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Research article

Novel Heronian mean based m-polar fuzzy power geometric aggregation operators and their application to urban transportation management

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Abstract: An m-polar fuzzy (mF) model offers a practical framework for decision-making by providing higher flexibility in handling uncertainties and preferences. The ability of mF sets to tackle multiple reference points permits for a more nuanced analysis, leading to more accurate results in complex decision scenarios. This study was mainly devoted to introducing three novel aggregation operators (AGOs) for multi-criteria decision-making (MCDM) based on generalized geometric Heronian mean (GGHM) operations comprise the concept of mF sets. The presented operators consisted of the weighted mF power GGHM (WmFPGGHM), ordered weighted mF power GGHM averaging (OWmFPGGHM), and hybrid mF power GGHM (HmFPGGHM) operators. Some essential fundamental properties of the proposed AGOs were investigated: idempotency, monotonicity, boundedness, and Abelian property. Furthermore, an algorithm based on the initiated WmFPGGHM operators was developed to address diverse daily-life MCDM scenarios. Next, to validate the efficiency of the established algorithm, it was implemented in a daily-life MCDM problem involving urban transportation management. At last, a sensitivity analysis of the initiated AGOs was provided with existing mF set-based operators involving Dombi, Yager, and Aczel-Alsina's operations-based AGOs.

Keywords: m-polar fuzzy sets; power geometric operators; Heronian mean; urban transportation; sensitivity analysis; multi-criteria decision-making
Mathematics Subject Classification: 03E72, 91B06

1. Introduction

Multi-criteria decision-making (MCDM) is a field of decision theory that refers to the procedure of evaluating and prioritizing objects based on multiple criteria when making decisions. MCDM approaches aim to balance trade-offs among criteria to reach the optimal decision and are widely used in several fields, including engineering, economics, and medical, to evaluate complicated decision-making related problems involving conflicting alternatives. Some ordinary MCDM approaches include the Analytical Hierarchy Process (AHP) [1], and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [2]. One major hindrance of these techniques is the involvement of qualitative and quantitative factors without dealing with subjective judgments and incomplete information. Integrating fuzzy sets for MCDM facilitate to deal with the uncertainties occurred in several decision-making problems. The idea of fuzzy sets (FSs) was developed by Zadeh [3] in 1965. FSs allow alternatives to be evaluated with belongingness degrees as compared to crisp values, which better describes imprecision in real-world problems. Later, Bellman and Zadeh [4] were the first who provided the phenomenon of decision-making under fuzzy information. Since the inception of this powerful concept, FSs has received significant attention from experts around the world, who have anticipated its real and theoretical aspects. For the most relevant research efforts on the theory and applications of FSs, readers are referred to [5].

To date, several generalizations of the FS model have been proposed to better handle complex real-life problems, such as intuitionistic fuzzy sets (IFSs) [6] and Pythagorean fuzzy sets (PFSs) [7], both of which involve two separate degrees of membership and non-membership with specific summation constraints. Many human decision-making situations involve bipolar judgmental information, i.e., positive and negative aspects. For example, friendship and hostility, likelihood and unlikelihood, or effect and side effect. Similarly, in Chinese medicine, Yang (positive) and Yin (negative) are considered two parts of a system. Motivated by this, Zhang [8] introduced the idea of bipolar fuzzy (BF) sets, also known as Yin-Yang BF sets, which naturally extend the fuzzy set model. In the case of a BF set, the co-domain is extended from the closed unit interval $[0,1]$ used in fuzzy sets to the product space $[-1,0]\times[0,1]$. Many significant contributions have been made to BF theory to improve decision-making methods (see references [9, 10]). However, among the various extensions of FSs, m-polar fuzzy (or mF) sets, proposed by Chen et al. [11], have emerged as a powerful mathematical tool for addressing decision problems where each criterion must be evaluated from multiple perspectives or poles. The presence of practical datasets involving multi-polar information was the primary motivation behind the development of mF sets. For instance, consider the statement, "Spain is a good territory". This statement cannot be adequately explained by a truth value belonging to $[0,1]$ because different properties of a good country (e.g., good in education, good in economic stability, good in agriculture) should be evaluated to provide a truth degree regarding the country's goodness. Each property may be described by a value (membership) in $[0,1]$. If we have m such properties to evaluate, then the truth degree of the statement is an m-tuple of numbers in $[0,1]$, i.e., a member of $[0,1]^m$. Existing FSs and their hybrid structures are inefficient in tackling with this variety of multi-polar information.

These days, the integration of various aggregation operators (AGOs) with FS-based MCDM techniques plays an important role in numerous domains, including medicine, environmental sciences, engineering, and economics. As a result, several MCDM approaches based on AGOs have been developed to improve the precision of optimal decision-making, and they continue to evolve with more advancements. For example, Asif et al. [12] presented Hamacher operations-based AGOs for Pythagorean fuzzy information and its application in multi-attribute decision-making problem. Imran et al. [13] introduced a MCDM method for robot selection by combining interval-valued IFSs and Aczel-Alsina Bonferroni mean operations. Hussain and Ullah [14] proposed spherical fuzzy Sugeno-

Weber AGOs and investigated their daily-life applications. Yager [15] proposed the idea of power geometric operators. To generalize these operators, Xu and Yager [16] formulated power geometric AGOs and discussed their applications in MCDM. In continuation of this effort, Xu [17] presented the intuitionistic fuzzy power geometric AGOs and explored their applications in group decision-making. Over the past decade, significant studies have emerged to aggregate bipolar data employing well-established operators. For example, Jana et al. [18] employed Dombi's operations to propose AGOs for bipolar information, effectively addressing real-life issues.

Heronian mean (HM) operators have gained considerable attention in MCDM and group decision-making (GDM) frameworks due to their ability to aggregate data while maintaining relationships between attributes. For instance, Wang and Feng [19] proposed generalized intuitionistic fuzzy Yager weighted HM-based AGOs and applied them to MCDM situations. Wang et al. [20] developed power HM AGOs based on q-rung orthopair hesitant fuzzy data for MCDM. In a similar manner, Javed et al. [21] proposed T-spherical fuzzy Dombi power HM-based AGOs for MCDM. Thilagavathy and Mohanaselvi [22] introduced a T-spherical fuzzy TOPSIS method, integrating Hamacher HM-based AGOs with distance measures, and implemented it to waste treatment. In another application, Kakati et al. [23] studied the Fermatean fuzzy Archimedean HM-based MCDM method for sustainable urban transport solutions. Zang et al. [24] generalized the scope of HM operators by developing the linguistic complex T-spherical fuzzy HM operator and applying it to emergency information quality assessment. Additionally, Hussain et al. [25] explored the selection of educational institutes using spherical fuzzy HM operators combined with the Aczel-Alsina triangular norm. Further expanding the framework, Yaacob et al. [26] introduced bipolar neutrosophic Dombi-based HM operators for MCDM. Thilagavathy and Mohanaselvi [27] proposed T-spherical fuzzy Hamacher HM geometric operators for decision-making, utilizing the SMART-based TODIM method. Naz et al. [28] applied 2-tuple linguistic q-rung orthopair fuzzy power HM operators to evaluate historical sites. Similarly, Li et al. [29] introduced generalized q-rung orthopair fuzzy interactive Hamacher power average and HM operators for MCDM. Zhang et al. [30] focused on spherical fuzzy Dombi power HM-based AGOs. Mo and Huang [31] proposed Archimedean geometric HM-based AGOs based on dual hesitant fuzzy sets. Hu et al. [32] extended the application of HM operators by introducing a three-parameter generalized weighted HM. Shi et al. [33] explored intuitionistic fuzzy power geometric HM operators, integrating power geometric operations with intuitionistic FSs. Deveci et al. [34] utilized fuzzy trigonometric AGOs based MCDM model for the assessment of objects in urban transportation. Faizi et al. [35] presented a new MCDM method by fusing HM and Bonferroni mean with hesitant 2-tuple linguistic term sets. Akram et al. [36] utilized generalized orthopair fuzzy Aczel-Alsina aggregation operators (AGOs) for energy resource selection.

With recent advancements, experts have noted a global shift towards multipolarity. Consequently, researchers have been investigating the aggregation of various datasets involving mF information using existing AGOs. For instance, Waseem et al. [37] introduced mF Hamacher AGOs, which were successfully applied in MCDM scenarios. Khameneh and Kilicman [38] proposed mF soft weighted AGOs, demonstrating their effectiveness in addressing MCDM problems. Akram et al. [39] examined mF Dombi AGOs and highlighted their applicability in MCDM. Later, Naz et al. [40] developed 2-tuple linguistic BF Heronian mean AGOs specifically for MCDM. Furthermore, Ali et al. [41] established specialized geometric and arithmetic AGOs for aggregating mF information using Yager's t-norm and t-conorm. Recently, Rehman et al. [42] proposed Aczel-Alsina operation based AGOs and studied their applications in the identification of wind power and desalination plants sites. For additional insights into MCDM scenarios utilizing AGOs, readers may refer to [43]. Given the versatility and broad applicability of Heronian mean operators, coupled with the growing demand for multipolar fuzzy aggregation, we aim to develop *m*F set-based power geometric Heronian mean AGOs.

1.1. Research gaps

Inspection of the published studies on urban transportation concluded that the transport type and route are crucial factors in the management of urban transportation [44–46]. To find a reasonable route in transportation management, it is important to consider the multi-polar characteristics of each crucial factor to avoid data loss. The preexisting literature did not consider multiple features of each attribute, which are very important for effective decision-making. However, in view of the criticality of the route selection problem in urban transportation, it is necessary to evaluate each information pole regarding every attribute.

Besides, to enhance individual and societal welfare in urban transportation, the development of new methods is the need of the hour, particularly when the construction of new routes is under consideration to facilitate urban transportation. Due to the involvement of different stakeholders, any project related to the improvement of urban transportation suffers from conflicts of interest among the various associated organizations, including private and public transport companies, municipal bodies, and government authorities [47]. To tackle the above-mentioned complexities, the suggested methodology provides a reliable tool for all stakeholders. Several useful mathematical tools have been reported to date for handling such complicated situations, like *m*F set-based AGOs, while there is a need for a more powerful tool that provides accurate decisions by elaborating on the interrelationships of attributes in the *m*F environment compared to existing operators. To sum up, the suggested AGOs, which integrate the *m*F set model with the power geometric Heronian mean, make significant contributions to MCDM methods. From the analysis of the above-discussed literature, some major research gaps are observed as follows:

(1) All the integrated *m*F set-based AGOs with *t*-norm and *t*-conorm operations like Hamacher [37], Dombi [39], Yager [41], Aczel-Alsina [42], etc., are not capable of effectively maintaining or considering the interrelationships among attributes, and this issue can be easily overcome by the Heronian mean (or HM).

(2) The existing HM-based AGOs, such as picture fuzzy interactional partitioned HM-AGOs [48], Archimedean HM operators based on complex IFSs [55], etc., fail to demonstrate the multi-polar sub-characteristics of each attribute, and this issue can be easily addressed by integrating the *m*F set theory.

For more details on urban transportation techniques, the readers are refereed to Table 1.

Table 1. Summary of urban transportation published works with research gaps.

References	MCDM methods/AGOs	Problem descriptions	Research gaps
Kakati et al. [23]	Fermatean fuzzy Archimedean Heronian Mean-Based Model	Estimation of sustainable urban transport solutions	Unable to tackle *m*F dataset
Sarkar [44]	Dual hesitant q-rung orthopair fuzzy Frank power partitioned Heronian mean AGOs	Estimation of the sustainable urban transport solutions	Inadequate to deal with *m*F information
Deveci et al. [45]	fuzzy trigonometric based decision-making method	Accelerating the integration of the metaverse into urban transportation	Not considered both interrelationships among attributes and their sub-features
Hezam et al. [46]	Intuitionistic fuzzy gained and lost dominance score based on symmetric point criterion	Prioritization of zero-carbon measures for sustainable urban transportation	Not capable to consider interrelationships among attributes in *m*F environment
Görçün et al. [49]	Modified WASPAS approach based on Heronian operators	Selection of tramcars for sustainable urban transportation	Not effective when dealing with *m*F information
Seker and Aydin [50]	IVIF-AHP and CODAS method	Evaluation of sustainable public transportation system	Inadequate to tackle data involving multi-polar(*m*-polar) properties of objects and interrelationship among attributes
Erdogan et al. [51]	Hybrid power Heronian function-based model	Charging of scheduling algorithms for workplace	Fail to tackle data involving multi-polar properties of objects
Deveci et al. [52]	Fuzzy Einstein WASPAS approach	Climate change mitigation strategies in urban mobility planning	Handicap to tackle sub-characteristics and interrelationship of attributes
Pamucar et al. [53]	Integrated DIBR and fuzzy Dombi CoCoSo model	Concept of Circular economy in urban mobility alternatives	Unable to deal with sub-features and interrelationship of attributes
Li et al. [54]	Modified spherical fuzzy partitioned Maclaurin symmetric mean operator	Sustainability assessment of regional transportation	Inefficient to deal with data having multiple sub-characteristics of attributes

1.2. Motivations

The motivations of the presented research study are outlined below:

(1) To date, several well-known AGOs based on HM has been introduced to aggregate different types of information. Among them, HM operators offer greater flexibility and accuracy in aggregation compared to other operators due to their ability to handle interrelationship among input arguments, which makes them very effective in decision-making situations. The literature consistently highlights the superior accuracy of HM based AGOs.

(2) Multi-polar FSs address multi-index ambiguities by accounting for multiple different features of an object. This approach enables more accurate, flexible, and appropriate MCDM in complex, multi-attributed situations compared to existing decision-making approaches. To understand this useful concept, suppose a group of students wishes to plan a summer holiday tour but is uncertain about the location. This situation cannot be explained well by a membership value in $[0, 1]$, as different characteristics of a suitable location need to be evaluated, such as having lakes and waterfalls, the availability of food and other services, and favorable weather conditions. These are sub-characteristics of the location, with each characteristic having a membership value in $[0, 1]$. If we use fuzzy set theory to deal with this information, we would have to choose a fuzzy set for each sub-characteristic, which is not a precise way to represent this information. Hence, multi-polar FSs, as an efficient extension of FSs, are more flexible and reliable.

(3) In a variety of MCDM problems, due to the interrelationships among attributes and existence of their multi-polar sub-features, the experts may provide unreasonable data. However, existing *m*F AGOs, including *m*F Dombi AGOs, *m*F Hamacher AGOs, *m*F Yager AGOs, and *m*F Aczel-Alsina AGOs fail to effectively demonstrate the interrelationship between attributes, and thus fail to mitigate the impact of such type of uncertain data. Moreover, other published literature of HM operators, such as spherical fuzzy HM operators based on Aczel-Alsina operations has ability to deal with interrelationship between attributes but fail to reflect multi-polar sub-characteristics of attributes. Therefore, to overcome these issues, the fusion of *m*F sets and power GGHM operators is initiated in this research article.

(4) Many MCDM problems, such as determining the best plan in urban transportation management, require adequate handling due to involvement of complex, multi-faceted information with multi-polar, multi-attribute, and multi-agent uncertainties. Conventional fuzzy, BF, and IFS models are insufficient for effectively handling these complexities. However, the *m*F set-based AGOs, such as *m*F Dombi, Hamacher, Yager, and Aczel-Alsina AGOs, are also ineffective when dealing with interrelationships among attributes.

(5) The strong aggregation capabilities of the power geometric Heronian mean, combined with the modeling of *m*F sets, can enhance MCDM paradigms in multi-polar uncertain scenarios. However, existing literature has not sufficiently explored this powerful combination.

Motivated by these factors, the proposed work aims to develop multi-polar fuzzy (*m*F) power geometric HM aggregation operators (AGOs) and demonstrate their effectiveness in MCDM. The following list highlights the key contributions of this work:

(1) Three novel power geometric Heronian mean-based AGOs for *m*-polar fuzzy information are introduced: the W*m*FPGGHM, OW*m*FPGHHM, and H*m*FPGGHM operators.

(2) Some basic properties of the initiated AGOs, including idempotency, monotonicity, boundedness, and commutativity, are investigated.
(3) An MCDM algorithm is developed for the aggregation of multi-polar fuzzy information under the introduced W*m*FPGGHM operators.
(4) The newly designed algorithm is applied to a practical scenario: selecting the best plan for urban transportation management: a case study of Saudi Arabia.
(5) A brief comparative analysis of the proposed technique is conducted with *m*F Yager AGOs [41], *m*F Dombi AGOs [39], and *m*F Aczel-Alsina [42].

The forthcoming work is structured as follows: Section 2 reviews key *m*-polar fuzzy concepts and revisits the geometric Heronian mean and power geometric operators. Section 3 presents three new power geometric AGOs based on the Heronian mean, namely, the W*m*FPGGHM, OW*m*FPGGHM, and H*m*FPGGHM operators. Some essential properties of the developed AGOs are also investigated in this section. Section 4 develops a novel MCDM algorithm based on weighted *m*F power GHM operators and explores a case study in Saudi Arabia for selecting the best plan for urban transportation management. Section 5 provides a comparison of the introduced AGOs with certain preexisting operators, including *m*F Yager AGOs [41], *m*F Dombi AGOs [39], and *m*F Aczel-Alsina [42]. Additionally, some advantages and limitations of the presented work are demonstrated in this section. Section 6 concludes this work and provides some suitable future directions.

2. Preliminaries

In this section, we first revisit the key concepts of *m*F sets, followed by a discussion of basic operations on *m*F numbers. Additionally, the definitions of the geometric HM and power geometric HM operators are reviewed.

Definition 2.1. *[11] An m-polar fuzzy (or mF) set* $\mathcal{T}$ *is a function* $\varphi : \mathcal{T} \to [0,1]^m$ *where the membership value of each object is defined by* $\varphi(\mathfrak{t}) = (\mathfrak{p}_1 \circ \varphi(\mathfrak{t}), \mathfrak{p}_2 \circ \varphi(\mathfrak{t}), \ldots, \mathfrak{p}_m \circ \varphi(\mathfrak{t}))$ *where* $\mathfrak{t} \in \mathcal{T}$*, and for* $q = 1, 2, \ldots, m$*,* $\mathfrak{p}_q \circ \varphi : [0,1]^m \to [0,1]$ *is a function that represents q-th projection.*

For every *m*F number (henceforth, *m*FN) $\tilde{\varphi} = (\mathfrak{p}_1 \circ \varphi, \ldots, \mathfrak{p}_m \circ \varphi)$ with $\mathfrak{p}_q \circ \varphi \in [0,1]$ for all $q = 1, 2, \ldots, m$, the functions (score and accuracy) $\tilde{\varphi}$ are provided as below:

Definition 2.2. *[37] Suppose that* $\tilde{\varphi} = (\mathfrak{p}_1 \circ \varphi, \ldots, \mathfrak{p}_m \circ \varphi)$ *is an mFN, then we define its score* $\mathfrak{U}$ *and accuracy* $\mathfrak{V}$ *functions as below:*

$$\mathfrak{U}(\tilde{\varphi}) = \frac{1}{m}\Big(\sum_{q=1}^{m}(\mathfrak{p}_q \circ \varphi)\Big),$$

$$\mathfrak{A}(\tilde{\varphi}) = \frac{1}{m}\Big(\sum_{q=1}^{m}(-1)^{q+1}(\mathfrak{p}_q \circ \varphi - 1)\Big),$$

where $\mathfrak{U}(\tilde{\varphi}) \in [0,1]$ *and* $\mathfrak{V}(\tilde{\varphi}) \in [-1,1]$.

Based on the above definition, we present an ordering criterion for *m*FNs as follows:

Definition 2.3. *[37] Suppose* $\tilde{\varphi}_1 = (\mathfrak{p}_1 \circ \varphi_1, \ldots, \mathfrak{p}_m \circ \varphi_1)$*, and* $\tilde{\varphi}_2 = (\mathfrak{p}_1 \circ \varphi_2, \ldots, \mathfrak{p}_m \circ \varphi_2)$ *are two mFNs, then*

(1) $\tilde{\varphi}_1 < \tilde{\varphi}_2$, *if* $\mathfrak{U}(\tilde{\varphi}_1) < \mathfrak{U}(\tilde{\varphi}_2)$,
(2) $\tilde{\varphi}_1 > \tilde{\varphi}_2$, *if* $\mathfrak{U}(\tilde{\varphi}_1) > \mathfrak{U}(\tilde{\varphi}_2)$,
(3) If $\mathfrak{U}(\tilde{\varphi}_1) = \mathfrak{U}(\tilde{\varphi}_2)$, *then*

- $\tilde{\varphi}_1 < \tilde{\varphi}_2$, *if* $\mathfrak{A}(\tilde{\varphi}_1) < \mathfrak{A}(\tilde{\varphi}_2)$,
- $\tilde{\varphi}_1 > \tilde{\varphi}_2$, *if* $\mathfrak{A}(\tilde{\varphi}_1) > \mathfrak{A}(\tilde{\varphi}_2)$,
- $\tilde{\varphi}_1 = \tilde{\varphi}_2$, *if* $\mathfrak{B}(\tilde{\varphi}_1) = \mathfrak{A}(\tilde{\varphi}_2)$.

We now revisit some fundamental notions of *m*FNs [37] as follows:

(1) $\tilde{\varphi}_1 \boxplus \tilde{\varphi}_2 = \Big(\mathfrak{p}_1 \circ \varphi_1 + \mathfrak{p}_1 \circ \varphi_2 - \mathfrak{p}_1 \circ \varphi_1 . \mathfrak{p}_1 \circ \varphi_2, \ldots, \mathfrak{p}_m \circ \varphi_1 + \mathfrak{p}_m \circ \varphi_2 - \mathfrak{p}_m \circ \varphi_1 . \mathfrak{p}_m \circ \varphi_2\Big)$,
(2) $\tilde{\varphi}_1 \boxtimes \tilde{\varphi}_2 = \Big(\mathfrak{p}_1 \circ \varphi_1 . \mathfrak{p}_1 \circ \varphi_2, \ldots, \mathfrak{p}_m \circ \varphi_1 . \mathfrak{p}_m \circ \varphi_2\Big)$,
(3) $\alpha\tilde{\varphi} = (1 - (1 - \mathfrak{p}_1 \circ \varphi)^{\alpha}), \ldots, 1 - (1 - \mathfrak{p}_m \circ \varphi)^{\alpha}),\ \alpha > 0$,
(4) $(\tilde{\varphi})^{\alpha} = ((\mathfrak{p}_1 \circ \varphi)^{\alpha}, \ldots, (\mathfrak{p}_m \circ \varphi)^{\alpha}),\ \alpha > 0$,
(5) $\tilde{\varphi}^c = (1 - \mathfrak{p}_1 \circ \varphi, \ldots, 1 - \mathfrak{p}_m \circ \varphi)$,
(6) $\tilde{\varphi}_1 \subseteq \tilde{\varphi}_2$, if and only if $\mathfrak{p}_1 \circ \varphi_1 \leq \mathfrak{p}_1 \circ \varphi_2, \ldots, \mathfrak{p}_m \circ \varphi_1 \leq \mathfrak{p}_m \circ \varphi_2$,
(7) $\tilde{\varphi}_1 \cup \tilde{\varphi}_2 = (\max(\mathfrak{p}_1 \circ \varphi_1, \mathfrak{p}_1 \circ \varphi_2), \ldots, \max(\mathfrak{p}_m \circ \varphi_1, \mathfrak{p}_m \circ \varphi_2))$,
(8) $\tilde{\varphi}_1 \cap \tilde{\varphi}_2 = (\min(\mathfrak{p}_1 \circ \varphi_1, \mathfrak{p}_1 \circ \varphi_2), \ldots, \min(\mathfrak{p}_m \circ \varphi_1, \mathfrak{p}_m \circ \varphi_2))$.

Theorem 2.1. *Suppose* $\tilde{\varphi}_1 = (\mathfrak{p}_1 \circ \varphi_1, \ldots, \mathfrak{p}_m \circ \varphi_1)$ *and* $\tilde{\varphi}_2 = (\mathfrak{p}_1 \circ \varphi_2, \ldots, \mathfrak{p}_m \circ \varphi_2)$ *are two mFNs, and* $\alpha, \alpha_1, \alpha_2 > 0$. *Then*

(1) $\tilde{\varphi}_1 \boxplus \tilde{\varphi}_2 = \tilde{\varphi}_2 \boxplus \tilde{\varphi}_1$,
(2) $\tilde{\varphi}_1 \boxtimes \tilde{\varphi}_2 = \tilde{\varphi}_2 \boxtimes \tilde{\varphi}_1$,
(3) $\alpha(\tilde{\varphi}_1 \boxplus \tilde{\varphi}_2) = \alpha(\tilde{\varphi}_1) \boxplus \alpha(\tilde{\varphi}_2)$,
(4) $(\tilde{\varphi}_1 \boxtimes \tilde{\varphi}_2)^{\alpha} = (\tilde{\varphi}_1)^{\alpha} \boxplus (\tilde{\varphi}_2)^{\alpha}$,
(5) $\alpha_1\tilde{\varphi}_1 \boxplus \alpha_2\tilde{\varphi}_1 = (\alpha_1 + \alpha_2)\tilde{\varphi}_1$,
(6) $(\tilde{\varphi}_1)^{\alpha_1} \boxtimes (\tilde{\varphi}_2)^{\alpha_2} = (\tilde{\varphi}_1)^{\alpha_1+\alpha_2}$,
(7) $((\tilde{\varphi}_1)^{\alpha_1})^{\alpha_2} = (\tilde{\varphi}_1)^{\alpha_1\alpha_2}$.

Proof. It can be immediately followed from the above properties and Definition 2.1. □
The HM operator is a standard tool, which is utilized to compute the relation between decision objects. Yu [43] extended this concept by presenting the notion of generalized geometric HM operator.

Definition 2.4. *[43] Suppose that* ξ, η, *and* $\mathfrak{t}_i$ *where i varies from 1 to n, are non-negative real numbers. If*

$$GHM^{\xi,\eta}(\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_n) = \frac{1}{\xi + \eta} \prod_{i=1, j=i}^{n} (\xi\mathfrak{t}_i + \eta\mathfrak{t}_j)^{\frac{2}{n(n+1)}}, \tag{2.1}$$

then $GHM^{\xi,\eta}$ *is referred to as geometric Heronian mean (or GHM) operator.*

Now, we recall the definition of power geometric operator, which was proposed by Xu and Yager [16].

Definition 2.5. *[16] Suppose that* $\mathfrak{t}_i$ *where i varies from 1 to n, are non-negative real numbers. The power geometric operator is given by*

$$PG(\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n) = \prod_{i=1}^{n} \mathrm{t}_i^{\frac{1+T(\mathrm{t}_i)}{\sum_{i=1}^{n}(1+T(\mathrm{t}_i))}}, \tag{2.2}$$

where $T(\mathrm{t}_i) = \sum_{j=1, j\neq i}^{n} Supp(\mathrm{t}_i, \mathrm{t}_j)$, $Supp(\mathrm{t}_i, \mathrm{t}_j)$ *is the support that satisfies the following constraints:*

(1) $Supp(\mathrm{t}_i, \mathrm{t}_j)$ *belongs to closed unit interval.*
(2) $Supp(\mathrm{t}_i, \mathrm{t}_j) = Supp(\mathrm{t}_j, \mathrm{t}_i)$.
(3) If $d(\mathrm{t}_i, \mathrm{t}_j) \leq d(\mathrm{t}_l, \mathrm{t}_k)$, *then* $Supp(\mathrm{t}_i, \mathrm{t}_j) \geq Supp(\mathrm{t}_l, \mathrm{t}_k)$, *where* $d(\mathrm{t}_i, \mathrm{t}_j)$ *serve as the distance between* t_i *and* t_j, *which is computed as*

$$d(\mathrm{t}_i, \mathrm{t}_j) = \sum_{j=1, j\neq i}^{n} \sqrt{\frac{1}{2}((\mathrm{t}_i - \mathrm{t}_j)^2 + (\mathrm{t}_i - \mathrm{t}_j)^2 + (\mathrm{t}_i - \mathrm{t}_j)^2)}. \tag{2.3}$$

3. *m*F generalized geometric Heronian mean operators

In this section, we present novel *m*F set-based operators, including weighted *m*F generalized geometric Heronian mean (W*m*FGGHM) operators, ordered W*m*FGGHM operators, and hybrid *m*FGGHM operators. Moreover, we investigate their essential properties with illustrative numerical examples.

Definition 3.1. *Suppose that* $\tilde{\varphi}_t = (\mathfrak{p}_1 \circ \varphi_t, \ldots, \mathfrak{p}_m \circ \varphi_t)$ *is a finite set of n mFNs, then a function* $WmFGGHM_\theta : \tilde{\varphi}^n \to \tilde{\varphi}$ *is called weighted mF generalized geometric Heronian mean (or WmFGGHM), which is provided by*

$$WmFGGHM_\theta^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes (n\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}, \tag{3.1}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ *includes the weights for each* $\tilde{\varphi}_{\mathrm{t}}$, $\forall\ \mathrm{t} = 1, \ldots, n$ *and* $\theta_t > 0$ *with* $\sum_{t=1}^{n} \theta_t = 1$. *In Eq (3.1),* λ_i *are the power weights of each* $\tilde{\varphi}_{\mathrm{t}}$, *which are given by*

$$\lambda_i = \frac{1 + T(\mathrm{t}_i)}{\sum_{s=1}^{n}(1 + T(\mathrm{t}_s))}. \tag{3.2}$$

Following is the key result in the generalized theory of weighted *m*F power GHM operators.

Theorem 3.1. *Suppose that* $\tilde{\varphi}_i = (\mathfrak{p}_1 \circ \varphi_i, \ldots, \mathfrak{p}_m \circ \varphi_i)$ *is a family of n mFNs where* $i = 1, 2, \ldots, n$, *then by applying the WmFGGHM operators, an accumulated degree of these mFNs is computed as follows:*

$$WmFGGHM_\theta^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes (n\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}$$

$$= \left(\left(1 - \prod_{i=1, j=i}^{n} (1 - (1 - (1 - p_1 \circ \varphi_i)^{n\lambda_i\theta_i})^{\xi} (1 - (1 - p_1 \circ \varphi_j)^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}, \ldots,\right.$$

$$\left.\left(1 - \prod_{i=1, j=i}^{n} (1 - (1 - (1 - p_m \circ \varphi_i)^{n\lambda_i\theta_i})^{\xi} (1 - (1 - p_m \circ \varphi_j)^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\right). \tag{3.3}$$

Now, the implementation of above theorem is performed in the following numerical example:

Example 3.1. *Suppose that* $\varphi_1 = (0.13, 0.34, 0.76)$, $\varphi_2 = (0.15, 0.50, 0.88)$, $\varphi_3 = (0.44, 0.27, 0.18)$, *and* $\varphi_4 = (0.11, 17, 0.26)$ *are 4FNs, and* $\theta = (0.2, 0.5, 0.1, 0.2)$ *contains weights associated to these 4FNs. Then, for* $\xi = 3$ *and* $\eta = 5$*, we have*

$$WmFGGHM_{\theta}^{3,5}(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_4) = \Big(\frac{2}{4(4+1)} \bigoplus_{i=1}^{4} \bigoplus_{j=i}^{4} (4\theta_i\lambda_i\tilde{\varphi}_i)^3 \otimes (4\theta_j\lambda_j\tilde{\varphi}_j)^5\Big)^{\frac{1}{3+5}}$$

$$= \left(\left(1 - \prod_{i=1,j=i}^{4} (1-(1-(1-p_1\circ\varphi_i)^{4\lambda_i\theta_i})^3(1-(1-p_1\circ\varphi_j)^{4\lambda_j\theta_j})^5)^{\frac{1}{10}}\right)^{\frac{1}{8}}, \ldots,\right.$$

$$\left.\left(1 - \prod_{i=1,j=i}^{4} (1-(1-(1-p_3\circ\varphi_i)^{4\lambda_i\theta_i})^3(1-(1-p_3\circ\varphi_j)^{4\lambda_j\theta_j})^5)^{\frac{1}{10}}\right)^{\frac{1}{8}}\right). \tag{3.4}$$

Before finding the values λ_1, λ_2, λ_3 *and* λ_4 *using Eq (3.2), we first compute the distances by the Eq (2.3) as*

$$d(\varphi_1, \varphi_2) = \sqrt{\frac{1}{2}((0.13-0.15)^2 + (0.34-0.50)^2 + (0.76-0.88)^2)} = 0.14213,$$

$$d(\varphi_1, \varphi_3) = \sqrt{\frac{1}{2}((0.13-0.44)^2 + (0.34-0.27)^2 + (0.76-0.18)^2)} = 0.46765,$$

$$d(\varphi_1, \varphi_4) = \sqrt{\frac{1}{2}((0.13-0.11)^2 + (0.34-0.17)^2 + (0.76-0.26)^2)} = 0.3737,$$

$$d(\varphi_2, \varphi_1) = \sqrt{\frac{1}{2}((0.15-0.13)^2 + (0.50-0.34)^2 + (0.88-0.76)^2)} = 0.14213,$$

$$d(\varphi_2, \varphi_3) = \sqrt{\frac{1}{2}((0.15-0.44)^2 + (0.50-0.27)^2 + (0.88-0.18)^2)} = 0.55991,$$

$$d(\varphi_2, \varphi_4) = \sqrt{\frac{1}{2}((0.15-0.11)^2 + (0.50-0.17)^2 + (0.88-0.26)^2)} = 0.49744,$$

$$d(\varphi_3, \varphi_1) = \sqrt{\frac{1}{2}((0.44-0.13)^2 + (0.27-0.34)^2 + (0.18-0.76)^2)} = 0.46765,$$

$$d(\varphi_3, \varphi_2) = \sqrt{\frac{1}{2}((0.44-0.15)^2 + (0.27-0.50)^2 + (0.18-0.88)^2)} = 0.55991,$$

$$d(\varphi_3, \varphi_4) = \sqrt{\frac{1}{2}((0.44-0.11)^2 + (0.27-0.17)^2 + (0.18-0.26)^2)} = 0.2503,$$

$$d(\varphi_4, \varphi_1) = \sqrt{\frac{1}{2}((0.11-0.13)^2 + (0.17-0.34)^2 + (0.26-0.76)^2)} = 0.3737,$$

$$d(\varphi_4, \varphi_2) = \sqrt{\frac{1}{2}((0.11-0.15)^2 + (0.17-0.50)^2 + (0.26-0.88)^2)} = 0.49744,$$

$$d(\varphi_4, \varphi_3) = \sqrt{\frac{1}{2}((0.11-0.44)^2 + (0.17-0.27)^2 + (0.26-0.18)^2)} = 0.3737.$$

Now, we are ready to calculate the values of $T(\varphi_i)$ where i varies from 1 to 4 as below:

$$T(\varphi_1) = 2.0165,\ T(\varphi_2) = 1.8005,\ T(\varphi_3) = 1.7221,\ \ T(\varphi_4) = 1.8786.$$

Consequently, we get the values each λ_i as

$$\lambda_1 = 0.2642,\ \lambda_2 = 0.24528,\ \lambda_3 = 0.23841,\ \textit{and}\ \lambda_4 = 0.25211.$$

Finally, by putting all values in Eq (3.4), we have

$$\begin{aligned}
&WmFGGHM_{\theta}^{3,5}(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_4)\\
&= \Bigg(\Bigg((1-(1-(1-(1-0.13)^{4\times0.20\times0.2642})^3(1-(1-0.13)^{4\times0.2\times0.2642})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.13)^{4\times0.20\times0.2642})^3(1-(1-0.15)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.13)^{4\times0.20\times0.2642})^3(1-(1-0.44)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.13)^{4\times0.20\times0.2642})^3(1-(1-0.11)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.15)^{4\times0.50\times0.24528})^3(1-(1-0.13)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.15)^{4\times0.50\times0.24528})^3(1-(1-0.15)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.15)^{4\times0.50\times0.24528})^3(1-(1-0.44)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.15)^{4\times0.50\times0.24528})^3(1-(1-0.11)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.44)^{4\times0.10\times0.23841})^3(1-(1-0.13)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.44)^{4\times0.10\times0.23841})^3(1-(1-0.15)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.44)^{4\times0.10\times0.23841})^3(1-(1-0.44)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.44)^{4\times0.10\times0.23841})^3(1-(1-0.11)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.11)^{4\times0.20\times0.25211})^3(1-(1-0.13)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.11)^{4\times0.20\times0.25211})^3(1-(1-0.15)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.11)^{4\times0.20\times0.25211})^3(1-(1-0.44)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.11)^{4\times0.20\times0.25211})^3(1-(1-0.11)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\Bigg)^{\frac{1}{8}},\\
&\quad\Bigg(1-(1-(1-(1-0.34)^{4\times0.20\times0.2642})^3(1-(1-0.34)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.34)^{4\times0.20\times0.2642})^3(1-(1-0.50)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.34)^{4\times0.20\times0.2642})^3(1-(1-0.27)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.34)^{4\times0.20\times0.2642})^3(1-(1-0.17)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.50)^{4\times0.50\times0.24528})^3(1-(1-0.34)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.50)^{4\times0.50\times0.24528})^3(1-(1-0.50)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}\\
&\quad\times(1-(1-(1-0.50)^{4\times0.50\times0.24528})^3(1-(1-0.27)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}
\end{aligned}$$

$$\times(1-(1-(1-0.50)^{4\times0.50\times0.24528})^3(1-(1-0.17)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.27)^{4\times0.10\times0.23841})^3(1-(1-0.34)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.27)^{4\times0.10\times0.23841})^3(1-(1-0.50)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.27)^{4\times0.10\times0.23841})^3(1-(1-0.27)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.27)^{4\times0.10\times0.23841})^3(1-(1-0.17)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.17)^{4\times0.20\times0.25211})^3(1-(1-0.34)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.17)^{4\times0.20\times0.25211})^3(1-(1-0.50)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.17)^{4\times0.20\times0.25211})^3(1-(1-0.27)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.17)^{4\times0.20\times0.25211})^3(1-(1-0.17)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\Bigg)^{\frac{1}{8}},$$
$$\Bigg(1-(1-(1-(1-0.76)^{4\times0.20\times0.2642})^3(1-(1-0.76)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.76)^{4\times0.20\times0.2642})^3(1-(1-0.88)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.76)^{4\times0.20\times0.2642})^3(1-(1-0.18)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.76)^{4\times0.20\times0.2642})^3(1-(1-0.26)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.88)^{4\times0.50\times0.24528})^3(1-(1-0.76)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.88)^{4\times0.50\times0.24528})^3(1-(1-0.88)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.88)^{4\times0.50\times0.24528})^3(1-(1-0.18)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.88)^{4\times0.50\times0.24528})^3(1-(1-0.26)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.18)^{4\times0.10\times0.23841})^3(1-(1-0.76)^{4\times0.20\times0.26411})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.18)^{4\times0.10\times0.23841})^3(1-(1-0.88)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.18)^{4\times0.10\times0.23841})^3(1-(1-0.18)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.18)^{4\times0.10\times0.23841})^3(1-(1-0.26)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.26)^{4\times0.20\times0.25211})^3(1-(1-0.76)^{4\times0.20\times0.2642})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.26)^{4\times0.20\times0.25211})^3(1-(1-0.88)^{4\times0.50\times0.24528})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.26)^{4\times0.20\times0.25211})^3(1-(1-0.18)^{4\times0.10\times0.23841})^5)^{\frac{1}{10}}$$
$$\times(1-(1-(1-0.26)^{4\times0.20\times0.25211})^3(1-(1-0.26)^{4\times0.20\times0.25211})^5)^{\frac{1}{10}}\Bigg)^{\frac{1}{8}}\Bigg),$$
$$=(0.0613,0.2170,0.4902).$$

We are now prepared to explore some fundamental concepts of W*m*FGGHM operators such as monotonicity, idempotency, and boundedness. We start with monotonicity.

Theorem 3.2. *(Monotonicity) For suppose that $\tilde{\varphi}_t$ and $\tilde{\varphi}'_t$ are two collections of n mFNs, if each $\tilde{\varphi}_t \leq \tilde{\varphi}'_t$, then*

$$WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) \leq WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1', \tilde{\varphi}_2', \ldots, \tilde{\varphi}_n'). \tag{3.5}$$

Proof. Its proof directly followed by Definition 3.1 and Theorem 3.1. □

Theorem 3.3. *(Idempotency) For a collection of mF numbers which are 'm' in number given as* $\tilde{\varphi}_t = (\mathfrak{p}_1 \circ \varphi_t, \ldots, \mathfrak{p}_m \circ \varphi_t)$ *such that* $\tilde{\varphi}_t = \tilde{\varphi}$*, we get*

$$WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \tilde{\varphi}. \tag{3.6}$$

Hence, the result $WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \tilde{\varphi}$ *is verified when* $\tilde{\varphi}_t = \tilde{\varphi}$*, where the range of 't' is from 1 to n.*

Theorem 3.4. *(Boundedness) Suppose that* $\tilde{\varphi}_t = (\mathfrak{p}_1 \circ \varphi_t, \ldots, \mathfrak{p}_m \circ \varphi_t)$ *is a set of 'n' mFNs if* $\tilde{\varphi}^l = \bigcap_{t=1}^{n}(\varphi_t)$ *and* $\tilde{\varphi}^u = \bigcup_{t=1}^{n}(\varphi_t)$*, then*

$$\tilde{\varphi}^l \leq WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) \leq \tilde{\varphi}^u. \tag{3.7}$$

Proof. It can be readily proved by Theorem 3.1 and Definition 3.1. □

Next, we propose the idea of ordered W*m*FGGHM operators, whose main objective is to rank *m*FNs first, and then apply the W*m*FGGHM operators. Moreover, we explain their phenomenon with a numerical example and useful results.

Definition 3.2. *Suppose* $\tilde{\varphi}_t = (\mathfrak{p}_1 \circ \varphi_t, \ldots, \mathfrak{p}_m \circ \varphi_t)$*,* $t = 1, 2, \ldots, n$*, is a collection of mFNs, then we define an ordered WmFGGHM (OWmFGGHM) operator as a mapping* $OWmFGGHM_{\theta}^{\xi,\eta} : \tilde{\varphi}^n \to \tilde{\varphi}$*, which is provided as*

$$OWmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_{\delta(i)})^{\xi} \otimes (n\theta_j\lambda_j\tilde{\varphi}_{\delta(j)})^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}, \tag{3.8}$$

here $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ *consists of the weights, and each* $\theta_{\mathrm{t}} \in (0, 1]$ *with* $\sum_{\mathrm{t}=1}^{n} \theta_{\mathrm{t}} = 1$*. Further, in Eq (3.8),* $\delta(\mathrm{t})$ *serve as the permutation which satisfies the inequality* $\tilde{\varphi}_{\delta(t-1)} \geq \tilde{\varphi}_{\delta(t)}$.

In the following theorem, the implementation process of OW*m*FGGHM operators on the collection of *m*FNs is provided.

Theorem 3.5. *Suppose that* $\tilde{\varphi}_t = (\mathfrak{p}_1 \circ \varphi_t, \ldots, \mathfrak{p}_m \circ \varphi_t)$ *is a family of 'n' mFNs, then their aggregation using an OWmFGGHM operator provide again an mFN, which is computed by the following equation:*

$$\begin{aligned} OWmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) &= \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_{\delta(i)})^{\xi} \otimes (n\theta_j\lambda_j\tilde{\varphi}_{\delta(j)})^{\eta}\Big)^{\dfrac{1}{\xi+\eta}} \\ &= \left(\left(1 - \prod_{i=1,j=i}^{n} (1 - (1 - (1 - p_1 \circ \varphi_{\delta(i)})^{n\lambda_i\theta_i})^{\xi} (1 - (1 - p_1 \circ \varphi_{\delta(j)})^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}, \ldots, \right. \\ &\left. \left(1 - \prod_{i=1,j=i}^{n} (1 - (1 - (1 - p_m \circ \varphi_{\delta(i)})^{n\lambda_i\theta_i})^{\xi} (1 - (1 - p_m \circ \varphi_{\delta(j)})^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\right), \end{aligned} \tag{3.9}$$

Proof. Its proof followed same arguments as used in Theorem 3.1. Thus, we omit it. □

Now, we illustrate the above result with an example as below:

Example 3.2. *Consider* $\tilde{\varphi}_1 = (0.52, 0.61, 0.67)$, $\tilde{\varphi}_2 = (0.31, 0.91, 0.83)$, $\tilde{\varphi}_3 = (0.47, 0.36, 0.30)$ *and* $\tilde{\varphi}_4 = (0.59, 0.99, 0.12)$ *as four* 3*PFNs with respective weights* $\theta = (0.55, 0.23, 0.11, 0.11)$. *Then, first, we find the scores as below:*

$$\begin{aligned}
\mathfrak{U}(\tilde{\varphi}_1) &= \frac{0.52 + 0.61 + 0.67}{3} = 0.6,\\
\mathfrak{U}(\tilde{\varphi}_2) &= \frac{0.31 + 0.91 + 0.83}{3} = 0.68,\\
\mathfrak{U}(\tilde{\varphi}_3) &= \frac{0.47 + 0.36 + 0.30}{3} = 0.37,\\
\mathfrak{U}(\tilde{\varphi}_4) &= \frac{0.59 + 0.99 + 0.12}{3} = 0.56.
\end{aligned}$$

This implies $\mathfrak{U}(\tilde{\varphi}_2) > \mathfrak{U}(\tilde{\varphi}_1) > \mathfrak{U}(\tilde{\varphi}_4) > \mathfrak{U}(\tilde{\varphi}_3)$, *therefore, the new ordering is given as follows:*

$$\begin{aligned}
\tilde{\varphi}_{\delta(1)} &= \tilde{\varphi}_2 = (0.31, 0.91, 0.83),\\
\tilde{\varphi}_{\delta(2)} &= \tilde{\varphi}_1 = (0.52, 0.61, 0.67),\\
\tilde{\varphi}_{\delta(3)} &= \tilde{\varphi}_4 = (0.59, 0.99, 0.12),\\
\varphi_{\delta(2)} &= \tilde{\varphi}_3 = (0.47, 0.36, 0.30).
\end{aligned}$$

Now for $\xi = 5$ *and* $\eta = 6$, *using Definition 3.2, we get*

$$\begin{aligned}
OWmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_4) &= \Big(\frac{2}{4(4+1)} \bigoplus_{i=1}^{4} \bigoplus_{j=i}^{4} (4\theta_i\lambda_i\tilde{\varphi}_{\delta(i)})^5 \otimes (4\theta_j\lambda_j\tilde{\varphi}_{\delta(j)})^6\Big)^{\frac{1}{5+6}}\\
= \Bigg(\Bigg(1 - \prod_{i=1, j=i}^{4} &(1 - (1 - (1 - p_1 \circ \varphi_{\delta(i)})^{4\lambda_i\theta_i})^5 (1 - (1 - p_1 \circ \varphi_{\delta(j)})^{4\lambda_j\theta_j})^6)^{\frac{1}{10}}\Bigg)^{\frac{1}{11}}, \ldots,\\
\Bigg(1 - \prod_{i=1, j=i}^{4} &(1 - (1 - (1 - p_3 \circ \varphi_{\delta(i)})^{4\lambda_i\theta_i})^5 (1 - (1 - p_3 \circ \varphi_{\delta(j)})^{4\lambda_j\theta_j})^6)^{\frac{1}{10}}\Bigg)^{\frac{1}{11}}\Bigg),
\end{aligned} \tag{3.10}$$

Next, for the implementation of desired ordered weighted operator, we first determine the values λ_1, λ_2, λ_3 *and* λ_4 *using Eq (3.2) and Definition 2.5 as below:*

$$\begin{aligned}
d(\varphi_{\delta(1)}, \varphi_{\delta(2)}) &= \sqrt{\frac{1}{2}((0.31 - 0.52)^2 + (0.91 - 0.61)^2 + (0.83 - 0.67)^2)} = 0.2826,\\
d(\varphi_{\delta(1)}, \varphi_{\delta(3)}) &= \sqrt{\frac{1}{2}((0.31 - 0.59)^2 + (0.91 - 0.99)^2 + (0.83 - 0.12)^2)} = 0.5426,\\
d(\varphi_{\delta(1)}, \varphi_{\delta(4)}) &= \sqrt{\frac{1}{2}((0.31 - 0.47)^2 + (0.91 - 0.36)^2 + (0.83 - 0.30)^2)} = 0.5518,\\
d(\varphi_{\delta(2)}, \varphi_{\delta(1)}) &= \sqrt{\frac{1}{2}((0.52 - 0.31)^2 + (0.61 - 0.91)^2 + (0.67 - 0.83)^2)} = 0.2826,
\end{aligned}$$

$$d(\varphi_{\delta(2)},\varphi_{\delta(3)}) = \sqrt{\frac{1}{2}((0.52-0.59)^2+(0.61-0.99)^2+(0.67-0.12)^2)} = 0.4753,$$
$$d(\varphi_{\delta(2)},\varphi_{\delta(4)}) = \sqrt{\frac{1}{2}((0.52-0.47)^2+(0.61-0.36)^2+(0.67-0.30)^2)} = 0.3177,$$
$$d(\varphi_{\delta(3)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.59-0.31)^2+(0.99-0.91)^2+(0.12-0.83)^2)} = 0.5426,$$
$$d(\varphi_{\delta(3)},\varphi_{\delta(2)}) = \sqrt{\frac{1}{2}((0.59-0.52)^2+(0.99-0.61)^2+(0.12-0.67)^2)} = 0.4753,$$
$$d(\varphi_{\delta(3)},\varphi_{\delta(4)}) = \sqrt{\frac{1}{2}((0.59-0.47)^2+(0.99-0.36)^2+(0.12-0.30)^2)} = 0.4710,$$
$$d(\varphi_{\delta(4)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.47-0.31)^2+(0.36-0.91)^2+(0.30-0.83)^2)} = 0.5518,$$
$$d(\varphi_{\delta(4)},\varphi_{\delta(2)}) = \sqrt{\frac{1}{2}((0.47-0.52)^2+(0.36-0.61)^2+(0.30-0.67)^2)} = 0.3177,$$
$$d(\varphi_{\delta(4)},\varphi_{\delta(3)}) = \sqrt{\frac{1}{2}((0.47-0.59)^2+(0.36-0.99)^2+(0.30-0.12)^2)} = 0.4710.$$

This implies

$$T(\varphi_{\delta(1)}) = 1.6230, \quad T(\varphi_{\delta(2)}) = 1.9244, \quad T(\varphi_{\delta(3)}) = 1.5111 \ \ \textit{and} \ \ T(\varphi_{\delta(4)}) = 1.6594.$$

Then, we compute the value of each λ_i as follows:

$$\lambda_1 = 0.2447, \quad \lambda_2 = 0.2729, \quad \lambda_3 = 0.2343 \quad \textit{and} \quad \lambda_4 = 0.2481.$$

Finally, by putting all these values in Eq (3.10), we have

$$
\begin{aligned}
&OWmFGGHM_{\theta}^{5,6}(\tilde{\varphi}_{\delta(1)},\tilde{\varphi}_{\delta(2)},\tilde{\varphi}_{\delta(3)},\tilde{\varphi}_{\delta(4)})\\
&=\Bigg(\Bigg(\Big(1-(1-(1-(1-(1-0.31)^{4\times0.55\times0.2447})^5(1-(1-0.31)^{4\times0.55\times0.2447})^6\Big)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.31)^{4\times0.55\times0.2447})^5(1-(1-0.52)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.31)^{4\times0.55\times0.2447})^5(1-(1-0.59)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.31)^{4\times0.55\times0.2447})^5(1-(1-0.47)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.52)^{4\times0.23\times0.2729})^5(1-(1-0.31)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.52)^{4\times0.23\times0.2729})^5(1-(1-0.52)^{4\times0.52\times0.2729})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.52)^{4\times0.23\times0.2729})^5(1-(1-0.59)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.52)^{4\times0.23\times0.2729})^5(1-(1-0.47)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.59)^{4\times0.11\times0.2343})^5(1-(1-0.31)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.59)^{4\times0.11\times0.2343})^5(1-(1-0.52)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}\\
&\times(1-(1-(1-(1-0.59)^{4\times0.11\times0.2343})^5(1-(1-0.59)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}
\end{aligned}
$$

$$\times(1-(1-(1-(1-0.59)^{4\times0.11\times0.2343})^5(1-(1-0.47)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.47)^{4\times0.11\times0.2481})^5(1-(1-0.31)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.47)^{4\times0.11\times0.2481})^5(1-(1-0.52)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.47)^{4\times0.11\times0.2481})^5(1-(1-0.59)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.47)^{4\times0.11\times0.2481})^5(1-(1-0.47)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}\Big)^{\frac{1}{11}},$$
$$\Big((1-(1-(1-(1-(1-0.91)^{4\times0.55\times0.2447})^5(1-(1-0.91)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.91)^{4\times0.55\times0.2447})^5(1-(1-0.61)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.91)^{4\times0.55\times0.2447})^5(1-(1-0.99)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.91)^{4\times0.55\times0.2447})^5(1-(1-0.36)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.61)^{4\times0.23\times0.2729})^5(1-(1-0.91)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.61)^{4\times0.23\times0.2729})^5(1-(1-0.61)^{4\times0.52\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.61)^{4\times0.23\times0.2729})^5(1-(1-0.99)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.61)^{4\times0.23\times0.2729})^5(1-(1-0.36)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.99)^{4\times0.11\times0.2343})^5(1-(1-0.91)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.99)^{4\times0.11\times0.2343})^5(1-(1-0.61)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.99)^{4\times0.11\times0.2343})^5(1-(1-0.99)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.99)^{4\times0.11\times0.2343})^5(1-(1-0.36)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.36)^{4\times0.11\times0.2481})^5(1-(1-0.91)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.36)^{4\times0.11\times0.2481})^5(1-(1-0.61)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.36)^{4\times0.11\times0.2481})^5(1-(1-0.99)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.36)^{4\times0.11\times0.2481})^5(1-(1-0.36)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}\Big)^{\frac{1}{11}},$$
$$\Big((1-(1-(1-(1-(1-0.83)^{4\times0.55\times0.2447})^5(1-(1-0.83)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.83)^{4\times0.55\times0.2447})^5(1-(1-0.67)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.83)^{4\times0.55\times0.2447})^5(1-(1-0.12)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.83)^{4\times0.55\times0.2447})^5(1-(1-0.30)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.67)^{4\times0.23\times0.2729})^5(1-(1-0.83)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.67)^{4\times0.23\times0.2729})^5(1-(1-0.67)^{4\times0.52\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.67)^{4\times0.23\times0.2729})^5(1-(1-0.12)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.67)^{4\times0.23\times0.2729})^5(1-(1-0.30)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$

$$\times(1-(1-(1-(1-0.12)^{4\times0.11\times0.2343})^5(1-(1-0.83)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.12)^{4\times0.11\times0.2343})^5(1-(1-0.67)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.12)^{4\times0.11\times0.2343})^5(1-(1-0.12)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.12)^{4\times0.11\times0.2343})^5(1-(1-0.30)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.30)^{4\times0.11\times0.2481})^5(1-(1-0.83)^{4\times0.55\times0.2447})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.30)^{4\times0.11\times0.2481})^5(1-(1-0.67)^{4\times0.23\times0.2729})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.30)^{4\times0.11\times0.2481})^5(1-(1-0.12)^{4\times0.11\times0.2343})^6)^{\frac{1}{10}}$$
$$\times(1-(1-(1-(1-0.30)^{4\times0.11\times0.2481})^5(1-(1-0.30)^{4\times0.11\times0.2481})^6)^{\frac{1}{10}}\Big)^{\frac{1}{11}}\Big)\Big),$$
$$=(0.1616, 0.5932, 0.4994).$$

The properties, including monotonicity, idempotency and boundedness as provided in Theorems 3.2–3.4 are satisfied by OW*m*FGGHM operators. Moreover, the OW*m*FGGHM operator verify commutative law, which is given as:

Theorem 3.6. *(Commutative Law) Suppose that* $\tilde{\varphi}_{\mathrm{t}}$ *and* $\tilde{\varphi_{\mathrm{t}}'}$ *are two finite families having 'n' mFNs, then*

$$OWmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1,\tilde{\varphi}_2,\ldots,\tilde{\varphi}_n)=OWmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi_1'},\tilde{\varphi_2'},\ldots,\tilde{\varphi_n'}), \tag{3.11}$$

here $\tilde{\varphi_i'}$ *serves as an arbitrary permutation of* $\tilde{\varphi}_i$*.*

Proof. Its proof is straightforward by Definition 3.2. □

One can easily observe from the Definitions 3.1 and 3.2 that both operators (W*m*FGGHM and OW*m*FGGHM) are significant in aggregation of *m*FNs. The key distinction is that W*m*FGGHM AGOs perform aggregation *m*F data without finding ranking orders of given *m*FNs while *m*FHOWG operators submitted for their order. Focusing on the features of above-studied operators, we introduce another general class of AGOs, called H*m*FGGHM operators, which retain the characteristics of both W*m*FGGHM and OW*m*FGGHM operators.

Definition 3.3. *Suppose that* $\tilde{\varphi}_i=(\mathfrak{p}_1\circ\varphi_i,\mathfrak{p}_2\circ\varphi_i,\ldots,\mathfrak{p}_m\circ\varphi_i)$ *be a finite family of 'n' mFNs, then a HmFGGHM operator is defined as*

$$HmFGGHM_{\theta,\Omega}^{\xi,\eta}(\tilde{\varphi}_1,\tilde{\varphi}_2,\ldots,\tilde{\varphi}_n)=\Big(\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}(n\theta_i\lambda_i\tilde{\tilde{\varphi}}_{\delta(i)})^{\xi}\otimes(n\theta_j\lambda_j\tilde{\tilde{\varphi}}_{\delta(j)})^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}, \tag{3.12}$$

here $\theta=(\theta_1,\theta_2,\ldots,\theta_n)$ *consists of the weights, and each* $\theta_{\mathrm{t}}\in(0,1]$ *with* $\sum_{\mathrm{t}=1}^{n}\theta_{\mathrm{t}}=1$*. Moreover, in Eq (3.12),* $\tilde{\tilde{\varphi}}_{\delta(\mathrm{t})}$ *is the biggest mFNs which is given by* $\tilde{\tilde{\varphi}}_{\delta(\mathrm{t})}=(n\Omega_{\mathrm{t}})\tilde{\varphi}_{\mathrm{t}}$*,* $\forall$ t $=1,2,\ldots,n$*, where* $\Omega=(\Omega_1,\Omega_2,\ldots,\Omega_n)$ *includes weights that satisfy* $\Omega_i\in(0,1]$*,* $\sum_{\mathrm{t}=1}^{n}\Omega_i=1$*.*

Observe that for $\theta = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, H$m$FGGHM operators convert into WmFGGHM AGOs. When $\Omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then H$m$FGGHM operator converts into OWmFGGHM operator. Therefore, HmFGGHM operators are generalization of both WmFGGHM and OWmFGGHM operators.

The main result that provide the process of execution of the HmFGGHM operators is given as below:

Theorem 3.7. *Suppose that $\tilde{\varphi}_t = (\mathfrak{p}_1 \circ \varphi_t, \ldots, \mathfrak{p}_m \circ \varphi_t)$ is a family of 'n' mFNs, then their aggregation using an HmFGGHM operator provide again an mFN, which is computed as*

$$HmFGGHM^{\xi,\eta}_{\theta,\Omega}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\tilde{\varphi}}_{\delta(i)})^{\xi} \otimes (n\theta_j\lambda_j\tilde{\tilde{\varphi}}_{\delta(j)})^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}$$

$$= \left(\left(1 - \prod_{i=1,j=i}^{n} (1-(1-(1-p_1 \circ \tilde{\tilde{\varphi}}_{\delta(i)})^{n\lambda_i\theta_i})^{\xi}(1-(1-p_1 \circ \tilde{\tilde{\varphi}}_{\delta(j)})^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}, \ldots,\right.$$

$$\left.\left(1 - \prod_{i=1,j=i}^{n} (1-(1-(1-p_m \circ \tilde{\tilde{\varphi}}_{\delta(i)})^{n\lambda_i\theta_i})^{\xi}(1-(1-p_m \circ \tilde{\tilde{\varphi}}_{\delta(j)})^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\right). \tag{3.13}$$

Proof. It is same as the proof of Theorem 3.1. So, we omit it. □

The following example demonstrates the utilization of the above theorem.

Example 3.3. *Assume that $\tilde{\varphi}_1 = (0.77, 0.33, 0.55)$, $\tilde{\varphi}_2 = (0.22, 0.55, 0.73)$, $\tilde{\varphi}_3 = (0.81, 0.23, 0.11)$ and $\tilde{\varphi}_4 = (0.67, 0.74, 0.93)$ are four 3FNs having weights $\Omega = (0.20, 0.33, 0.18, 0.29)$, and $\theta = (0.33, 0.18, 0.29, 0.20)$ is another weight-vector containing weights from expert. Then, by Definition 3.3, for $\xi = 4$ and $\eta = 7$, we have*

$$HmFGGHM^{4,7}_{\theta,\Omega}(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_4) = \Big(\frac{2}{4(4+1)} \bigoplus_{i=1}^{4} \bigoplus_{j=i}^{4} (4\theta_i\lambda_i\tilde{\tilde{\varphi}}_{\delta(i)})^{4} \otimes (4\theta_j\lambda_j\tilde{\tilde{\varphi}}_{\delta(j)})^{7}\Big)^{\dfrac{1}{4+7}}$$

$$= \left(\left(1 - \prod_{i=1,j=i}^{4} (1-(1-(1-p_1 \circ \tilde{\tilde{\varphi}}_{\delta(i)})^{4\lambda_i\theta_i})^{4}(1-(1-p_1 \circ \tilde{\tilde{\varphi}}_{\delta(j)})^{4\lambda_j\theta_j})^{7})^{\frac{1}{10}}\right)^{\frac{1}{11}}, \ldots,\right.$$

$$\left.\left(1 - \prod_{i=1,j=i}^{4} (1-(1-(1-p_m \circ \tilde{\tilde{\varphi}}_{\delta(i)})^{4\lambda_i\theta_i})^{4}(1-(1-p_m \circ \tilde{\tilde{\varphi}}_{\delta(j)})^{4\lambda_j\theta_j})^{7})^{\frac{1}{10}}\right)^{\frac{1}{11}}\right). \tag{3.14}$$

Next, for the implementation of desired ordered weighted operator, we first determine the values λ_1, λ_2, λ_3, and λ_4 using Eq (3.2) and Definition 2.5 as below:

$$d(\varphi_1, \varphi_2) = \sqrt{\frac{1}{2}((0.77-0.22)^2 + (0.33-0.55)^2 + (0.55-0.73)^2)} = 0.4378,$$

$$d(\varphi_1, \varphi_3) = \sqrt{\frac{1}{2}((0.77-0.81)^2 + (0.33-0.23)^2 + (0.55-0.11)^2)} = 0.3203,$$

$$d(\varphi_1, \varphi_4) = \sqrt{\frac{1}{2}((0.77-0.67)^2 + (0.33-0.74)^2 + (0.55-0.93)^2)} = 0.4016,$$

$$d(\varphi_2,\varphi_1) = \sqrt{\frac{1}{2}((0.22-0.77)^2+(0.55-0.33)^2+(0.73-0.55)^2)} = 0.4378,$$

$$d(\varphi_2,\varphi_3) = \sqrt{\frac{1}{2}((0.22-0.81)^2+(0.55-0.23)^2+(0.73-0.11)^2)} = 0.6461,$$

$$d(\varphi_2,\varphi_4) = \sqrt{\frac{1}{2}((0.22-0.67)^2+(0.55-0.74)^2+(0.73-0.93)^2)} = 0.3732,$$

$$d(\varphi_3,\varphi_1) = \sqrt{\frac{1}{2}((0.81-0.77)^2+(0.23-0.33)^2+(0.11-0.55)^2)} = 0.3203,$$

$$d(\varphi_3,\varphi_2) = \sqrt{\frac{1}{2}((0.81-0.22)^2+(0.23-0.55)^2+(0.11-0.73)^2)} = 0.6461,$$

$$d(\varphi_3,\varphi_4) = \sqrt{\frac{1}{2}((0.81-0.67)^2+(0.23-0.74)^2+(0.11-0.93)^2)} = 0.6900,$$

$$d(\varphi_4,\varphi_1) = \sqrt{\frac{1}{2}((0.67-0.77)^2+(0.74-0.33)^2+(0.93-0.55)^2)} = 0.4016,$$

$$d(\varphi_4,\varphi_2) = \sqrt{\frac{1}{2}((0.67-0.22)^2+(0.74-0.55)^2+(0.93-0.73)^2)} = 0.3732,$$

$$d(\varphi_4,\varphi_3) = \sqrt{\frac{1}{2}((0.67-0.81)^2+(0.74-0.23)^2+(0.93-0.11)^2)} = 0.6900.$$

This implies

$$T(\varphi_1) = 1.8403,\ \ T(\varphi_2) = 1.5429,\ \ T(\varphi_3) = 1.3436\ \ \textit{and}\ \ T(\varphi_4) = 1.5352.$$

Then, we compute the value of each λ_i as follows:

$$\lambda_1 = 0.2768,\quad \lambda_2 = 0.2478,\quad \lambda_3 = 0.2284\quad \textit{and}\quad \lambda_4 = 0.2470.$$

By putting all these values in Eq (3.10), we get

$$\begin{aligned}\tilde{\tilde{\varphi}}_1 = \Bigg(&\Big(1-(1-(1-(1-0.77)^{4\times0.2768\times0.33})^4(1-(1-0.77)^{4\times0.2768\times0.33})^7)^{\frac{1}{10}}\Big)^{\frac{1}{11}},\\ &\Big(1-(1-(1-(1-0.33)^{4\times0.2768\times0.33})^4(1-(1-0.33)^{4\times0.2768\times0.33})^7)^{\frac{1}{10}}\Big)^{\frac{1}{11}},\\ &\Big(1-(1-(1-(1-0.55)^{4\times0.2768\times0.33})^4(1-(1-0.55)^{4\times0.2768\times0.33})^7)^{\frac{1}{10}}\Big)^{\frac{1}{11}}\Bigg)\\ =\ &(0.3370, 0.1104, 0.2053).\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{\tilde{\varphi}}_2 = \Bigg(&\Big(1-(1-(1-(1-0.22)^{4\times0.2478\times0.29})^4(1-(1-0.22)^{4\times0.2478\times0.29})^7)^{\frac{1}{10}}\Big)^{\frac{1}{11}},\\ &\Big(1-(1-(1-(1-0.55)^{4\times0.2478\times0.29})^4(1-(1-0.55)^{4\times0.2478\times0.29})^7)^{\frac{1}{10}}\Big)^{\frac{1}{11}},\end{aligned}$$

$$\left(1-(1-(1-(1-0.73)^{4\times0.2478\times0.29})^4(1-(1-0.73)^{4\times0.2478\times0.29})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}}\Bigg)$$
$$=(0.0559, 0.1664, 0.2544).$$

$$\tilde{\tilde{\varphi}}_3=\Bigg(\left(1-(1-(1-(1-0.81)^{4\times0.2284\times0.18})^4(1-(1-0.81)^{4\times0.2284\times0.18})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}},$$
$$\left(1-(1-(1-(1-0.23)^{4\times0.2284\times0.18})^4(1-(1-0.23)^{4\times0.2284\times0.18})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}},$$
$$\left(1-(1-(1-(1-0.11)^{4\times0.2284\times0.18})^4(1-(1-0.11)^{4\times0.2284\times0.18})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}}\Bigg)$$
$$=(0.1938, 0.0354, 0.0000).$$

$$\tilde{\tilde{\varphi}}_4=\Bigg(\left(1-(1-(1-(1-0.67)^{4\times0.2470\times0.20})^4(1-(1-0.67)^{4\times0.2470\times0.20})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}},$$
$$\left(1-(1-(1-(1-0.74)^{4\times0.2470\times0.20})^4(1-(1-0.74)^{4\times0.2470\times0.20})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}},$$
$$\left(1-(1-(1-(1-0.93)^{4\times0.2470\times0.20})^4(1-(1-0.93)^{4\times0.2470\times0.20})^7)^{\frac{1}{10}}\right)^{\frac{1}{11}}\Bigg)$$
$$=(0.1597, 0.1896, 0.3315).$$

Now the scores of these 3FNs for $\xi = 4$ and $\eta = 7$ are calculated as below:

$$\mathfrak{U}(\tilde{\tilde{\varphi}}_1)=\frac{0.3370+0.1104+0.2053}{3}=0.2177,$$
$$\mathfrak{U}(\tilde{\tilde{\varphi}}_2)=\frac{0.0559+0.1664+0.2544}{3}=0.1589,$$
$$\mathfrak{U}(\tilde{\tilde{\varphi}}_3)=\frac{0.1938+0.0354+0.0000}{3}=0.0764,$$
$$\mathfrak{U}(\tilde{\tilde{\varphi}}_4)=\frac{0.1597+0.1896+0.3315}{3}=0.2269.$$

Since, $\mathfrak{U}(\tilde{\tilde{\varphi}}_4) > \mathfrak{U}(\tilde{\tilde{\varphi}}_1) > \mathfrak{U}(\tilde{\tilde{\varphi}}_2) > \mathfrak{U}(\tilde{\tilde{\varphi}}_3)$, *thus*

$$\tilde{\tilde{\varphi}}_{\delta(1)}=\tilde{\tilde{\varphi}}_4=(0.1597, 0.1896, 0.3315), \qquad \tilde{\tilde{\varphi}}_{\delta(2)}=\tilde{\tilde{\varphi}}_1=(0.3370, 0.1104, 0.2053),$$
$$\tilde{\tilde{\varphi}}_{\delta(3)}=\tilde{\tilde{\varphi}}_2=(0.0559, 0.1664, 0.2544), \qquad \tilde{\tilde{\varphi}}_{\delta(4)}=\tilde{\tilde{\varphi}}_3=(0.1938, 0.0354, 0.0000).$$

Reconsider Eq (3.2) and Definition 2.5 to compute the values of λ_1^*, λ_2^*, λ_3^* and λ_3^* as follows:

$$d(\varphi_{\delta(1)},\varphi_{\delta(2)})=\sqrt{\frac{1}{2}((0.1597-0.3370)^2+(0.1896-0.1104)^2+(0.3315-0.2053)^2)}=0.1638,$$
$$d(\varphi_{\delta(1)},\varphi_{\delta(2)})=\sqrt{\frac{1}{2}((0.1597-0.0559)^2+(0.1896-0.1664)^2+(0.3315-0.2544)^2)}=0.0929,$$
$$d(\varphi_{\delta(1)},\varphi_{\delta(2)})=\sqrt{\frac{1}{2}((0.1597-0.1938)^2+(0.1896-0.0354)^2+(0.3315-0.0000)^2)}=0.2596,$$
$$d(\varphi_{\delta(2)},\varphi_{\delta(1)})=\sqrt{\frac{1}{2}((0.3370-0.1597)^2+(0.1104-0.1896)^2+(0.2053-0.3315)^2)}=0.1638,$$

$$d(\varphi_{\delta(2)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.3370-0.0559)^2+(0.1104-0.1664)^2+(0.2053-0.2544)^2)} = 0.2056,$$

$$d(\varphi_{\delta(2)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.3370-0.1938)^2+(0.1104-0.0354)^2+(0.2053-0.0000)^2)} = 0.1848,$$

$$d(\varphi_{\delta(3)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.0559-0.1597)^2+(0.1664-0.1896)^2+(0.2544-0.3315)^2)} = 0.0929,$$

$$d(\varphi_{\delta(3)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.0559-0.3370)^2+(0.1664-0.1104)^2+(0.2544-0.2053)^2)} = 0.2056,$$

$$d(\varphi_{\delta(3)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.0559-0.1938)^2+(0.1664-0.0354)^2+(0.2544-0.0000)^2)} = 0.2246,$$

$$d(\varphi_{\delta(4)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.1938-0.1597)^2+(0.0354-0.1896)^2+(0.0000-0.3315)^2)} = 0.2596,$$

$$d(\varphi_{\delta(4)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.1938-0.3370)^2+(0.0354-0.1104)^2+(0.0000-0.2053)^2)} = 0.1848,$$

$$d(\varphi_{\delta(4)},\varphi_{\delta(1)}) = \sqrt{\frac{1}{2}((0.1938-0.0559)^2+(0.0354-0.1664)^2+(0.0000-0.2544)^2)} = 0.2246.$$

Using these calculated distances values, we have

$$\lambda_1^* = 0.2536, \quad \lambda_2^* = 0.2508, \quad \lambda_3^* = 0.2531 \quad \textit{and} \quad \lambda_4^* = 0.2425.$$

Finally, by putting all these values in Eq (3.10), we get

$$\begin{aligned}
&HmFGGHM^{4,7}_{\theta,\Omega}(\tilde{\varphi}_1,\tilde{\varphi}_2,\tilde{\varphi}_3,\tilde{\varphi}_4)\\
&= \Bigg(\Bigg((1-(1-(1-(1-0.1597)^{4\times0.20\times0.2536})^4(1-(1-0.1597)^{4\times0.20\times0.2536})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1597)^{4\times0.20\times0.2536})^4(1-(1-0.3370)^{4\times0.33\times0.2508})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1597)^{4\times0.20\times0.2536})^4(1-(1-0.0559)^{4\times0.18\times0.2531})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1597)^{4\times0.20\times0.2536})^4(1-(1-0.1938)^{4\times0.29\times0.2425})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3370)^{4\times0.33\times0.2508})^4(1-(1-0.1597)^{4\times0.20\times0.2536})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3370)^{4\times0.33\times0.2508})^4(1-(1-0.3370)^{4\times0.33\times0.2508})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3370)^{4\times0.33\times0.2508})^4(1-(1-0.0559)^{4\times0.18\times0.2531})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3370)^{4\times0.33\times0.2508})^4(1-(1-0.1938)^{4\times0.29\times0.2425})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.0559)^{4\times0.18\times0.2531})^4(1-(1-0.1597)^{4\times0.20\times0.2536})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.0559)^{4\times0.18\times0.2531})^4(1-(1-0.3370)^{4\times0.33\times0.2508})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.0559)^{4\times0.18\times0.2531})^4(1-(1-0.0559)^{4\times0.18\times0.2531})^7)^{\frac{1}{10}}\\
&\times(1-(1-(1-0.0559)^{4\times0.18\times0.2531})^4(1-(1-0.1938)^{4\times0.29\times0.2425})^7)^{\frac{1}{10}}\\
&\times((1-(1-(1-0.1938)^{4\times0.29\times0.2425})^4(1-(1-0.1597)^{4\times0.20\times0.2536})^7)^{\frac{1}{10}}
\end{aligned}$$

$$
\begin{aligned}
&\times(1-(1-(1-0.1938)^{4\times0.29\times0.2425})^{4}(1-(1-0.3370)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1938)^{4\times0.29\times0.2425})^{4}(1-(1-0.0559)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times((1-(1-(1-0.1938)^{4\times0.29\times0.2425})^{4}(1-(1-0.1938)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\Big)^{\frac{1}{11}},\\
\Big(&(1-(1-(1-(1-0.1896)^{4\times0.20\times0.2536})^{4}(1-(1-0.1896)^{4\times0.20\times0.2538})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1896)^{4\times0.20\times0.2536})^{4}(1-(1-0.1104)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1896)^{4\times0.20\times0.2536})^{4}(1-(1-0.1664)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1896)^{4\times0.20\times0.2536})^{4}(1-(1-0.0354)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1104)^{4\times0.33\times0.2508})^{4}(1-(1-0.1896)^{4\times0.20\times0.2536})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1104)^{4\times0.33\times0.2508})^{4}(1-(1-0.1104)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1104)^{4\times0.33\times0.2508})^{4}(1-(1-0.1664)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1104)^{4\times0.33\times0.2508})^{4}(1-(1-0.0354)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1664)^{4\times0.18\times0.2531})^{4}(1-(1-0.1896)^{4\times0.20\times0.2536})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1664)^{4\times0.18\times0.2531})^{4}(1-(1-0.1104)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1664)^{4\times0.18\times0.2531})^{4}(1-(1-0.1664)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.1664)^{4\times0.18\times0.2531})^{4}(1-(1-0.0354)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\\
&\times((1-(1-(1-0.0354)^{4\times0.29\times0.2425})^{4}(1-(1-0.1896)^{4\times0.20\times0.2536})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.0354)^{4\times0.29\times0.2425})^{4}(1-(1-0.1104)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.0354)^{4\times0.29\times0.2425})^{4}(1-(1-0.1664)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times((1-(1-(1-0.0354)^{4\times0.29\times0.2425})^{4}(1-(1-0.0354)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\Big)^{\frac{1}{11}},\\
\Big(&(1-(1-(1-(1-0.3315)^{4\times0.20\times0.2536})^{4}(1-(1-0.3315)^{4\times0.20\times0.2538})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3315)^{4\times0.20\times0.2536})^{4}(1-(1-0.2053)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3315)^{4\times0.20\times0.2536})^{4}(1-(1-0.2544)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.3315)^{4\times0.20\times0.2536})^{4}(1-(1-0.0000)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.2053)^{4\times0.33\times0.2508})^{4}(1-(1-0.3315)^{4\times0.20\times0.2536})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.2053)^{4\times0.33\times0.2508})^{4}(1-(1-0.2053)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.2053)^{4\times0.33\times0.2508})^{4}(1-(1-0.2544)^{4\times0.18\times0.2531})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.2053)^{4\times0.33\times0.2508})^{4}(1-(1-0.0000)^{4\times0.29\times0.2425})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.2544)^{4\times0.18\times0.2531})^{4}(1-(1-0.3315)^{4\times0.20\times0.2536})^{7})^{\frac{1}{10}}\\
&\times(1-(1-(1-0.2544)^{4\times0.18\times0.2531})^{4}(1-(1-0.2053)^{4\times0.33\times0.2508})^{7})^{\frac{1}{10}}
\end{aligned}
$$

$$\times (1-(1-(1-0.2544)^{4\times0.18\times0.2531})^4(1-(1-0.2544)^{4\times0.18\times0.2531})^7)^{\frac{1}{10}}$$
$$\times (1-(1-(1-0.2544)^{4\times0.18\times0.2531})^4(1-(1-0.0000)^{4\times0.29\times0.2425})^7)^{\frac{1}{10}}$$
$$\times ((1-(1-(1-0.0000)^{4\times0.29\times0.2425})^4(1-(1-0.3315)^{4\times0.20\times0.2536})^7)^{\frac{1}{10}}$$
$$\times (1-(1-(1-0.0000)^{4\times0.29\times0.2425})^4(1-(1-0.2053)^{4\times0.33\times0.2508})^7)^{\frac{1}{10}}$$
$$\times (1-(1-(1-0.0000)^{4\times0.29\times0.2425})^4(1-(1-0.2544)^{4\times0.18\times0.2531})^7)^{\frac{1}{10}}$$
$$\times ((1-(1-(1-0.0000)^{4\times0.29\times0.2425})^4(1-(1-0.0000)^{4\times0.29\times0.2425})^7)^{\frac{1}{10}}\Big)^{\frac{1}{11}}\Big),$$
$$= (0.1037, 0.0354, 0.0709).$$

4. Application to MCDM using aggregation *m*F information

In this section, we first provide an algorithm based on the suggested weighted *m*F power generalized geometric Heronian mean (or W*m*FPGGHM) AGOs. Then, we apply it on a practical case study problem, that is, selection of best transportation plan in Saudi Arabia. For a better understanding, we display the steps of Algorithm 1 in a flowchart diagram (see Figure 1).

Algorithm 1 Selecting a suitable alternative using W*m*FPGGHM operators.

Step I(Input):

(i) a universe of discourse containing 'n' alternatives,
(ii) a set of attributes $\mathcal{N}_k$ where k varies from 1 to $\mathfrak{t}$,
(iii) a $\mathfrak{t}$-tuple containing weights $\theta_1, \theta_2, \ldots, \theta_\mathfrak{t}$ where $\sum_{k=1}^{\mathfrak{t}} \theta_k = 1$,
(iv) an *m*F decision matrix regarding each alternative, which is given as:

$$\tilde{\mathfrak{Q}} = (\tilde{\mathfrak{r}}_{is})_{n\times\mathfrak{t}} = (\mathfrak{p}_1 \circ \varphi_{is}, \mathfrak{p}_2 \circ \varphi_{is}, \ldots, \mathfrak{p}_m \circ \varphi_{is})_{n\times\mathfrak{t}}.$$

Step II: Compute the aggregated/preference value ($\tilde{\mathfrak{r}}_s$) for every object of the universe of discourse using W*m*FGGHM operators as provided by Eq (3.3), which can be calculated by the following formula:

$$\tilde{\mathfrak{r}}_s = WmFGGHM_\theta^{\xi,\eta}(\tilde{\varphi}_{s1}, \tilde{\varphi}_{s2}, \ldots, \tilde{\varphi}_{s\mathfrak{t}}) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_i)^\xi \otimes (n\theta_j\lambda_j\tilde{\varphi}_j)^\eta\Big)^{\dfrac{1}{\xi+\eta}}$$
$$= \left(\left(1 - \prod_{i=1,j=i}^{n} (1-(1-(1-p_1\circ\varphi_i)^{n\lambda_i\theta_i})^\xi(1-(1-p_1\circ\varphi_j)^{n\lambda_j\theta_j})^\eta)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}, \ldots,\right.$$
$$\left.\left(1 - \prod_{i=1,j=i}^{n} (1-(1-(1-p_m\circ\varphi_i)^{n\lambda_i\theta_i})^\xi(1-(1-p_m\circ\varphi_j)^{n\lambda_j\theta_j})^\eta)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\right). \quad (4.1)$$

Step III: By Definition 2.2, find the final scores $\mathfrak{U}(\tilde{\mathfrak{r}}_s)$ of each alternative of the universal set.

Step IV: Last, rank the available objects in descending order regarding their score values.

Output: The option having highest position in ranking will be the optimal choice. In the case if multiple options have the same maximum score value, then anyone of them could be selected as the best option.

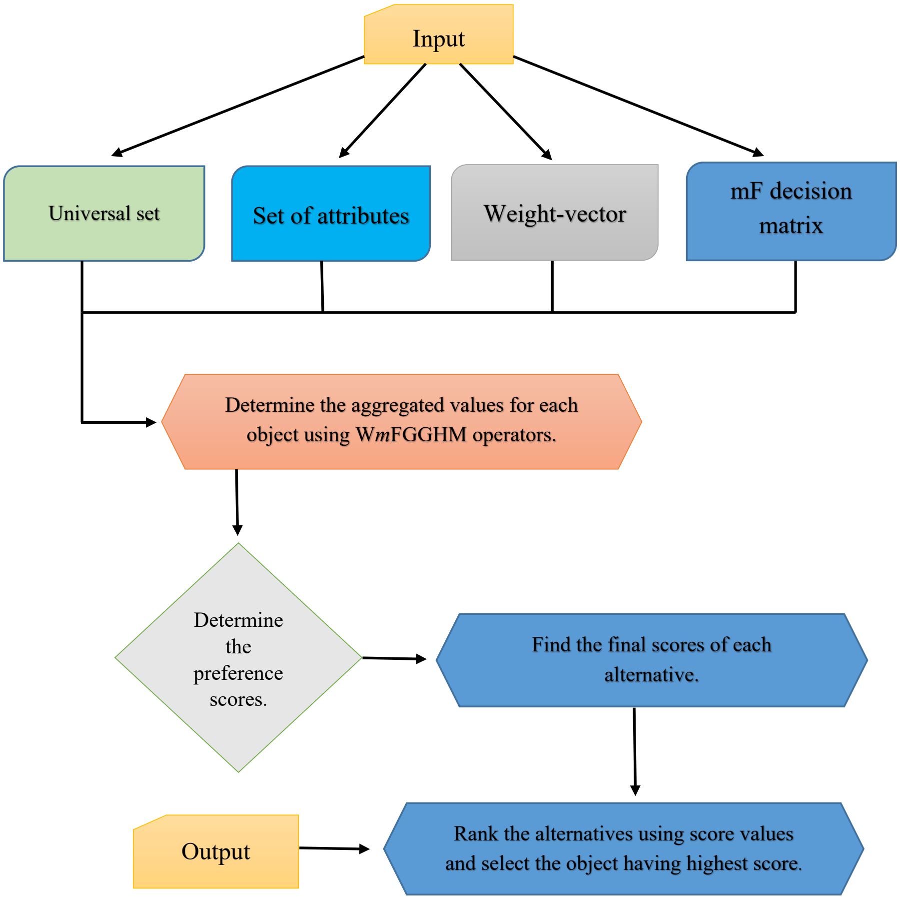

Figure 1. Flowchart diagram of Algorithm 1.

4.1. *Urban transportation management: a case study of Saudi Arabia*

Urban transportation management plays a crucial role in the environmental and economic stability of every country. Efficient transportation is essential for enabling the movement of people and goods for trade. Well-designed transportation systems provide benefits such as improved accessibility and reduced congestion, which, in turn, contribute to a higher quality of urban life. A significant contributor to greenhouse gas emissions in urban areas is vehicle usage. To reduce the carbon footprint, developed countries promote public transportation and cycling. With the population increasing daily, effective transportation is becoming a critical issue in major cities. Poor transportation management is a primary cause of overcrowding in urban areas. Therefore, proper investment in efficient solutions is necessary to address the challenges faced in urban transportation.

In the last decade, a rapid increase in urbanization and vehicle ownership has been observed in Saudi Arabia, presenting a major challenge for sustainable transportation management, particularly in large cities like Riyadh. As the capital of Saudi Arabia, Riyadh's population has now exceeded seven million. Rapid urbanization is a primary cause of road accidents, air pollution, and traffic congestion. To address these issues, the government has launched various projects, such as the Riyadh Metro, to support public transportation. However, the available transportation options are insufficient to meet public needs. Consequently, the government of Saudi Arabia has proposed additional initiatives to manage Riyadh's transportation system. For example, they are enhancing public transportation infrastructure by designing various alternative transport modes to attract the public, such as advancing dedicated cycling lanes and expanding bus routes. Additionally, the development of pedestrian-friendly footpaths is being fully considered to enable people to move from one area to another within the city without needing transportation. Furthermore, the government of Saudi Arabia is addressing these transportation-related issues by announcing the construction of a new metro train in Riyadh. They have twelve route plans in mind; however, to determine the best route for the metro train, they decided to invite senior experts in this field for optimal solutions. After a detailed discussion, the experts agreed to consider the following favorable parameters for selecting the optimal route.

$\mathcal{N}_1$ denotes the 'Safety Issues'.

$\mathcal{N}_2$ denotes the 'Environmental Issues'.

$\mathcal{N}_3$ denotes the 'Travel Demand Forecasting'.

$\mathcal{N}_4$ denotes the 'Transportation Cost'.

To help you better understand how mF numbers are constructed some additional sub parameters are listed below.

- The parameter 'Safety Issues' consists of traffic congestion, inadequate infrastructure, lack of proper safety regulations, and enforcement.
- The parameter 'Environmental Issues' consists of air pollution, noise, and temperature.
- The parameter 'Travel Demand Forecasting' consists of predict travel behavior and resulting demand for a certain future time period based on the nature of transportation system, the number and character of trip-makers, and assumptions dealing with land-use.
- The parameter 'Transportation Cost' includes the expenses related to the transportation of raw materials, finished products, and employees.

The final evaluations of all twelve route plans with respect to the favorable parameters is described in Table 2 by 3F decision matrix.

Table 2. 3F decision matrix.

	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$
$\mathcal{S}_1$	(0.35, 0.57, 0.68)	(0.46, 0.97, 0.55)	(0.81, 0.15, 0.44)	(0.17, 0.24, 0.12)
$\mathcal{S}_2$	(0.65, 0.47, 0.78)	(0.44, 0.18, 0.67)	(0.68, 0.74, 0.20)	(0.70, 0.16, 0.19)
$\mathcal{S}_3$	(0.18, 0.41, 0.39)	(0.44, 0.38, 0.74)	(0.67, 0.58, 0.96)	(0.13, 0.15, 0.27)
$\mathcal{S}_4$	(0.76, 0.54, 0.45)	(0.29, 0.37, 0.66)	(0.78, 0.86, 0.75)	(0.57, 0.54, 0.66)
$\mathcal{S}_5$	(0.68, 0.75, 0.47)	(0.37, 0.52, 0.55)	(0.79, 0.47, 0.88)	(0.76, 0.92, 0.16)
$\mathcal{S}_6$	(0.45, 0.67, 0.74)	(0.66, 0.78, 0.67)	(0.68, 0.54, 0.66)	(0.71, 0.78, 0.79)
$\mathcal{S}_7$	(0.58, 0.73, 0.77)	(0.78, 0.17, 0.33)	(0.55, 0.49, 0.28)	(0.17, 0.16, 0.15)
$\mathcal{S}_8$	(0.27, 0.46, 0.42)	(0.85, 0.89, 0.11)	(0.48, 0.12, 0.15)	(0.66, 0.77, 0.90)
$\mathcal{S}_9$	(0.76, 0.26, 0.22)	(0.51, 0.49, 0.41)	(0.98, 0.62, 0.95)	(0.76, 0.97, 0.91)
$\mathcal{S}_{10}$	(0.59, 0.96, 0.22)	(0.57, 0.79, 0.11)	(0.28, 0.42, 0.75)	(0.96, 0.27, 0.89)
$\mathcal{S}_{11}$	(0.67, 0.16, 0.12)	(0.15, 0.59, 0.11)	(0.18, 0.21, 0.51)	(0.61, 0.71, 0.93)
$\mathcal{S}_{12}$	(0.79, 0.46, 0.32)	(0.65, 0.92, 0.81)	(0.88, 0.20, 0.50)	(0.96, 0.73, 0.90)

According to the importance of attributes, the experts assign a suitable weight to each attribute as below:

$$\theta_1 = 0.24,\ \theta_2 = 0.35,\ \theta_3 = 0.10,\ \theta_4 = 0.31.$$

Clearly $\sum_{t=1}^{4} \theta_t = 1$. We now compute the most suitable ranking of the available alternatives under the WmFGGHM operators:

Step II: For $\xi = 3$ and $\eta = 5$, by implementing the WmFGGHM operator as provided in Eq (3.3), we compute the preference value $\tilde{\mathfrak{r}}_s$ $(s = 1, 2, \ldots, 12)$ of each alternative as below:

$$\begin{aligned}
\tilde{\mathfrak{r}}_1 &= (0.1576, 0.5271, 0.2237), & \tilde{\mathfrak{r}}_2 &= (0.2566, 0.1215, 0.2858),\\
\tilde{\mathfrak{r}}_3 &= (0.1503, 0.1349, 0.3123), & \tilde{\mathfrak{r}}_4 &= (0.2472, 0.1990, 0.2722),\\
\tilde{\mathfrak{r}}_5 &= (0.2798, 0.4084, 0.2037), & \tilde{\mathfrak{r}}_6 &= (0.2879, 0.3617, 0.3346),\\
\tilde{\mathfrak{r}}_7 &= (0.3191, 0.1879, 0.2099), & \tilde{\mathfrak{r}}_8 &= (0.3637, 0.4125, 0.3677),\\
\tilde{\mathfrak{r}}_9 &= (0.3263, 0.4937, 0.3974), & \tilde{\mathfrak{r}}_{10} &= (0.4417, 0.4592, 0.3349),\\
\tilde{\mathfrak{r}}_{11} &= (0.2048, 0.2499, 0.3851), & \tilde{\mathfrak{r}}_{12} &= (0.4956, 0.4470, 0.4281).
\end{aligned}$$

Step III: Determine the score value $\mathfrak{U}(\tilde{\mathfrak{r}}_s)$ of each above calculated 3FN by Definition 2.2 as follows:

$$\begin{aligned}
&\mathfrak{U}(\tilde{\mathfrak{r}}_1) = 0.3028,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_2) = 0.2213,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_3) = 0.1992,\\
&\mathfrak{U}(\tilde{\mathfrak{r}}_4) = 0.2395,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_5) = 0.2973,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_6) = 0.3281,\\
&\mathfrak{U}(\tilde{\mathfrak{r}}_7) = 0.2390,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_8) = 0.3813,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_9) = 0.4058,\\
&\mathfrak{U}(\tilde{\mathfrak{r}}_{10}) = 0.4119,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_{11}) = 0.2799,\ \ \mathfrak{U}(\tilde{\mathfrak{r}}_{12}) = 0.4569.
\end{aligned}$$

Step IV: Next, find the ranking of alternatives with respect to scores values $\mathfrak{U}(\tilde{\mathfrak{r}}_i)$, where $i = 1, 2, \ldots, 12$ as:

$$\mathcal{S}_{12} > \mathcal{S}_{10} > \mathcal{S}_9 > \mathcal{S}_8 > \mathcal{S}_6 > \mathcal{S}_1 > \mathcal{S}_5 > \mathcal{S}_{11} > \mathcal{S}_4 > \mathcal{S}_7 > \mathcal{S}_2 > \mathcal{S}_3.$$

Output: The alternative $\mathcal{S}_{12}$ has highest score, thus, it is the best route for the new urban transportation project.

5. Discussion

One of the significant factors in urban transportation is the route selection, which is influenced by a number of factors, such as technological, environmental, operational, and socio-economics. It is clearly a MCDM problem due to the involvement of multiple attributes, and to obtain an optimal result, it is important to deeply evaluate each attribute, which is only possible by applying mF set theory. Besides, aggregation operators (AGOs) are designed to accumulate information in systems that require the integration of multiple datasets to accomplish a specific objective. However, these days, weighted AGOs are playing a significant role in MCDM situations by integrating the actions of different attributes, along with their partial preferences, into a single outcome. Additionally, the mF sets provide an appropriate mechanism for handling multi-dimensional parameterized information. Since different AGOs yield different outcomes when applied to a dataset, this variability presents a challenge in preexisting mF set-based weighted averaging and geometric AGOs. To address this issue, we introduced WmFPGGHM operators.

Further, to validate this comparative analysis, Section 4.1 (Selection of the best route plan in urban transportation management: a case study of Saudi Arabia) is reconsidered using preexisting AGOs-based MCDM methodologies. Consequently, the results of the proposed WmFGGHM AGOs are compared with those obtained by applying the preexisting *m*F Yager weighted averaging (*m*FYWA), *m*F Yager weighted geometric (*m*FYWG), *m*F Dombi weighted averaging (*m*FDWA), *m*F Dombi weighted geometric (*m*FDWG) operators, *m*F Aczel-Alsina weighted averaging (*m*FAAWA), *m*F Aczel-Alsina weighted geometric (*m*FAAWG) AGOs, and *m*F TOPSIS approach. Tables 3 and 4 provide the final score values and objects' rankings, respectively. This comparison is graphically represented in Figure 2. It is clear from Tables 3 and 4, and Figure 2 that the outcomes determined by applying the proposed AGOs are largely consistent with those obtained using the existing *m*F set-based operators, that is, the optimal objects obtained by implementing the proposed AGOs and existing approaches is same, while there is a minor change in the rankings of sub-optimal alternatives. It is important to note that the existing *m*F set-based models as discussed above, fail to depict the interrelationship among the input values, while the suggested AGOs hold this feature. To pose a better comparison analysis regarding features between the innovative W*m*FPGGHM AGOs and some HM based operators, a feature-based comparison is provided by Table 5.

Table 3. Comparison between the scores of the developed W*m*FPGGHM AGOs with *m*FYWA [41] and *m*FYWG [41], *m*FDWA [39], *m*FDWG [39], *m*FAAWA [42], *m*FAAWG [42] operators, and *m*F TOPSIS approach [5].

Score Values\ AGOs	*m*FYWA	*m*FYWG	*m*FDWA	*m*FDWG
$\mathfrak{U}(\tilde{\mathfrak{r}}_1)$	0.6504	0.4437	0.7383	0.7620
$\mathfrak{U}(\tilde{\mathfrak{r}}_2)$	0.6060	0.4638	0.6441	0.6829
$\mathfrak{U}(\tilde{\mathfrak{r}}_3)$	0.5399	0.3915	0.7114	0.7604
$\mathfrak{U}(\tilde{\mathfrak{r}}_4)$	0.6403	0.5429	0.6939	0.7674
$\mathfrak{U}(\tilde{\mathfrak{r}}_5)$	0.7006	0.5547	0.7943	0.5859
$\mathfrak{U}(\tilde{\mathfrak{r}}_6)$	0.7120	0.6832	0.7250	0.3532
$\mathfrak{U}(\tilde{\mathfrak{r}}_7)$	0.6191	0.4115	0.6740	0.8028
$\mathfrak{U}(\tilde{\mathfrak{r}}_8)$	0.7554	0.5357	0.8380	0.7527
$\mathfrak{U}(\tilde{\mathfrak{r}}_9)$	0.7877	0.5649	0.9412	0.5823
$\mathfrak{U}(\tilde{\mathfrak{r}}_{10})$	0.7843	0.5298	0.9085	0.6859
$\mathfrak{U}(\tilde{\mathfrak{r}}_{11})$	0.6536	0.4372	0.7115	0.8162
$\mathfrak{U}(\tilde{\mathfrak{r}}_{12})$	0.8167	0.6887	0.9517	0.8227
Score Values\ AGOs	*m*FAAWA	*m*FAAWG	*m*F TOPSIS	Proposed W*m*FPGGHM
$\mathfrak{U}(\tilde{\mathfrak{r}}_1)$	0.6793	0.2709	0.4037	0.3028
$\mathfrak{U}(\tilde{\mathfrak{r}}_2)$	0.5979	0.3478	0.3354	0.2213
$\mathfrak{U}(\tilde{\mathfrak{r}}_3)$	0.5511	0.2780	0.4523	0.1992
$\mathfrak{U}(\tilde{\mathfrak{r}}_4)$	0.6487	0.4783	0.6314	0.2395
$\mathfrak{U}(\tilde{\mathfrak{r}}_5)$	0.7276	0.4527	0.3102	0.2973
$\mathfrak{U}(\tilde{\mathfrak{r}}_6)$	0.7145	0.6603	0.4020	0.3281
$\mathfrak{U}(\tilde{\mathfrak{r}}_7)$	0.6142	0.2494	0.6051	0.2390
$\mathfrak{U}(\tilde{\mathfrak{r}}_8)$	0.7873	0.3262	0.2680	0.3813
$\mathfrak{U}(\tilde{\mathfrak{r}}_9)$	0.8742	0.4592	0.5076	0.4058
$\mathfrak{U}(\tilde{\mathfrak{r}}_{10})$	0.8508	0.3669	0.3226	0.4119
$\mathfrak{U}(\tilde{\mathfrak{r}}_{11})$	0.6711	0.2294	0.6022	0.2799
$\mathfrak{U}(\tilde{\mathfrak{r}}_{12})$	0.8779	0.6612	0.6842	0.4569

Table 4. Comparison between the rankings of the developed W*m*FPGGHM operators with *m*FYWA [41] and *m*FYWG [41], *m*FDWA [39], *m*FDWG [39], *m*FAAWA [42], *m*FAAWG [42] AGOs, and *m*F TOPSIS approach [5].

AGOs/MCDM method	Ranking order	Choice
W*m*FPGGHM	$\mathcal{S}_{12} > \mathcal{S}_{10} > \mathcal{S}_{9} > \mathcal{S}_{8} > \mathcal{S}_{6} > \mathcal{S}_{1} > \mathcal{S}_{5} > \mathcal{S}_{11} > \mathcal{S}_{4} > \mathcal{S}_{7} > \mathcal{S}_{2} > \mathcal{S}_{3}$	$\mathcal{S}_{12}$
*m*FYWA	$\mathcal{S}_{12} > \mathcal{S}_{9} > \mathcal{S}_{10} > \mathcal{S}_{8} > \mathcal{S}_{6} > \mathcal{S}_{5} > \mathcal{S}_{11} > \mathcal{S}_{1} > \mathcal{S}_{4} > \mathcal{S}_{2} > \mathcal{S}_{7} > \mathcal{S}_{3}$	$\mathcal{S}_{12}$
*m*FYWG	$\mathcal{S}_{12} > \mathcal{S}_{6} > \mathcal{S}_{9} > \mathcal{S}_{5} > \mathcal{S}_{4} > \mathcal{S}_{8} > \mathcal{S}_{10} > \mathcal{S}_{2} > \mathcal{S}_{1} > \mathcal{S}_{11} > \mathcal{S}_{97} > \mathcal{S}_{3}$	$\mathcal{S}_{12}$
*m*FAAWA	$\mathcal{S}_{12} > \mathcal{S}_{9} > \mathcal{S}_{10} > \mathcal{S}_{8} > \mathcal{S}_{5} > \mathcal{S}_{6} > \mathcal{S}_{1} > \mathcal{S}_{11} > \mathcal{S}_{4} > \mathcal{S}_{7} > \mathcal{S}_{2} > \mathcal{S}_{3}$	$\mathcal{S}_{12}$
*m*FAAWG	$\mathcal{S}_{12} > \mathcal{S}_{6} > \mathcal{S}_{4} > \mathcal{S}_{9} > \mathcal{S}_{5} > \mathcal{S}_{10} > \mathcal{S}_{2} > \mathcal{S}_{8} > \mathcal{S}_{3} > \mathcal{S}_{1} > \mathcal{S}_{7} > \mathcal{S}_{11}$	$\mathcal{S}_{12}$
*m*FDWA	$\mathcal{S}_{12} > \mathcal{S}_{10} > \mathcal{S}_{9} > \mathcal{S}_{8} > \mathcal{S}_{5} > \mathcal{S}_{1} > \mathcal{S}_{6} > \mathcal{S}_{11} > \mathcal{S}_{3} > \mathcal{S}_{4} > \mathcal{S}_{7} > \mathcal{S}_{2}$	$\mathcal{S}_{12}$
*m*FDWG	$\mathcal{S}_{12} > \mathcal{S}_{11} > \mathcal{S}_{7} > \mathcal{S}_{4} > \mathcal{S}_{1} > \mathcal{S}_{3} > \mathcal{S}_{8} > \mathcal{S}_{10} > \mathcal{S}_{2} > \mathcal{S}_{5} > \mathcal{S}_{9} > \mathcal{S}_{6}$	$\mathcal{S}_{12}$
*m*F TOPSIS	$\mathcal{S}_{12} > \mathcal{S}_{4} > \mathcal{S}_{7} > \mathcal{S}_{11} > \mathcal{S}_{9} > \mathcal{S}_{3} > \mathcal{S}_{1} > \mathcal{S}_{6} > \mathcal{S}_{2} > \mathcal{S}_{10} > \mathcal{S}_{5} > \mathcal{S}_{8}$	$\mathcal{S}_{12}$

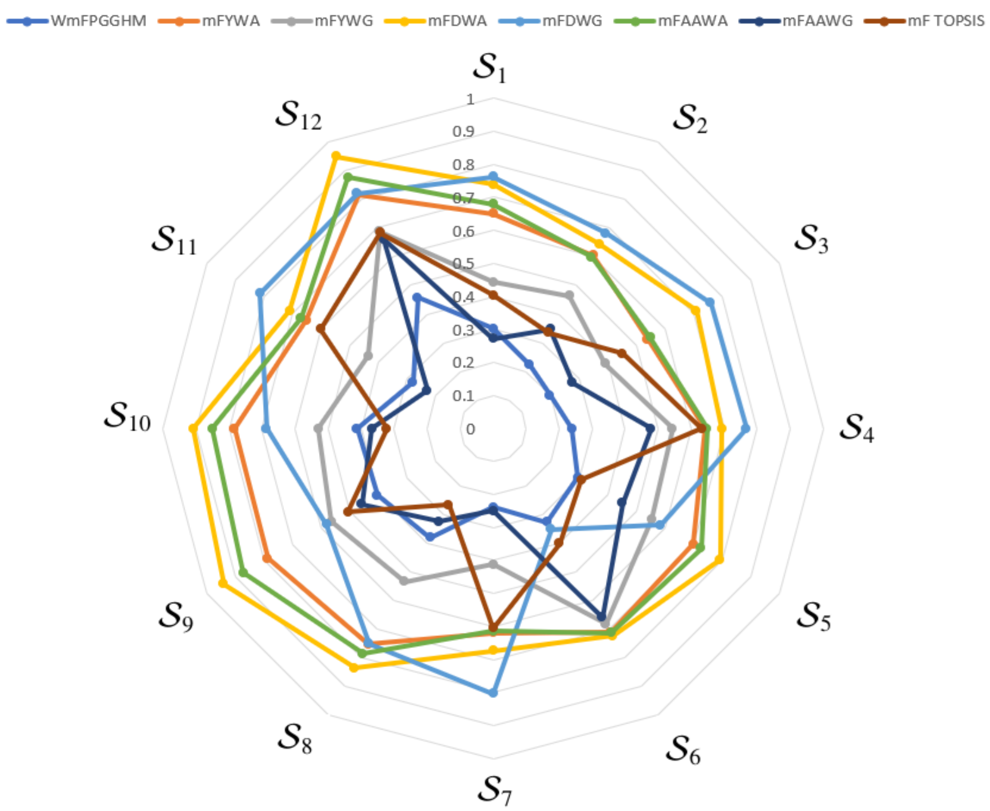

Figure 2. Comparison between the rankings of the developed W*m*FPGGHM operators with *m*FYWA [41] and *m*FYWG [41], *m*FDWA [39], *m*FDWG [39], *m*FAAWA [42], *m*FAAWG [42] AGOs, and *m*F TOPSIS approach [5].

Table 5. Comparison between the features of AGOs.

AGOs	Consider sub-characteristics of attributes	Consider interrelationship among attributes	Consider criteria weights from decision-makers	Whether model uncertainty is more powerful
*m*F Hamacher AGOs [37]	Yes	No	Yes	No
*m*F soft weighted AGOs [38]	Yes	No	Yes	No
*m*F Dombi AGOs [39]	Yes	No	Yes	No
*m*F Yager AGOs [41]	Yes	No	Yes	No
*m*F Aczel-Alsina AGOs [42]	Yes	No	Yes	No
Picture fuzzy interactional partitioned HM-AGOs [48]	No	Yes	No	No
Archimedean HM operators based on complex IFSs [55]	No	Yes	Yes	No
Cubic *m*F TOPSIS approach [5]	Yes	No	No	No
Cubic *m*F ELECTRE-I method [5]	Yes	No	No	No
T-spherical fuzzy Aczel Alsina HM operators [56]	No	Yes	Yes	Yes
Bipolar complex fuzzy partition HM operators [57]	No	Yes	Yes	No
Interval-valued IF-HM AGOs [58]	No	Yes	Yes	No
Interval-valued picture fuzzy geometric HM operators [59]	No	Yes	Yes	Yes
q-Rung orthopair fuzzy Aczel-Alsina power HM-AGOs [60]	No	No	No	Yes
T-spherical uncertain linguistic MARCOS method based on HM [61]	No	Yes	Yes	No
Proposed W*m*FGGHM AGOs	Yes	Yes	Yes	Yes

5.1. Advantages of suggested operators

These days, experts accept the fact that various daily-life problems contain or are affected by multi-polar information. Following this belief, a number of studies have been carried after the introduction of *m*F set theory, while the integration of *m*F sets with AGOs is limited to date, for instance, *m*F sets are integrated only with Dombi, Yager, Hamacher, and Aczel-Alsina operations. These AGOs are not capable of effectively maintaining or considering the interrelationships among attributes, and this issue can be easily overcome by the Heronian mean (or HM). Moreover, all other theories fail to demonstrate the multi-polar sub-characteristics of each attribute, and this issue can be easily addressed by integrating the *m*F set theory. Therefore, to support MCDM methods, an innovative fusion of power geometric HM and *m*F sets is emerged as *m*FPGGHM operators.

On the other hand, in MCDM problems like selecting best route to support urban transportation, it is important to consider every piece of information in addition to efficient uncertainty depiction (in terms of both the multi-polar attributes and interrelationships of those attributes). However, the initiated *m*F-HM based operators considered both multi-polar attributes and interrelationships of those attributes, to approach an optimal decision or to compute the ranking of objects.

5.2. Limitations of proposed operators

Despite the merits of the proposed Heronian mean-based AGOs, the methodology presented in Algorithm 1 has certain limitations. A key limitation is the complexity of the calculations, which becomes more time-consuming as the volume of information increases. This can make the process challenging when dealing with large datasets. In such scenarios, software tools like MATLAB can be utilized to streamline the computations. Another limitation is that the suggested model fails to tackle complicated scenarios like independent uncertainty depiction for non-membership, or interval fuzzy values. To overcome this limitation, the proposed AGOs can be integrated with powerful structures like interval-valued fuzzy sets, intuitionistic fuzzy sets, etc., which can improve the applicability to more complicated problems. The third limitation is the known weights given by the experts, which may provided a bias result. To overcome this issue, advanced approaches such as incorporating unknown weights techniques, like *m*F-AHP approach [62] may be used for finding criteria weights.

6. Conclusions

With the rapid rise in population, urban areas are becoming increasingly congested, leading to significant challenges in managing transportation systems. Efficient transportation plans are essential for every city and country to address these growing issues. Development of improved solutions of transportation needs consideration of multiple factors such as time efficiency, road infrastructure, environmental impact, and traffic density. Some other factors like sustainability, cost-effectiveness, and safety, also play an important role in finding the most suitable routes. To tackle this type of MCDM problems, in this paper, we initiated three novel aggregation operators AGOs for MCDM based on generalized geometric Heronian mean (GGHM) operations, integrating with the concept of *m*F sets. The presented operators are: Weighted *m*F power GGHM (W*m*FPGGHM), ordered weighted *m*F power GGHM averaging (OW*m*FPGGHM), and hybrid *m*F power GGHM (H*m*FPGGHM) operators. We investigated some fundamental properties of the proposed AGOs, including idempotency, monotonicity, boundedness, and Abelian property. Further, we presented an algorithm based on the initiated WmFPGGHM operators in order to address diverse daily-life MCDM scenarios. Next, to validate the efficiency of developed algorithm, we implemented it to a daily-life MCDM problem involving the urban transportation management: a case study of Saudi Arabia. Last, we compared the developed AGOs with some preexisting mF set-based operators involving Dombi, Yager, and Aczel-Alsina's operations. For future research, this work can be easily expanded to (1) Weighted m-polar fuzzy power generalized geometric Bonferroni mean operators, (2) Weighted hesitant m-polar fuzzy power generalized geometric Bonferroni mean operators, and (3) m-polar fuzzy power generalized geometric Bonferroni mean operators.

Author contributions

Ghous Ali: Conceptualization, data curation, formal analysis, supervision, validation, methodology, visualization, writing-original draft, writing-review and editing; Kholood Alsager: Funding acquisition, investigation, writing-review and editing, methodology. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

A.1. Proof of Theorem 3.1

Proof. It can be easily proved by the induction principle.
(1). By putting $n = 1$ in the Eq (3.3), we get $\theta_1 = 1$ and $\lambda_1 = 1$, thus

$$WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes (n\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}$$

$$= \left(\left(1 - (1 - (1 - (1 - p_1 \circ \varphi_1)^{n\lambda_1})^{\xi}(1 - (1 - p_1 \circ \varphi_1)^{n\lambda_1\theta_1})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}, \ldots,\right.$$

$$\left.\left(1 - (1 - (1 - (1 - p_m \circ \varphi_1)^{n\lambda_1})^{\xi}(1 - (1 - p_m \circ \varphi_1)^{n\lambda_1\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\right)$$

$$= (p_1 \circ \varphi_1, \ldots, p_1 \circ \varphi_m).$$

Therefore, when $n = 1$, the Eq (3.3) is verified.
(2). Now we suppose that the Eq (3.3) holds when $n = ŧ$, here $k \in \mathbb{N}$ (set of natural numbers), then

$$WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_ŧ) = \Big(\frac{2}{ŧ(ŧ+1)} \bigoplus_{i=1}^{ŧ} \bigoplus_{j=i}^{ŧ} (ŧ\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes (ŧ\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\dfrac{1}{\xi+\eta}}$$

$$= \left(\left(1 - \prod_{i=1,j=i}^{t} (1 - (1 - (1 - p_1 \circ \varphi_i)^{ŧ\lambda_i\theta_i})^{\xi}(1 - (1 - p_1 \circ \varphi_j)^{ŧ\lambda_j\theta_j})^{\eta})^{\frac{2}{ŧ(ŧ+1)}}\right)^{\frac{1}{\xi+\eta}}, \ldots,\right.$$

$$\left.\left(1 - \prod_{i=1,j=i}^{ŧ} (1 - (1 - (1 - p_m \circ \varphi_i)^{ŧ\lambda_i\theta_i})^{\xi}(1 - (1 - p_m \circ \varphi_j)^{ŧ\lambda_j\theta_j})^{\eta})^{\frac{2}{ŧ(ŧ+1)}}\right)^{\frac{1}{\xi+\eta}}\right).$$

For $n = \mathfrak{k} + 1$, we get

$$WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_{\mathfrak{k}+1}) = \Big(\frac{2}{\mathfrak{k}(\mathfrak{k}+1)} \bigoplus_{i=1}^{\mathfrak{k}} \bigoplus_{j=i}^{\mathfrak{k}} (\mathfrak{k}\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes (\mathfrak{k}\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\frac{1}{\xi+\eta}},$$

$$\oplus \Big(\frac{2}{(\mathfrak{k}+1)((\mathfrak{k}+1)+1)} \bigoplus_{i=\mathfrak{k}+1} \bigoplus_{j=\mathfrak{k}+1} ((\mathfrak{k}+1)\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes ((\mathfrak{k}+1)\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\frac{1}{\xi+\eta}}$$

$$= \Bigg(\Bigg(1 - \prod_{i=1,j=i}^{\mathfrak{k}} (1-(1-(1-p_1\circ\varphi_i)^{\mathfrak{k}\lambda_i\theta_i})^{\xi}(1-(1-p_1\circ\varphi_j)^{\mathfrak{k}\lambda_j\theta_j})^{\eta})^{\frac{2}{\mathfrak{k}(\mathfrak{k}+1)}}\Bigg)^{\frac{1}{\xi+\eta}}, \ldots,$$

$$\Bigg(1 - \prod_{i=1,j=i}^{\mathfrak{k}} (1-(1-(1-p_m\circ\varphi_i)^{\mathfrak{k}\lambda_i\theta_i})^{\xi}(1-(1-p_m\circ\varphi_j)^{\mathfrak{k}\lambda_j\theta_j})^{\eta})^{\frac{2}{\mathfrak{k}(\mathfrak{k}+1)}}\Bigg)^{\frac{1}{\xi+\eta}}\Bigg)$$

$$\oplus \Bigg(\Bigg(1-(1-(1-(1-p_1\circ\varphi_{\mathfrak{k}+1})^{(\mathfrak{k}+1)\lambda_{\mathfrak{k}+1}\theta_{\mathfrak{k}+1}})^{\xi}$$

$$(1-(1-p_1\circ\varphi_{\mathfrak{k}+1})^{(\mathfrak{k}+1)\lambda_{\mathfrak{k}+1}\theta_{\mathfrak{k}+1}})^{\eta})^{\frac{2}{(\mathfrak{k}+1)((\mathfrak{k}+1)+1)}}\Bigg)^{\frac{1}{\xi+\eta}}, \ldots,$$

$$\Bigg(1-(1-(1-(1-p_m\circ\varphi_{\mathfrak{k}+1})^{(\mathfrak{k}+1)\lambda_{\mathfrak{k}+1}\theta_{\mathfrak{k}+1}})^{\xi}$$

$$(1-(1-p_m\circ\varphi_{\mathfrak{k}+1})^{(\mathfrak{k}+1)\lambda_{\mathfrak{k}+1}\theta_{\mathfrak{k}+1}})^{\eta})^{\frac{2}{(\mathfrak{k}+1)((\mathfrak{k}+1)+1)}}\Bigg)^{\frac{1}{\xi+\eta}}\Bigg)$$

$$= \Bigg(\Bigg(1 - \prod_{i=1,j=i}^{(\mathfrak{k}+1)} (1-(1-(1-p_1\circ\varphi_i)^{(\mathfrak{k}+1)\lambda_i\theta_i})^{\xi}(1-(1-p_1\circ\varphi_j)^{(\mathfrak{k}+1)\lambda_j\theta_j})^{\eta})^{\frac{2}{(\mathfrak{k}+1)((\mathfrak{k}+1)+1)}}\Bigg)^{\frac{1}{\xi+\eta}}, \ldots,$$

$$\Bigg(1 - \prod_{i=1,j=i}^{(\mathfrak{k}+1)} (1-(1-(1-p_m\circ\varphi_i)^{(\mathfrak{k}+1)\lambda_i\theta_i})^{\xi}(1-(1-p_m\circ\varphi_j)^{(\mathfrak{k}+1)\lambda_j\theta_j})^{\eta})^{\frac{2}{(\mathfrak{k}+1)((\mathfrak{k}+1)+1)}}\Bigg)^{\frac{1}{\xi+\eta}}\Bigg).$$

Thus, Eq (3.3) is satisfied when $n = \mathfrak{k} + 1$. Subsequently, the result in Eq (3.3) is verified for all natural numbers. □

A.2. Proof of Theorem 3.3

Proof.

$$WmFGGHM_{\theta}^{\xi,\eta}(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n) = \Big(\frac{2}{n(n+1)} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} (n\theta_i\lambda_i\tilde{\varphi}_i)^{\xi} \otimes (n\theta_j\lambda_j\tilde{\varphi}_j)^{\eta}\Big)^{\frac{1}{\xi+\eta}}$$

$$= \Bigg(\Bigg(1 - \prod_{i=1,j=i}^{n} (1-(1-(1-p_1\circ\varphi_i)^{n\lambda_i\theta_i})^{\xi}(1-(1-p_1\circ\varphi_j)^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\Bigg)^{\frac{1}{\xi+\eta}}, \ldots,$$

$$\left(1-\prod_{i=1,j=i}^{n}(1-(1-(1-p_m\circ\varphi_i)^{n\lambda_i\theta_i})^{\xi}(1-(1-p_m\circ\varphi_j)^{n\lambda_j\theta_j})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\Bigg)$$

$$=\left(\left((1-(1-(1-(1-p_1\circ\varphi)^{n\lambda})^{\xi}(1-(1-p_1\circ\varphi)^{n\lambda})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}},\ldots,\right.$$

$$\left.\left((1-(1-(1-(1-p_m\circ\varphi)^{n\lambda})^{\xi}(1-(1-p_m\circ\varphi)^{n\lambda})^{\eta})^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\xi+\eta}}\right),$$

$$=(\mathfrak{p}_1\circ\varphi,\ldots,\mathfrak{p}_m\circ\varphi)\ \ for\ \ \xi+\eta=1.$$

□