AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 34147–34183.
DOI: 10.3934/math.20241627
Received: 17 October 2024
Revised: 24 November 2024
Accepted: 28 November 2024
Published: 04 December 2024

Research article

On the Cauchy problem of compressible Micropolar fluids subjected to Hall current

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Abstract: In this paper, the large-time behavior of global strong solutions is justified for the three dimensional compressible micropolar fluids subjected to Hall current. Both the global existence and the optimal decay rates of strong solutions are obtained when the smooth initial data are sufficiently close to the non-vacuum equilibrium in H^1. In addition, the vanishing limit of the Hall coefficient is also justified.

Keywords: micropolar fluids; Hall current; optimal decay rates; vanishing Hall limit
Mathematics Subject Classification: 35A01, 35Q35, 76W05

1. Introduction

In this paper, we are interested in the mathematical analysis of the equations in a viscous, electrically conducting, micropolar fluids in the presence of a magnetic field, taking into account the effect of Hall current. When the strength of the magnetic field is sufficiently large, Ohm's law $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \boldsymbol{b})$ needs to be modified to include Hall current so that the electric current density $\mathbf{J}$ satisfies the relation

$$\mathbf{J} + \omega_e \tau_e (\mathbf{J} \times \boldsymbol{b}) = \sigma\Big(\mathbf{E} + \mathbf{u} \times \boldsymbol{b} + \frac{1}{e n_e} \nabla p_e\Big),$$

where $\mathbf{E}, \boldsymbol{b}$, and $\mathbf{u}$ stand for the electric field, magnetic induction, and the fluid velocity, respectively. $\omega_e, \tau_e, \sigma, e$, and p_e are the cyclotron frequency of electrons, electron collision time, electric conductivity, electron charge, and electron pressure, respectively (see [7, p.101]). Then the electric field $\mathbf{E}$ can be written as

$$\mathbf{E} = \frac{1}{\sigma}\Big(\mathbf{J} + \omega_e \tau_e (\mathbf{J} \times \boldsymbol{b})\Big) - \mathbf{u} \times \boldsymbol{b} - \frac{1}{e n_e} \nabla p_e.$$

Faraday's and Ampère's laws yield

$$\partial_t \boldsymbol{b} + \nabla \times \mathbf{E} = 0, \quad \nabla \times \boldsymbol{b} = \mu_e \mathbf{J},$$

where μ_e is the magnetic permeability constant. Thus, the generalized magnetic induction equation with the Hall effect has the following form:

$$\partial_t \boldsymbol{b} + \nabla \times \Big(\frac{1}{\mu_e \sigma}(\nabla \times \boldsymbol{b} + \omega_e \tau_e \nabla \times \boldsymbol{b} \times \boldsymbol{b})\Big) = \nabla \times (\mathbf{u} \times \boldsymbol{b}).$$

The theory of micropolar fluids was developed by Eringen [10–12] and a generalization including the effects of magnetic fields has been developed by Ahmadi and Shahinpoor [2]. It has important engineering applications, such as in the extraction of oils/gas from oil fields, fluid flow in chemical engineering and magnetohydrodynamic (MHD) generators with neutral fluid seeding in the form of rigid microinclusions (see [12] and the references therein).

This paper is concerned with the mathematical analysis of the equations describing the flow in viscous, electrically conducting, micropolar fluids in the presence of a magnetic field, taking into account the effect of Hall current. The three-dimensional (3D) compressible, viscous micropolar fluids subject to Hall current occupying a domain $\Omega \subset \mathbb{R}^3$ are given by

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) \\ \qquad = (\mu_1 + \zeta)\Delta\mathbf{u} + (\mu_1 + \lambda_1 - \zeta)\nabla\mathrm{div}\mathbf{u} + 2\zeta\nabla \times \mathbf{w} + (\nabla \times \boldsymbol{b}) \times \boldsymbol{b}, \\ (\rho\mathbf{w})_t + \mathrm{div}(\rho\mathbf{u} \otimes \mathbf{w}) + 4\zeta\mathbf{w} = \mu_2\Delta\mathbf{w} + (\mu_2 + \lambda_2)\nabla\mathrm{div}\mathbf{w} + 2\zeta\nabla \times \mathbf{u}, \\ \boldsymbol{b}_t - \nabla \times (\mathbf{u} \times \boldsymbol{b}) + \beta\nabla \times \Big(\dfrac{(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}}{\rho}\Big) = \nu\Delta\boldsymbol{b}, \\ \mathrm{div}\boldsymbol{b} = 0. \end{cases} \tag{1.1}$$

Here, $\rho \geqslant 0, \mathbf{u} = (u^1, u^3, u^3), \mathbf{w} = (w^1, w^2, w^3)$ and $\boldsymbol{b} = (b^1, b^2, b^3)$ stand for the density, velocity, micro-rotational velocity and magnetic field, respectively. $p(\rho) = a\rho^\gamma$ $(a > 0,\ \gamma > 1)$ is the pressure. The constants μ_1 and λ_1 denote the shear and bulk viscosity coefficients, ζ is the dynamics micro-rotation viscosity, μ_2, and λ_2 are the angular viscosities, ν is the magnetic diffusivity of the magnetic field and β is the Hall coefficient, and they satisfy

$$\mu_1 > 0, \mu_2 > 0, \zeta > 0, \nu > 0, \beta > 0, \quad 2\mu_1 + 3\lambda_1 - 4\zeta \geqslant 0, \quad 2\mu_2 + 3\lambda_2 \geqslant 0.$$

Let $\Omega = \mathbb{R}^3$. We consider the Cauchy problem of (1.1) with the far-field behavior

$$(\rho, \mathbf{u}, \mathbf{w}, \boldsymbol{b})(x, t)\Big|_{|x|\to\infty} \to (1, 0, 0, 0), \qquad t \geqslant 0, \tag{1.2}$$

and the initial data

$$(\rho, \mathbf{u}, \mathbf{w}, \boldsymbol{b})(x, t)\Big|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)(x), \qquad x \in \mathbb{R}^3. \tag{1.3}$$

Micropolar fluids are fluids with microstructures belonging to a class of fluids with a nonsymmetric stress tensor. They exhibit micro-rotation effects and micro-rotational inertia. Liquids, fluids with additives, some polymeric fluids, colloidal fluids, and animal blood are a few examples of micropolar fluids. The theory of micropolar fluids was introduced by Eringen [10] and subsequently extended widely to the case of electrically conducting fluids in the magnetic field and to polarized fluids in an electric field. With the deepening of the research of micropolar fluid theory, the fluids considering the

Hall effect attracted more and more attention. It has important engineering applications, such as in the extraction of oils/gas from oil fields, fluid flow in chemical engineering, and magnetohydrodynamic generators with neutral fluids seeding in the form of microinclusions.

There is much literature on the Cauchy problems of the micropolar system. If the magnetic field $\mathbf{b} = 0$, then the system (1.1) reduces to the classical micropolar fluid system, which has been successfully applied for modeling rheologically complex liquids such as blood and suspensions (see, e.g., [10–12]). Physically it may represent the fluids consisting of bar-like elements. Mujaković [24, 25] studied the one-dimensional problem. Amirat, Hamdache [3] proved the existence of a global weak solution in a bounded domain in $\mathbb{R}^3$. Later, Chen et al. [5] obtained the global weak solutions of 3D compressible micropolar fluids with discontinuous initial data and vacuum.

From the mathematical viewpoint, the system (1.1) becomes the classical compressible magneto-micropolar fluids provided $\beta = 0$. Compared with the classical magneto-micropolar fluids, the system (1.1) has the Hall term $\frac{(\nabla\times \boldsymbol{b})\times \boldsymbol{b}}{\rho}$ in $(1.1)_4$, which is very important in describing many phenomena such as magnetic reconnection in space plasmas, star formation, neutron stars and geo-dynamo (see [16, 23, 27, 29] and references therein). When the Hall term is neglected, the system (1.1) reduces to the well-known compressible magneto-micropolar fluid system, which has received many studies (see [17, 28, 30, 32, 33, 35, 36]). Wei et al. [30] established the existence of global-in-time smooth solutions under the condition of the small perturbations of initial data in H^3-norm and also obtained the long-time behavior of magneto-micropolar fluids. Based on the time-weighted energy estimate, Zhang in [35] proved the asymptotic stability of steady state with the strictly positive constant density, vanishing velocity, micro-rotational velocity, and magnetic field. Their results were later improved by Tong and Tan [28], where they showed that the solution of the magneto-micropolar fluids converges to its constant equilibrium state at the exact same L^2-decay rate as the linearized equations, which shows that the convergence rate is optimal (see [6, 8, 35] for related results).

If we consider the effects of the Hall term in the MHD system, Xiang [31] investigated the large-time behavior of solutions to the 3D compressible Hall magneto-hydrodynamics equations in addition, he also obtained that the smooth solution of the compressible Hall-magneto-hydrodynamics system converges globally in time to the smooth solution of the compressible magneto-hydro-dynamics system as $\beta \to 0$. Later, Lai et al. [18] considered the 3D compressible full Hall-MHD equations, and they obtained the global existence and optimal decay rates when the initial data are appropriately small. Lai and Xu [19] established the global existence of strong solutions for planar compressible, viscous, heat-conductive Hall-MHD equations with large initial data (see also [13, 15, 34] for related results).

For the magneto-micropolar fluids with Hall term, it has attracted the attention of many physicists and mathematicians due to their important background, rich phenomena, mathematical complexity, and challenges (see [4, 21, 26]). Mekheimer and El Kot [21] investigated the influence of magnetic field and Hall current on blood flow through a stenotic artery. Rani and Tomar [26] investigated the thermal instability of a micropolar fluid layer heated from below in the presence of the Hall current, and showed that the Hall current parameter has a destabilizing effect on the system. Amirat and Hamdache [4] studied the system, which is a combination of the generalized magnetic induction, the equations of micropolar fluids, and the temperature equation, and obtained the global existence of weak solutions in a bounded domain of $\mathbb{R}^3$.

Motivated by the results as in [4, 18, 31], the aim of this paper is to study the large-time behavior of solutions to the 3D compressible micropolar fluids subjected to Hall current. Before stating the main results, we explain the notation and conventions used throughout this paper. We denote

$$\int f(x)\,dx = \int_{\mathbb{R}^3} f(x)\,dx.$$

For $1 \leqslant r \leqslant \infty, k \in \mathbb{Z}$ and $\alpha > 0$, we denote the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \quad D^{k,r} = \{\mathbf{u} \in L^1_{loc} \mid \|\nabla^k \mathbf{u}\|_{L^r} < \infty\}, \quad \|\mathbf{u}\|_{D^{k,r}} = \|\nabla^k \mathbf{u}\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D^1 = \{\mathbf{u} \in L^6 \mid \|\nabla \mathbf{u}\|_{L^2} < \infty\}. \end{cases}$$

We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space $L^2(\mathbb{R}^3)$, i.e.,

$$\langle f, g \rangle \triangleq \int_{\mathbb{R}^3} f(x) g(x)\,dx,$$

and set

$$\|(f, g)\|_{H^p} \triangleq \|f\|_{H^p(\mathbb{R}^3)} + \|g\|_{H^p(\mathbb{R}^3)}, \quad \text{for } p \geqslant 0.$$

We now state the definition of strong solutions of (1.1)–(1.3) as follows.

Definition 1.1. *A pair of functions* $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})$ *is called a strong solution to the problem* (1.1)–(1.3) *in* $\mathbb{R}^3 \times (0, \infty)$*, if and only if* $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})$ *satisfies* (1.1) *almost everywhere in* $\mathbb{R}^3 \times (0, \infty)$*, and belongs to the following class of functions in which the uniqueness can be shown to hold*

$$\begin{cases} \rho - 1 \in C([0, T]; H^3) \cap L^2(0, T; H^4), \quad \inf_{(x,t) \in \mathbb{R}^3 \times (0,\infty)} \rho(x, t) > 0, \\ (\mathbf{u}, \mathbf{w}, \mathbf{b}) \in C([0, T]; H^3) \cap L^2(0, T; H^4), \end{cases}$$

for any $0 < T < \infty$.

Now, we state our main results as follows. First, we will give the global existence and decay rates of strong solutions of the system (1.1) as follows.

Theorem 1.1. I. Global existence. *Suppose that* $(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^3$*. Then there exists a positive constant* ε_0*, depending only on* $\|\nabla^2(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1}$*,* $\mu_1, \lambda_1, \zeta, \mu_1, \lambda_2, \nu, \gamma$*, and a, such that the system* (1.1)–(1.3) *possesses a unique global strong solution* $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})$ *on* $\mathbb{R}^3 \times (0, \infty)$ *satisfying*

$$\begin{aligned} &\|(\rho - 1, \mathbf{u}, \mathbf{w}, \mathbf{b})(t)\|^2_{H^3} + \int_0^t \Big(\|\nabla \rho(s)\|^2_{H^2} + \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(s)\|^2_{H^3} \Big) ds \\ &\quad \leqslant C \|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|^2_{H^3}, \end{aligned} \tag{1.4}$$

for all $t \geqslant 0$*, provided*

$$\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1} \leqslant \varepsilon_0, \tag{1.5}$$

where the positive constant C is independent of β *and t.*

II. Decay rates. *Assume further that* $(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in L^1$*. Then there exists a positive constant* $\varepsilon_1 \in (0, \varepsilon_0]$*, depending on* $\|\nabla^2(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1}$ *and* $\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^1}$*, such that for any* $t \geqslant 0$

$$\|\nabla^m(\rho - 1, \mathbf{u}, \mathbf{w}, \mathbf{b})(t)\|_{L^2} \leqslant C(1 + t)^{-\left(\frac{3}{4} + \frac{m}{2}\right)}, \quad m = 0, 1, \tag{1.6}$$

and that for any $t \geqslant \tilde{T}$ *with* $\tilde{T} > 0$ *being large enough and depending on* $\|\nabla^2(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1}$ *and* $\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^1}$,

$$\begin{cases} \|(\nabla^2(\rho - 1, \mathbf{u}, \mathbf{w})(t)\|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}}, \\ \|\nabla^m \mathbf{b}(t)\|_{L^2} \leqslant C(1+t)^{-\left(\frac{3}{4}+\frac{m}{2}\right)}, \quad m = 2, 3, \end{cases} \tag{1.7}$$

provided $\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1} \leqslant \varepsilon_1$.

Remark 1.1. It was worth noting that all of our estimates are uniform in the Hall coefficient β in Theorem 1.1.

Remark 1.2. Compared with the decay estimates of the linear system (cf. Lemma 2.1), the decay rates stated in the second part of Theorem 1.1 are optimal, except for the one of $\|\nabla^3(\rho - 1, \mathbf{u}, \mathbf{w}, \mathbf{b})\|_{L^2}$. Indeed, if $(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^m$ with $m \geqslant 4$, then the optimal decay rates of the solutions can be obtained up to the $(m-1)$-th order derivatives of $(\rho-1, \mathbf{u}, \mathbf{w}, \mathbf{b})$ and the m-th order derivatives of $\mathbf{b}$. The lack of the optimal decay estimates of $\|\nabla^m(\rho-1, \mathbf{u}, \mathbf{w}, \mathbf{b})\|_{L^2}$ is mainly due to the insufficient dissipation of density and the strong coupling of fluid quantities.

Next, we will show that the unique smooth solution of the 3D compressible micropolar fluids subjected to Hall current converges globally in time to a smooth solution of the 3D compressible magneto-micropolar system as the Hall coefficient $\beta \to 0$.

Theorem 1.2 (Vanishing Hall limit)**.** *Suppose that* $(\rho^\beta, \mathbf{u}^\beta, \mathbf{w}^\beta, \mathbf{b}^\beta)$ *and* $(\rho^0, \mathbf{u}^0, \mathbf{w}^0, \mathbf{b}^0)$ *are two smooth solutions to Eq* (1.1) *obtained in Theorem 1.1 corresponding to* $\beta > 0$ *and* $\beta = 0$, *respectively. Then for any* $T \in (0, \infty)$, *it holds that*

$$\rho^\beta \to \rho^0, \quad \mathbf{u}^\beta \to \mathbf{u}^0, \quad \mathbf{w}^\beta \to \mathbf{w}^0 \quad \text{and} \quad \mathbf{b}^\beta \to \mathbf{b}^0 \qquad \text{in } C([0,T]; H^2), \tag{1.8}$$

as $\beta \to 0$. *Moreover, there exists a positive constant c depending on T such that*

$$\sup_{t\in[0,T]} \|(\rho^\beta - \rho^0, \mathbf{u}^\beta - \mathbf{u}^0, \mathbf{w}^\beta - \mathbf{w}^0, \mathbf{b}^\beta - \mathbf{b}^0)(t)\|_{H^2}^2 \leqslant c\beta^2. \tag{1.9}$$

The proofs of Theorems 1.1 and 1.2 are similar to the ones in [18, 31, 37], based on the standard L^2-method and the origin ideas developed by Matsumura and Nishida [22]. It is worth noting that though the H^1-perturbation of initial data is small, the higher-order derivatives could be of large oscillations. Compared with the results in [37], where the authors only obtained the optimal decay estimates for the L^p-norm ($p \in [2, 6]$) of the solution and the L^2-norm of its first-order derivative, the decay rates of both $\|(\nabla^2\rho, \nabla^2\mathbf{u}, \nabla^2\mathbf{w}, \nabla^2\mathbf{b})(t)\|_{L^2}$ and $\|\nabla^3\mathbf{b}\|_{L^2}$ for large $t > 0$ are also optimal in the present paper. This will be achieved by making full use of the H^1-decay estimates and the Sobolev interpolation inequalities (see (3.81) and (3.82)). The key point here is that all the estimates are uniform in the Hall coefficient β.

2. Reformulation

In this section, the global strong solutions near the state $(1, 0, 0, 0)$ to the Cauchy problem (1.1)–(1.3) will be constructed. Define

$$\varrho \triangleq \rho - 1, \quad \boldsymbol{v} \triangleq \mathbf{u}, \quad \omega \triangleq \mathbf{w}, \quad \boldsymbol{h} \triangleq \boldsymbol{b}.$$

Then, the quantities $(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})$ satisfy

$$\begin{cases}\varrho_t + \mathrm{div}\boldsymbol{v} = R_1,\\ \boldsymbol{v}_t - (\mu_1+\zeta)\Delta\boldsymbol{v} - (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}\boldsymbol{v} + a\gamma\nabla\varrho - 2\zeta\nabla\times\omega = R_2,\\ \omega_t - \mu_2\Delta\omega - (\mu_2+\lambda_2)\nabla\mathrm{div}\omega + 4\zeta\omega - 2\zeta\nabla\times\boldsymbol{v} = R_3,\\ \boldsymbol{h}_t - \nu\Delta\boldsymbol{h} = R_4, \quad \mathrm{div}\boldsymbol{h} = 0,\\ (\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\big|_{t=0} = (\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0) = (\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \boldsymbol{b}_0),\\ (\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0) \to (0,0,0,0), \quad \text{as } |x|\to\infty,\end{cases} \tag{2.1}$$

where the functions R_1, R_2, R_3, and R_4 are defined as

$$\begin{cases}R_1 \triangleq -\varrho\mathrm{div}\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\varrho,\\ R_2 \triangleq -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - f(\varrho)\Big[(\mu_1+\zeta)\Delta\boldsymbol{v} + (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}\boldsymbol{v}\Big]\\ \qquad\quad + g(\varrho)(\nabla\times\boldsymbol{h})\times\boldsymbol{h} - h(\varrho)\nabla\varrho - 2\zeta f(\varrho)\nabla\times\omega,\\ R_3 \triangleq -\boldsymbol{v}\cdot\nabla\omega - f(\varrho)\Big[\mu_2\Delta\omega + (\mu_2+\lambda_2)\nabla\mathrm{div}\omega\Big]\\ \qquad\quad + 4\zeta f(\varrho)\omega - 2\zeta f(\varrho)\nabla\times\boldsymbol{v},\\ R_4 \triangleq -\boldsymbol{v}\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\mathrm{div}\boldsymbol{v} + \boldsymbol{h}\cdot\nabla\boldsymbol{v} - \beta\nabla\times\Big[g(\varrho)[(\nabla\times\boldsymbol{h})\times\boldsymbol{h}]\Big],\end{cases} \tag{2.2}$$

and $f(\varrho), g(\varrho)$, and $h(\varrho)$ given by

$$f(\varrho) \triangleq \frac{\varrho}{1+\varrho}, \qquad g(\varrho) \triangleq \frac{1}{1+\varrho}, \quad \text{and} \quad h(\varrho) \triangleq a\gamma\Big((\varrho+1)^{\gamma-2} - 1\Big). \tag{2.3}$$

The left-hand side of (2.1) is indeed the linearized magneto-micropolar fluid equations. Thus, the $L^p - L^p$ time decay property of the linearized magneto-micropolar fluid equations of (2.1) can be obtained in similar arguments as used in [35, Theorem 2.1]. Thus, we can rewrite the solution of (2.1) as

$$V(t) = e^{-t\mathcal{L}}V_0 + \int_0^t e^{-(t-\tau)\mathcal{L}}(R_1, R_2, R_3, R_4)d\tau,$$

where $V \triangleq (\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})$, $V_0 \triangleq (\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)$, and $\mathcal{L}$ is a matrix-valued differential operator given by

$$\mathcal{L} \triangleq \begin{pmatrix} 0 & \mathrm{div} & 0 & 0\\ \gamma\nabla & -(\mu_1+\zeta)\Delta - (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div} & -2\zeta\nabla\times & 0\\ 0 & -2\zeta\nabla\times & 4\zeta - \mu_2\Delta - (\mu_2+\lambda_2)\nabla\mathrm{div} & 0\\ 0 & 0 & 0 & \nu\Delta\end{pmatrix}. \tag{2.4}$$

Therefore, due to [35], we have the following lemma.

Lemma 2.1. *Suppose that $V_0 \in L^1 \cap H^3$. Let $V \triangleq V(x,t)$, be the smooth solution of $V_t + \mathcal{L}V = 0$. Then for any $m \in \{0,1,2,3,4\}$,*

$$\|\nabla^m V(t)\|_{L^2} \leqslant C(1+t)^{-\left(\frac{3}{4}+\frac{m}{2}\right)}\Big(\|V_0\|_{L^1} + \|\nabla^m V_0\|_{L^2}\Big), \tag{2.5}$$

where $C > 0$ is a generic constant depending only on $\mu_1, \lambda_1, \zeta, \mu_2, \lambda_2$ and ν.

Lemma 2.1, together with the Duhamel principle, gives rise to the following lemma.

Lemma 2.2. *Assume that a quadruple of* $(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})$ *is the smooth solution of* (2.1) *with the initial data* $(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0) \in L^1 \cap H^3$. *Then for any* $m \in 0, 1, 2, 3$,

$$\begin{aligned}\|\nabla^m(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{L^2} \leqslant\, & C(1+t)^{-\left(\frac{3}{4}+\frac{m}{2}\right)}\|(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|_{L^1\cap H^m} \\ & + C\int_0^t (1+t-\tau)^{-\left(\frac{3}{4}+\frac{m}{2}\right)}\|(R_1, R_2, R_3, R_4)\|_{L^1\cap H^m}d\tau,\end{aligned} \tag{2.6}$$

where $C > 0$ *is a generic constant depending only on* $\mu_1, \lambda_1, \zeta, \mu_2, \lambda_2$ *and* ν.

Next, we will give the product and commutator estimates, which can be found in [20].

Lemma 2.3. *Suppose that* f *and* g *are the smooth functions in the Schwartz class. Then, for any* $s > 0$ *and* $1 < p < +\infty$, *there exists a generic positive constant* C *such that*

$$\|D^s(fg)\|_{L^p} \leqslant C\Big(\|f\|_{L^{p_1}}\|D^s g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\Big), \tag{2.7}$$

and

$$\|D^s(fg) - fD^s g\|_{L^p} \leqslant C\Big(\|\nabla f\|_{L^{p_1}}\|D^{s-1} g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\Big), \tag{2.8}$$

where $p_1, p_2 > 1$ *satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}. \tag{2.9}$$

The following lemma, which can be found in [37], is essential for deriving the decay rates.

Lemma 2.4. *Suppose that* $a, b, c \in \mathbb{R}$ *satisfy* $a \in [0, b], b \in (1, \infty)$ *and* $c \in (0, \infty)$. *Then, there exists a positive constant C, depending only on* a, b, *and* c, *such that for any* $t > 0$,

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}d\tau + \int_0^t (1+\tau)^{-a}e^{-c(t-\tau)}d\tau \leqslant C(1+t)^{-a}. \tag{2.10}$$

Finally, we recall the local existence theorem of (1.1)–(1.3) (also cf. Eqs (2.1)–(2.3)), which can be proved in a similar way to [13].

Lemma 2.5. *Suppose that the initial data satisfies*

$$(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^3, \qquad \inf_{x\in\mathbb{R}^3} \rho_0(x) > 0, \quad \mathrm{div}\mathbf{b}_0 = 0. \tag{2.11}$$

Then there exists a small positive time T_* *such that the problem* (1.1)–(1.3) *possesses a unique classical solution* $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})$ *on* $\mathbb{R}^3 \times [0, T_*]$ *satisfying*

$$(\rho - 1, \mathbf{u}, \mathbf{w}, \mathbf{b}) \in C([0, T_*]; H^3) \cap L^2(0, T_*; H^4), \qquad \inf_{(x,t)\in\mathbb{R}^3\times[0,T_*]} \rho(x, t) > 0. \tag{2.12}$$

Proof. In order to prove Lemma 2.5, we denote the Banach space

$$\mathcal{B} \triangleq \{\tilde{\mathbf{u}} \,\big|\, \|\tilde{\mathbf{u}}\|_{\mathcal{B}} \leqslant K\}$$

with the form

$$\|\tilde{\mathbf{u}}\|_{\mathcal{B}} \triangleq \|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^3)} + \|\tilde{\mathbf{u}}\|_{L^2(0,T;H^4)} + \|\partial_t\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)} + \|\partial_t\tilde{\mathbf{u}}\|_{L^2(0,T;H^2)}. \tag{2.13}$$

Let $\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\boldsymbol{b}}$ be given, the linear problem of (1.1)–(1.3) can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho\tilde{\mathbf{u}}) = 0, \qquad \lim\limits_{|x|\to\infty} \rho(x,t) = 1, \\ \rho(\cdot,0) = \rho_0, \\ \rho\mathbf{u}_t + \rho\tilde{\mathbf{u}}\cdot\nabla\mathbf{u} + \nabla p(\rho) \\ \qquad = (\mu_1+\zeta)\Delta\mathbf{u} + (\mu_1+\lambda_1-\zeta)\nabla\operatorname{div}\mathbf{u} + 2\zeta\nabla\times\mathbf{w} + (\nabla\times\boldsymbol{b})\times\boldsymbol{b}, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, \qquad \lim\limits_{|x|\to\infty} \mathbf{u}(x,t) = 0, \\ \rho\mathbf{w}_t + \rho\tilde{\mathbf{u}}\cdot\nabla\mathbf{w} + 4\zeta\mathbf{w} = \mu_2\Delta\mathbf{w} + (\mu_2+\lambda_2)\nabla\operatorname{div}\mathbf{w} + 2\zeta\nabla\times\tilde{\mathbf{u}}, \\ \mathbf{w}(\cdot,0) = \mathbf{w}_0, \qquad \lim\limits_{|x|\to\infty} \mathbf{w}(x,t) = 0, \\ \boldsymbol{b}_t - \nabla\times(\tilde{\mathbf{u}}\times\boldsymbol{b}) + \beta\nabla\times\Big(\dfrac{(\nabla\times\boldsymbol{b})\times\tilde{\boldsymbol{b}}}{\rho}\Big) = \nu\Delta\boldsymbol{b}, \qquad \operatorname{div}\boldsymbol{b} = 0, \\ \boldsymbol{b}(\cdot,0) = \boldsymbol{b}_0, \qquad \lim\limits_{|x|\to\infty} \boldsymbol{b}(x,t) = 0. \end{cases} \tag{2.14}$$

Let $(\mathbf{u}, \mathbf{w}, \boldsymbol{b})$ be the unique strong solution to the problem (2.14). We define the fixed point map:

$$\mathcal{F} : (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{b}}) \in \mathcal{B}\times\mathcal{B}\times\mathcal{B} \to (\mathbf{u}, \mathbf{w}, \boldsymbol{b}) \in \mathcal{B}\times\mathcal{B}\times\mathcal{B}$$

with

$$\tilde{\mathbf{u}}(\cdot,0) = \mathbf{u}_0, \tilde{\mathbf{w}}(\cdot,0) = \mathbf{w}_0, \tilde{\boldsymbol{b}}(\cdot,0) = \boldsymbol{b}_0, \operatorname{div}\tilde{\boldsymbol{b}} = 0, \lim_{|x|\to\infty}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\boldsymbol{b}}) = (0,0,0).$$

We will prove the map $\mathcal{F}$ mapping $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ into $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ for suitable constant K and small T, and $\mathcal{F}$ is a contraction mapping on $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$, and thus $\mathcal{F}$ has a unique fixed point in $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$. In order to do this, we will divide the proof into five steps.

Step I. For given $\tilde{\mathbf{u}} \in \mathcal{B}$, we will prove that for some small $0 < T \leqslant 1$, the problems $(2.14)_1$ and $(2.14)_2$ has a unique solution ρ satisfying

$$C^{-1} \leqslant \rho \leqslant C, \quad \|\nabla\rho\|_{L^\infty(0,T;H^2)} \leqslant C, \quad \|\rho_t\|_{L^\infty(0,T;H^2)} \leqslant CK, \tag{2.15}$$

here and later on, C will denote a constant independent of K.

Since $(2.14)_1$ is linear with regular $\tilde{\mathbf{u}}$, the existence and uniqueness are well-known. We only need to establish (2.15). In order to prove (2.15), we know from $(2.14)_1$ that

$$\rho(x,t) = \rho_0 \exp\Big(-\int_0^t \operatorname{div}\tilde{\mathbf{u}}\,ds\Big), \tag{2.16}$$

which yields

$$\inf\rho_0\exp\Big(-\int_0^T \|\operatorname{div}\tilde{\mathbf{u}}\|_{L^\infty}dt\Big) \leqslant \rho(x,t) \leqslant \sup\rho_0\exp\Big(\int_0^T \|\operatorname{div}\tilde{\mathbf{u}}\|_{L^\infty}dt\Big),$$

thus

$$C\inf\rho_0 \leqslant \inf\rho_0 \exp\big(-CKT^{1/2}\big) \leqslant \rho(x,t) \leqslant \sup\rho_0 \exp\big(CKT^{1/2}\big) \leqslant C\|\rho_0\|_{L^\infty},$$

provided that $KT^{1/2} \leqslant 1$ and $T \leqslant 1$.

In a similar way, we use (2.16) to denote the expression of $\nabla\rho, \Delta\rho$ and $\nabla\Delta\rho$, and then give the estimation of $\nabla\rho$, $\Delta\rho$, and $\nabla\Delta\rho$ as follows:

$$\begin{aligned}\|\nabla\rho\|_{L^\infty(0,T;L^2)} &\leqslant C\exp\Big(\int_0^t \|\mathrm{div}\tilde{\mathbf{u}}\|_{L^\infty} ds\Big)\Big(1+\int_0^T \|\tilde{\mathbf{u}}\|_{H^2} dt\Big)\\ &\leqslant C(1+TK) \leqslant C,\end{aligned}$$

$$\begin{aligned}\|\Delta\rho\|_{L^\infty(0,T;L^2)} &\leqslant C\exp\Big(\int_0^T \|\mathrm{div}\tilde{\mathbf{u}}\|_{L^\infty} dt\Big)\Big[1+\int_0^T \|\tilde{\mathbf{u}}\|_{H^3} dt+\Big(\int_0^T \|\tilde{\mathbf{u}}\|_{H^2} dt\Big)^2\Big]\\ &\leqslant C(1+T^{1/2}K+T^2K^2) \leqslant C,\end{aligned}$$

and

$$\begin{aligned}\|\nabla\Delta\rho\|_{L^\infty(0,T;L^2)} &\leqslant C\exp\Big(\int_0^T \|\mathrm{div}\tilde{\mathbf{u}}\|_{L^\infty} dt\Big)\Big[1+\int_0^T \|\tilde{\mathbf{u}}\|_{H^4} dt+\Big(\int_0^T \|\tilde{\mathbf{u}}\|_{H^3} dt\Big)^2\Big]\\ &\leqslant C(1+T^{1/2}K+T^2K^2) \leqslant C,\end{aligned}$$

provided that $T^{1/2}K \leqslant 1$ and $T \leqslant 1$.

Similarly,

$$\begin{aligned}\|\rho_t\|_{L^\infty(0,T;L^2)} &\leqslant \|\tilde{\mathbf{u}}\|_{L^\infty(0,T;L^\infty)}\|\nabla\rho\|_{L^\infty(0,T;L^2)} + \|\rho\|_{L^\infty(0,T;L^\infty)}\|\mathrm{div}\tilde{\mathbf{u}}\|_{L^\infty(0,T;L^2)} \leqslant CK,\\ \|\nabla\rho_t\|_{L^\infty(0,T;L^2)} &\leqslant \|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^2)}\|\nabla\rho\|_{L^\infty(0,T;H^1)} + \|\rho\|_{L^\infty(0,T;L^\infty)}\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^2)} \leqslant CK,\\ \|\nabla^2\rho_t\|_{L^\infty(0,T;L^2)} &\leqslant \|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^2)}\|\nabla\rho\|_{L^\infty(0,T;H^2)} + \|\rho\|_{L^\infty(0,T;L^\infty)}\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^3)} \leqslant CK,\end{aligned}$$

provided that $KT^{1/2} \leqslant 1$ and $T \leqslant 1$, and thus (2.15) hold.

Step II. For given $\tilde{\mathbf{u}}, \tilde{\boldsymbol{b}} \in \mathcal{B}$, we will prove that for some small $0 < T \leqslant 1$, the problem $(2.14)_7$ and $(2.14)_8$ have a unique solution $\boldsymbol{b}$ satisfying

$$\|\boldsymbol{b}\|_{L^\infty(0,T;H^3)} + \|\boldsymbol{b}\|_{L^2(0,T;H^4)} + \|\boldsymbol{b}_t\|_{L^\infty(0,T;H^1)} + \|\boldsymbol{b}_t\|_{L^2(0,T;H^2)} \leqslant C_1. \tag{2.17}$$

Since $(2.14)_7$ is linear with regular $\rho, \tilde{\mathbf{u}}$, and $\tilde{\boldsymbol{b}}$, the existence and uniqueness are well-known we only need to establish (2.17). In order to prove (2.17), we multiply $(2.14)_7$ by $\boldsymbol{b}$ and integrate over $\mathbb{R}^3$, after integration by parts, we have from the Gronwall inequality that

$$\|\boldsymbol{b}\|_{L^\infty(0,T;L^2)} + \|\boldsymbol{b}\|_{L^2(0,T;H^1)} \leqslant C_1,$$

provided that $K^2T \leqslant 1$.

Applying Δ to $(2.14)_7$, and then multiplying it by $\Delta\boldsymbol{b}$, after integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |\Delta\boldsymbol{b}|^2 dx + \int |\nabla\Delta\boldsymbol{b}|^2 dx\\ &= \int \Delta(\tilde{\mathbf{u}}\times\boldsymbol{b})\cdot\Delta(\nabla\times\boldsymbol{b})dx - \int \Delta\frac{(\nabla\times\boldsymbol{b})\times\tilde{\boldsymbol{b}}}{\rho}\cdot\Delta(\nabla\times\boldsymbol{b})dx\\ &:= J_1 + J_2.\end{aligned}$$

We bound J_1 as follows:

$$J_1 \leqslant \|\Delta(\tilde{\mathbf{u}} \times \boldsymbol{b})\|_{L^2} \|\Delta(\nabla \times \boldsymbol{b})\|_{L^2} \leqslant CK\|\boldsymbol{b}\|_{H^2} \|\nabla\Delta\boldsymbol{b}\|_{L^2} \leqslant \frac{1}{4}\|\nabla\Delta\boldsymbol{b}\|_{L^2}^2 + CK^2\|\boldsymbol{b}\|_{H^2}^2.$$

Using (2.15), we bound J_2 as follows:

$$\begin{aligned}
J_2 &= -\int \Big(\frac{(\nabla \times \boldsymbol{b}) \times \Delta\tilde{\boldsymbol{b}}}{\rho} + \frac{2\sum_i \partial_i(\nabla \times \boldsymbol{b}) \times \partial_i \tilde{\boldsymbol{b}}}{\rho} \Big) \Delta(\nabla \times \boldsymbol{b}) dx \\
&\quad - \int \Big[\Delta\frac{1}{\rho} \cdot ((\nabla \times \boldsymbol{b}) \times \tilde{\boldsymbol{b}}) + 2\nabla\frac{1}{\rho} \cdot \nabla((\nabla \times \boldsymbol{b}) \times \tilde{\boldsymbol{b}}) \Big] \cdot \Delta(\nabla \times \boldsymbol{b}) dx \\
&\leqslant CK\|\Delta\boldsymbol{b}\|_{L^2}^{1/2} \|\nabla\Delta\boldsymbol{b}\|_{L^2}^{1/2} + CK\|\Delta\boldsymbol{b}\|_{L^2} \|\nabla\Delta\boldsymbol{b}\|_{L^2} \\
&\leqslant \frac{1}{4}\|\nabla\Delta\boldsymbol{b}\|_{L^2}^2 + CK^4\|\Delta\boldsymbol{b}\|_{L^2}^2.
\end{aligned}$$

Due to the Gronwall inequality, one has

$$\|\boldsymbol{b}\|_{L^\infty(0,T;H^2)} + \|\boldsymbol{b}\|_{L^2(0,T;H^3)} \leqslant C_1,$$

provided that $K^4T + K^2T \leqslant 1$.

Applying $\nabla\Delta$ to $(2.14)_7$, then multiplying it by $\nabla\Delta\boldsymbol{b}$, after integration by parts, we get from Gronwall inequality that

$$\|\boldsymbol{b}\|_{L^\infty(0,T;H^3)} + \|\boldsymbol{b}\|_{L^2(0,T;H^4)} \leqslant C_1,$$

provided that $K^6T + K^4T + K^2T \leqslant 1$.

Applying ∂_t to $(2.14)_7$, then multiplying it by $\partial_t\boldsymbol{b}$, after integration by parts, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\boldsymbol{b}_t|^2 dx + \int |\nabla\boldsymbol{b}_t|^2 dx \\
&= \int \partial_t(\tilde{\mathbf{u}} \times \boldsymbol{b}) \cdot (\nabla \times \boldsymbol{b}_t) dx - \int \partial_t \frac{(\tilde{\boldsymbol{b}} \times (\nabla \times \boldsymbol{b}))}{\rho} \cdot (\nabla \times \boldsymbol{b}_t) dx \\
&\leqslant C\Big(\|\partial_t\tilde{\mathbf{u}}\|_{L^2}\|\boldsymbol{b}\|_{L^\infty} + \|\tilde{\mathbf{u}}\|_{L^\infty}\|\boldsymbol{b}_t\|_{L^2}\Big)\|\nabla \times \boldsymbol{b}_t\|_{L^2} \\
&\quad + C\Big(\|\partial_t\rho\|_{L^6}\|\tilde{\boldsymbol{b}}\|_{L^6}\|\nabla \times \boldsymbol{b}\|_{L^6} + \|\partial_t\tilde{\boldsymbol{b}}\|_{L^3}\|\nabla \times \boldsymbol{b}\|_{L^6}\Big)\|\nabla \times \boldsymbol{b}_t\|_{L^2} \\
&\leqslant \frac{1}{2}\|\nabla\boldsymbol{b}_t\|_{L^2}^2 + CK^2 + CK^4 + CK\Big(\|\partial_t\tilde{\boldsymbol{b}}\|_{L^2}^2 + \|\nabla\partial_t\tilde{\boldsymbol{b}}\|_{L^2}^2\Big).
\end{aligned}$$

Using Gronwall's inequality, one has

$$\|\boldsymbol{b}_t\|_{L^\infty(0,T;L^2)} + \|\boldsymbol{b}_t\|_{L^2(0,T;H^1)} \leqslant C_1,$$

provided that $K^4T \leqslant 1$. In the same way, applying $\Delta\partial_t$ to $(2.14)_7$, then multiplying it by $\Delta\boldsymbol{b}_t$, after integration by parts, we get from Gronwall's inequality that

$$\|\boldsymbol{b}_t\|_{L^\infty(0,T;H^1)} + \|\boldsymbol{b}_t\|_{L^2(0,T;H^2)} \leqslant C_1,$$

provided that $K^6T \leqslant 1$, which yields (2.17).

Step III. For given $\rho, \tilde{\mathbf{u}} \in \mathscr{B}$, we will prove that for some small $0 < T \leqslant 1$, the problems $(2.14)_5$ and $(2.14)_6$ have a unique solution $\mathbf{w}$ satisfying

$$\|\mathbf{w}\|_{L^\infty(0,T;H^3)} + \|\mathbf{w}\|_{L^2(0,T;H^4)} + \|\mathbf{w}_t\|_{L^\infty(0,T;H^1)} + \|\mathbf{w}_t\|_{L^2(0,T;H^2)} \leqslant C_2. \tag{2.18}$$

Since $(2.14)_5$ in linear with regular ρ and $\tilde{\mathbf{u}}$, the existence and uniqueness are well-known, we only need to establish (2.18). In order to prove (2.18), we multiply $(2.14)_5$ by $\mathbf{w}$ and integrate over $\mathbb{R}^3$. After integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{w}|^2 dx + \int \Big[\mu_2|\nabla \mathbf{w}|^2 + (\mu_2+\lambda_2)|\mathrm{div}\mathbf{w}|^2 + 4\zeta|\mathbf{w}|^2\Big]dx \\
&= 2\zeta\int \nabla\times\tilde{\mathbf{u}}\cdot\mathbf{w}dx \leqslant C\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\mathbf{w}\|_{L^2} \leqslant C\|\mathbf{w}\|_{L^2},
\end{aligned}$$

which, yields

$$\|\mathbf{w}\|_{L^\infty(0,T;L^2)} + \|\mathbf{w}\|_{L^2(0,T;H^1)} \leqslant C_2.$$

Multiplying $(2.14)_5$ by $\mathbf{w}_t$ and integrating over $\mathbb{R}^3$, after integrating by parts, one deduces from (2.15) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \Big[\mu_2|\nabla \mathbf{w}|^2 + (\mu_2+\lambda_2)|\mathrm{div}\mathbf{w}|^2 + 4\zeta|\mathbf{w}|^2\Big]dx + \int \rho|\mathbf{w}_t|^2 dx \\
&= \int \rho\tilde{\mathbf{u}}\cdot\nabla\mathbf{w}\cdot\mathbf{w}_t dx + 2\zeta\int \nabla\times\tilde{\mathbf{u}}\cdot\mathbf{w}_t dx \\
&\leqslant C\|\tilde{\mathbf{u}}\|_{L^\infty}\|\nabla\mathbf{w}\|_{L^2}\|\rho^{1/2}\mathbf{w}_t\|_{L^2} + C\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\mathbf{w}_t\|_{L^2} \\
&\leqslant \frac{1}{2}\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2 + C + CK^2\|\nabla\mathbf{w}\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|\mathbf{w}\|_{L^\infty(0,T;H^1)} + \|\mathbf{w}_t\|_{L^2(0,T;L^2)} \leqslant C_2,$$

provided $K^2T \leqslant 1$.

Applying ∂_t to $(2.14)_5$, then multiplying it by $\partial_t\mathbf{w}$ and integrating over $\mathbb{R}^3$, after integrating by parts, one deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{w}_t|^2 dx + \int \Big[\mu_2|\nabla \mathbf{w}_t|^2 + (\mu_2+\lambda_2)|\mathrm{div}\mathbf{w}_t|^2 + 4\zeta|\mathbf{w}_t|^2\Big]dx \\
&= 2\zeta\int \nabla\times\tilde{\mathbf{u}}_t\cdot\mathbf{w}_t dx - \int (\rho\tilde{\mathbf{u}}\cdot\nabla\mathbf{w})_t\cdot\mathbf{w}_t dx - \int \rho_t|\mathbf{w}_t|^2 dx \\
&\leqslant CK^2\|\rho^{1/2}\mathbf{w}_t\|_{L^2} + CK^2,
\end{aligned}$$

which gives

$$\|\mathbf{w}_t\|_{L^\infty(0,T;L^2)} + \|\mathbf{w}_t\|_{L^2(0,T;H^1)} \leqslant C_2,$$

provided $K^2T \leqslant 1$.

In a similar way, we deduce from the $\dot{H}^3$-theory of the elliptic system that

$$\|\mathbf{w}\|_{L^\infty(0,T;H^3)} + \|\mathbf{w}\|_{L^2(0,T;H^4)} \leqslant C_2,$$

and

$$\|\mathbf{w}_t\|_{L^\infty(0,T;H^1)} + \|\mathbf{w}_t\|_{L^2(0,T;H^2)} \leqslant C_2,$$

provided $K^2T \leqslant 1$.

Step IV. For given $\rho, \tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\boldsymbol{b}} \in \mathscr{B}$, we will prove that for some small $0 < T \leqslant 1$, the problem $(2.14)_3$ and $(2.14)_4$ have a unique solution $\mathbf{u}$ satisfying

$$\|\mathbf{u}\|_{L^\infty(0,T;H^3)} + \|\mathbf{u}\|_{L^2(0,T;H^4)} + \|\mathbf{u}_t\|_{L^\infty(0,T;H^1)} + \|\mathbf{u}_t\|_{L^2(0,T;H^2)} \leqslant C_3. \tag{2.19}$$

Since $(2.14)_3$ is linear with regular $\rho, \tilde{\mathbf{u}}, \tilde{\boldsymbol{b}}$ and $\tilde{\mathbf{w}}$, the existence and uniqueness are well-known, we only need to establish (2.19). In order to prove (2.19), we multiply $(2.14)_3$ by $\mathbf{u}$ and integrate over $\mathbb{R}^3$. After integration by parts and taking (2.16)–(2.18) into consideration, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}|^2 dx + \int \left[(\mu_1+\zeta)|\nabla\mathbf{u}|^2 + (\mu_1+\lambda_1-\zeta)|\mathrm{div}\mathbf{u}|^2\right]dx \\
&= \int (\nabla\times\boldsymbol{b})\times\boldsymbol{b}\cdot\mathbf{u}dx + \int \nabla p(\rho)\cdot\mathbf{u}dx + 2\zeta\int \nabla\times\mathbf{w}\cdot\mathbf{u}dx \\
&\leqslant C\|\boldsymbol{b}\|_{L^\infty}\|\nabla\boldsymbol{b}\|_{L^2}\|\mathbf{u}\|_{L^2} + C\|\nabla\rho\|_{L^2}\|\mathbf{u}\|_{L^2} + C\|\nabla\mathbf{w}\|_{L^2}\|\mathbf{u}\|_{L^2} \\
&\leqslant C\|\mathbf{u}\|_{L^2},
\end{aligned}$$

which, yields

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}\|_{L^2(0,T;H^1)} \leqslant C_2.$$

Multiplying $(2.14)_3$ by $\mathbf{u}_t$ and integrating over $\mathbb{R}^3$, after integrating by parts, one deduces

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left[(\mu_1+\zeta)|\nabla\mathbf{u}|^2 + (\mu_1+\lambda_1-\zeta)|\mathrm{div}\mathbf{u}|^2\right]dx + \int \rho|\mathbf{u}_t|^2 dx \\
&= -\int \rho\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx - \int \nabla p(\rho)\cdot\mathbf{u}_t dx + 2\zeta\int \nabla\times\mathbf{w}\cdot\mathbf{u}_t dx + \int (\nabla\times\boldsymbol{b})\times\boldsymbol{b}\cdot\mathbf{u}_t dx \\
&\leqslant C\|\tilde{\mathbf{u}}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\rho^{1/2}\mathbf{u}_t\|_{L^2} + C\|\nabla\rho\|_{L^2}\|\mathbf{u}_t\|_{L^2} + C\|\nabla\mathbf{w}\|_{L^2}\|\mathbf{u}_t\|_{L^2} + \|\boldsymbol{b}\|_{L^\infty}\|\nabla\boldsymbol{b}\|_{L^2}\|\mathbf{u}_t\|_{L^2} \\
&\leqslant \frac{1}{2}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + C + CK^2\|\nabla\mathbf{u}\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|\mathbf{u}\|_{L^\infty(0,T;H^1)} + \|\mathbf{u}_t\|_{L^2(0,T;L^2)} \leqslant C_2,$$

provided $K^2T \leqslant 1$.

Applying ∂_t to $(2.14)_3$, then multiplying it by $\partial_t\mathbf{u}$ and integrating over $\mathbb{R}^3$, after integrating by parts, one deduces

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}_t|^2 dx + \int \left[(\mu_1+\zeta)|\nabla\mathbf{u}_t|^2 + (\mu_1+\lambda_1-\zeta)|\mathrm{div}\mathbf{u}_t|^2\right]dx \\
&= 2\zeta\int \nabla\times\tilde{\mathbf{w}}_t\cdot\mathbf{u}_t dx - \int (\rho\tilde{\mathbf{u}}\cdot\nabla\mathbf{u})_t\cdot\mathbf{u}_t dx - \int \rho_t|\mathbf{u}_t|^2 dx - \int \nabla p(\rho)_t\cdot\mathbf{u}_t dx \\
&\quad + \int [(\nabla\times\boldsymbol{b})\times\boldsymbol{b}]_t\cdot\mathbf{u}_t dx \leqslant CK^2\|\rho^{1/2}\mathbf{u}_t\|_{L^2} + CK^2,
\end{aligned}$$

which gives

$$\|\mathbf{u}_t\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}_t\|_{L^2(0,T;H^1)} \leqslant C_2,$$

provided $K^2T \leqslant 1$.

In a similar way, we deduce from the $\dot{H}^3$-theory of the elliptic system that

$$\|\mathbf{u}\|_{L^\infty(0,T;H^3)} + \|\mathbf{u}\|_{L^2(0,T;H^4)} \leqslant C_2,$$

and

$$\|\mathbf{u}_t\|_{L^\infty(0,T;H^1)} + \|\mathbf{u}_t\|_{L^2(0,T;H^2)} \leqslant C_2,$$

provided $K^2T \leqslant 1$.

Step V. Due to the above analysis, we can take $K = \max\{C_1, C_2, C_3\}$, and thus $\mathcal{F}$ maps $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ into $\mathcal{B}\times\mathcal{B}\times\mathcal{B}$. Therefore, in this step, we will prove that $\mathcal{F}$ is contracted in the sense of a weaker norm, that is, there is a constant $\alpha \in (0,1)$ such that for any $(\tilde{\mathbf{u}}_i, \tilde{\boldsymbol{b}}_i, \tilde{\mathbf{w}}_i)$ $(i = 1,2)$ and some small $0 < T \leqslant 1$, the following estimate holds:

$$\|\mathcal{F}(\tilde{\mathbf{u}}_1, \tilde{\mathbf{w}}_1, \tilde{\boldsymbol{b}}_1) - \mathcal{F}(\tilde{\mathbf{u}}_2, \tilde{\mathbf{w}}_2, \tilde{\boldsymbol{b}}_2)\|_{L^2(0,T;H^1)} \leqslant \alpha\|(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2, \tilde{\mathbf{w}}_1 - \tilde{\mathbf{w}}_2, \tilde{\boldsymbol{b}}_1 - \tilde{\boldsymbol{b}}_2)\|_{L^2(0,T;H^1)}. \tag{2.20}$$

In order to obtain (2.20), we suppose that $(\rho_i, \mathbf{u}_i, \mathbf{w}_i, \boldsymbol{b}_i)$ $(i = 1,2)$ are the solutions to the problem (2.14) corresponding to $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i, \tilde{\boldsymbol{b}}_i)$. Denote

$$\begin{aligned} &\rho = \rho_1 - \rho_2, \quad \mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \quad \mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2, \quad \boldsymbol{b} = \boldsymbol{b}_1 - \boldsymbol{b}_2, \\ &\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2, \quad \tilde{\mathbf{w}} = \tilde{\mathbf{w}}_1 - \tilde{\mathbf{w}}_2, \quad \tilde{\boldsymbol{b}} = \tilde{\boldsymbol{b}}_1 - \tilde{\boldsymbol{b}}_2. \end{aligned}$$

Then, we obtain that

$$\begin{cases} \rho_t + \operatorname{div}(\rho\tilde{\mathbf{u}}_1) = -\operatorname{div}(\rho_2\tilde{\mathbf{u}}), \\ \rho_1\mathbf{u}_t + \rho_1\tilde{\mathbf{u}}_1\cdot\nabla\mathbf{u} + \nabla[p(\rho_1) - p(\rho_2)] - (\mu_1+\zeta)\Delta\mathbf{u} - (\mu_1+\lambda_1-\zeta)\nabla\operatorname{div}\mathbf{u} \\ \qquad +2\zeta\nabla\times\mathbf{w} + (\nabla\times\boldsymbol{b})\times\boldsymbol{b}_1 + (\nabla\times\boldsymbol{b}_2)\times\boldsymbol{b} - \rho\mathbf{u}_{2t} - (\rho_1\tilde{\mathbf{u}}_1 - \rho_2\tilde{\mathbf{u}}_2)\cdot\nabla\mathbf{u}_2 \\ \rho_1\mathbf{w}_t + \rho_1\tilde{\mathbf{u}}_1\cdot\nabla\mathbf{w} + 4\zeta\mathbf{w} - \mu_2\Delta\mathbf{w} - (\mu_2+\lambda_2)\nabla\operatorname{div}\mathbf{w} \\ \qquad = 2\zeta\nabla\times\tilde{\mathbf{u}} - \rho\mathbf{w}_{2t} - (\rho_1\tilde{\mathbf{w}}_1 - \rho_2\tilde{\mathbf{w}}_2)\cdot\nabla\mathbf{w}_2, \\ \boldsymbol{b}_t + \nabla\times(\boldsymbol{b}_1\times\tilde{\mathbf{u}}) + \nabla\times(\boldsymbol{b}\times\tilde{\mathbf{u}}_2) + \beta\nabla\times\Big[\Big(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\Big)((\nabla\times\boldsymbol{b}_1)\times\tilde{\boldsymbol{b}}_1)\Big] \\ \qquad +\beta\nabla\times\Big(\dfrac{1}{\rho_2}(\nabla\times\boldsymbol{b})\times\tilde{\boldsymbol{b}}_1\Big) + \beta\nabla\times\Big(\dfrac{1}{\rho_2}(\nabla\times\boldsymbol{b}_2)\times\tilde{\boldsymbol{b}}\Big) = \nu\Delta\boldsymbol{b}. \end{cases} \tag{2.21}$$

Texting $(2.21)_1$ by ρ, we obtain that for any $\varepsilon \in (0,1)$,

$$\begin{aligned} \frac{1}{2}\int|\rho|^2dx &\leqslant C\|\nabla\tilde{\mathbf{u}}_1\|_{L^\infty}\|\rho\|_{L^2}^2 + C\Big(\|\nabla\tilde{\mathbf{u}}\|_{L^\infty}\|\rho_2\|_{L^\infty} + \|\tilde{\mathbf{u}}\|_{L^6}\|\nabla\rho_2\|_{L^3}\Big)\|\rho\|_{L^2} \\ &\leqslant C\|\tilde{\mathbf{u}}_1\|_{H^3}\|\rho\|_{L^2}^2 + C\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\rho\|_{L^2} \\ &\leqslant \varepsilon\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2 + C\Big(1 + \|\tilde{\mathbf{u}}_1\|_{H^3}\Big)\|\rho\|_{L^2}^3. \end{aligned}$$

Multiplying $(2.21)_4$ by $\boldsymbol{b}$, after integration by parts, we see that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}\|_{L^2}^2 + \nu\|\nabla \boldsymbol{b}\|_{L^2}^2 \\
&= \int (\tilde{\mathbf{u}} \times \boldsymbol{b}_1 + \tilde{\mathbf{u}}_2 \times \boldsymbol{b})(\nabla \times \boldsymbol{b})dx + \beta \int \Big(\frac{1}{\rho_1} - \frac{1}{\rho_2}\Big)((\nabla \times \boldsymbol{b}_1) \times \tilde{\boldsymbol{b}}_1)(\nabla \times \boldsymbol{b})dx \\
&\quad - \beta \int \frac{1}{\rho_2}(\nabla \times \boldsymbol{b}_2) \times \tilde{\boldsymbol{b}} \cdot (\nabla \times \boldsymbol{b})dx \\
&\leqslant C\|\nabla \tilde{\mathbf{u}}\|_{L^2}\|\boldsymbol{b}\|_{L^2} + C\|\boldsymbol{b}\|_{L^2}\|\nabla \boldsymbol{b}\|_{L^2} + C\|\rho\|_{L^2}\|\nabla \boldsymbol{b}\|_{L^2}\|\boldsymbol{b}_1\|_{H^3}^{1/2} + C\|\nabla \tilde{\boldsymbol{b}}\|_{L^2}\|\boldsymbol{b}\|_{L^2}\|\boldsymbol{b}_2\|_{H^3}^{1/2} \\
&\leqslant \frac{1}{2}\|\nabla \boldsymbol{b}\|_{L^2}^2 + \varepsilon\|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 + \varepsilon\|\nabla \tilde{\boldsymbol{b}}\|_{L^2}^2 + C\|\boldsymbol{b}\|_{L^2}^2 + C\|\rho\|_{L^2}^2\|\boldsymbol{b}_1\|_{H^3} + \|\boldsymbol{b}_2\|_{H^3}\|\boldsymbol{b}\|_{L^2}^2.
\end{aligned}$$

Multiplying $(2.21)_3$ by $\mathbf{w}$ and integrating over $\mathbb{R}^3$, we obtain that for any $\varepsilon \in (0,1)$ and $\epsilon \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\rho_1^{1/2}\mathbf{w}\|_{L^2}^2 + \mu_2\|\nabla \mathbf{w}\|_{L^2}^2 + (\mu_2 + \lambda_2)\|\mathrm{div}\mathbf{w}\|_{L^2}^2 + 4\zeta\|\mathbf{w}\|_{L^2}^2 \\
&= -\int \rho \mathbf{w}_{2t} \cdot \mathbf{w}dx - \int (\rho \tilde{\mathbf{w}}_1 + \rho_2 \tilde{\mathbf{w}}) \cdot \nabla \mathbf{w}_2 \cdot \mathbf{w}dx + 2\zeta \int (\nabla \times \tilde{\mathbf{u}}) \cdot \mathbf{w}dx \\
&\leqslant C\|\mathbf{w}_{2t}\|_{L^3}\|\rho\|_{L^2}\|\mathbf{w}\|_{L^6} + C\|\rho\|_{L^2}\|\mathbf{w}\|_{L^3} + C\|\nabla \tilde{\mathbf{w}}\|_{L^2}\|\mathbf{w}\|_{L^2} + C\|\nabla \tilde{\mathbf{u}}\|_{L^2}\|\mathbf{w}\|_{L^2} \\
&\leqslant \frac{\mu_2}{2}\|\nabla \mathbf{w}\|_{L^2}^2 + \epsilon\|\nabla \tilde{\mathbf{w}}\|_{L^2}^2 + \varepsilon\|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 \\
&\quad + C\Big(\|\rho\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2 + \|\mathbf{w}_{2t}\|_{L^6}\|\rho\|_{L^2}^2\Big).
\end{aligned}$$

Multiplying $(2.21)_2$ by $\mathbf{u}$ and integrating over $\mathbb{R}^3$, we obtain that for any $\varepsilon \in (0,1)$ and $\epsilon \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\rho_1^{1/2}\mathbf{u}\|_{L^2}^2 + (\mu_1 + \zeta)\|\nabla \mathbf{u}\|_{L^2}^2 + (\mu_1 + \lambda_1 - \zeta)\|\mathrm{div}\mathbf{u}\|_{L^2}^2 \\
&= \int [p(\rho_1) - p(\rho_2)\mathrm{div}\mathbf{u}]dx + \int [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}_1 + (\nabla \times \boldsymbol{b}_2) \times \boldsymbol{b}] \cdot \mathbf{u}dx \\
&\quad - \int \rho \mathbf{u}_{2t} \cdot \mathbf{u}dx - \int (\rho \tilde{\mathbf{u}}_1 + \rho_2 \tilde{\mathbf{u}}) \cdot \nabla \mathbf{u}_2 \cdot \mathbf{u}dx + 2\zeta \int (\nabla \times \mathbf{w}) \cdot \mathbf{u}dx \\
&\leqslant C\|\rho\|_{L^2}\|\nabla \mathbf{u}\|_{L^2} + C\|\nabla \boldsymbol{b}\|_{L^2}\|\mathbf{u}\|_{L^2} + C\|\boldsymbol{b}\|_{L^2}\|\mathbf{u}\|_{L^3} + C\|\nabla \mathbf{w}\|_{L^2}\|\mathbf{u}\|_{L^2} \\
&\quad + C\|\mathbf{u}_{2t}\|_{L^3}\|\rho\|_{L^2}\|\mathbf{u}\|_{L^6} + C\|\rho\|_{L^2}\|\mathbf{u}\|_{L^3} + C\|\nabla \tilde{\mathbf{u}}\|_{L^2}\|\mathbf{u}\|_{L^2} \\
&\leqslant \frac{\mu_1 + \zeta}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \epsilon\|\nabla \boldsymbol{b}\|_{L^2}^2 + \varepsilon\|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 \\
&\quad + C\Big(\|\rho\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^2 + \|\boldsymbol{b}\|_{L^2}^2 + \|\mathbf{u}_{2t}\|_{L^6}\|\rho\|_{L^2}^2\Big).
\end{aligned}$$

Combining the above inequalities and using the Gronwall inequality, and taking ε and ϵ suitably small, we conclude that (2.20) holds true.

Due to the above steps and the Banach fixed point theorem, we finish the proof of Lemma 2.5. □

Lemma 2.6. *Let $p \in [2, \frac{3s}{3-s}]$ for $s \in [2,3)$, or $p \in [2,\infty]$ for $s = 3$, and let $q \in (1,\infty)$, $r \in (3,\infty)$. There exists some generic constant $C > 0$ which may depend on s and r such that for $f \in L^2 \cap D_0^{1,s}$ and $g \in L^q \cap D_0^{1,r}$, we have*

$$\|f\|_{L^p}^p \leqslant C\|f\|_{L^2}^{p-3s(p-2)/(5s-6)}\|\nabla f\|_{L^s}^{3s(p-2)/(5s-6)}, \tag{2.22}$$

and

$$\|g\|_{L^\infty} \leqslant C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.23}$$

3. Proof of Theorem 1.1

3.1. Global existence of classical solutions

In this section, we will establish some necessary a priori bounds for smooth solution to the system (2.1)–(2.3) to extend the local classical solution guaranteed by Lemma 2.5. Thus, let $T > 0$ be a fixed time, and $(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})$ be the smooth solution to (2.1)–(2.3) on $\mathbb{R}^3 \times [0, T]$ in the class (2.12) with smooth initial data $(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)$ satisfying (2.11). To estimate this solution, we make the following a priori assumptions. For any given $L > 1$,(not necessarily small), suppose that

$$\sup_{t\in[0,T]} \|\nabla^2(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{H^1} \leqslant L, \tag{3.1}$$

and

$$\sup_{t\in[0,T]} \|(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{H^1} \leqslant \alpha, \tag{3.2}$$

where α is a positive constant, depending on L, and satisfies

$$0 < \alpha \leqslant \alpha_0 \triangleq \frac{1}{4C^2L}. \tag{3.3}$$

Here C is the Sobolev embedding constant of (3.8).

We have the following key a priori estimates on $(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})$.

Proposition 3.1. *For any given positive constant $M > 0$ (not necessarily small), suppose that*

$$(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0) \in H^3, \qquad \|\nabla^2(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|_{H^1} \leqslant M. \tag{3.4}$$

Then there exists a positive constant ε, depending only on $\mu_1, \lambda_1, \zeta, \mu_2, \lambda_2, \nu$ and M, such that for any $t > 0$, the system (2.1) *possesses a unique global classical solution $(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h}) \in \mathbb{R}^3 \times [0, \infty)$ satisfying*

$$\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|^2_{H^2} + \int_0^t \left(\|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|^2_{H^2} + \|\nabla^2\varrho\|^2_{H^1}\right)d\tau \leqslant C\|\nabla(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|^2_{H^2}, \tag{3.5}$$

and

$$\|(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|^2_{H^1} + \int_0^t \left(\|\nabla(\boldsymbol{v}, \omega, \boldsymbol{h})\|^2_{H^1} + \|\nabla\varrho\|^2_{L^2}\right)d\tau \leqslant C\|(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|^2_{H^1}, \tag{3.6}$$

provided

$$\|(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|_{H^1} \leqslant \varepsilon. \tag{3.7}$$

In order to prove Theorem 1.1, it was suffices to prove Proposition 3.1. However, the assumptions of (3.1) and (3.2) are crucial to prove Proposition 3.1. Therefore, our main aim in the next is to close the a priori assumptions (3.1) and (3.2). Obviously, we can infer from (3.1)–(3.3) and the Sobolev embedding inequality (cf. [1]) that

$$\|\varrho(t)\|_{L^\infty} \leqslant C\|\nabla\varrho\|^{1/2}_{L^2}\|\nabla^2\varrho\|^{1/2}_{L^2} \leqslant \frac{1}{2}, \qquad \forall\ t \in [0, T], \tag{3.8}$$

which implies that

$$\frac{1}{2} \leqslant \inf_{(x,t)\in\mathbb{R}^3\times[0,T]} \varrho(x, t) + 1 \leqslant \sup_{(x,t)\in\mathbb{R}^3\times[0,T]} \varrho(x, t) + 1 \leqslant \frac{3}{2}, \tag{3.9}$$

moreover

$$|f(\varrho)| \leqslant C|\varrho|, \qquad |g(\varrho)| \leqslant C, \qquad |h(\varrho)| \leqslant C|\varrho|, \tag{3.10}$$

and

$$|f^{(m)}(\varrho)| \leqslant C, \qquad |g^{(m)}(\varrho)| \leqslant C, \qquad |h^{(m)}(\varrho)| \leqslant C, \tag{3.11}$$

for any $m \geqslant 1$. We now begin to derive a series of a priori estimates.

Lemma 3.1. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$\frac{d}{dt}\|(a\gamma\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \|\nabla(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 \leqslant C\alpha^{1/2}L^{1/2}\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2. \tag{3.12}$$

Proof. Multiplying $(2.1)_1$–$(2.1)_4$ by $a\gamma\varrho, \boldsymbol{v}, \omega$, and $\boldsymbol{h}$ in L^2, respectively, and integrating by parts, we obtain after adding them together that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(a\gamma\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \Big(\mu_1\|\nabla\boldsymbol{v}\|_{L^2}^2 + (\mu_1 + \lambda_1)\|\mathrm{div}\boldsymbol{v}\|_{L^2}^2 + \nu\|\nabla\boldsymbol{h}\|_{L^2}^2 \\
&\qquad + \mu_2\|\nabla\omega\|_{L^2}^2 + (\mu_2 + \lambda_2)\|\mathrm{div}\omega\|_{L^2}^2 + \zeta\|\nabla \times \boldsymbol{v} - 2\omega\|_{L^2}^2\Big) \\
&= \langle R_1, a\gamma\varrho\rangle + \langle R_2, \boldsymbol{v}\rangle + \langle R_3, \omega\rangle + \langle R_4, \boldsymbol{h}\rangle.
\end{aligned} \tag{3.13}$$

To deal with the right-hand side of (3.13), we notice from (2.2) and (3.2) that

$$\begin{aligned}
\langle R_1, a\gamma\varrho\rangle &\leqslant C\|\varrho\|_{L^3}\|\nabla\boldsymbol{v}\|_{L^2}\|\nabla\varrho\|_{L^2} \leqslant C\|\varrho\|_{H^1}\|\nabla\boldsymbol{v}\|_{L^2}\|\nabla\varrho\|_{L^2} \\
&\leqslant C\alpha\Big(\|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2\Big).
\end{aligned} \tag{3.14}$$

By using (3.1), (3.2), (3.9)–(3.11), and the Sobolev inequality (2.22), we deduce from (2.2) and the integration by parts that

$$\begin{aligned}
\langle R_2, \boldsymbol{v}\rangle &\leqslant C\|\boldsymbol{v}\|_{L^3}\|\nabla\boldsymbol{v}\|_{L^2}\|\boldsymbol{v}\|_{L^6} + C\|\nabla\varrho\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^3}\|\boldsymbol{v}\|_{L^6} + C\|\nabla\boldsymbol{v}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^3}\|\varrho\|_{L^6} \\
&\quad + C\|\nabla\boldsymbol{h}\|_{L^2}\|\boldsymbol{h}\|_{L^3}\|\boldsymbol{v}\|_{L^6} + C\|\varrho\|_{L^3}\|\nabla\varrho\|_{L^2}\|\boldsymbol{v}\|_{L^6} + C\|\varrho\|_{L^3}\|\nabla\omega\|_{L^2}\|\boldsymbol{v}\|_{L^6} \\
&\leqslant C\Big(\|(\varrho, \boldsymbol{v}, \boldsymbol{h})\|_{H^1} + \|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\Big)\|(\nabla\varrho, \nabla\boldsymbol{v}, \nabla\omega, \nabla\boldsymbol{h})\|_{L^2}^2 \\
&\leqslant C\alpha^{1/2}L^{1/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla\boldsymbol{h}\|_{L^2}^2\Big),
\end{aligned} \tag{3.15}$$

where we have used the fact that $\alpha \in (0, 1)$ and $L > 1$. According to the observation of (3.13), one has

$$\|\omega\|_{L^2}^2 \leqslant C\Big(\|\nabla \times \boldsymbol{v} - 2\omega\|_{L^2}^2 + \|\nabla\boldsymbol{v}\|_{L^2}^2\Big),$$

thus, by using a similar manner to (3.15) and choosing $\delta_1 = \min\{\frac{\zeta}{C}, \frac{\mu_1}{4C}\}$ in (3.16), we can deduce from the above inequality that

$$\begin{aligned}
\langle R_3, \omega\rangle &\leqslant C\|\boldsymbol{v}\|_{L^3}\|\nabla\omega\|_{L^2}\|\omega\|_{L^6} + C\|\nabla\varrho\|_{L^2}\|\nabla\omega\|_{L^3}\|\omega\|_{L^6} \\
&\quad + C\|\nabla\omega\|_{L^2}\|\nabla\omega\|_{L^3}\|\varrho\|_{L^6} + C\|\varrho\|_{L^3}\|\omega\|_{L^2}\|\omega\|_{L^6} + C\|\varrho\|_{L^3}\|\nabla\boldsymbol{v}\|_{L^2}\|\omega\|_{L^6} \\
&\leqslant C\Big(\|(\varrho, \boldsymbol{v}, \omega)\|_{H^1} + \|\nabla\omega\|_{L^2}^{1/2}\|\nabla^2\omega\|_{L^2}^{1/2}\Big)\Big(\|(\nabla\varrho, \nabla\boldsymbol{v}, \nabla\omega)\|_{L^2}^2\Big) + \delta_1\|\omega\|_{L^2}^2 \\
&\leqslant C\alpha^{1/2}L^{1/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla\boldsymbol{h}\|_{L^2}^2\Big) \\
&\quad + \zeta\|\nabla \times \boldsymbol{v} - 2\omega\|_{L^2}^2 + \frac{\mu_1}{4}\|\nabla\boldsymbol{v}\|_{L^2}^2,
\end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
\langle R_4, \boldsymbol{h}\rangle &\leqslant C\|\boldsymbol{v}\|_{L^3}\|\nabla \boldsymbol{h}\|_{L^2}\|\boldsymbol{h}\|_{L^6} + C\|\boldsymbol{h}\|_{L^3}\|\nabla \boldsymbol{v}\|_{L^2}\|\boldsymbol{h}\|_{L^6} + C\|\nabla \boldsymbol{h}\|_{L^3}\|\boldsymbol{h}\|_{L^6}\|\nabla \boldsymbol{h}\|_{L^2}\\
&\leqslant C\Big(\|(\boldsymbol{v}, \boldsymbol{h})\|_{H^1} + \|\nabla \boldsymbol{h}\|_{L^2}^{1/2}\|\nabla^2 \boldsymbol{h}\|_{L^2}^{1/2}\Big)\|(\nabla \boldsymbol{v}, \nabla \boldsymbol{h})\|_{L^2}^2 \\
&\leqslant C\alpha^{1/2}L^{1/2}\Big(\|\nabla \boldsymbol{v}\|_{L^2}^2 + \|\nabla \boldsymbol{h}\|_{L^2}^2\Big).
\end{aligned}
\tag{3.17}
$$

Substituting (3.14)–(3.17) into (3.13) immediately yields (3.12). □

Lemma 3.2. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla(a\gamma\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \|\nabla(\nabla\times \boldsymbol{v} - 2\omega)\|_{L^2}^2\\
&\quad\leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2\Big).
\end{aligned}
\tag{3.18}
$$

Proof. Operating ∇ to $(2.1)_1$–$(2.1)_4$, multiplying them by $a\gamma\nabla\varrho$, $\nabla \boldsymbol{v}$, $\nabla\omega$, and $\nabla \boldsymbol{h}$ in L^2, respectively, and integrating by parts, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(a\gamma\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \Big(\mu_1\|\nabla^2 \boldsymbol{v}\|_{L^2}^2 + (\mu_1+\lambda_1)\|\nabla \mathrm{div}\boldsymbol{v}\|_{L^2}^2 + \nu\|\nabla^2 \boldsymbol{h}\|_{L^2}^2\\
&\qquad + \mu_2\|\nabla^2\omega\|_{L^2}^2 + (\mu_2+\lambda_2)\|\nabla\mathrm{div}\omega\|_{L^2}^2 + \zeta\|\nabla(\mathrm{rot}\boldsymbol{v} - 2\omega)\|_{L^2}^2\Big)\\
&= \langle\nabla R_1, a\gamma\nabla\varrho\rangle - \langle R_2, \Delta \boldsymbol{v}\rangle - \langle R_3, \Delta\omega\rangle - \langle R_4, \Delta \boldsymbol{h}\rangle.
\end{aligned}
\tag{3.19}
$$

It follows from (3.1), (3.2), and the integration by parts that

$$
\begin{aligned}
\langle\nabla R_1, a\gamma\nabla\varrho\rangle &\leqslant C\|\nabla\varrho\|_{L^3}\|\nabla\varrho\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^6} + C\|\varrho\|_{L^6}\|\nabla^2 \boldsymbol{v}\|_{L^2}\|\nabla\varrho\|_{L^3}\\
&\leqslant C\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla^2 \boldsymbol{v}\|_{L^2}^2\Big)\\
&\leqslant C\alpha^{1/2}L^{1/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla^2 \boldsymbol{v}\|_{L^2}^2\Big).
\end{aligned}
\tag{3.20}
$$

Due to the Sobolev inequalities (2.22), (2.23), (3.10), and (3.11), it can be obtained by direct calculation that

$$
\begin{aligned}
\|(R_2, R_3, R_4)\|_{L^2} &\leqslant C\|\boldsymbol{v}\|_{L^3}\|\nabla \boldsymbol{v}\|_{L^6} + C\|\varrho\|_{L^\infty}\|\nabla^2 \boldsymbol{v}\|_{L^2} + C\|\boldsymbol{h}\|_{L^3}\|\nabla \boldsymbol{h}\|_{L^6}\\
&\quad + C\|\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^2} + C\|\varrho\|_{L^3}\|\nabla\omega\|_{L^6} + C\|\boldsymbol{v}\|_{L^3}\|\nabla\omega\|_{L^6}\\
&\quad + C\|\varrho\|_{L^\infty}\|\nabla^2\omega\|_{L^2} + C\|\varrho\|_{L^\infty}\|\omega\|_{L^2} + C\|\varrho\|_{L^3}\|\nabla \boldsymbol{v}\|_{L^6}\\
&\quad + C\|\boldsymbol{v}\|_{L^3}\|\nabla \boldsymbol{h}\|_{L^6} + C\|\boldsymbol{h}\|_{L^3}\|\nabla \boldsymbol{v}\|_{L^6} + C\|\nabla\varrho\|_{L^2}\|\nabla \boldsymbol{h}\|_{L^\infty}\|\boldsymbol{h}\|_{L^\infty}\\
&\quad + C\|\nabla \boldsymbol{h}\|_{L^4}^2 + C\|\boldsymbol{h}\|_{L^\infty}\|\nabla^2 \boldsymbol{h}\|_{L^2}\\
&\leqslant C\Big(\|(\varrho, \boldsymbol{v}, \boldsymbol{h})\|_{L^3}\|\nabla(\boldsymbol{v}, \boldsymbol{h}, \omega)\|_{L^6} + \|(\varrho, \boldsymbol{h})\|_{L^\infty}\|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}\Big)\\
&\quad + C\Big(\|\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^2} + \|\nabla\varrho\|_{L^2}\|\nabla \boldsymbol{h}\|_{L^\infty}\|\boldsymbol{h}\|_{L^\infty} + \|\nabla \boldsymbol{h}\|_{L^4}^2\Big)\\
&\leqslant C\Big(\|(\varrho, \boldsymbol{v}, \boldsymbol{h})\|_{H^1} + \|\nabla(\varrho, \boldsymbol{h})\|_{L^2}^{1/2}\|\nabla^2(\varrho, \boldsymbol{h})\|_{L^2}^{1/2}\Big)\|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}\\
&\quad + C\Big(\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2} + \|\nabla^2 \boldsymbol{h}\|_{L^2}\|\nabla^3 \boldsymbol{h}\|_{L^2}^{1/2}\|\nabla \boldsymbol{h}\|_{L^2}^{1/2}\Big)\|\nabla\varrho\|_{L^2}\\
&\leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2} + \|\nabla\varrho\|_{L^2}\Big),
\end{aligned}
\tag{3.21}
$$

thus, using (3.21), one has

$$\left|\langle R_2, \Delta \boldsymbol{v}\rangle\right| + \left|\langle R_3, \Delta\omega\rangle\right| + \left|\langle R_4, \Delta \boldsymbol{h}\rangle\right| \leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2\Big). \tag{3.22}$$

Putting (3.20) and (3.22) into (3.19), we obtain (3.18). □

Remark 3.1. Obviously, we can infer from (3.20) that

$$\begin{aligned}\langle \nabla R_1, a\gamma\nabla\varrho\rangle &\leqslant C\|\nabla\varrho\|_{L^3}^2\|\nabla\boldsymbol{v}\|_{L^3} + C\|\varrho\|_{L^3}\|\nabla^2\boldsymbol{v}\|_{L^2}\|\nabla\varrho\|_{L^6}\\ &\leqslant C\|\nabla\varrho\|_{L^2}\|\nabla^2\varrho\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2} + C\|\varrho\|_{H^1}\|\nabla^2\varrho\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}\\ &\leqslant C\alpha\Big(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^2\boldsymbol{v}\|_{L^2}^2\Big).\end{aligned} \tag{3.23}$$

Similarly,

$$\|(R_2,R_3,R_4)\|_{L^2} \leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2} + \|\nabla^2\varrho\|_{L^2}\Big),$$

which, combined with (3.19) and (3.23), yields

$$\frac{d}{dt}\|\nabla(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 + \|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 \leqslant C\alpha^{1/2}L^{3/2}\|\nabla^2(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2, \tag{3.24}$$

which will be used later to close the estimate of $\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2$.

Lemma 3.3. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$\begin{aligned}\frac{d}{dt}\langle\boldsymbol{v},\nabla\varrho\rangle + \|\nabla\varrho\|_{L^2}^2 &\leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 + \|\nabla\varrho\|_{L^2}^2\Big)\\ &\quad + C\Big(\|\nabla\boldsymbol{v}\|_{H^1}^2 + \|\nabla\omega\|_{L^2}^2\Big).\end{aligned} \tag{3.25}$$

Proof. Multiplying $(2.1)_2$ by $\nabla\varrho$ in L^2, one has

$$\begin{aligned}&\frac{d}{dt}\langle\boldsymbol{v},\nabla\varrho\rangle + a\gamma\|\nabla\varrho\|_{L^2}^2\\ &= \langle\boldsymbol{v},\nabla\varrho_t\rangle + \langle(\mu_1+\zeta)\Delta\boldsymbol{v} + (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}\boldsymbol{v} + 2\zeta\nabla\times\omega, \nabla\varrho\rangle + \langle R_2,\nabla\varrho\rangle.\end{aligned} \tag{3.26}$$

Thanks to $(2.1)_1$, we deduce from integration by parts that

$$\begin{aligned}\langle\boldsymbol{v},\nabla\varrho_t\rangle &= -\langle\boldsymbol{v},\nabla[\mathrm{div}\boldsymbol{v} + \mathrm{div}(\varrho\boldsymbol{v})]\rangle = \langle\mathrm{div}\boldsymbol{v}, \mathrm{div}\boldsymbol{v} + \mathrm{div}(\varrho\boldsymbol{v})\rangle\\ &\leqslant C\|\nabla\boldsymbol{v}\|_{L^2}^2 + C\|\nabla\boldsymbol{v}\|_{L^3}\Big(\|\nabla\varrho\|_{L^2}\|\boldsymbol{v}\|_{L^6} + \|\nabla\boldsymbol{v}\|_{L^2}\|\varrho\|_{L^6}\Big)\\ &\leqslant C\|\nabla\boldsymbol{v}\|_{L^2}^2 + C\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla\boldsymbol{v}\|_{L^2}^2\Big)\\ &\leqslant C\|\nabla\boldsymbol{v}\|_{L^2}^2 + C\alpha^{1/2}L^{1/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla\boldsymbol{v}\|_{L^2}^2\Big),\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}&\langle(\mu_1+\zeta)\Delta\boldsymbol{v} + (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}\boldsymbol{v} + 2\zeta\nabla\times\omega,\nabla\varrho\rangle\\ &\leqslant C\frac{a\gamma}{4}\|\nabla\varrho\|_{L^2}^2 + C\|\nabla^2\boldsymbol{v}\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2.\end{aligned} \tag{3.28}$$

The Cauchy-Schwarz inequality, together with (3.21), gives

$$\langle R_2,\nabla\varrho\rangle \leqslant C\|R_2\|_{L^2}\|\nabla\varrho\|_{L^2} \leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2\Big). \tag{3.29}$$

Substituting (3.27)–(3.29) into (3.26) yields (3.25). □

Lemma 3.4. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2+\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2\\&\leqslant C\alpha^{1/4}L^{3/2}\Big(\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2+\|\nabla^2\varrho\|_{L^2}^2\Big).\end{aligned}\tag{3.30}$$

Proof. Similar to the proof of Lemma 3.2, we infer from (2.1) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^2(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2+\Big(\mu_1\|\nabla^3\boldsymbol{v}\|_{L^2}^2+(\mu_1+\lambda_1)\|\nabla^2\mathrm{div}\boldsymbol{v}\|_{L^2}^2+\nu\|\nabla^3\boldsymbol{h}\|_{L^2}^2\\&\qquad+\mu_2\|\nabla^3\omega\|_{L^2}^2+(\mu_2+\lambda_2)\|\nabla^2\mathrm{div}\omega\|_{L^2}^2+\zeta\|\nabla^2(\mathrm{rot}\boldsymbol{v}-2\omega)\|_{L^2}^2\Big)\\&=\langle\nabla^2R_1,a\gamma\nabla^2\varrho\rangle-\langle\nabla^2R_2,\Delta\boldsymbol{v}\rangle-\langle\nabla^2R_3,\Delta\omega\rangle-\langle\nabla^2R_4,\Delta\boldsymbol{h}\rangle.\end{aligned}\tag{3.31}$$

Keeping in mind that $L>1$, thus, the inequalities (2.7) and (2.8), together with Gagliardo-Nirenberg inequality (cf. [1]) show that

$$\begin{aligned}\langle\nabla^2R_1,a\gamma\nabla^2\varrho\rangle&\leqslant C\Big(\|\nabla^2\varrho\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^\infty}+\|\varrho\|_{L^\infty}\|\nabla^3\boldsymbol{v}\|_{L^2}+\|\nabla\varrho\|_{L^3}\|\nabla^2\boldsymbol{v}\|_{L^6}\Big)\|\nabla^2\varrho\|_{L^2}\\&\leqslant C\|\nabla^2\varrho\|_{L^2}\Big(\|\nabla\boldsymbol{v}\|_{L^2}^{1/4}\|\nabla^3\boldsymbol{v}\|_{L^2}^{3/4}\|\nabla^2\varrho\|_{L^2}+\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^3\boldsymbol{v}\|_{L^2}\Big)\\&\leqslant C\alpha^{1/4}L^{3/4}\Big(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3\boldsymbol{v}\|_{L^2}^2\Big).\end{aligned}\tag{3.32}$$

Due to the Sobolev inequalities (2.22) and (2.23), one has

$$\begin{aligned}\|\nabla(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\|_{L^2}&\leqslant C\|\nabla^2\boldsymbol{v}\|_{L^6}\|\boldsymbol{v}\|_{L^3}+C\|\nabla\boldsymbol{v}\|_{L^6}\|\nabla\boldsymbol{v}\|_{L^3}\\&\leqslant C\Big(\|\nabla^3\boldsymbol{v}\|_{L^2}\|\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}+\|\nabla^3\boldsymbol{v}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\boldsymbol{v}\|_{L^2}^{1/2}\Big)\\&\leqslant C\alpha\|\nabla^3\boldsymbol{v}\|_{L^2}.\end{aligned}\tag{3.33}$$

Similar to the derivation of (3.33), we can obtain that

$$\begin{aligned}&\|\nabla(\boldsymbol{v}\cdot\nabla\omega)\|_{L^2}+\|\nabla(\boldsymbol{v}\cdot\nabla\boldsymbol{h})\|_{L^2}+\|\nabla(\boldsymbol{h}\cdot\mathrm{div}\boldsymbol{v})\|_{L^2}+\|\nabla(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\|_{L^2}\\&\leqslant C\|(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^{1/2}\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^{1/2}\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}\\&\leqslant C\alpha\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}.\end{aligned}\tag{3.34}$$

The combination of (3.8)–(3.11) and the interpolation inequality shows that

$$\begin{aligned}&\|\nabla(h(\varrho)\nabla\varrho)\|_{L^2}+\|\nabla(f(\varrho)\nabla\times\omega)\|_{L^2}+\|\nabla(f(\varrho)\nabla\times\boldsymbol{v})\|_{L^2}\\&\leqslant C\|\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}+C\|\varrho\|_{L^3}\|\nabla^2(\omega,\boldsymbol{v})\|_{L^6}+C\|\nabla(\omega,\boldsymbol{v})\|_{L^3}\|\nabla\varrho\|_{L^6}\\&\leqslant C\Big(\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}+\|\nabla\omega\|_{L^2}^{1/2}\|\nabla^2\omega\|_{L^2}^{1/2}+\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\Big)\|\nabla^2\varrho\|_{L^2}\\&\quad+C\|\varrho\|_{H^1}\|\nabla^3(\omega,\boldsymbol{v})\|_{L^2}\\&\leqslant C\alpha^{1/2}L^{1/2}\|\nabla^2\varrho\|_{L^2}+C\alpha\|\nabla^3(\boldsymbol{v},\omega)\|_{L^2}\\&\leqslant C\alpha^{1/2}L^{1/2}\Big(\|\nabla^2\varrho\|_{L^2}+\|\nabla^3(\boldsymbol{v},\omega)\|_{L^2}\Big),\end{aligned}\tag{3.35}$$

and

$$\left|\left\langle\nabla^2(f(\varrho)\omega),\Delta\omega\right\rangle\right| \leqslant C\int\Big(|\nabla^2\varrho||\omega||\nabla^2\omega|+|\nabla\varrho||\nabla\omega||\nabla^2\omega|+|\varrho||\nabla^2\omega|^2\Big)dx$$
$$\leqslant C\|\nabla^2\varrho\|_{L^2}\|\omega\|_{L^3}\|\nabla^2\omega\|_{L^6}+C\|\nabla\varrho\|_{L^6}\|\nabla\omega\|_{L^{3/2}}\|\nabla^2\omega\|_{L^6}$$
$$+C\|\varrho\|_{L^2}\|\nabla^2\omega\|_{L^6}\|\nabla^2\omega\|_{L^2}$$
$$\leqslant C\alpha\Big(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3\omega\|_{L^2}^2\Big)+\delta_2\|\nabla^2\omega\|_{L^2}^2, \tag{3.36}$$

where δ_2 is an undetermined positive constant. According to the observation of (3.31), we know that

$$\|\nabla^2\omega\|_{L^2}^2 \leqslant C\Big(\|\nabla^2(\nabla\times\boldsymbol{v}-2\omega)\|_{L^2}^2+\|\nabla^2(\nabla\times\boldsymbol{v})\|_{L^2}^2\Big)$$
$$\leqslant C\Big(\|\nabla^2(\nabla\times\boldsymbol{v}-2\omega)\|_{L^2}^2+\|\nabla^3\boldsymbol{v}\|_{L^2}^2\Big), \tag{3.37}$$

thus, putting (3.37) into (3.36) and choosing $\delta_2=\{\frac{\zeta}{C},\frac{\mu_1}{4C}\}$, one has

$$\left|\left\langle\nabla^2(f(\varrho)\omega),\Delta\omega\right\rangle\right| \leqslant C\alpha\Big(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3\omega\|_{L^2}^2\Big)+\zeta\|\nabla^2(\nabla\times\boldsymbol{v}-2\omega)\|_{L^2}^2$$
$$+\frac{\mu_1}{4}\|\nabla^3\boldsymbol{v}\|_{L^2}^2. \tag{3.38}$$

Similar to the derivation of (3.35) and (3.38), one has

$$\|\nabla[h(\varrho)(\Delta\boldsymbol{v},\nabla\mathrm{div}\boldsymbol{v},\Delta\omega,\nabla\mathrm{div}\omega)]\|_{L^2} \leqslant C\|\nabla\varrho\|_{L^3}\|\nabla^2(\boldsymbol{v},\omega)\|_{L^6}+C\|\varrho\|_{L^\infty}\|\nabla^3(\boldsymbol{v},\omega)\|_{L^2}$$
$$\leqslant C\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^3(\boldsymbol{v},\omega)\|_{L^2}$$
$$\leqslant C\alpha^{1/2}L^{1/2}\|\nabla^3(\boldsymbol{v},\omega)\|_{L^2}, \tag{3.39}$$

$$\left|\left\langle\nabla^2[g(\varrho)(\nabla\times\boldsymbol{h})\times\boldsymbol{h}],\nabla^2\boldsymbol{v}\right\rangle\right| \leqslant C\|\nabla^2\varrho\|_{L^2}\|\nabla\boldsymbol{h}\|_{L^3}\|\boldsymbol{h}\|_{L^\infty}\|\nabla^2\boldsymbol{v}\|_{L^6}+C\|\nabla\varrho\|_{L^2}\|\nabla^2\boldsymbol{h}\|_{L^6}\|\boldsymbol{h}\|_{L^6}\|\nabla^2\boldsymbol{v}\|_{L^6}$$
$$+C\|\nabla\varrho\|_{L^6}\|\nabla\boldsymbol{h}\|_{L^6}\|\nabla\boldsymbol{h}\|_{L^2}\|\nabla^2\boldsymbol{v}\|_{L^6}+C\|\nabla^3\boldsymbol{h}\|_{L^2}\|\boldsymbol{h}\|_{L^3}\|\nabla^2\boldsymbol{v}\|_{L^6}$$
$$+C\|\nabla^2\boldsymbol{h}\|_{L^6}\|\nabla\boldsymbol{h}\|_{L^{3/2}}\|\nabla^2\boldsymbol{v}\|_{L^6}$$
$$\leqslant C\|\nabla^2\varrho\|_{L^2}\|\nabla\boldsymbol{h}\|_{L^2}\|\nabla^2\boldsymbol{h}\|_{L^2}\|\nabla^3\boldsymbol{v}\|_{L^2}+C\|\nabla\varrho\|_{L^2}\|\nabla^3\boldsymbol{h}\|_{L^3}\|\nabla\boldsymbol{h}\|_{L^2}\|\nabla^3\boldsymbol{v}\|_{L^2}$$
$$+C\|\nabla^2\varrho\|_{L^2}\|\nabla^2\boldsymbol{h}\|_{L^2}\|\nabla\boldsymbol{h}\|_{L^2}\|\nabla^3\boldsymbol{v}\|_{L^2}+C\|\nabla^3\boldsymbol{h}\|_{L^2}\|\boldsymbol{h}\|_{H^1}\|\nabla^3\boldsymbol{v}\|_{L^2}$$
$$+C\|\nabla^3\boldsymbol{h}\|_{L^2}\|\boldsymbol{h}\|_{H^1}\|\nabla^3\boldsymbol{v}\|_{L^2}$$
$$\leqslant C\alpha L\Big(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3(\boldsymbol{v},\boldsymbol{h})\|_{L^2}^2\Big), \tag{3.40}$$

and

$$\|\nabla^2[g(\varrho)(\nabla\times\boldsymbol{h})\times\boldsymbol{h}]\|_{L^2} \leqslant C\|\nabla^2\varrho\|_{L^2}\|\nabla\boldsymbol{h}\|_{L^\infty}\|\boldsymbol{h}\|_{L^\infty}+C\|\nabla^2\boldsymbol{h}\|_{L^3}\|\boldsymbol{h}\|_{L^\infty}\|\nabla\varrho\|_{L^6}$$
$$+C\|\nabla\varrho\|_{L^6}\|\nabla\boldsymbol{h}\|_{L^6}^2+C\|\boldsymbol{h}\|_{L^\infty}\|\nabla^3\boldsymbol{h}\|_{L^2}+C\|\nabla^2\boldsymbol{h}\|_{L^6}\|\nabla\boldsymbol{h}\|_{L^3}$$
$$\leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2\varrho\|_{L^2}+\|\nabla^3\boldsymbol{h}\|_{L^2}\Big). \tag{3.41}$$

The combination of (3.33)–(3.35) and (3.38)–(3.41), gives

$$\left|\left\langle\nabla^2R_2,\Delta\boldsymbol{v}\right\rangle+\left\langle\nabla^2R_3,\Delta\omega\right\rangle+\left\langle\nabla^2R_4,\Delta\boldsymbol{h}\right\rangle\right| \leqslant C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2\Big). \tag{3.42}$$

Therefore, substituting (3.32) and (3.42) into (3.31), we can obtain (3.30) by using Cauchy-Schwarz inequality. □

Lemma 3.5. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$\frac{d}{dt}\langle \mathrm{div}\boldsymbol{v}, \Delta\varrho\rangle + \|\nabla^2\varrho\|_{L^2}^2 \leqslant C\|\nabla^2\boldsymbol{v}\|_{H^1}^2 + C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 + \|\nabla^2\varrho\|_{L^2}^2\Big). \tag{3.43}$$

Proof. Operating div to $(2.1)_2$ and multiplying the resulting equation by $\Delta\varrho$ in L^2, after integrating by parts, one has from $\nabla\cdot(\nabla\times\omega) = 0$ that

$$\begin{aligned}\frac{d}{dt}\langle \mathrm{div}\boldsymbol{v}, \Delta\varrho\rangle + a\gamma\|\nabla^2\varrho\|_{L^2}^2 &= -\langle\nabla\mathrm{div}\boldsymbol{v}, \nabla\varrho_t\rangle + \langle \mathrm{div}R_2, \Delta\varrho\rangle\\ &+ \langle(\mu_1+\zeta)\mathrm{div}\Delta\boldsymbol{v} + (\mu_1+\lambda_1-\zeta)\mathrm{div}(\nabla\mathrm{div}\boldsymbol{v}), \Delta\varrho\rangle.\end{aligned} \tag{3.44}$$

Due to $(2.1)_1$, one has

$$\begin{aligned}\Big|\langle\nabla\mathrm{div}\boldsymbol{v}, \nabla\varrho_t\rangle\Big| &\leqslant C\Big(\|\nabla^2\boldsymbol{v}\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla\boldsymbol{v}\|_{L^6} + \|\varrho\|_{L^\infty}\|\nabla^2\boldsymbol{v}\|_{L^2}\\ &\quad + \|\boldsymbol{v}\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}\Big)\|\nabla^2\boldsymbol{v}\|_{L^2}\\ &\leqslant C\|\nabla^2\boldsymbol{v}\|_{L^2}^2 + C\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^2\\ &\quad + \|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}\|\nabla^2\boldsymbol{v}\|_{L^2}\\ &\leqslant C\|\nabla^2\boldsymbol{v}\|_{L^2}^2 + C\alpha^{1/2}L^{1/2}\|\nabla^2(\varrho,\boldsymbol{v})\|_{L^2}^2.\end{aligned} \tag{3.45}$$

On the other hand, it follows from (3.42) that

$$\begin{aligned}&\Big|\langle \mathrm{div}R_2, \Delta\varrho\rangle + \langle(\mu_1+\zeta)\mathrm{div}\Delta\boldsymbol{v} + (\mu_1+\lambda_1-\zeta)\mathrm{div}(\nabla\mathrm{div}\boldsymbol{v}), \Delta\varrho\rangle\Big|\\ &\leqslant \frac{a\gamma}{4}\|\nabla^2\varrho\|_{L^2}^2 + C\|\nabla^3\boldsymbol{v}\|_{L^2}^2 + C\alpha^{1/2}L^{3/2}\Big(\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2\Big),\end{aligned}$$

which, together with (3.44) and (3.45), gives (3.43). □

Lemma 3.6. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$\frac{d}{dt}\|\nabla^3(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 + \|\nabla^4(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 \leqslant C\alpha^{1/4}L^{5/2}\Big(\|\nabla^4(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 + \|\nabla^3\varrho\|_{L^2}^2\Big). \tag{3.46}$$

Proof. Similar to the proof of Lemma 3.4, we infer from (2.1) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^3(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2 + \Big(\mu_1\|\nabla^4\boldsymbol{v}\|_{L^2}^2 + (\mu_1+\lambda_1)\|\nabla^3\mathrm{div}\boldsymbol{v}\|_{L^2}^2 + \nu\|\nabla^4\boldsymbol{h}\|_{L^2}^2\\ &\quad + \mu_2\|\nabla^4\omega\|_{L^2}^2 + (\mu_2+\lambda_2)\|\nabla^3\mathrm{div}\omega\|_{L^2}^2 + \zeta\|\nabla^3(\mathrm{rot}\boldsymbol{v} - 2\omega)\|_{L^2}^2\Big)\\ &= \langle\nabla^3R_1, a\gamma\nabla^3\varrho\rangle - \langle\nabla^3R_2, \nabla^3\boldsymbol{v}\rangle - \langle\nabla^3R_3, \nabla^3\omega\rangle - \langle\nabla^3R_4, \nabla^3\boldsymbol{h}\rangle.\end{aligned} \tag{3.47}$$

Keeping in mind that $L > 1$, thus, Lemma 2.3, together with Gagliardo-Nirenberg inequality (cf. [1]) shows that

$$\begin{aligned}\langle\nabla^3R_1, a\gamma\nabla^3\varrho\rangle &\leqslant C\Big(\|\nabla^3\varrho\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^\infty} + \|\varrho\|_{L^\infty}\|\nabla^4\boldsymbol{v}\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla^3\boldsymbol{v}\|_{L^6}\Big)\|\nabla^3\varrho\|_{L^2}\\ &\leqslant C\|\nabla^3\varrho\|_{L^2}\Big(\|\nabla\boldsymbol{v}\|_{L^2}^{1/4}\|\nabla^3\boldsymbol{v}\|_{L^2}^{3/4}\|\nabla^3\varrho\|_{L^2} + \|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^4\boldsymbol{v}\|_{L^2}\Big)\\ &\leqslant C\alpha^{1/4}L^{3/4}\Big(\|\nabla^3\varrho\|_{L^2}^2 + \|\nabla^4\boldsymbol{v}\|_{L^2}^2\Big).\end{aligned} \tag{3.48}$$

Due to the Sobolev interpolation inequalities (2.22) and (2.23) (cf. [9, 14]), one has from (2.7) that

$$\begin{aligned}\|\nabla^2(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\|_{L^2} &\leqslant C\|\nabla^3\boldsymbol{v}\|_{L^6}\|\boldsymbol{v}\|_{L^3}+C\|\nabla^2\boldsymbol{v}\|_{L^6}\|\nabla\boldsymbol{v}\|_{L^3}\\&\leqslant C\Big(\|\nabla^4\boldsymbol{v}\|_{L^2}\|\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}+\|\nabla^3\boldsymbol{v}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\Big)\\&\leqslant C\alpha\|\nabla^4\boldsymbol{v}\|_{L^2}.\end{aligned}\tag{3.49}$$

Similar to the derivation of (3.49), we can get

$$\begin{aligned}&\|\nabla^2(\boldsymbol{v}\cdot\nabla\omega)\|_{L^2}+\|\nabla^2(\boldsymbol{v}\cdot\nabla\boldsymbol{h})\|_{L^2}+\|\nabla^2(\boldsymbol{h}\cdot\mathrm{div}\boldsymbol{v})\|_{L^2}+\|\nabla^2(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\|_{L^2}\\&\leqslant C\|(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^{1/2}\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^{1/2}\|\nabla^4(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}\\&\leqslant C\alpha\|\nabla^4(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}.\end{aligned}\tag{3.50}$$

The combination of (3.8)–(3.11) and the interpolation inequality shows that

$$\begin{aligned}&\|\nabla^2(h(\varrho)\nabla\varrho)\|_{L^2}+\|\nabla^2(f(\varrho)\nabla\times\omega)\|_{L^2}+\|\nabla^2(f(\varrho)\nabla\times\boldsymbol{v})\|_{L^2}\\&\leqslant C\|\varrho\|_{L^\infty}\|\nabla^3\varrho\|_{L^2}+C\|\nabla(\varrho,\boldsymbol{v},\omega)\|_{L^3}\|\nabla^2\varrho\|_{L^6}+C\|\varrho\|_{L^3}\|\nabla^3(\omega,\boldsymbol{v})\|_{L^6}\\&\leqslant C\Big(\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}+\|\nabla\omega\|_{L^2}^{1/2}\|\nabla^2\omega\|_{L^2}^{1/2}+\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\Big)\|\nabla^3\varrho\|_{L^2}\\&\quad+C\|\varrho\|_{H^1}\|\nabla^4(\omega,\boldsymbol{v})\|_{L^2}\\&\leqslant C\alpha^{1/2}L^{1/2}\|\nabla^3\varrho\|_{L^2}+C\alpha\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2}\\&\leqslant C\alpha^{1/2}L^{1/2}\Big(\|\nabla^3\varrho\|_{L^2}+\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2}\Big),\end{aligned}\tag{3.51}$$

and

$$\begin{aligned}&\Big|\big\langle\nabla^3(f(\varrho)\omega),\nabla^3\omega\big\rangle\Big|\\&\leqslant C\int\Big(|\nabla^3\varrho||\omega||\nabla^3\omega|+|\nabla^2\varrho||\nabla\omega||\nabla^3\omega|+|\nabla\varrho||\nabla^2\omega||\nabla^3\omega|+|\varrho||\nabla^3\omega|^2\Big)dx\\&\leqslant C\|\nabla^3\varrho\|_{L^2}\|\omega\|_{L^3}\|\nabla^3\omega\|_{L^6}+C\|\nabla^2\varrho\|_{L^6}\|\nabla\omega\|_{L^{3/2}}\|\nabla^3\omega\|_{L^6}\\&\quad+C\|\nabla\varrho\|_{L^2}^{3/2}\|\nabla^2\omega\|_{L^6}\|\nabla^3\omega\|_{L^6}+C\|\varrho\|_{L^2}\|\nabla^3\omega\|_{L^6}\|\nabla^3\omega\|_{L^2}\\&\leqslant C\alpha\Big(\|\nabla^3\varrho\|_{L^2}^2+\|\nabla^4\omega\|_{L^2}^2\Big)+\delta_3\|\nabla^3\omega\|_{L^2}^2,\end{aligned}\tag{3.52}$$

where δ_3 is an undetermined positive constant. According to the observation of (3.47), we know that

$$\begin{aligned}\|\nabla^3\omega\|_{L^2}^2&\leqslant C\Big(\|\nabla^3(\nabla\times\boldsymbol{v}-2\omega)\|_{L^2}^2+\|\nabla^3(\nabla\times\boldsymbol{v})\|_{L^2}^2\Big)\\&\leqslant C\Big(\|\nabla^3(\nabla\times\boldsymbol{v}-2\omega)\|_{L^2}^2+\|\nabla^4\boldsymbol{v}\|_{L^2}^2\Big),\end{aligned}\tag{3.53}$$

thus, putting (3.53) into (3.52) and choosing $\delta_3=\{\frac{\zeta}{C},\frac{\mu_1}{4C}\}$, one has

$$\begin{aligned}\Big|\big\langle\nabla^3(f(\varrho)\omega),\nabla^3\omega\big\rangle\Big|&\leqslant C\alpha\Big(\|\nabla^3\varrho\|_{L^2}^2+\|\nabla^4\omega\|_{L^2}^2\Big)+\zeta\|\nabla^3(\nabla\times\boldsymbol{v}-2\omega)\|_{L^2}^2\\&\quad+\frac{\mu_1}{4}\|\nabla^4\boldsymbol{v}\|_{L^2}^2.\end{aligned}\tag{3.54}$$

Similar to the derivation of (3.51) and (3.54), one has from (2.7) that

$$\begin{aligned}
&\|\nabla^2[h(\varrho)(\Delta \boldsymbol{v}, \nabla \mathrm{div}\boldsymbol{v}, \Delta\omega, \nabla\mathrm{div}\omega)]\|_{L^2}\\
&\leqslant C\|\nabla^2\varrho\|_{L^3}\|\nabla^2(\boldsymbol{v},\omega)\|_{L^6} + C\|\varrho\|_{L^\infty}\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2}\\
&\leqslant C\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^3\varrho\|_{L^2}^{1/2}\|\nabla^3(\boldsymbol{v},\omega)\|_{L^2} + C\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2}\\
&\leqslant C\alpha^{1/2}L^{1/2}\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2} + C\|\nabla\varrho\|_{L^2}^{1/4}\|\nabla^3\varrho\|_{L^2}^{3/4}\|\nabla^2(\boldsymbol{v},\omega)\|_{L^2}^{1/2}\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2}^{1/2}\\
&\leqslant C\alpha^{1/4}L\Big(\|\nabla^4(\boldsymbol{v},\omega)\|_{L^2} + \|\nabla^3\varrho\|_{L^2}\Big),
\end{aligned}\tag{3.55}$$

and

$$\begin{aligned}
\|\nabla^2[g(\varrho)(\nabla\times\boldsymbol{h})\times\boldsymbol{h}]\|_{L^2} &\leqslant C\|\nabla^2\varrho\|_{L^3}\|\nabla\boldsymbol{h}\|_{L^6}\|\boldsymbol{h}\|_{L^\infty} + C\|\nabla\boldsymbol{h}\|_{L^6}\|\boldsymbol{h}\|_{L^\infty}\|\nabla\varrho\|_{L^6}^2\\
&\quad + C\|\boldsymbol{h}\|_{L^3}\|\nabla^3\boldsymbol{h}\|_{L^6} + C\|\nabla\boldsymbol{h}\|_{L^3}\|\nabla^2\boldsymbol{h}\|_{L^6}\\
&\leqslant C\alpha L^{3/4}\Big(\|\nabla^3\varrho\|_{L^2} + \|\nabla^4\boldsymbol{h}\|_{L^2}\Big),
\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}
\|\nabla^3[g(\varrho)(\nabla\times\boldsymbol{h})\times\boldsymbol{h}]\|_{L^2} &\leqslant C\Big(\|\nabla^3\varrho\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla^2\varrho\|_{L^6} + \|\nabla\varrho\|_{L^6}^3\Big)\|\nabla\boldsymbol{h}\|_{L^\infty}\|\boldsymbol{h}\|_{L^\infty}\\
&\quad + C\|\nabla\boldsymbol{h}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{h}\|_{L^2}^{1/2}\|\nabla^4\boldsymbol{h}\|_{L^2}\\
&\leqslant C\alpha^{1/2}L^{5/2}\Big(\|\nabla^3\varrho\|_{L^2} + \|\nabla^4\boldsymbol{h}\|_{L^2}\Big).
\end{aligned}\tag{3.57}$$

The combination of (3.49)–(3.51) and (3.54)–(3.57) yields

$$\Big|\langle\nabla^3R_2,\nabla^3\boldsymbol{v}\rangle + \langle\nabla^3R_3,\nabla^3\omega\rangle + \langle\nabla^3R_4,\nabla^3\boldsymbol{h}\rangle\Big| \leqslant C\alpha^{1/4}L^{5/2}\Big(\|\nabla^3\varrho\|_{L^2} + \|\nabla^4(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}\Big),\tag{3.58}$$

which, together with (3.47) and (3.48), gives (3.46). □

Lemma 3.7. *Let the assumptions* (3.1) *and* (3.2) *be in force. Then*

$$\frac{d}{dt}\langle\nabla\mathrm{div}\boldsymbol{v},\nabla\Delta\varrho\rangle + \|\nabla^3\varrho\|_{L^2}^2 \leqslant C\|\nabla^3\boldsymbol{v}\|_{H^1}^2 + C\alpha^{1/4}L^{5/2}\Big(\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 + \|\nabla^3\varrho\|_{L^2}^2\Big).\tag{3.59}$$

Proof. Operating ∇div to $(2.1)_2$ and multiplying the resulting equation by $\nabla\Delta\varrho$ in L^2, after integrating by parts, one has from $\nabla\cdot(\nabla\times\omega) = 0$ that

$$\begin{aligned}
\frac{d}{dt}\langle\nabla\mathrm{div}\boldsymbol{v},\nabla\Delta\varrho\rangle + a\gamma\|\nabla^3\varrho\|_{L^2}^2 &= \langle\nabla\mathrm{div}\boldsymbol{v},\nabla\Delta\varrho_t\rangle + \langle\nabla\mathrm{div}R_2,\nabla\Delta\varrho\rangle\\
&\quad + \Big\langle(\mu_1+\zeta)\nabla\mathrm{div}\Delta\boldsymbol{v} + (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}(\nabla\mathrm{div}\boldsymbol{v}),\nabla\Delta\varrho\Big\rangle.
\end{aligned}\tag{3.60}$$

Due to $(2.1)_1$, one has

$$\begin{aligned}
\langle\nabla\mathrm{div}\boldsymbol{v},\nabla\Delta\varrho_t\rangle &= -\langle\Delta\mathrm{div}\boldsymbol{v},\Delta\varrho_t\rangle\\
&\leqslant C\Big(\|\nabla^3\boldsymbol{v}\|_{L^2} + \|\varrho\|_{L^\infty}\|\nabla^3\boldsymbol{v}\|_{L^2} + \|\boldsymbol{v}\|_{L^\infty}\|\nabla^3\varrho\|_{L^2}\Big)\|\nabla^3\boldsymbol{v}\|_{L^2}\\
&\leqslant C\|\nabla^3\boldsymbol{v}\|_{L^2}^2 + C\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^3\boldsymbol{v}\|_{L^2}^2\\
&\quad + \|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{v}\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}\|\nabla^3\boldsymbol{v}\|_{L^2}\\
&\leqslant C\|\nabla^3\boldsymbol{v}\|_{L^2}^2 + C\alpha^{1/2}L^{1/2}\|\nabla^3(\varrho,\boldsymbol{v})\|_{L^2}^2.
\end{aligned}\tag{3.61}$$

On the other hand, it follows from (3.58) that

$$\begin{aligned}&\Big|\big\langle\nabla\mathrm{div}R_2,\nabla\Delta\varrho\big\rangle+\big\langle(\mu_1+\zeta)\nabla\mathrm{div}\Delta\boldsymbol{v}+(\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}(\nabla\mathrm{div}\boldsymbol{v}),\nabla\Delta\varrho\big\rangle\Big|\\&\leqslant\frac{a\gamma}{4}\|\nabla^3\varrho\|_{L^2}^2+C\|\nabla^3\boldsymbol{v}\|_{H^1}^2+C\alpha^{1/4}L^{5/2}\Big(\|\nabla^3\varrho\|_{L^2}^2+\|\nabla^4(\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}^2\Big),\end{aligned}$$

which, together with (3.60) and (3.61), gives (3.59). □

With all the a priori estimates obtained in Lemmas 3.1–3.7 at hand, we are ready to prove Proposition 3.1 next.

Proof of Proposition 3.1. Due to the inequalities of (3.12), (3.18), and (3.25), one has

$$\begin{aligned}&\frac{d}{dt}\|(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2\\&\leqslant C_1\alpha^{1/2}L^{3/2}\Big(\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\|\nabla\varrho\|_{L^2}^2\Big)\\&\leqslant\frac{1}{2}\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+C_1\alpha^{1/2}L^{3/2}\|\nabla\varrho\|_{L^2}^2,\end{aligned}\tag{3.62}$$

and

$$\begin{aligned}\frac{d}{dt}\langle\boldsymbol{v},\nabla\varrho\rangle+\|\nabla\varrho\|_{L^2}^2&\leqslant C_2\alpha^{1/2}L^{3/2}\Big(\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\|\nabla\varrho\|_{L^2}^2\Big)\\&\quad+C_2\Big(\|\nabla\boldsymbol{v}\|_{H^1}^2+\|\nabla\omega\|_{L^2}^2\Big)\\&\leqslant 2C_2\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+C_2\alpha^{1/2}L^{3/2}\|\nabla\varrho\|_{L^2}^2,\end{aligned}\tag{3.63}$$

provided $\alpha>0$ is chosen to be small enough such that

$$0<\alpha\leqslant\alpha_1\triangleq\min\Big\{\varepsilon,\Big(\frac{1}{2C_1L^{3/2}}\Big)^2,\frac{1}{L^3}\Big\}.$$

Therefore, we infer from (3.62) that

$$\frac{d}{dt}\|(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\frac{1}{2}\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2\leqslant C_1\alpha^{1/2}L^{3/2}\|\nabla\varrho\|_{L^2}^2,$$

which, multiply $M_1\triangleq\max\{4,8C_2\}$ and added to (3.63), gives

$$\frac{d}{dt}\Big(M_1\|(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\langle\boldsymbol{v},\nabla\varrho\rangle\Big)+2C_2\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\|\nabla\varrho\|_{L^2}^2\leqslant C_3(M_1)\alpha^{1/2}L^{3/2}\|\nabla\varrho\|_{L^2}^2,$$

where $C_3(M_1)$ is a positive number depending on M_1. Then, if $\alpha>0$ is chosen to be small such that

$$0<\alpha\leqslant\alpha_2\triangleq\min\Big\{\alpha_1,\Big(\frac{1}{2C_3(M_1)L^{3/2}}\Big)^2\Big\},$$

then, we have

$$\frac{d}{dt}\Big(M_1\|(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\langle\boldsymbol{v},\nabla\varrho\rangle\Big)+2C_2\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\frac{1}{2}\|\nabla\varrho\|_{L^2}^2\leqslant 0.$$

Integrating the above inequality over $[0,T]$, one has

$$\sup_{t\in[0,T]}\|(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})(t)\|_{H^1}^2+\int_0^T\Big(\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2+\|\nabla\varrho\|_{L^2}^2\Big)dt\leqslant\check{C}\|(\varrho_0,\boldsymbol{v}_0,\omega_0,\boldsymbol{h}_0)\|_{H^1}^2, \tag{3.64}$$

moreover,

$$\langle\boldsymbol{v},\nabla\varrho\rangle\leqslant\frac{M_1}{2}\|(\varrho,\boldsymbol{v})\|_{H^1}^2.$$

It follows from (3.24), (3.30), and (3.46) that

$$\begin{aligned}&\frac{d}{dt}\|\nabla(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2\\&\leqslant C_4\alpha^{1/4}L^{5/2}\Big(\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+\|\nabla^2\varrho\|_{H^1}^2\Big)\\&\leqslant\frac{1}{2}\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+C_4\alpha^{1/4}L^{5/2}\|\nabla^2\varrho\|_{H^1}^2,\end{aligned}$$

thus

$$\frac{d}{dt}\|\nabla(a\gamma\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+\frac{1}{2}\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2\leqslant C_4\alpha^{1/4}L^{5/2}\|\nabla^2\varrho\|_{H^1}^2, \tag{3.65}$$

provided $\alpha>0$ is chosen to be small enough such that

$$0<\alpha\leqslant\alpha_3\triangleq\min\Big\{\alpha_2,\Big(\frac{1}{2C_4L^{5/2}}\Big)^4\Big\}.$$

Next, we infer from (3.43) and (3.59) that

$$\begin{aligned}&\frac{d}{dt}\Big(\langle\mathrm{div}\boldsymbol{v},\Delta\varrho\rangle+\langle\nabla\mathrm{div}\boldsymbol{v},\nabla\Delta\varrho\rangle\Big)+\|\nabla^2\varrho\|_{H^1}^2\\&\leqslant C_5\|\nabla^2\boldsymbol{v}\|_{H^2}^2+C_5\alpha^{1/4}L^{5/2}\Big(\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+\|\nabla^2\varrho\|_{H^1}^2\Big)\\&\leqslant 2C_5\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+\frac{1}{2}\|\nabla^2\varrho\|_{H^1}^2,\end{aligned}$$

then

$$\frac{d}{dt}\Big(\langle\mathrm{div}\boldsymbol{v},\Delta\varrho\rangle+\langle\nabla\mathrm{div}\boldsymbol{v},\nabla\Delta\varrho\rangle\Big)+\frac{1}{2}\|\nabla^2\varrho\|_{H^1}^2\leqslant 2C_5\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2, \tag{3.66}$$

provided $\alpha>0$ is chosen to be small enough such that

$$0<\alpha\leqslant\alpha_4\triangleq\min\Big\{\alpha_3,\Big(\frac{1}{2C_5L^{5/2}}\Big)^4,\Big(\frac{1}{L^{5/2}}\Big)^4\Big\}.$$

Multiplying (3.65) by $M_2\triangleq\{8,8C_5\}$, and adding the resulting inequality to (3.66), after integrating the resulting inequality over $[0,T]$, one has

$$\begin{aligned}&\sup_{t\in[0,T]}\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})(t)\|_{H^2}^2+\int_0^T\Big(\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2+\|\nabla^2\varrho\|_{H^1}^2\Big)dt\\&\leqslant\hat{C}\|\nabla(\varrho_0,\boldsymbol{v}_0,\omega_0,\boldsymbol{h}_0)\|_{H^2}^2.\end{aligned} \tag{3.67}$$

Taking $L^2\triangleq 4\hat{C}\|\nabla(\varrho_0,\boldsymbol{v}_0,\omega_0,\boldsymbol{h}_0)\|_{H^2}^2$ in (3.67), and choosing $\varepsilon>0$ sufficiently small such that $4\check{C}\|(\varrho_0,\boldsymbol{v}_0,\omega_0,\boldsymbol{h}_0)\|_{H^1}^2\leqslant\alpha^2$ in (3.64), then, we can close the a priori assumptions (3.1) and (3.2) by bootstrap arguments. This, together with the local existence result (cf. Lemma 2.5), finishes the proof of Proposition 3.1, and thus, the proof of the first part of Theorem 1.1 is complete. □

3.2. Decay rates

In this subsection, our main aim is to derive the decay rates of the solutions $(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})$ obtained in the first part of Theorem 1.1.

Lemma 3.8. *Let the conditions of Proposition 3.1 be in force. Assume that $\varepsilon > 0$ is small enough and $\|(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|_{L^1}$ is bounded. Then for any $t \geqslant 0$,*

$$\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{H^2}^2 \leqslant C(1+t)^{-5/2}, \tag{3.68}$$

and

$$\|(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{L^2}^2 \leqslant C(1+t)^{-3/2}. \tag{3.69}$$

Proof. Similar to the derivation of (3.64), we can infer from (3.24), (3.30), (3.43), (3.46), and (3.59) that there exist some positive constants $\tilde{M}$ and $\tilde{c}$ such that if $\varepsilon > 0$ is small enough, then

$$\mathcal{A}'(t) + \tilde{c}\Big(\|\nabla^2(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^2}^2 + \|\nabla^2 \varrho\|_{H^1}^2\Big) \leqslant 0,$$

where

$$\mathcal{A}'(t) \triangleq \tilde{M}\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^2}^2 + \langle \operatorname{div}\boldsymbol{v}, \Delta\varrho\rangle + \langle \nabla\operatorname{div}\boldsymbol{v}, \nabla\Delta\varrho\rangle.$$

Therefore

$$\mathcal{A}'(t) + \tilde{c}\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^2}^2 \leqslant C\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2. \tag{3.70}$$

Due to Cauchy-Schwarz inequality, it holds that

$$\Big|\langle \operatorname{div}\boldsymbol{v}, \Delta\varrho\rangle + \langle \nabla\operatorname{div}\boldsymbol{v}, \nabla\Delta\varrho\rangle\Big| \leqslant C\Big(\|\nabla\boldsymbol{v}\|_{H^1}^2 + \|\nabla^2\varrho\|_{H^1}^2\Big),$$

thus

$$-C_6\Big(\|\nabla\boldsymbol{v}\|_{H^1}^2 + \|\nabla^2\varrho\|_{H^1}^2\Big) \leqslant \langle \operatorname{div}\boldsymbol{v}, \Delta\varrho\rangle + \langle \nabla\operatorname{div}\boldsymbol{v}, \nabla\Delta\varrho\rangle \leqslant C_6\Big(\|\nabla\boldsymbol{v}\|_{H^1}^2 + \|\nabla^2\varrho\|_{H^1}^2\Big).$$

For suitable large number $\tilde{M} \geqslant C_6 + 1 > 0$, then

$$\mathcal{A}(t) \sim \|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^2}^2,$$

then, we can infer from (3.70) that there exists a positive constant c such that

$$\mathcal{A}'(t) + c\mathcal{A}(t) \leqslant C\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2,$$

thus

$$\mathcal{A}(t) \leqslant \mathcal{A}(0)e^{-ct} + C\int_0^t e^{-c(t-s)}\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(s)\|_{L^2}^2 ds. \tag{3.71}$$

Thanks to (2.6), we have

$$\begin{aligned} \|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{L^2} \leqslant{}& C(1+t)^{-5/4}\|\nabla(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|_{L^1 \cap H^1} \\ &+ C\int_0^t (1+t-s)^{-5/4}\|(R_1, R_2, R_3, R_4)\|_{L^1 \cap H^1} ds. \end{aligned} \tag{3.72}$$

The Cauchy-Schwarz inequality, together with (3.4)–(3.7), and (3.8)–(3.11), shows that

$$
\begin{aligned}
\|(R_1,R_2,R_3,R_4)\|_{L^1} &\leqslant C\|\varrho\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2} + C\|\nabla\varrho\|_{L^2}\|\boldsymbol{v}\|_{L^2} + C\|\boldsymbol{v}\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2} + C\|\varrho\|_{L^2}\|\nabla^2 \boldsymbol{v}\|_{L^2} \\
&\quad + C\|\boldsymbol{h}\|_{L^2}\|\nabla \boldsymbol{h}\|_{L^2} + C\|\varrho\|_{L^2}\|\nabla\varrho\|_{L^2} + C\|\varrho\|_{L^2}\|\nabla\omega\|_{L^2} + C\|\boldsymbol{v}\|_{L^2}\|\nabla\omega\|_{L^2} \\
&\quad + C\|\varrho\|_{L^2}\|\nabla^2\omega\|_{L^2} + C\|\varrho\|_{L^2}\|\omega\|_{L^2} + C\|\boldsymbol{v}\|_{L^2}\|\nabla \boldsymbol{h}\|_{L^2} + \|\boldsymbol{h}\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2} \\
&\quad + C\|\nabla\varrho\|_{L^3}\|\nabla \boldsymbol{h}\|_{L^6}\|\boldsymbol{h}\|_{L^2} + C\|\varrho\|_{L^\infty}\|\nabla^2 \boldsymbol{h}\|_{L^2}\|\boldsymbol{h}\|_{L^2} + C\|\varrho\|_{L^\infty}\|\nabla \boldsymbol{h}\|_{L^2}^2 \\
&\leqslant C\Big(\|(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{L^2}\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1} + \|\nabla \boldsymbol{h}\|_{L^2}^2 + \|\nabla\varrho\|_{H^1}\|\nabla \boldsymbol{h}\|_{L^6}\|\boldsymbol{h}\|_{L^2}\Big) \\
&\leqslant \varepsilon C(M)\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1} \\
&\leqslant \varepsilon C(M)\mathcal{A}^{1/2}(t),
\end{aligned}
\tag{3.73}
$$

and similarly,

$$
\begin{aligned}
\|(R_1,R_2,R_3,R_4)\|_{H^1} &\leqslant C(\|R_1\|_{H^1} + \|(R_1,R_2,R_3,R_4)\|_{H^1}) \\
&\leqslant C\|(\varrho,\boldsymbol{v})\|_{H^1}\|\nabla(\varrho,\boldsymbol{v})\|_{H^2} + \varepsilon^{1/2}C(M)\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2 \\
&\leqslant \varepsilon^{1/2}C(M)\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2} \\
&\leqslant \varepsilon^{1/2}C(M)\mathcal{A}^{1/2}(t).
\end{aligned}
\tag{3.74}
$$

Substituting (3.73) and (3.74) into (3.72), one has from (3.4) that

$$
\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})(t)\|_{L^2} \leqslant C(1+t)^{-5/4} + \varepsilon^{1/2}C\int_0^t (1+t-s)^{-5/4}\mathcal{A}^{1/2}(s)ds. \tag{3.75}
$$

Set

$$
\mathcal{F}(t) \triangleq \sup_{s\in[0,t]}\Big((1+s)^{5/2}\mathcal{A}(s)\Big).
$$

So that, the inequality (2.10), together with (3.75), gives

$$
\begin{aligned}
\|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})(t)\|_{L^2} &\leqslant C(1+t)^{-5/4} + \varepsilon^{1/2}C\int_0^t (1+t-s)^{-5/4}\mathcal{A}^{1/2}(s)ds \\
&\leqslant C(1+t)^{-5/4} + C\varepsilon^{1/2}\mathcal{F}^{1/2}(t)\int_0^t (1+t-s)^{-5/4}(1+s)^{-5/4}ds \\
&\leqslant C(1+t)^{-5/4}\Big(1+\varepsilon^{1/2}\mathcal{F}^{1/2}(t)\Big),
\end{aligned}
$$

which, combined with (3.71), shows that

$$
\begin{aligned}
(1+t)^{5/2}\mathcal{A}(t) &\leqslant C(1+t)^{5/2}e^{-ct} + C\Big(1+\varepsilon\mathcal{F}(t)\Big)(1+t)^{5/2} \\
&\quad \times \int_0^t e^{-c(t-s)}(1+s)^{-5/2}(s)ds \\
&\leqslant C + C\varepsilon\mathcal{F}(t).
\end{aligned}
$$

Thus, if $\varepsilon > 0$ is small enough, then we obtain that $\mathcal{F}(t) \leqslant C$. This, together with $\mathcal{A}(t) \sim \|\nabla(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^2}^2$, yields (3.68).

In order to prove (3.69), we use (2.6), (2.10), (3.68), (3.73), and (3.74) to obtain that

$$\begin{aligned}
\|(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{L^2} &\leqslant C(1+t)^{-3/4}\|(\varrho_0, \boldsymbol{v}_0, \omega_0, \boldsymbol{h}_0)\|_{L^1\cap H^1} \\
&\quad + C\int_0^t (1+t-s)^{-3/4}\|(R_1, R_2, R_3, R_4)\|_{L^1\cap L^1} ds \\
&\leqslant C(1+t)^{-3/4} + C\int_0^t (1+t-s)^{-3/4}\mathcal{A}^{1/2}(s) ds \\
&\leqslant C(1+t)^{-3/4} + C\mathcal{F}^{1/2}(t)\int_0^t (1+t-s)^{-3/4}(1+s)^{-5/4}(s) ds \\
&\leqslant C(1+t)^{-3/4},
\end{aligned} \tag{3.76}$$

which implies the desired estimate (3.69). Therefore, we complete the proof of Lemma 3.8. □

Compared with (2.5) in Lemma 2.1, the decay rates of the H^1-norm of solutions stated in Lemma 3.8 are optimal. Next, our main aim is to show the decay estimates of higher derivatives.

Lemma 3.9. *Let the conditions of Proposition 3.1 be in force. Then there exists a positive time T_1 such that if $\varepsilon > 0$ is small enough, the following estimate holds.*

$$\|\nabla^2(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{H^1}^2 \leqslant C(1+t)^{-7/2}, \tag{3.77}$$

for any $t \geqslant T_1$.

Proof. In terms of (2.7)–(2.9) in Lemma 2.3, one has from (2.2) and (3.6) that

$$\begin{aligned}
\|\nabla R_1\|_{L^2} \leqslant \|\nabla^2(\varrho \boldsymbol{v})\|_{L^2} &\leqslant C\Big(\|\varrho\|_{L^3}\|\nabla^2 \boldsymbol{v}\|_{L^6} + \|\boldsymbol{v}\|_{L^3}\|\nabla^2\varrho\|_{L^6}\Big) \\
&\leqslant C\|(\varrho, \boldsymbol{v})\|_{H^1}\|\nabla^3(\varrho, \boldsymbol{v})\|_{L^2} \\
&\leqslant C\varepsilon\|\nabla^3(\varrho, \boldsymbol{v})\|_{L^2}.
\end{aligned} \tag{3.78}$$

Due to Proposition 3.1 and Lemma 3.8, one has

$$\begin{aligned}
\|\nabla(R_2, R_3, R_4)\|_{L^2} &\leqslant C\varepsilon\|\nabla^3(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2} + C(M)\|\nabla(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^2}^2 \\
&\leqslant C\varepsilon\|\nabla^3(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2} + C(M)(1+t)^{-5/2}.
\end{aligned} \tag{3.79}$$

Taking (3.78) and (3.79) into (3.31), one has from integration by parts that

$$\frac{d}{dt}\|\nabla^2(a\gamma\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 + \|\nabla^3(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{L^2}^2 \leqslant C\varepsilon\|\nabla^3\varrho\|_{L^2}^2 + C(1+t)^{-5},$$

which, together with (3.46) and (3.59), gives that there exist some positive constants $\bar{M}$ (suitably large) and $\bar{c}$ (suitably small) such that if $\varepsilon > 0$ is small enough, then

$$\mathcal{A}_1'(t) + \bar{c}\Big(\|\nabla^3(\boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^1}^2 + \|\nabla^3\varrho\|_{L^2}^2\Big) \leqslant C(1+t)^{-5}, \tag{3.80}$$

where

$$\mathcal{A}_1'(t) \triangleq \bar{M}\|\nabla^2(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^1}^2 + \big\langle \nabla \mathrm{div}\boldsymbol{v}, \nabla\Delta\varrho \big\rangle \sim \|\nabla^2(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})\|_{H^1}^2.$$

The combination of the Sobolev interpolation inequalities (2.22) and (2.23) and Cauchy-Schwarz inequality shows

$$\|\nabla^2\varrho\|_{L^2}^2 \leqslant C\|\nabla\varrho\|_{L^2}\|\nabla^3\varrho\|_{L^2} \leqslant \frac{1}{\delta}(1+t)\|\nabla^3\varrho\|_{L^2}^2 + C(\delta)(1+t)^{-1}\|\nabla\varrho\|_{L^2}^2,$$

where $\delta > 0$ is a positive number to be chosen later. Then

$$\|\nabla^3\varrho\|_{L^2}^2 \geqslant \delta(1+t)^{-1}\|\nabla^2\varrho\|_{L^2}^2 - C(\delta)(1+t)^{-2}\|\nabla\varrho\|_{L^2}^2. \tag{3.81}$$

Similarly,

$$\|\nabla^3(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 \geqslant \delta(1+t)^{-1}\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 - C(\delta)(1+t)^{-2}\|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2. \tag{3.82}$$

Putting (3.81) and (3.82) into (3.80), one has

$$\begin{aligned}
&\mathcal{A}_1'(t) + \frac{\delta\bar{c}}{2}(1+t)^{-1}\|\nabla^2(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 + \frac{\bar{c}}{2}\Big(\delta(1+t)^{-1}\|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^3\varrho\|_{L^2}^2\Big)\\
&\leqslant C(1+t)^{-5} + C(\delta)(1+t)^{-2}\Big(\|\nabla\varrho\|_{L^2}^2 + \|\nabla(\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2\Big)\\
&\leqslant C(\delta)(1+t)^{-9/2}.
\end{aligned} \tag{3.83}$$

If $t \geqslant \delta > 0$, then $\|\nabla^3\varrho\|_{L^2}^2 \geqslant \delta(1+t)^{-1}\|\nabla^3\varrho\|_{L^2}^2$, so that , we infer from (3.83) that

$$\mathcal{A}_1'(t) + \frac{\delta\bar{c}}{2}(1+t)^{-1}\|\nabla^2(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2 \leqslant C(\delta)(1+t)^{-9/2}. \tag{3.84}$$

Furthermore, due to $\mathcal{A}_1'(t) \sim \|\nabla^2(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2$ for suitable large $\bar{M}$, then there exists a positive constant c_1 depending only on $\bar{M}$ and $\bar{c}$, such that

$$\mathcal{A}_1'(t) + \delta c_1(1+t)^{-1}\mathcal{A}_1(t) \leqslant C(\delta)(1+t)^{-9/2}. \tag{3.85}$$

If $\delta = 4c_1^{-1}$, then it follows from (3.85) that

$$\mathcal{A}_1'(t) + 4(1+t)^{-1}\mathcal{A}_1(t) \leqslant C(\delta)(1+t)^{-9/2},$$

thus

$$\frac{d}{dt}\Big((1+t)^4\mathcal{A}_1(t)\Big) = (1+t)^4\Big(\mathcal{A}_1'(t) + 4(1+t)^{-1}\mathcal{A}_1(t)\Big) \leqslant C(1+t)^{-1/2},$$

which, integrated over $(0,t)$, gives

$$(1+t)^4\mathcal{A}_1(t) \leqslant \mathcal{A}_1(0) + C_7(1+t)^{1/2} \leqslant 2C_7(1+t)^{1/2}, \tag{3.86}$$

provided $t > 0$ is large enough such that

$$t \geqslant T_1 \triangleq \max\Big\{\delta, \Big(\frac{\mathcal{A}_1(0)}{C_7}\Big)^2 - 1\Big\}.$$

Dur to $\mathcal{A}_1'(t) \sim \|\nabla^2(\varrho,\boldsymbol{v},\omega,\boldsymbol{h})\|_{H^1}^2$, thus, we can obtain (3.77) from (3.86). Therefore, we complete the proof of Lemma 3.9. □

Lemma 3.10. *Let the conditions of Proposition 3.1 be in force. Then there exists a positive time T_2 such that if $\varepsilon > 0$ is small enough, the following estimate holds*

$$\|\nabla^3(\varrho, \boldsymbol{v}, \omega, \boldsymbol{h})(t)\|_{H^1}^2 \leqslant C(1+t)^{-9/2}, \tag{3.87}$$

for any $t \geqslant T_2$.

Proof. Operating ∇^2 to $(2.1)_4$, multiplying it by $\nabla\Delta\boldsymbol{h}$ in L^2, and integrating by parts, we infer from Cauchy-Schwarz inequality that

$$\frac{d}{dt}\|\nabla^3\boldsymbol{h}\|_{L^2}^2 + \frac{3\nu}{2}\|\nabla^4\boldsymbol{h}\|_{L^2}^2 \leqslant C\|\nabla^2 R_4\|_{L^2}^2. \tag{3.88}$$

The right-hand side terms of (3.88) can be estimated as follows. Due to (2.7)–(2.9), (3.48), we infer from (3.68), (3.69), and (3.77) that

$$\begin{aligned}
&\|\nabla^2(\boldsymbol{v}\cdot\nabla\boldsymbol{h})\|_{L^2} + \|\nabla^2(\boldsymbol{h}\mathrm{div}\boldsymbol{v})\|_{L^2} + \|\nabla^2(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\|_{L^2}\\
&\leqslant C\Big(\|(\boldsymbol{v},\boldsymbol{h})\|_{L^\infty}\|\nabla^3(\boldsymbol{v},\boldsymbol{h})\|_{L^2} + \|\nabla^2(\boldsymbol{v},\boldsymbol{h})\|_{L^6}\|\nabla(\boldsymbol{v},\boldsymbol{h})\|_{L^3}\Big)\\
&\leqslant C\|\nabla(\boldsymbol{v},\boldsymbol{h})\|_{L^2}^{1/2}\|\nabla^2(\boldsymbol{v},\boldsymbol{h})\|_{L^2}^{1/2}\|\nabla^3(\boldsymbol{v},\boldsymbol{h})\|_{L^2}\\
&\leqslant C(1+t)^{-13/4},
\end{aligned} \tag{3.89}$$

and

$$\begin{aligned}
\|\nabla^3[g(\varrho)(\nabla\times\boldsymbol{h})\times\boldsymbol{h})]\|_{L^2} &\leqslant C\Big(\|g(\varrho)\|_{L^\infty}\|\nabla^3((\nabla\times\boldsymbol{h})\times\boldsymbol{h})\|_{L^2} + \|\nabla^3\varrho\|_{L^2}\|(\nabla\times\boldsymbol{h})\times\boldsymbol{h}\|_{L^\infty}\Big)\\
&\leqslant C\|\boldsymbol{h}\|_\infty\|\nabla^4\boldsymbol{h}\|_{L^2} + C\|\nabla^3\boldsymbol{h}\|_{L^6}\|\nabla\boldsymbol{h}\|_{L^3} + C\|\nabla^3\varrho\|_{L^2}\|\nabla\boldsymbol{h}\|_{L^\infty}\|\boldsymbol{h}\|_{L^\infty}\\
&\leqslant C\|\nabla\boldsymbol{h}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{h}\|_{L^2}^{1/2}\|\nabla^4\boldsymbol{h}\|_{L^2} + C\|\nabla^3\varrho\|_{L^2}\|\nabla\boldsymbol{h}\|_{L^2}^{1/2}\|\nabla^2\boldsymbol{h}\|_{L^2}\|\nabla^3\boldsymbol{h}\|_{L^2}^{1/2}\\
&\leqslant \frac{\nu}{2}\|\nabla^4\boldsymbol{h}\|_{L^2}^2 + C(1+t)^{-13/4}.
\end{aligned} \tag{3.90}$$

Substituting (3.89) and (3.90) into (3.88), we obtain that

$$\frac{d}{dt}\|\nabla^3\boldsymbol{h}\|_{L^2}^2 + \|\nabla^4\boldsymbol{h}\|_{L^2}^2 \leqslant C(1+t)^{-13/4}. \tag{3.91}$$

Similar to the derivation of (3.81), one has from (3.77) that

$$\begin{aligned}
\|\nabla^4\boldsymbol{h}\|_{L^2}^2 &\geqslant 5(1+t)^{-1}\|\nabla^3\boldsymbol{h}\|_{L^2}^2 - C(1+t)^{-2}\|\nabla^2\boldsymbol{h}\|_{L^2}^2\\
&\geqslant 5(1+t)^{-1}\|\nabla^3\boldsymbol{h}\|_{L^2}^2 - C(1+t)^{-11/2},
\end{aligned}$$

which, together with (3.91), yields

$$\frac{d}{dt}\|\nabla^3\boldsymbol{h}\|_{L^2}^2 + 5(1+t)^{-1}\|\nabla^3\boldsymbol{h}\|_{L^2}^2 \leqslant C(1+t)^{-11/2}. \tag{3.92}$$

Therefore, we deduce from (3.92) that

$$\begin{aligned}
\frac{d}{dt}\Big((1+t)^5\|\nabla^3\boldsymbol{h}\|_{L^2}^2\Big) &= (1+t)^5\frac{d}{dt}\Big(\|\nabla^3\boldsymbol{h}\|_{L^2}^2 + 5(1+t)^{-1}\|\nabla^3\boldsymbol{h}\|_{L^2}^2\Big)\\
&\leqslant C(1+t)^{-1/2},
\end{aligned} \tag{3.93}$$

which, implies

$$(1+t)^5\|\nabla^3 \boldsymbol{h}\|_{L^2}^2 \leqslant \|\nabla^3 \boldsymbol{h}_0\|_{L^2}^2 + C_8(1+t)^{1/2} \leqslant 2C_8(1+t)^{1/2},$$

provided

$$t \geqslant T_2 \triangleq \max\Big\{T_1, \Big(\frac{\|\nabla^3 \boldsymbol{h}_0\|_{L^2}^2}{C_8}\Big)^2 - 1\Big\}.$$

Thus, we can obtain (3.87) from (3.93). Therefore, we complete the proof of Lemma 3.10. □

Proof of decay rates. Collecting Lemmas 3.8–3.10 together, one immediately obtains the desired decay rates stated in the section part of Theorem 1.1. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first consider the standard MHD equations without Hall effects (i.e., $\beta = 0$) as follows:

$$\begin{cases} \rho_t^0 + \mathrm{div}(\rho^0\mathbf{u}^0) = 0, \\ (\rho^0\mathbf{u}^0)_t + \mathrm{div}(\rho^0\mathbf{u}^0 \otimes \mathbf{u}^0) + \nabla p^0 \\ \qquad = (\mu_1+\zeta)\Delta\mathbf{u}^0 + (\mu_1+\lambda_1-\zeta)\nabla\mathrm{div}\mathbf{u}^0 + 2\zeta\nabla\times\mathbf{w}^0 + (\nabla\times\boldsymbol{b}^0)\times\boldsymbol{b}^0, \\ (\rho^0\mathbf{w}^0)_t + \mathrm{div}(\rho^0\mathbf{u}^0\otimes\mathbf{w}^0) + 4\zeta\mathbf{w}^0 = \mu_2\Delta\mathbf{w}^0 + (\mu_2+\lambda_2)\nabla\mathrm{div}\mathbf{w}^0 + 2\zeta\nabla\times\mathbf{u}^0, \\ \boldsymbol{b}_t^0 - \nabla\times(\mathbf{u}^0\times\boldsymbol{b}^0) = \nu\Delta\boldsymbol{b}^0, \\ \mathrm{div}\boldsymbol{b}^0 = 0, \end{cases} \tag{4.1}$$

with far-field boundary conditions and initial conditions:

$$\begin{cases} (\rho^0, \mathbf{u}^0, \mathbf{w}^0, \boldsymbol{b}^0)\Big|_{|x|\to\infty} \to (1,0,0,0), \\ (\rho^0, \mathbf{u}^0, \mathbf{w}^0, \boldsymbol{b}^0)\Big|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)(x), \quad x\in\mathbb{R}^3, \end{cases} \tag{4.2}$$

where $p^0 = p(\rho^0) = a(\rho^0)^\gamma$.

According to the observation, we know that all of the global estimates and decay rates established in Section 3 hold for the system (4.1)-(4.2). Therefore, we give the following global existence result for the problem (4.1)-(4.2).

Proposition 4.1. *Assume that the initial data* $(\rho_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$ *satisfy*

$$(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^3, \qquad \|\nabla^2(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1} \leqslant L_0, \tag{4.3}$$

for any given constants L_0 *(not necessary small). Then, there exists a positive constant* $\bar{\varepsilon}_0$ *depending on* $L_0, \mu_1, \lambda_1, \zeta, \mu_2, \lambda_2, \nu, a,$ *and* γ *such that if*

$$\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^1} \leqslant \bar{\varepsilon}_0, \tag{4.4}$$

then the Cauchy problem (4.1)-(4.2) *has a unique global classical solution* $(\rho^0, \boldsymbol{u}^0, \boldsymbol{w}^0, \boldsymbol{b}^0)$ *on* $\mathbb{R}^3\times(0,\infty)$ *satisfying*

$$\begin{aligned} &\|(\rho^0 - 1, \mathbf{u}^0, \mathbf{w}^0, \mathbf{b}^0)(t)\|_{H^3}^2 + \int_0^t \Big(\|\nabla\rho^0(s)\|_{H^2}^2 + \|(\nabla\mathbf{u}^0, \nabla\mathbf{w}^0, \nabla\mathbf{b}^0)(s)\|_{H^3}^2\Big)ds \\ &\quad \leqslant C\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_{H^3}^2, \end{aligned} \tag{4.5}$$

for all $t \geqslant 0$*, where* C *is positive constant independent of* t*.*

In order to prove the convergence rates stated in Theorem 1.2, we define

$$\pi \triangleq \rho - \rho^0, \quad \boldsymbol{v} \triangleq \mathbf{u} - \mathbf{u}^0, \quad \varpi \triangleq \mathbf{w} - \mathbf{w}^0, \quad \boldsymbol{h} \triangleq \boldsymbol{b} - \boldsymbol{b}^0,$$

where $(\rho, \mathbf{u}, \mathbf{w}, \boldsymbol{b})$ is the solution of the problem (1.1)–(1.3), and $(\rho^0, \mathbf{u}^0, \mathbf{w}^0, \boldsymbol{b}^0)$ is the solution of problem (4.1)-(4.2). Then, the quadruple $(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})$ satisfies

$$\begin{cases} \pi_t + \boldsymbol{v} \cdot \nabla \rho + \mathbf{u}^0 \cdot \nabla \pi + \rho \mathrm{div} \boldsymbol{v} + \pi \mathrm{div} \mathbf{u}^0 = 0, \\ \rho \boldsymbol{v}_t + \rho \mathbf{u} \cdot \nabla \boldsymbol{v} - (\mu_1 + \zeta) \Delta \boldsymbol{v} - (\mu_1 + \lambda_1 - \zeta) \nabla \mathrm{div} \boldsymbol{v} = -\pi \mathbf{u}^0_t - \rho \boldsymbol{v} \cdot \nabla \mathbf{u}^0 \\ \qquad -\pi \mathbf{u}^0 \cdot \nabla \mathbf{u}^0 - \nabla (p - p^0) + 2\zeta \nabla \times \varpi + \boldsymbol{b} \cdot \nabla \boldsymbol{h} + \boldsymbol{h} \cdot \nabla \boldsymbol{b}^0 - \frac{1}{2} \nabla (|\boldsymbol{b}|^2 - |\boldsymbol{b}^0|^2), \\ \rho \varpi_t + \rho \mathbf{u} \cdot \nabla \varpi - \mu_2 \Delta \varpi - (\mu_2 + \lambda_2) \nabla \mathrm{div} \varpi = -\pi \mathbf{w}^0_t - \rho \boldsymbol{v} \cdot \nabla \mathbf{w}^0 - \pi \mathbf{u}^0 \cdot \nabla \mathbf{w}^0 \\ \qquad -4\zeta \varpi + 2\zeta \nabla \times \boldsymbol{v}, \\ \boldsymbol{h}_t - \nu \Delta \boldsymbol{h} = -\mathbf{u} \cdot \nabla \boldsymbol{h} - \boldsymbol{v} \cdot \nabla \boldsymbol{b}^0 + \boldsymbol{b} \cdot \nabla \boldsymbol{v} + \boldsymbol{h} \cdot \nabla \mathbf{u}^0 - \boldsymbol{b} \mathrm{div} \boldsymbol{v} - \boldsymbol{h} \mathrm{div} \mathbf{u}^0 \\ \qquad -\beta \nabla \times \Big(\frac{(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}}{\rho} \Big), \quad \mathrm{div} \boldsymbol{h} = 0, \\ (\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})\big|_{t=0} = (0, 0, 0, 0), \\ (\pi_0, \boldsymbol{v}_0, \varpi_0, \boldsymbol{h}_0) \to (0, 0, 0, 0), \quad \text{as } |x| \to \infty. \end{cases} \tag{4.6}$$

Lemma 4.1. *Let $(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})$ be the solution of system (4.6). There exists a positive constant $C = C(T)$, independent of β, such that for any $T \in (0, \infty)$, then*

$$\sup_{t \in [0,T]} \|(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})(t)\|_{L^2}^2 + \int_0^T \|\nabla (\boldsymbol{v}, \varpi, \boldsymbol{h})\|_{L^2}^2 dt \leqslant C\beta^2. \tag{4.7}$$

Proof. Multiplying $(4.6)_1$ by π in L^2 and integrating by parts, we infer from (1.4), (4.5), and Cauchy-Schwarz inequality that

$$\Big(\|\pi\|_{L^2}^2\Big)_t \leqslant C\|\nabla(\rho, \mathbf{u}^0)\|_{H^2}\Big(\|(\pi, \boldsymbol{v})\|_{L^2}^2 + \|\nabla \boldsymbol{v}\|_{L^2}^2\Big) \leqslant C\Big(\|(\pi, \boldsymbol{v})\|_{L^2}^2 + \|\nabla \boldsymbol{v}\|_{L^2}^2\Big). \tag{4.8}$$

With the aid of (1.4) and (4.5), we can easily deduce from (4.6) that

$$\|(\mathbf{u}_t, \mathbf{u}^0_t, \mathbf{w}_t, \mathbf{w}^0_t, \boldsymbol{b}_t, \boldsymbol{b}^0_t)\|_{H^1}^2 + \int_0^T \|(\mathbf{u}_t, \mathbf{u}^0_t, \mathbf{w}_t, \mathbf{w}^0_t, \boldsymbol{b}_t, \boldsymbol{b}^0_t)\|_{H^2}^2 dt \leqslant C. \tag{4.9}$$

Since it holds that

$$|p - p^0| + |\boldsymbol{b}|^2 - |\boldsymbol{b}^0|^2 \leqslant C\Big(|\pi| + |\boldsymbol{h}|\Big), \tag{4.10}$$

thus, multiplying $(4.6)_2$ by $\boldsymbol{v}$ in L^2 and integrating by parts, we infer from (1.4), (4.5), (4.9), and (4.10) that

$$\Big(\|\rho^{1/2} \boldsymbol{v}\|_{L^2}^2\Big)_t + \|\nabla \boldsymbol{v}\|_{L^2}^2 \leqslant C\|(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})\|_{L^2}^2. \tag{4.11}$$

Similarly, multiplying $(4.6)_3$ by ϱ in L^2 and integrating by parts, one can deduce from (1.4), (4.5) and (4.9) that

$$\Big(\|\rho^{1/2} \varpi\|_{L^2}^2\Big)_t + \|\nabla \varpi\|_{L^2}^2 + \|\varpi\|_{L^2}^2 \leqslant C\|(\pi, \boldsymbol{v}, \varpi)\|_{L^2}^2. \tag{4.12}$$

Multiplying $(4.6)_4$ by $\boldsymbol{h}$ and integrating by parts, we can obtain from (1.4), (4.5) and (4.9) that

$$\begin{aligned}\left(\|\boldsymbol{h}\|_{L^2}^2\right)_t + \|\nabla \boldsymbol{h}\|_{L^2}^2 &\leqslant C\|(\boldsymbol{v},\boldsymbol{h})\|_{L^2}^2 + C\beta^2\|\rho^{-1}(\nabla\times\boldsymbol{b})\times\boldsymbol{b}\|_{L^2}^2 \\ &\leqslant C\|(\boldsymbol{v},\boldsymbol{h})\|_{L^2}^2 + C\beta^2.\end{aligned} \tag{4.13}$$

Multiplying (4.11) by a suitably large constant and adding the result to (4.8), we obtain from (4.12) and (4.13) that

$$\left(\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2\right)_t + \|\nabla(\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 \leqslant C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 + C\beta^2,$$

which, together with the Gronwall inequality and $\rho > 0$, gives rise to (4.7). □

Lemma 4.2. *Let* $(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})$ *be the solution of system* (4.6)*. There exists a positive constant* $C = C(T)$*, independent of* β*, such that for any* $T \in (0,\infty)$*, then*

$$\sup_{t\in[0,T]} \|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})(t)\|_{L^2}^2 + \int_0^T \|\nabla^2(\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 dt \leqslant C\beta^2. \tag{4.14}$$

Proof. Operating ∇ on both sides of $(4.6)_1$, and multiplying it by $\nabla\pi$ in L^2, after integrating by parts, we deduce from (4.7) and the Cauchy-Schwarz inequality that

$$\left(\|\nabla\pi\|_{L^2}^2\right)_t \leqslant C(\sigma)\|(\pi,\boldsymbol{v})\|_{H^1}^2 + \sigma\|\nabla^2\boldsymbol{v}\|_{L^2}^2 \leqslant C\beta^2 + \sigma\|\nabla^2\boldsymbol{v}\|_{L^2}^2 + C(\sigma)\|\nabla(\pi,\boldsymbol{v})\|_{L^2}^2, \tag{4.15}$$

where σ is an undetermined positive constant.

Multiplying $(4.6)_2$ by $\boldsymbol{v}_t$ in L^2 and integrating by parts, then, by virtue of (1.4), (4.5), (4.7), and (4.9) that

$$\begin{aligned}\left(\|\nabla\boldsymbol{v}\|_{L^2}^2\right)_t + \|\rho^{1/2}\boldsymbol{v}_t\|_{L^2}^2 &\leqslant C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 \\ &\leqslant C\beta^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\end{aligned} \tag{4.16}$$

Multiplying $(4.6)_3$ by ϖ_t in L^2 and integrating by parts, then, by virtue of (1.4), (4.5), (4.7), and (4.9) that

$$\begin{aligned}\left(\|\nabla\varpi\|_{L^2}^2 + \|\varpi\|_{L^2}^2\right)_t + \|\rho^{1/2}\varpi_t\|_{L^2}^2 &\leqslant C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 \\ &\leqslant C\beta^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\end{aligned} \tag{4.17}$$

Similarly,

$$\begin{aligned}\left(\|\nabla\boldsymbol{h}\|_{L^2}^2\right)_t + \|\boldsymbol{h}_t\|_{L^2}^2 &\leqslant C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 \\ &\leqslant C\beta^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\end{aligned} \tag{4.18}$$

Due to (3.9), we know that ρ is strictly lower-bounded. Thus, it follows from (4.15)–(4.18) that

$$\begin{aligned}&\left(\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 + \|\varpi\|_{L^2}^2\right)_t + \|(\boldsymbol{v}_t,\varpi_t,\boldsymbol{h}_t)\|_{L^2}^2 \\ &\leqslant C\beta^2 + \sigma\|\nabla^2\boldsymbol{v}\|_{L^2}^2 + C(\sigma)\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\end{aligned} \tag{4.19}$$

The combination of (1.4), (4.5)–(4.7), and (4.9) gives

$$\begin{aligned}\|\nabla^2(\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 &\leqslant C\|(\boldsymbol{v}_t,\varpi_t,\boldsymbol{h}_t)\|_{L^2}^2 + C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{H^1}^2 + C\beta^2\\ &\leqslant C\beta^2 + C_9\|(\boldsymbol{v}_t,\varpi_t,\boldsymbol{h}_t)\|_{L^2}^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\end{aligned}\tag{4.20}$$

Multiplying (4.19) by (C_9+1) and choosing $\sigma=(C_9+1)/2$ in (4.19), adding the resulting inequality to (4.20), one has

$$\left(\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 + \|\varpi\|_{L^2}^2\right)_t + \|\nabla^2(\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2 \leqslant C\beta^2 + C\|\nabla(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2,$$

which, together with the Gronwall inequality, yields (4.14). □

Lemma 4.3. *Let $(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})$ be the solution of system* (4.6). *There exists a positive constant $C=C(T)$, independent of β, such that for any $T\in(0,\infty)$, then*

$$\sup_{t\in[0,T]}\|\nabla^2(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})(t)\|_{L^2}^2 + \int_0^T\|\nabla^3(\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2dt \leqslant C\beta^2.\tag{4.21}$$

Proof. Due to (1.4), (2.7)–(2.9), (4.7), and (4.14), we deduce from $(4.6)_1$, Cauchy-Schwarz inequality, and the Sobolev inequality (2.22) that

$$\left(\|\nabla^2\pi\|_{L^2}^2\right)_t \leqslant C\|(\pi,\boldsymbol{v})\|_{H^2}^2 + \frac{1}{2}\|\nabla^3\boldsymbol{v}\|_{L^2}^2 \leqslant C\beta^2 + \frac{1}{2}\|\nabla^3\boldsymbol{v}\|_{L^2}^2 + C\|\nabla^2(\pi,\boldsymbol{v})\|_{L^2}^2.\tag{4.22}$$

Thanks to $(4.6)_2$, we have

$$\begin{aligned}\boldsymbol{v}_t + \mathbf{u}\cdot\nabla\boldsymbol{v} - \frac{\mu_1+\zeta}{\rho}\Delta\boldsymbol{v} - \frac{\mu_1+\lambda_1-\zeta}{\rho}\nabla\mathrm{div}\boldsymbol{v} = -\frac{1}{\rho}\Big(\pi\mathbf{u}_t^0 + \rho\boldsymbol{v}\cdot\nabla\mathbf{u}^0\\ + \pi\mathbf{u}^0\cdot\nabla\mathbf{u}^0 + \nabla(p-p^0) - 2\zeta\nabla\times\varpi - \boldsymbol{b}\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\cdot\nabla\boldsymbol{b}^0 + \frac{1}{2}\nabla(|\boldsymbol{b}|^2-|\boldsymbol{b}^0|^2)\Big).\end{aligned}\tag{4.23}$$

Operating ∇^2 on both sides of (4.23), multiplying the resulting equation by $\nabla^2\boldsymbol{v}$ in L^2, and integrating by parts, we get from (2.7)–(2.9) and (4.7)–(4.14) that

$$\left(\|\nabla^2\boldsymbol{v}\|_{L^2}^2\right)_t + \frac{3}{2}\|\nabla^3\boldsymbol{v}\|_{L^2}^2 \leqslant C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{H^2}^2 \leqslant C\beta^2 + C\|\nabla^2(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\tag{4.24}$$

Due to $(4.6)_3$, one has

$$\begin{aligned}&\varpi_t + \mathbf{u}\cdot\nabla\varpi - \frac{\mu_2}{\rho}\Delta\varpi - \frac{\mu_2+\lambda_2}{\rho}\nabla\mathrm{div}\varpi\\ &= -\frac{1}{\rho}\Big(\pi\mathbf{w}_t^0 + \rho\boldsymbol{v}\cdot\nabla\mathbf{w}^0 + \pi\mathbf{u}^0\cdot\nabla\mathbf{w}^0 + 4\zeta\varpi - 2\zeta\nabla\times\boldsymbol{v}\Big).\end{aligned}\tag{4.25}$$

Operating ∇^2 on both sides of (4.25), multiplying the resulting equation by $\nabla^2\varpi$ in L^2, and integrating by parts, we get from (2.7)–(2.9) and (4.7)–(4.14) that

$$\left(\|\nabla^2\varpi\|_{L^2}^2\right)_t + \|\nabla^3\varpi\|_{L^2}^2 \leqslant C\|(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{H^2}^2 \leqslant C\beta^2 + C\|\nabla^2(\pi,\boldsymbol{v},\varpi,\boldsymbol{h})\|_{L^2}^2.\tag{4.26}$$

Operating ∇^2 on both sides of $(4.6)_4$, multiplying the resulting equation by $\nabla^2\boldsymbol{h}$ in L^2, and integrating by parts, we get from (2.7)–(2.9) and (4.7)–(4.14) that

$$\left(\|\nabla^2\boldsymbol{h}\|_{L^2}^2\right)_t + \|\nabla^3\boldsymbol{h}\|_{L^2}^2 \leqslant C\|(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})\|_{H^2}^2 \leqslant C\beta^2 + C\|\nabla^2(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})\|_{L^2}^2. \tag{4.27}$$

The combination of (4.22), (4.24), (4.26), and (4.27) gives that

$$\left(\|\nabla^2(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})\|_{L^2}^2\right)_t + \|\nabla^3(\boldsymbol{v}, \varpi, \boldsymbol{h})\|_{L^2}^2 \leqslant C\beta^2 + C\|\nabla^2(\pi, \boldsymbol{v}, \varpi, \boldsymbol{h})\|_{L^2}^2,$$

which, together with the Gronwall inequality, yields (4.21). □

Proof of Theorem 1.2. Now, the convergence rates of the vanishing limit of the Hall coefficient stated in Theorem 1.2 readily follows from Lemmas 4.1–4.3. □

5. Conclusions

This paper is concerned with the Cauchy problem of the compressible magneto-micropolar fluids subjected to Hall current in three-dimensional whole space. Both the global existence and optimal decay rates of strong solutions are obtained when the smooth initial data are sufficiently close to the non-vacuum equilibrium in H^1. As a by-product of the uniform estimates, the vanishing limit of the Hall coefficient is also justified. We refer to Theorems 1.1 and 1.2 for details.

Acknowledgments

This project is partially supported by Shandong Provincial Natural Science Foundation (Grant No. ZR2024MA033) and the Science and Technology Project of Weifang (2022GX006).

The author is very grateful to the experts who gave some guidance and suggestions during the writing process of this paper, and also to the anonymous reviewers for their comments on this paper, which greatly improved the quality of this paper.

Conflict of interest

The author declares that she has no conflict of interest.

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