - \alpha(\pi)b'$ for all $\pi \in \mathcal{R}$. Substituting this into Eq (4.1), we obtain:

$$\begin{aligned} p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b\left(b'[\pi,\xi] - \alpha([\pi,\xi])b'\right) + a[\pi,\xi]^2q \\ +b\left(b'[\pi,\xi] - \alpha([\pi,\xi])b'\right)[\pi,\xi]q - c[\pi,\xi]^2 - bg([\pi,\xi]^2) = 0 \end{aligned} \tag{4.2}$$

for all $\pi, \xi \in \mathcal{R}$.

Applying the definition of g, we have:

$$\begin{aligned} p[\pi,\xi]a[\pi,\xi] - c[\pi,\xi]^2 + p[\pi,\xi]b\left(b'[\pi,\xi] - \alpha([\pi,\xi])b'\right) + a[\pi,\xi]^2q + b(b'[\pi,\xi] \\ -\alpha([\pi,\xi])b')[\pi,\xi]q - b\Big\{(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))[\pi,\xi] \\ +\alpha([\pi,\xi])(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))\Big\} = 0, \end{aligned} \tag{4.3}$$

for all $\pi, \xi \in \mathcal{R}$. Since g is a skew outer derivation, applying Chuang's theorem (see Fact 2.4) to Eq (4.3), we obtain:

$$\begin{aligned} p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b\left(b'[\pi,\xi] - \alpha([\pi,\xi])b'\right) + a[\pi,\xi]^2q + b(b'[\pi,\xi] \\ -\alpha([\pi,\xi])b')[\pi,\xi]q - c[\pi,\xi]^2 - b\Big\{(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1)[\pi,\xi] \\ +\alpha([\pi,\xi])(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1)\Big\} = 0, \end{aligned} \tag{4.4}$$

for all $\pi, \xi, s_1, s_2 \in \mathcal{R}$. Specifically, setting $s_2 = 0$ in Eq (4.4), we obtain:

$$b\{(\alpha(\pi)s_2 - s_2\pi)[\pi,\xi] + \alpha([\pi,\xi])(\alpha(\pi)s_2 - s_2\pi)\} = 0, \tag{4.5}$$

for all $\pi, \xi, s_2 \in \mathcal{R}$.

Now, if the automorphism α is not inner, then according to [30], Eq (4.5) simplifies to:

$$b\{(s_3s_2 - s_2\pi)[\pi,\xi] + [s_3, s_4](s_3s_2 - s_2\pi)\} = 0,$$

for all $\pi, \xi, s_3, s_2 \in \mathcal{R}$. In particular, we have:

$$2b[\pi,\xi]^2 = 0 \implies b = 0,$$

which implies that both Δ_1 and Δ_2 are inner b-generalized skew derivations, contradicting our initial assumption.

Furthermore, if α is an inner automorphism, then there exists some $t \in Q_r$ such that $\alpha(\pi) = t\pi t^{-1}$, and Eq (4.5) simplifies to:

$$b\left\{\left(t\pi t^{-1}s_2 - s_2\pi\right)[\pi,\xi] + t[\pi,\xi]t^{-1}\left(t\pi t^{-1}s_2 - s_2\pi\right)\right\} = 0, \tag{4.6}$$

for all $\pi, \xi, s_2 \in \mathcal{R}$.

In particular, setting $s_2 = t\xi$, we get:

$$2bt[\pi,\xi]^2 = 0 \implies bt = 0 \Rightarrow b = 0,$$

which again leads to a contradiction.

d is skew outer and g is skew inner derivation.

In this scenario, there exists an element $c \in Q_r$ such that $g(\pi) = c\pi - \alpha(\pi)c$ for all $\pi \in \mathcal{R}$. Consequently, Eq (4.1) simplifies to:

$$\begin{aligned} p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b\,(d(\pi)\xi + \alpha(\pi)d(\xi) - d(\xi)\pi - \alpha(\xi)d(\pi)) \\ +a[\pi,\xi]^2q + b\,(d(\pi)\xi + \alpha(\pi)d(\xi) - d(\xi)\pi - \alpha(\xi)d(\pi))[\pi,\xi]q \\ -c[\pi,\xi]^2 - b\left(c[\pi,\xi]^2 - \alpha([\pi,\xi]^2)c\right) = 0 \end{aligned} \tag{4.7}$$

for all $\pi, \xi \in \mathcal{R}$. Since d is an outer derivation, by applying Fact 2.4, Eq (4.7) further reduces to:

$$\begin{aligned} p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b\,(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1) - c[\pi,\xi]^2 \\ -b(c[\pi,\xi]^2 + a[\pi,\xi]^2q + b\,(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1)[\pi,\xi]q) - \alpha([\pi,\xi]^2)c = 0 \end{aligned} \tag{4.8}$$

for all $\pi, \xi, s_1, s_2 \in \mathcal{R}$. In particular, setting $s_2 = 0$ in Eq (4.8), Q_r satisfies the following:

$$p[\pi,\xi]b\,(\alpha(\pi)s_2 - s_2\pi) + b\,(\alpha(\pi)s_2 - s_2\pi)[\pi,\xi]q = 0 \tag{4.9}$$

for all $\pi, \xi, s_2 \in \mathcal{R}$.

Now, if the automorphism α is not inner, then according to [30], Eq (4.9) simplifies to:

$$p[\pi,\xi]b\,(z_1s_2 - s_2\pi) + b\,(z_1s_2 - s_2\pi)[\pi,\xi]q = 0$$

for all $\pi, \xi, z_1, s_2 \in \mathcal{R}$. Specifically, we have:

$$p[\pi,\xi]b[\pi,\xi] + b[\pi,\xi]^2 q = 0 \tag{4.10}$$

for all $\pi, \xi \in \mathcal{R}$. By Lemma 3.5, it follows that either $p, pb \in C$, or $b \in C$.

First, assume that $p, pb \in C$. If $p = 0$, then from Eq (4.10), we get $q = 0$, implying $p + q = 0$, which is a contradiction. If $p \neq 0$, then $b \in C$, and thus, from Eq (4.10) we obtain $[\pi,\xi]^2(p+q) = 0$, implying $p + q = 0$, which is also a contradiction.

Next, if $b \in C$, similar arguments lead to the conclusion that $p + q = 0$, again resulting in a contradiction.

Moreover, if the automorphism α is inner, there exists an element $t \in Q_r$ such that $\alpha(\pi) = t\pi t^{-1}$. In this case, Eq (4.9) reduces to:

$$p[\pi,\xi]b\left(t\pi t^{-1}s_2 - s_2\pi\right) + b\left(t\pi t^{-1}s_2 - s_2\pi\right)[\pi,\xi]q = 0 \tag{4.11}$$

for all $\pi, \xi, s_2 \in \mathcal{R}$. Substituting $s_2 = t\xi$ into Eq (4.11), we obtain:

$$p[\pi,\xi]bt[\pi,\xi] + bt[\pi,\xi]^2 q = 0 \tag{4.12}$$

for all $\pi, \xi \in \mathcal{R}$. Since Eq (4.12) is analogous to Eq (4.10), the previous arguments lead us to the same contradiction.

Now, let's consider the case where both d and g are skew outer derivations. Then, we have the following scenarios:

d and g are C-modulo independent.

In this case, after applying the definitions of d and g, Q_r satisfies the following equation:

$$\begin{aligned}
&p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b(d(\pi)\xi + \alpha(\pi)d(\xi) - d(\xi)\pi - \alpha(\xi)d(\pi)) \\
&+a[\pi,\xi]^2 q + b(d(\pi)\xi + \alpha(\pi)d(\xi) - d(\xi)\pi - \alpha(\xi)d(\pi))[\pi,\xi]q \\
&-c[\pi,\xi]^2 - b\Big\{(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))[\pi,\xi] \\
&+\alpha([\pi,\xi])(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))\Big\} = 0
\end{aligned} \tag{4.13}$$

for all $\pi, \xi \in \mathcal{R}$. Then, by Chuang's theorem (see Fact 2.4), Eq (4.13) reduces to

$$\begin{aligned}
&p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1) \\
&+a[\pi,\xi]^2 q - c[\pi,\xi]^2 + b(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1)[\pi,\xi]q \\
&-b\Big\{(s_3\xi + \alpha(\pi)s_4 - s_4\pi - \alpha(\xi)s_3)[\pi,\xi] \\
&+\alpha([\pi,\xi])(s_3\xi + \alpha(\pi)s_4 - s_4\pi - \alpha(\xi)s_3)\Big\} = 0
\end{aligned} \tag{4.14}$$

for all $\pi, \xi, s_1, s_2, s_3, s_4 \in \mathcal{R}$. Choosing $s_4 = 0$ in Eq (4.14), we obtain:

$$b\Big\{(\alpha(\pi)s_2 - s_2\pi)[\pi,\xi] + \alpha([\pi,\xi])(\alpha(\pi)s_2 - s_2\pi)\Big\} = 0 \tag{4.15}$$

for all $s_2, \pi, \xi \in \mathcal{R}$. Now, Eq (4.15) is analogous to Eq (4.5). Therefore, using similar arguments as in Case 1, we arrive at a contradiction.

d and g are C-modulo dependent.

Consider the case where $\Delta_1(\pi) = a\pi + bd(\pi)$ and $\Delta_2(\pi) = c\pi + bg(\pi)$ for all $\pi \in \mathcal{R}$, where $a, c \in Q_r$ are suitable constants, and d and g are nonzero skew derivations of $\mathcal{R}$ associated with the automorphism α. Additionally, assume that d and g are linearly C-dependent modulo inner skew derivations. Then, there exist $\eta, \tau \in C$, $v \in Q_r$, and an automorphism ϕ of $\mathcal{R}$ such that

$$\eta d(\pi) + \tau g(\pi) = v\pi - \phi(\pi)v, \quad \text{for all } \pi \in \mathcal{R}.$$

Case 1: $\eta \neq 0$ and $\tau \neq 0$. Then, we have

$$d(\pi) = a_1 g(\pi) + (a_2\pi - \phi(\pi)a_2), \quad \text{for all } \pi \in \mathcal{R}, \quad \text{where } a_1 = -\eta^{-1}\tau, \quad a_2 = \eta^{-1}v.$$

It is important to note that if d is an inner skew derivation, then, according to Fact 2.9, g also becomes an inner skew derivation. In this case, the conclusion follows directly from Proposition 3.7. Therefore, in the following analysis, we will assume that d is a nonzero outer skew derivation. Consequently, using Fact 2.8, we conclude that either $\phi = \alpha$ or $a_2 = 0$. Summarizing, we reach one of the following conclusions:

a) $d(\pi) = a_1 g(\pi) + (a_2\pi - \alpha(\pi)a_2)$,
b) $d(\pi) = a_1 g(\pi)$.

We now demonstrate that each of these conditions leads to a contradiction. For brevity, we focus on Case 1, as it can be shown that Case 2 follows from Case 1. Thus, let $d(\pi) = a_1 g(\pi) + (a_2\pi - \alpha(\pi)a_2)$ for all $\pi \in \mathcal{R}$.

Thus, from Eq (4.1), we have

$$\begin{aligned} &p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b(a_1 g([\pi,\xi]) + (a_2[\pi,\xi] - \alpha([\pi,\xi])a_2)) - c[\pi,\xi]^2 \\ &+a[\pi,\xi]^2 q + b(a_1 g([\pi,\xi]) + (a_2[\pi,\xi] - \alpha([\pi,\xi])a_2))[\pi,\xi]q - bg([\pi,\xi])^2 = 0 \end{aligned} \tag{4.16}$$

for all $\pi, \xi \in \mathcal{R}$. Using the definition of g in Eq (4.16), we have

$$\begin{aligned} &p[\pi,\xi]a[\pi,\xi] + a[\pi,\xi]^2 q + b\{a_1(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))\}[\pi,\xi]q \\ &+p[\pi,\xi]b\Big\{a_1(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi)) + (a_2[\pi,\xi] - \alpha([\pi,\xi])a_2)\Big\} \\ &-b\Big\{(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))[\pi,\xi] + \alpha([\pi,\xi])(g(\pi)\xi + \alpha(\pi)g(\xi) \\ &-g(\xi)\pi - \alpha(\xi)g(\pi))\Big\} + b\{a_2[\pi,\xi] - \alpha([\pi,\xi])a_2\}[\pi,\xi]q - c[\pi,\xi]^2 = 0 \end{aligned} \tag{4.17}$$

for all $\pi, \xi \in \mathcal{R}$. Applying Fact 2.4 in Eq (4.17), we obtain:

$$\begin{aligned} &p[\pi,\xi]a[\pi,\xi] + a[\pi,\xi]^2 q - c[\pi,\xi]^2 \\ &+p[\pi,\xi]b\Big\{a_1(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1) + (a_2[\pi,\xi] - \alpha([\pi,\xi])a_2)\Big\} \\ &-b\Big\{(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1)[\pi,\xi] + \alpha([\pi,\xi])(s_1\xi + \alpha(\pi)s_2 - s_2\pi - \alpha(\xi)s_1)\Big\} \end{aligned}$$

$$+b\Big\{a_1(s_1\xi+\alpha(\pi)s_2-s_2\pi-\alpha(\xi)s_1)+(a_2[\pi,\xi]-\alpha([\pi,\xi])a_2)\Big\}[\pi,\xi]q=0 \tag{4.18}$$

for all $\pi,\xi,s_1,s_2\in\mathcal{R}$. Now, choosing $s_1=0$ in Eq (4.18), we obtain:

$$\begin{aligned}&p[\pi,\xi]b(a_1(s_1\xi-\alpha(\xi)s_1))+b(a_1(s_1\xi-\alpha(\xi)s_1))[\pi,\xi]q\\&-b\{(s_1\xi-\alpha(\xi)s_1)[\pi,\xi]+[\alpha(\pi),\alpha(\xi)](s_1\xi-\alpha(\xi)s_1)\}=0\end{aligned} \tag{4.19}$$

for all $\pi,\xi,s_1\in\mathcal{R}$. If the automorphism α is not inner, then from [15]

$$\begin{aligned}&p[\pi,\xi]b(a_1(s_1\xi-s_3s_1))+b(a_1(s_1\xi-s_3s_1))[\pi,\xi]q\\&-b\{(s_1\xi-s_3s_1)[\pi,\xi]+[s_1,s_3](s_1\xi-s_3s_1)\}=0\end{aligned} \tag{4.20}$$

for all $\pi,\xi,s_1,s_3\in\mathcal{R}$. In particular, choosing $\pi=0$ and $\xi=s_3$ in Eq (4.20), we have

$$b[s_1,s_3]^2=0 \tag{4.21}$$

for all $s_1,s_3\in\mathcal{R}$, which implies that $b=0$, a contradiction. Suppose the automorphism α is inner, then there exists $t\in\mathcal{Q}_r$ such that $\alpha(\pi)=t\pi t^{-1}$, and Eq (4.19) takes the form

$$\begin{aligned}&p[\pi,\xi]b(a_1(s_1\xi-t\xi t^{-1}s_1))+b(a_1(s_1\xi-t\xi t^{-1}s_1))[\pi,\xi]q\\&-b\{(s_1\xi-t\xi t^{-1}s_1)[\pi,\xi]+[t\pi t^{-1},t\xi t^{-1}](s_1\xi-t\xi t^{-1}s_1)\}=0\end{aligned} \tag{4.22}$$

for all $\pi,\xi,s_1\in\mathcal{R}$. In particular, choosing $s_1=ts_1$ and $\pi=t\pi$ in Eq (4.22), we obtain:

$$\begin{aligned}&p[\pi,\xi]bt(a_1(s_1\xi-\xi s_1))+b(a_1t(s_1\xi-\xi s_1))[\pi,\xi]q\\&-bt\{(s_1\xi-\xi s_1)[\pi,\xi]+[\pi,\xi](s_1\xi-\xi s_1)\}=0\end{aligned} \tag{4.23}$$

for all $\pi,\xi,s_1\in\mathcal{R}$. In particular, we get:

$$p[\pi_1,\pi_2]bta_1[\pi_1,\pi_2]+bta_1[\pi_1,\pi_2]^2q-2bt[\pi_1,\pi_2]^2=0 \tag{4.24}$$

for all $\pi_1,\pi_2\in\mathcal{R}$. Thus, from Lemma (3.4), we get $p+q\in C$, a contradiction.
Case 2: $\eta=0$ and $\tau\neq0$. Then, we have

$$g(\pi)=a_2\pi-\phi(\pi)a_2,\ \text{for all}\ \pi\in\mathcal{R},\ \text{where}\ a_2=\tau^{-1}v.$$

In this case, we can assume that the skew derivation d is not inner. If it were inner, the conclusion would follow from Proposition 3.7. Additionally, since the automorphism associated with a skew derivation is unique, in this scenario, we have $\phi=\alpha$. Therefore, $\mathcal{Q}_r$ satisfies:

$$\begin{aligned}&p[\pi,\xi]a[\pi,\xi]+p[\pi,\xi]bd([\pi,\xi])+a[\pi,\xi]^2q+bd([\pi,\xi])[\pi,\xi]q\\&-c[\pi,\xi]^2-b(a_2[\pi,\xi])^2-\alpha([\pi,\xi]^2a_2))=0\end{aligned} \tag{4.25}$$

for all $\pi,\xi\in\mathcal{R}$. Applying the definition of d in Eq (4.25), we get

$$p[\pi,\xi]a[\pi,\xi]+p[\pi,\xi]b\,\{d(\pi)\xi+\alpha(\pi)d(\xi)-d(\xi)\pi-\alpha(\xi)d(\pi)\}+a[\pi,\xi]^2q$$

$$-c[\pi,\xi]^2 + b\{d(\pi)\xi + \alpha(\pi)d(\xi) - d(\xi)\pi - \alpha(\xi)d(\pi)\}[\pi,\xi]q \\ -b(a_2[\pi,\xi])^2 - \alpha([\pi,\xi]^2 a_2)) = 0 \tag{4.26}$$

for all $\pi, \xi \in \mathcal{R}$. Then by using Fact 2.4 in Eq (4.26), we obtain:

$$p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b(s_2\xi + \alpha(\pi)s_1 - s_1\pi - \alpha(\xi)s_2) + a[\pi,\xi]^2 q - c[\pi,\xi]^2 \\ +b(s_2\xi + \alpha(\pi)s_1 - s_1\pi - \alpha(\xi)s_2)[\pi,\xi]q - b(a_2[\pi,\xi])^2 - \alpha([\pi,\xi]^2 a_2)) = 0 \tag{4.27}$$

for all $\pi, \xi, s_2, s_1 \in \mathcal{R}$. In particular, choosing $s_1 = 0$ in Eq (4.27), Q_r satisfies the blended component

$$p[\pi,\xi]b(\alpha(\pi)s_1 - s_1\pi) + b(\alpha(\pi)s_1 - s_1\pi)[\pi,\xi]q = 0 \tag{4.28}$$

for all $\pi, \xi, s_1 \in \mathcal{R}$. The above Eq (4.28) is similar to Eq (4.9), therefore this case also leads to a contradiction.

Case 3: $\eta \neq 0$ and $\tau = 0$. Then, we have

$$d(\pi) = a_1\pi - \phi(\pi)a_1, \text{ for all } \pi \in \mathcal{R}, \text{ where } a_1 = \eta^{-1}v.$$

Similar to Case 2, here we are assuming that g is not inner and $\alpha = \phi$. Hence, Q_r satisfies:

$$p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b(a_1[\pi,\xi] - \alpha([\pi,\xi]a_1)) + a[\pi,\xi]^2 q + b(a_1[\pi,\xi] \\ -b\Big\{(g(\pi)\xi + \alpha(\pi)g(\xi) - g(\xi)\pi - \alpha(\xi)g(\pi))[\pi,\xi] + \alpha([\pi,\xi])(g(\pi)\xi + \alpha(\pi)g(\xi) \\ -g(\xi)\pi - \alpha(\xi)g(\pi))\Big\} - \alpha([\pi,\xi]a_1))[\pi,\xi]q - c[\pi,\xi]^2 = 0 \tag{4.29}$$

for all $\pi, \xi \in \mathcal{R}$. Then, from Fact 2.4, we have

$$p[\pi,\xi]a[\pi,\xi] + p[\pi,\xi]b(a_1[\pi,\xi] - \alpha([\pi,\xi]a_1)) - \alpha([\pi,\xi]a_1))[\pi,\xi]q - c[\pi,\xi]^2 \\ -b\Big\{(s_2\xi + \alpha(\pi)s_1 - s_1\pi - \alpha(\xi)s_2)[\pi,\xi] + \alpha([\pi,\xi])(s_2\xi + \alpha(\pi)s_1 - s_1\pi - \alpha(\xi)s_2)\Big\} \\ +a[\pi,\xi]^2 q + b(a_1[\pi,\xi]) = 0 \tag{4.30}$$

for all $\pi, \xi, s_2, s_1 \in \mathcal{R}$. Choosing $s_1 = 0$ in Eq (4.30), we have

$$b\{\alpha(\pi)s_1 - s_1\pi)[\pi,\xi] + \alpha([\pi,\xi])(\alpha(\pi)s_1 - s_1\pi)\} = 0 \tag{4.31}$$

for all $\pi, \xi, s_1 \in \mathcal{R}$. The above (4.31) is similar to Eq (4.5), therefore, this case also leads to a contradiction.

5. Conclusions

In this article, we characterize all possible forms of b-generalized skew derivations Δ_1 and Δ_2 that satisfy the identity $p\pi\Delta_1(\pi)+\Delta_1(\pi)\pi q = \Delta_2(\pi^2)$ for all $\pi \in \mathcal{L}$. The advantage of the methodology used in this article is that it can be applied to all additive maps for which Facts 2.3 and 2.4 hold. Unfortunately, however, it does not hold for many additive maps for example, it is not valid for (α,β)-derivations [31].

It would be an interesting problem to study this identity further by considering the case $p + q \in C$ or by examining the identity $p\pi\Delta_1(\pi) + \Delta_1(\pi)\Delta_3(\pi) = \Delta_2(\pi^2)$, for all $\pi \in \mathcal{L}$, where Δ_3 is another b-generalized skew derivation.

Author contributions

Ashutosh Pandey: contributed to the conceptualization and formulation of the problem, conducted a significant portion of the mathematical analysis, and participated in drafting the manuscript; Mani Shankar Pandey: played a central role in developing the theoretical framework, performing key computations, and revising the manuscript; Omaima Alshanqiti: provided guidance on the research methodology, assisted with the mathematical proofs, and was involved in reviewing and refining the manuscript. All authors read and approved the final version of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest or competing interests that could have influenced the results and/or discussion presented in this paper.

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