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Research article

Milne and Hermite-Hadamard's type inequalities for strongly multiplicative convex function via multiplicative calculus

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Abstract: In this paper, we take into account the notion of strongly multiplicative convex function and derive integral inequalities of Hermite-Hadamard ($H.H$) type for such a function in the frame of multiplicative calculus. We also develop integral inequalities of $H.H$ type for product and quotient of strongly multiplicative convex and strongly multiplicative concave functions via multiplicative calculus. All the results of the theorems are verified graphically by taking into account some reasonable examples. Additionally, we establish the inequalities of the Milne type for strongly multiplicative convex functions.

Keywords: convex function; multiplicative convex function; strongly multiplicative convex function; Hermite-Hadamard type inequality; Milne type inequality
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1. Introduction

Concept of convexity is very important in engineering and mathematics as it provides a comprehensive basis for a wide range of phenomena. Convex sets and functions are inherently an essential part of this theory because they can reduce the enormous complexity of this process and optimize complex mathematical models. Thus, these problems are extremely easily solved, as they have only one global minimum and are very important in optimization. In addition control theory, economics, and optimization, as well as other fields, convexity is an applied math concept. It is not only a theoretical issue, but engineers also have to deal with it to make the systems' development more stable and the efficiency problems less severe. As an example, convexity plays an extremely significant role in control theory since it guarantees that the systems are built to withstand different types of conditions. Moreover, convexity is utilized in economics for the analysis of market behaviors and the preferences of people, which facilitates the decision-making process about how to distribute resources in an efficient manner. One of the many aspects of convexity theory is its connectivity to integral inequalities, making it one of the most spectacular areas of research. In many fields of analysis or mathematical expressions that bound certain integral values, we apply the use of integral inequalities. These inequalities, which are critical to obtaining essential calculus conclusions such as the $H.H$ inequality bounding the average value of a convex function over an interval, often come up in connection with convex functions (see [1–4]).

Strongly convex functions are a more sophisticated form of regular convex functions, as first proposed by Karamardian in [5]. These functions are especially helpful in some mathematical situations because of additional features. As several researchers have noted, Polyak established the idea of strongly convex functions in his work [6]. This implies that Polyak made a substantial contribution to the development of the mathematical concept of strongly convex functions by possibly being the first to formalise and investigate the idea. By presenting the notion of higher-order strongly convex functions, Lin and Fukushima extended the conventional understanding of convexity [7]. They proved that a function is only higher-order highly convex if and only if its gradient demonstrates strong monotonicity of the same order. This holds true for every continuously differentiable function. Two important inequalities, the $H.H$ and Jensen inequalities, were refined and extended in a number of ways by Srivastava and colleagues in 2011, adding to the body of work on mathematical inequalities. By concentrating on these inequalities within the framework of n variables, their work [8] expanded the usefulness and accuracy of these fundamental mathematical instruments. Strongly generalised convex functions of higher order were first proposed by Mishra and Sharma in [9]. This sophisticated term, especially when analysed at a higher degree, enhances the classical notion of convexity by taking into account functions with more intricate structures and behaviours. They developed new $H.H$-type integral inequalities based on this idea. These inequalities offer more sophisticated and potent tools for mathematical study within this framework, as they are specifically designed for the class of strongly generalised convex functions of higher order. Strongly exponentially convex functions are a concept that Noor and Noor defined and introduced in [10], which was an important addition. The concept of exponential components has been added to the thought of classical convexity therefore, this idea has been expanded to enable more finery of functions and a wider range of applications. The authors of the same paper [11] made a major achievement in the mathematical analysis. They introduced and defined the concept of strongly multiplicative convex functions.

Multiplicative calculus, thus called non-Newtonian calculus, offers a new integration and differentiation technique based on the principle of arithmetic addition by division and multiplication, respectively, for integration and differentiation. This form of calculus provides a more general framework than the classical Newtonian calculus, which has been the dominant theory in mathematics since its introduction by Leibniz and Newton in the seventeenth century. The work of Grossman denoted one of the first investigations in this field, and thus the idea of multiplicative calculus gained currency in the 1970s. Grossman's work made a revolutionary shift from the classical theories of calculus and presented a new perspective of differential and integral calculus. Multiplicative calculus is less familiar than its uh, cousin, the Newtonian one, but it has a unique methodology. This is mainly due to its limited scope of applications, which are mainly concerned with the positive side of things. Nevertheless, multiplicative calculus is less popular than it is, and therefore, it has been the basis of many interesting discoveries and applications in various fields. As an example, a simple theorem of multiplicative calculus was given by Bashirov et al. in [14], and then a complicated version of the idea was developed by Riza and Bashirov in [15]. Integral inequalities in multiplicative calculus are extensively discussed in the literature. So far, Ali et al. came up with the *H.H* type inequalities utilizing the principles of the multiplicative calculus [16]. The author [17] has shown the multiplicative inequalities of *H.H* type for the multiplicatively harmonic convex functions. This work was further elaborated by Du and Peng in [18], where they employed multiplicative Riemann-Liouville fractional integrals to formulate the *H.H* inequalities in a fractional setting. In the article [19], Meftah et al. obtained Ostrowski type inequalities for multiplicatively convex functions. Fractional versions of inequalities of the Maclaurin-type for multiplicatively convex and P-functions are carried out by Peng and Du in [20]. In [21], Du and Long utilized multiplicative Riemann-Liouville fractional integral operators and obtained multi-parameterized integral inequalities. For Readers are urged to examine the publications referenced in the paragraph, including research by [22], for more applications and insights into multiplicative calculus. These references give a thorough history of the field's growth as well as an outline of its possible disciplinary uses.

We are driven to investigate novel integral inequalities, particularly for substantially multiplicative convex functions in the fields of generalised convex functions and multiplicative calculus. The well-known *H.H* inequality has been expanded upon and generalised by several scholars within the context of multiplicative calculus; nevertheless, the particular situation of strongly multiplicative convex functions has not been covered. Prior research on integral inequalities using multiplicative calculus has mostly concentrated on different kinds of convex functions. However, there is still more to learn about strongly multiplicative convex functions. Identifying this gap, we provide Milne-type and *H.H* integral inequalities specifically designed for strongly multiplicative convex functions. In spite of this tendency, the field does not have any inequalities designed especially for strongly multiplicative convex functions. Consequently, within the context of multiplicative calculus, we want to close this gap by creating new integral inequalities based on the strongly multiplicative convex functions.

2. Preliminaries

Definition 2.1. *[1] The mapping* $\mathrm{f} : \mathrm{I} \subset \Re \rightarrow \Re$ *is designated as convex, if*

$$\mathrm{f}\left(\mathrm{u}_o \mathrm{t}_o + (1-\mathrm{t}_o)\mathrm{v}_o\right) \leq \mathrm{t}_o \mathrm{f}(\mathrm{u}_o) + (1-\mathrm{t}_o)\mathrm{f}(\mathrm{v}_o), \tag{2.1}$$

$\forall\ \mathrm{t}_o \in [0,1]$ *and* $\mathrm{u}_o, \mathrm{v}_o \in \mathrm{I}$ *holds.*

An intriguing finding in the study of convex functions is the *H.H* inequality, named after the mathematicians Jacques Hadamard [24] and Charles Hermite [25]. The integral average of any convex function defined on a compact interval may be found with both lower and upper bounds thanks to this inequality, which is well-known in the mathematical literature. Hermite and Hadamard separately found the inequality, and their discoveries were crucial to the development of convex analysis. It provides a useful tool for estimating a convex function's integral, enabling mathematicians to ascertain approximations based on the characteristics of the function across a certain interval. This inequality is especially important because it sheds light on the behaviour of convex functions, which makes it a valuable tool for a variety of mathematical analytic applications, including numerical integration and optimisation.

These inequalities can be stated that if $\mathrm{f} : \mathrm{I} \rightarrow \Re$ is possessing convexity on $\mathrm{I} \subseteq \Re$ and $\mathrm{u}_o, \mathrm{v}_o \in \mathrm{I}$

with $u_o < v_o$, then

$$f\left(\frac{u_o + v_o}{2}\right) \leq \frac{1}{v_o - u_o} \int_{u_o}^{v_o} f(x)d(x) \leq \frac{f(u_o) + f(v_o)}{2}. \tag{2.2}$$

Definition 2.2. *[1] Multiplicatively convex function* $f : I \rightarrow \Re$ *is a positive function satisfying the below-mentioned inequality*

$$f(u_o t_o + (1 - t_o)v_o) \leq [f(u_o)]^{t_o}[f(v_o)]^{(1-t_o)}, \tag{2.3}$$

$\forall\, t_o \in [0, 1]$ *and* $u_o, v_o \in I$.

Definition 2.3. *[12] Strongly convex function* f *on the convex set I with a constant* $c > 0$, *is a function satisfying the below-mentioned inequality*

$$f(u_o t_o + (1 - t_o)v_o) \leq t_o f(u_o) + (1 - t_o)f(v_o) - c t_o(1 - t_o)(v_o - u_o)^2, \tag{2.4}$$

$\forall\, t_o \in [0, 1]$ *and* $u_o, v_o \in I$.

Definition 2.4. *[11] Strongly multiplicative-convex function* f *on the convex set I with a constant* $c > 0$, *is a function satisfying the below-mentioned inequality*

$$\ln f(u_o t_o + (1 - t_o)v_o) \leq t_o \ln f(u_o) + (1 - t_o) \ln f(v_o) - c t_o(1 - t_o)(v_o - u_o)^2, \tag{2.5}$$

$\forall\, t_o \in [0, 1]$ *and* $u_o, v_o \in I$.

2.1. Multiplicative calculus

We begin by going over several definitions, characteristics, and ideas associated with differentiation in this part. We also go over certain components of multiplicative integration.

Definition 2.5. *[14] Assume that the function* $f : \Re \rightarrow \Re$ *is positive. The function* f *denotes multiplicative derivative by* f^* *is defined as follows.*

$$\frac{d^* f}{d\varkappa} = f^*(\varkappa) = \lim_{h \to 0} \left(\frac{f(\varkappa + h)}{f(\varkappa)}\right)^{\frac{1}{h}}.$$

Remark 2.6. *Relationship between multiplicative and ordinary derivative of a positive differentiable function is given as below*

$$f^* = e^{(\ln f(\varkappa))'} = e^{\frac{f'(\varkappa)}{f(\varkappa)}}.$$

The following attributes are admitted by the multiplicative derivative:

Proposition 2.7. *[14] Let for two multiplicatively differentiable functions* f *and* g *with* c *as an arbitrary constant, the functions* cf, $f + g$, f/g *and* fg *are* $*$ *differentiable*

(1) $(cf)^*(\varkappa) = f^*(\varkappa)$.
(2) $(fg)^*(\varkappa) = f^*(\varkappa)g^*(\varkappa)$.
(3) $(f + g)^*(\varkappa) = f^*(\varkappa)^{\frac{f(\varkappa)}{f(\varkappa)+g(\varkappa)}} g^*(\varkappa)^{\frac{g(\varkappa)}{f(\varkappa)+g(\varkappa)}}$.

(4) $\left(\frac{f}{g}\right)^*(\varkappa) = \frac{f^*(\varkappa)}{g^*(\varkappa)}$.
(5) $(f^g)^*(\varkappa) = f^*(\varkappa)^{g(\varkappa)} f(\varkappa)^{g'(\varkappa)}$.

The multiplicative integral, occasionally referred to as the * integral, is symbolised by the symbol $\int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa}$. This mathematical model was proposed by Bashirov et al. in [14]. In defining the classical Riemann integral of f over the interval $[u_o, v_o]$ the approach involves employing the sum of product term the definition of the multiplicative integral of f over the interval $[u_o, v_o]$ involves raising the product of terms to a power.

The association that exists among the multiplicative integral and the Rimann integral is as outlined below [14]:

Proposition 2.8. *Riemann integrability of* f *on* $[u_o, v_o]$ *implies the multiplicative integrability of* f *on the same interval.*

$$\int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa} = e^{\int_{u_o}^{v_o} \ln(f(\varkappa))d\varkappa}.$$

Furthermore, as demonstrated by Bashirov et al. [14], multiplicative integrable has the subsequent characteristics and outcomes:

Proposition 2.9. *Riemann integrability of* f *on* $[u_o, v_o]$ *implies * integrability of* f *on the same interval*

(1) $\int_{u_o}^{v_o} ((f(\varkappa))^p)^{d\varkappa} = \int_{u_o}^{v_o} \left((f(\varkappa))^{d\varkappa}\right)^p$.
(2) $\int_{u_o}^{v_o} (f(\varkappa) g(\varkappa))^{d\varkappa} = \int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa} . \int_{u_o}^{v_o} (g(\varkappa))^{d\varkappa}$.
(3) $\int_{u_o}^{v_o} \left(\frac{f(\varkappa)}{g(\varkappa)}\right)^{d\varkappa} = \frac{\int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa}}{\int_{u_o}^{v_o} (g(\varkappa))^{d\varkappa}}$.
(4) $\int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa} = \int_{u_o}^{c} (f(\varkappa))^{d\varkappa} . \int_{c}^{v_o} (f(\varkappa))^{d\varkappa}, u_o \leq c \leq v_o$.
(5) $\int_{u_o}^{u_o} (f(\varkappa))^{d\varkappa} = 1, \int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa} = \left(\int_{v_o}^{u_o} (f(\varkappa_1))^{d\varkappa}\right)^{-1}$.

Theorem 2.10. *(Multiplicative integration by parts [14]) Let* $f : [u_o, v_o] \to \Re$ *and* $g : [u_o, v_o] \to \Re$ *are possessing* **differentiability and differentiability respectively, so the function* f^g *is* **integrable, then it implies that*

$$\int_{u_o}^{v_o} \left(f^*(\varkappa)^{g(\varkappa)}\right)^{d\varkappa} = \frac{f(v_o)^{g(v_o)}}{f(u_o)^{g(u_o)}} . \frac{1}{\int_{u_o}^{v_o} \left(f(\varkappa)^{g'(\varkappa)}\right)^{d\varkappa}}.$$

Lemma 2.11. *[16] Let* $f : [u_o, v_o] \to \Re$ *and* $g : [u_o, v_o] \to \Re$ *are possessing* **differentiability, and differentiability respectively, so the function* f^g *is* **integrable, then it implies that*

$$\int_{u_o}^{v_o} \left(f^*(h(\varkappa))^{h'(\varkappa)g(\varkappa)}\right)^{d\varkappa} = \frac{f(v_o)^{g(v_o)}}{f(u_o)^{g(u_o)}} . \frac{1}{\int_{u_o}^{v_o} \left(f(h(\varkappa))^{g'(\varkappa)}\right)^{d\varkappa}}.$$

Now, we take into account some key terminology and mathematical foundations for multiplicative calculus theory, which will be utilised throughout the rest of this paper.

Proposition 2.12. *The* log *convexity of* f *and* g*, implies the* log *convexity of* fg *and* $\frac{f}{g}$.

For convex functions, the standard *H.H* inequality is provided by the equality (2.2). In [16], Ali et al. demonstrated the *H.H* inequality for multiplicatively convex functions in the following way:

Theorem 2.13. *Let the positive function* f *is possessing multiplicatively convexity on* $[u_o, v_o]$ *then the subsequent disparities are true.*

$$f\left(\frac{u_o + v_o}{2}\right) \leq \left(\int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa}\right)^{\frac{1}{v_o - u_o}} \leq G(f(u_o), f(v_o)), \tag{2.6}$$

where G(., .) *is geometric mean.*

3. Hermite-Hadamard's inequalities for strongly multiplicative convex functions

To initiate our article, we aim to derive the *H.H* inequality pertaining to the strongly multiplicative convex function.

Theorem 3.1. *Let the function* $f : [u_o, v_o] \to R$ *with* $0 \leq u_o < v_o$ *and* $f \in L^1[u_o, v_o]$ *is strongly multiplicative convex function on* $[u_o, v_o]$*, then the below-mentioned inequalities hold:*

$$exp\left\{\frac{c}{12}(u_o - v_o)^2\right\} f\left(\frac{u_o + v_o}{2}\right) \leq \left(\int_{u_o}^{v_o} (f(\varkappa))^{d\varkappa}\right)^{\frac{1}{v_o - u_o}} \leq \frac{G(f(u_o), f(v_o))}{exp\{\frac{c}{6}(v_o - u_o)^2\}}. \tag{3.1}$$

Proof. Since f be a strongly multiplicative convex function, then

$$\begin{aligned}
&\ln f\left(\frac{u_o + v_o}{2}\right) = \ln f\left(\frac{t_o u_o + (1 - t_o)v_o + t_o v_o + (1 - t_o)u_o}{2}\right)\\
&= \ln f\left(\frac{t u_o + (1 - t_o)v_o}{2} + \frac{t_o v_o + (1 - t_o)u_o}{2}\right)\\
&\leq \ln\left[\frac{f(t_o u_o + (1 - t_o)v_o)^{\frac{1}{2}} f(t_o v_o + (1 - t_o)u_o)^{\frac{1}{2}}}{exp\{\frac{c}{4}((1 - 2t_o)(v_o - u_o))^2\}}\right]\\
&= \frac{1}{2}\ln f(t_o u_o + (1 - t_o)v_o) + \frac{1}{2} f(t_o v_o + (1 - t_o)u_o) - \frac{c}{4}((1 - 2t_o)(v_o - u_o))^2,
\end{aligned}$$

integrate the inequality with respect to t_o over [0, 1], we obtain

$$\begin{aligned}
&\ln f\left(\frac{u_o + v_o}{2}\right) \leq \frac{1}{2}\int_0^1 \ln f(t_o u_o + (1 - t_o)v_o) dt_o + \frac{1}{2}\int_0^1 \ln f(t_o v_o + (1 - t_o)u_o) dt_o\\
&- \int_0^1 \frac{c}{4}((1 - 2t_o)(v_o - u_o))^2 dt_o,
\end{aligned}$$

after simple calculations, we obtain

$$\begin{aligned}
&\ln f\left(\frac{u_o+v_o}{2}\right) \le \frac{1}{v_o-u_o}\int_{u_o}^{v_o}(\ln f(\varkappa))^{d\varkappa} - \frac{c}{4}(v_o-u_o)^2\\
&\Rightarrow f\left(\frac{u_o+v_o}{2}\right) \le exp\left\{\frac{1}{v_o-u_o}\int_{u_o}^{v_o}(\ln f(\varkappa))^{d\varkappa}\right\} - exp\left\{\frac{c}{4}(u_o-v_o)^2\right\}\\
&= \frac{\left(\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}\right)^{\frac{1}{v_o-u_o}}}{exp\left\{\frac{c}{4}(v_o-u_o)^2\right\}}.
\end{aligned}$$

Thus, we have

$$exp\left\{\frac{c}{4}(u_o-v_o)^2\right\} f\left(\frac{u_o+v_o}{2}\right) \le \left(\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}\right)^{\frac{1}{v_o-u_o}}. \tag{3.2}$$

This is the first part of inequality (3.1).

For the proof of second inequality, we consider the right-hand side of inequality (3.2), we have

$$\begin{aligned}
&\left(\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}\right)^{\frac{1}{v_o-u_o}} = exp\left\{\frac{1}{v_o-u_o}\int_{u_o}^{v_o}(\ln f(\varkappa))d\varkappa\right\} = exp\left\{\int_0^1 \ln f(t_o u_o + (1-t_o)v_o)dt\right\}\\
&\le exp\left\{\int_0^1 \ln\left[\frac{(f(u_o))^{t_o}(f(v_o))^{1-t_o}}{exp\{ct_o(1-t_o)(v_o-u_o)^2\}}\right]\right\}\\
&= exp\left\{\int_0^1 \left[t_o \ln f(u_o) + (1-t_o)\ln f(v_o) - ct_o(1-t_o)(v_o-u_o)^2\right] dt_o\right\}\\
&= exp\left\{\ln\sqrt{f(u_o)f(v_o)} - \frac{c}{6}(v_o-u_o)^2\right\}\\
&= \frac{G(f(u_o), f(v_o))}{exp\{\frac{c}{6}((v_o-u_o))^2\}}.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have what we want. □

Example 3.2. *The following Figure 1 describes the viability of Theorem 3.1 for* $f(\varkappa) = exp\left\{\left(1+\varkappa^2\right)^2\right\}$ *and* $c = 2$.

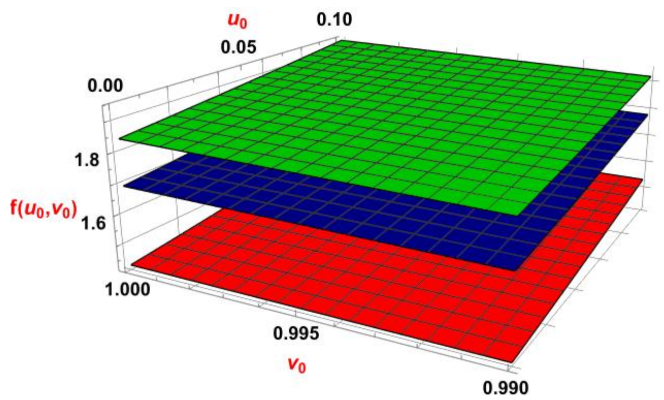

Figure 1. Graphical description for $u_o \in [0, 0.1]$ and $v_o \in [0.99, 1]$.

Theorem 3.3. *Let the functions* f *and* g *are strongly multiplicative convex functions on* $[\mathrm{u}_o, \mathrm{v}_o]$*, then the below-mentioned inequalities hold:*

$$\begin{aligned} &exp\left\{\frac{c}{6}(\mathrm{v}_o-\mathrm{u}_o)^2\right\}\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)g\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right) \\ &\leq \left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}} \leq \frac{G(\mathrm{f}(\mathrm{u}_o),\mathrm{f}(\mathrm{v}_o))\cdot G(\mathrm{g}(\mathrm{u}_o),\mathrm{g}(\mathrm{v}_o))}{exp\left\{\frac{c}{3}(\mathrm{v}_o-\mathrm{u}_o)^2\right\}}. \end{aligned} \tag{3.4}$$

Proof. Since f and g are strongly multiplicative convex function, then

$$\begin{aligned} &\ln\left(\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\mathrm{g}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\right)=\ln\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)+\ln\mathrm{g}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right) \\ &=\ln\mathrm{f}\left(\frac{\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o+\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o}{2}\right)+\ln\mathrm{g}\left(\frac{\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o+\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o}{2}\right) \\ &=\ln\mathrm{f}\left(\frac{\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o}{2}+\frac{\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o}{2}\right)+\ln\mathrm{g}\left(\frac{\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o}{2}+\frac{\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o}{2}\right) \\ &\leq\ln\left[\frac{\mathrm{f}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)^{\frac{1}{2}}\mathrm{f}(\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o)^{\frac{1}{2}}}{exp\left\{\frac{c}{4}((1-2\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o))^2\right\}}\right]+\ln\left[\frac{\mathrm{g}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)^{\frac{1}{2}}\mathrm{g}(\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o)^{\frac{1}{2}}}{exp\left\{\frac{c}{4}((1-2\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o))^2\right\}}\right] \\ &=\frac{1}{2}\ln\mathrm{f}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)+\frac{1}{2}\ln\mathrm{f}(\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o) \\ &+\frac{1}{2}\ln\mathrm{g}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)+\frac{1}{2}\ln\mathrm{g}(\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o)-\frac{c}{2}((1-2\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o))^2, \end{aligned}$$

integrate the inequality with respect to t_o over $[0, 1]$, we obtain

$$\begin{aligned} &\ln\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)+\ln\mathrm{g}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\leq\frac{1}{2}\int_0^1\ln\mathrm{f}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)d\mathrm{t}_o+\frac{1}{2}\int_0^1\ln\mathrm{f}(\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o)d\mathrm{t}_o \\ &+\frac{1}{2}\int_0^1\ln\mathrm{g}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)d\mathrm{t}_o+\frac{1}{2}\int_0^1\ln\mathrm{g}(\mathrm{t}_o\mathrm{v}_o+(1-\mathrm{t}_o)\mathrm{u}_o)d\mathrm{t}_o-\int_0^1\frac{c}{2}((1-2\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o))^2d\mathrm{t}_o, \end{aligned}$$

after simple calculations, we obtain

$$\begin{aligned} &\ln\left(\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\mathrm{g}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\right)\leq\frac{1}{\mathrm{v}_o-\mathrm{u}_o}\int_{\mathrm{u}_o}^{\mathrm{v}_o}\ln\mathrm{f}(\varkappa)\mathrm{d}\varkappa+\frac{1}{\mathrm{v}_o-\mathrm{u}_o}\int_{\mathrm{u}_o}^{\mathrm{v}_o}\ln\mathrm{g}(\varkappa)\mathrm{d}\varkappa-\frac{c}{6}(\mathrm{u}_o-\mathrm{v}_o)^2 \\ &\Rightarrow\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\mathrm{g}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\leq\frac{exp\left\{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\ln\mathrm{f}(\varkappa))\mathrm{d}\varkappa\right\}exp\left\{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\ln\mathrm{g}(\varkappa))\mathrm{d}\varkappa\right\}}{exp\left\{\frac{c}{6}(\mathrm{u}_o-\mathrm{v}_o)^2\right\}} \\ &=\frac{\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}}\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}}}{exp\left\{\frac{c}{6}(\mathrm{u}_o-\mathrm{v}_o)^2\right\}}. \end{aligned}$$

Thus, we have

$$exp\left\{\frac{c}{6}(\mathrm{u}_o-\mathrm{v}_o)^2\right\}\mathrm{f}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\mathrm{g}\left(\frac{\mathrm{u}_o+\mathrm{v}_o}{2}\right)\leq\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}}. \tag{3.5}$$

This is the first part of inequality (3.4).

Now, for the proof of the second inequality, we consider the right-hand side of (3.5), we have

$$\left(\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}\int_{u_o}^{v_o}(g(\varkappa))^{d\varkappa}\right)^{\frac{1}{v_o-u_o}} = exp\left\{\frac{1}{v_o-u_o}\left(\int_{u_o}^{v_o}(\ln f(\varkappa))d\varkappa+\int_{u_o}^{v_o}(\ln g(\varkappa))d\varkappa\right)\right\}$$
$$= exp\left\{\int_0^1 \ln f(t_o u_o+(1-t_o)v_o)dt_o+\int_0^1 \ln g(t_o u_o+(1-t_o)v_o)dt_o\right\}$$
$$\leq exp\left\{\int_0^1 \ln\left[\frac{(f(u_o))^{t_o}(f(v_o))^{1-t_o}}{exp\{ct_o(1-t_o)(v_o-u_o)^2\}}\right]dt_o+\int_0^1 \ln\left[\frac{(g(u_o))^{t_o}(g(v_o))^{1-t_o}}{exp\{ct_o(1-t_o)(u_o-v_o)^2\}}\right]dt_o\right\}$$
$$= exp\Bigg\{\int_0^1\left[t_o\ln f(u_o)+(1-t_o)\ln f(v_o)-ct_o(1-t_o)(v_o-u_o)^2\right]dt_o$$
$$+\int_0^1\left[t_o\ln g(u_o)+(1-t_o)\ln g(v_o)-ct_o(1-t_o)(v_o-u_o)^2\right]dt_o\Bigg\}$$
$$= exp\left\{\ln\sqrt{f(u_o)f(v_o)}+\ln\sqrt{g(u_o)g(v_o)}-\frac{c}{3}(v_o-u_o)^2\right\}$$
$$= \frac{G(f(u_o),f(v_o))G(g(u_o),g(v_o))}{exp\left\{\frac{c}{3}((v_o-u_o))^2\right\}}. \tag{3.6}$$

From (3.5) and (3.6), we have what we want. □

Example 3.4. *The following Figure 2 describes the viability of Theorem 3.3 for* $f(\varkappa) = exp\left\{\left(1+\varkappa^2\right)^2\right\}$, $g(\varkappa) = exp\left\{10\left(1+\varkappa^2\right)^2\right\}$ *and* $c = 2$.

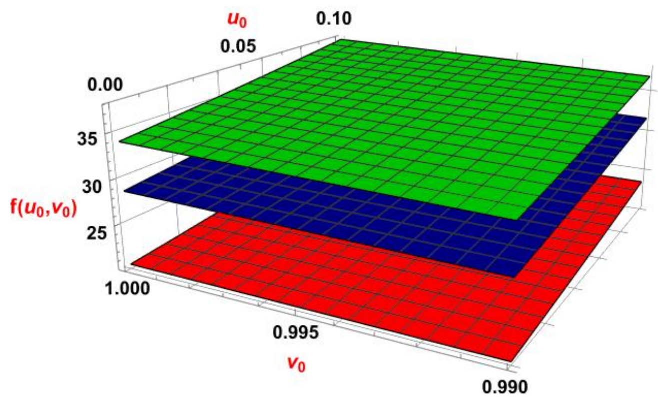

Figure 2. Graphical description for $u_o \in [0, 0.1]$ and $v_o \in [0.99, 1]$.

Theorem 3.5. *If* f *be positive and strongly multiplicative convex function on* $[u_o, v_o]$ *and* g *be positive and strongly multiplicative concave function, then the following inequalities hold:*

$$exp\left\{\frac{(c_2-c_1)(v_o-u_o)^2}{12}\right\}\frac{f\left(\frac{u_o+v_o}{2}\right)}{g\left(\frac{u_o+v_o}{2}\right)} \leq \left(\frac{\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}}{\int_{u_o}^{v_o}(g(\varkappa))^{d\varkappa}}\right)^{\frac{1}{v_o-u_o}} \leq \frac{G(f(u_o),f(v_o))}{G(g(u_o),g(v_o))}exp\left\{\frac{(c_2-c_1)(v_o-u_o)^2}{6}\right\}. \tag{3.7}$$

Proof. Since f and g are strongly multiplicative convex functions and strongly log-concave functions respectively, then we have

$$\begin{aligned}
&\ln\left(\frac{f\left(\frac{u_o+v_o}{2}\right)}{g\left(\frac{u_o+v_o}{2}\right)}\right) = \ln f\left(\frac{u_o+v_o}{2}\right) - \ln g\left(\frac{u_o+v_o}{2}\right)\\
&= \ln f\left(\frac{t_o u_o + (1-t_o)v_o + t_o v_o + (1-t_o)u_o}{2}\right) - \ln g\left(\frac{t_o u_o + (1-t_o)v_o + t_o v_o + (1-t_o)u_o}{2}\right)\\
&= \ln f\left(\frac{t_o u_o + (1-t_o)v_o}{2} + \frac{t_o v_o + (1-t_o)u_o}{2}\right) - \ln g\left(\frac{t_o u_o + (1-t_o)v_o}{2} + \frac{t_o v_o + (1-t_o)u_o}{2}\right)\\
&= \frac{1}{2}[\ln f(t_o u_o + (1-t_o)v_o) + \ln f(t_o v_o + (1-t_o)u_o)] - \frac{c_1}{4}((1-2t_o)(u_o - v_o))^2\\
&- \frac{1}{2}[\ln g(t_o u_o + (1-t_o)v_o) + \ln g(t_o v_o + (1-t_o)u_o)] + \frac{c_2}{4}((1-2t_o)(u_o - v_o))^2,
\end{aligned}$$

integrate the inequality with respect to t_o over $[0,1]$, we obtain

$$\begin{aligned}
&\ln\left(\frac{f\left(\frac{u_o+v_o}{2}\right)}{g\left(\frac{u_o+v_o}{2}\right)}\right)\\
&\le \frac{1}{2}\left[\int_0^1 \ln f(t_o u_o + (1-t_o)v_o)dt_o + \int_0^1 \ln f(t_o v_o + (1-t_o)u_o)dt_o\right] - \int_0^1 \frac{c_1}{4}((1-2t_o)(u_o - v_o))^2 dt_o\\
&- \frac{1}{2}\left[\int_0^1 \ln g(t_o u_o + (1-t_o)v_o)dt_o + \int_0^1 \ln g(t_o v_o + (1-t_o)u_o)dt_o\right] + \int_0^1 \frac{c_2}{4}((1-2t_o)(u_o - v_o))^2 dt_o,
\end{aligned}$$

after simple calculations, we obtain

$$\ln\left(\frac{f\left(\frac{u_o+v_o}{2}\right)}{g\left(\frac{u_o+v_o}{2}\right)}\right) \le \frac{1}{v_o - u_o}\int_{u_o}^{v_o} \ln f(\varkappa)d\varkappa - \frac{1}{v_o - u_o}\int_{u_o}^{v_o} \ln g(\varkappa)d\varkappa + \left(\frac{c_2}{12} - \frac{c_1}{12}\right)(u_o - v_o)^2,$$

thus, we have

$$exp\left\{\left(\frac{c_1}{12} - \frac{c_2}{12}\right)(u_o - v_o)^2\right\} \frac{f\left(\frac{u_o+v_o}{2}\right)}{g\left(\frac{u_o+v_o}{2}\right)} \le \left(\frac{\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}}{\int_{u_o}^{v_o}(g(\varkappa)d^{d\varkappa}}\right)^{\frac{1}{v_o - u_o}}. \tag{3.8}$$

This is the first part of inequality (3.7).

Now, for the proof of the second inequality, we consider the right hand side of (3.8), we have

$$\begin{aligned}
\left(\frac{\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}}{\int_{u_o}^{v_o}(g(\varkappa)d^{d\varkappa}}\right)^{\frac{1}{v_o - u_o}} &= exp\left\{\frac{1}{v_o - u_o}\left(\int_{u_o}^{v_o}(\ln f(\varkappa))d\varkappa - \int_{u_o}^{v_o}(\ln g(\varkappa))d\varkappa\right)\right\}\\
&= exp\left\{\int_0^1 \ln f(t_o u_o + (1-t_o)v_o)dt_o - \int_0^1 \ln g(t_o u_o + (1-t_o)v_o)dt_o\right\}
\end{aligned}$$

$$\leq exp\Big\{\int_0^1 [\mathrm{t}_o \ln \mathrm{f}(\mathrm{u}_o) + (1-\mathrm{t}_o)\ln \mathrm{f}(\mathrm{v}_o) - c_1\mathrm{t}_o(1-\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o)^2]d\mathrm{t}_o$$
$$-\int_0^1 [\mathrm{t}_o \ln \mathrm{g}(\mathrm{u}_o) + (1-\mathrm{t}_o)\ln \mathrm{g}(\mathrm{v}_o) - c_2\mathrm{t}_o(1-\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o)^2]d\mathrm{t}_o\Big\}$$
$$= exp\{\ln \sqrt{\mathrm{f}(\mathrm{u}_o)\mathrm{f}(\mathrm{v}_o)} - \ln \sqrt{\mathrm{g}(\mathrm{u}_o)\mathrm{g}(\mathrm{v}_o)}\}exp\Big\{\Big(\frac{c_2}{6}-\frac{c_1}{6}\Big)(\mathrm{u}_o-\mathrm{v}_o)^2\Big\}$$
$$= \frac{G(\mathrm{f}(\mathrm{u}_o),\mathrm{f}(\mathrm{v}_o))}{G(\mathrm{g}(\mathrm{u}_o),\mathrm{g}(\mathrm{v}_o))}exp\left\{\left(\frac{c_2}{6}-\frac{c_1}{6}\right)(\mathrm{u}_o-\mathrm{v}_o)^2\right\}, \tag{3.9}$$

from (3.8) and (3.9), we have what we want. □

Example 3.6. *The following Figure 3 describes the viability of Theorem 3.5 for* $\mathrm{f}(\varkappa) = exp\left\{\left(1+\varkappa^2\right)^2\right\}, g(\varkappa) = exp\left\{-\left(1+\varkappa^2\right)^2\right\}$, $c_1 = 2$ *and* $c_2 = 3$.

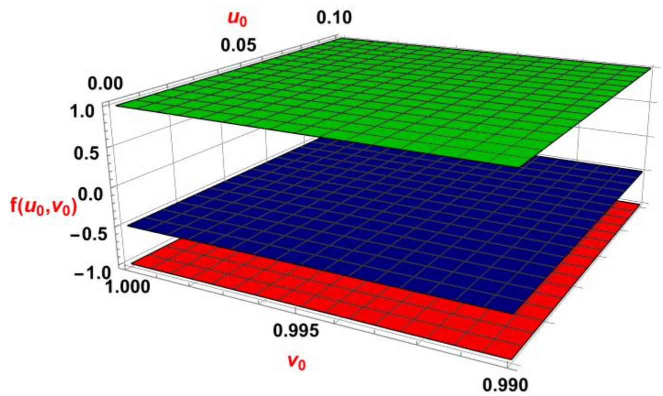

Figure 3. Graphical description for $\mathrm{u}_o \in [0, 0.1]$ and $\mathrm{v}_o \in [0.99, 1]$.

Theorem 3.7. *If* f *is a convex function and* g *is strongly multiplicative concave function, then the following inequality holds:*

$$\left(\frac{\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}}{\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}} \leq \frac{\left(\frac{(\mathrm{f}(\mathrm{v}_o))^{\mathrm{f}(\mathrm{v}_o)}}{\mathrm{f}(\mathrm{u}_o)^{\mathrm{f}(\mathrm{u}_o)}}\right)^{\frac{1}{\mathrm{f}(\mathrm{v}_o)-\mathrm{f}(\mathrm{u}_o)}} exp\left\{\frac{c}{6}(\mathrm{v}_o-\mathrm{u}_o)^2\right\}}{exp\{1\}\times G(\mathrm{g}(\mathrm{u}_o),\mathrm{g}(\mathrm{v}_o))}. \tag{3.10}$$

Proof. We consider the right-hand side of (3.10), we have

$$\left(\frac{\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}}{\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}} = exp\left\{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\ln \mathrm{f}(\varkappa))\mathrm{d}\varkappa - \int_{\mathrm{u}_o}^{\mathrm{v}_o}(\ln \mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)\right\}$$
$$= exp\left\{\int_0^1 \ln \mathrm{f}(\mathrm{t}_o\mathrm{u}_o + (1-\mathrm{t}_o)\mathrm{v}_o)\mathrm{dt}_o - \int_0^1 \ln \mathrm{g}(\mathrm{t}_o\mathrm{u}_o + (1-\mathrm{t}_o)\mathrm{v}_o)\mathrm{dt}_o\right\}$$
$$\leq exp\left\{\int_0^1 \ln \mathrm{f}(\mathrm{t}_o\mathrm{u}_o + (1-\mathrm{t}_o)\mathrm{v}_o)\mathrm{dt}_o - \int_0^1 \ln\left[\frac{(\mathrm{g}(\mathrm{u}_o))^{\mathrm{t}_o}(\mathrm{g}(\mathrm{v}_o))^{1-\mathrm{t}_o}}{c\mathrm{t}_o(1-\mathrm{t}_o)(\mathrm{u}_o-\mathrm{v}_o)^2}\right]\right\}$$
$$= exp\Big\{\int_0^1 \ln \mathrm{f}(\mathrm{t}_o\mathrm{u}_o + (1-\mathrm{t}_o)\mathrm{v}_o)\mathrm{dt}_o$$

$$-\int_0^1\left[t_o\ln g(u_o)+(1-t_o)\ln g(v_o)-ct_o(1-t_o)(v_o-u_o)^2\right]dt_o\Big\}$$

$$=exp\Big\{\ln\left(\frac{(\mathfrak{f}(v_o))f(v_o)}{f(u_o)f(u_o)}\right)^{\frac{1}{f(b)-f(u_o)}}-1-\left(\ln\sqrt{f(u_o)f(v_o)}-\frac{c}{6}(v_o-u_o)^2\right)\Big\}$$

$$=\frac{\left(\frac{(f(v_o))^{f(v_o)}}{f(u_o)^{f(u_o)}}\right)^{\frac{1}{f(v_o)-f(u_o)}}exp\left\{\frac{c}{6}(v_o-u_o)^2\right\}}{exp\{1\}\times G(g(u_o),g(v_o))}.$$

This ends the proof.

Example 3.8. *The following Figure 4 describes the viability of Theorem 3.7 for* $f(\varkappa)=exp\left\{\varkappa^2\right\}$, $g(\varkappa)=exp\left\{-\left(1+\varkappa^2\right)^2\right\}$ *and* $c=2$.

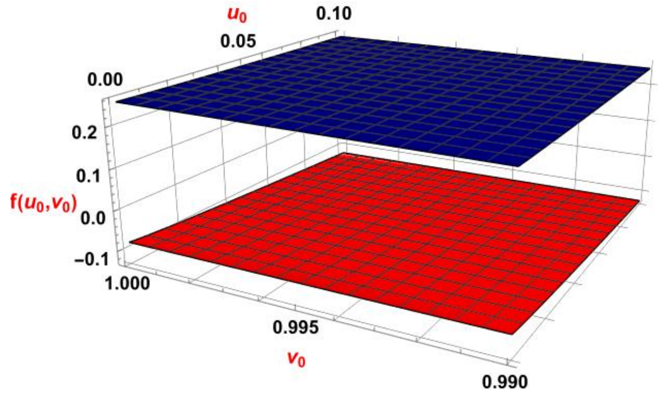

Figure 4. Graphical description for $u_o\in[0,0.1]$ and $v_o\in[0.99,1]$.

Theorem 3.9. *If* f *is strongly multiplicative convex function and* g *is concave function, then the following inequality hold:*

$$\left(\frac{\int_{u_o}^{v_o}\left(f(\varkappa)^{d\varkappa}\right.}{\int_{u_o}^{v_o}(g(\varkappa))^{d\varkappa}}\right)^{\frac{1}{v_o-u_o}}\leq\frac{exp\{1\}\times G(f(u_o),f(v_o))}{\left(\frac{(g(v_o))^{g(v_o)}}{g(u_o)^{g(u_o)}}\right)^{\frac{1}{g(v_o)-g(u_o)}}exp\left\{\frac{1}{6}(v_o-u_o)^2\right\}}.\tag{3.11}$$

Proof. We consider the right-hand side of (3.11), we have

$$\left(\frac{\int_{u_o}^{v_o}(f(\varkappa))^{d\varkappa}}{\int_{u_o}^{v_o}(g(\varkappa))^{d\varkappa}}\right)^{\frac{1}{v_o-u_o}}=exp\left\{\frac{1}{v_o-u_o}\left(\int_{u_o}^{v_o}(\ln f(\varkappa))d\varkappa-\int_{u_o}^{v_o}(\ln g(\varkappa))d\varkappa\right)\right\}$$

$$=exp\left\{\int_0^1\ln f(t_ou_o+(1-t_o)v_o)dt_o-\int_0^1\ln g(t_ou_o+(1-t_o)v_o)dt_o\right\}$$

$$\leq exp\left\{\int_0^1\left[t_o\ln f(u_o)+(1-t_o)\ln f(v_o)-ct_o(1-t_o)(v_o-u_o)^2\right]dt_o-\int_0^1\ln[t_og(u_o)+(1-t_o)g(v_o)]dt_o\right\}$$

$$=\frac{exp\left\{\ln\sqrt{f(u_o)f(v_o)}-\frac{c}{6}(v_o-u_o)^2\right\}}{exp\left\{\left[\ln\left(\frac{(g(v_o))^{g(v_o)}}{g(u_o)^{g(u_o)}}\right)^{\frac{1}{g(v_o)-g(u_o)}}\right]-1\right\}}=\frac{e\{1\}\times G(f(u_o),f(v_o))}{\left(\frac{(g(v_o))^{g(v_o)}}{g(u_o)^{g(u_o)}}\right)^{\frac{1}{g(v_o)-g(u_o)}}exp\left\{\frac{c}{6}(v_o-u_o)^2\right\}}.$$

This ends the proof. □

Example 3.10. *The following Figure 5 describes the viability of Theorem 3.9 for* $\mathrm{f}(\varkappa) = exp\{\left(1+\varkappa^2\right)^2\}$, $\mathrm{g}(\varkappa) = exp\{-\varkappa^2\}$ *and* $c = 2$.

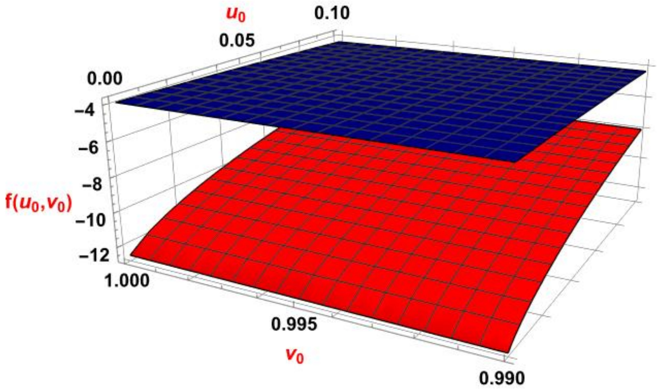

Figure 5. Graphical description for $\mathrm{u}_o \in [0, 0.1]$ and $\mathrm{v}_o \in [0.99, 1]$.

Theorem 3.11. *If* f *is convex function and* g *is strongly multiplicative convex function, then the following inequality hold:*

$$\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}} \le \frac{\left(\frac{(\mathrm{f}(\mathrm{v}_o))^{\mathrm{f}(\mathrm{v}_o)}}{f(\mathrm{u}_o)^{\mathrm{f}(\mathrm{u}_o)}}\right)^{\frac{1}{\mathrm{f}(\mathrm{v}_o)-\mathrm{f}(\mathrm{u}_o)}} G(\mathrm{g}(\mathrm{u}_o),\mathrm{g}(\mathrm{v}_o))}{exp\left\{\frac{c}{6}(\mathrm{v}_o-\mathrm{u}_o)^2\right\}}. \tag{3.12}$$

Proof. We consider the right-hand side of (3.12), we have

$$\begin{aligned}
&\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{f}(\varkappa))^{\mathrm{d}\varkappa}\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)^{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}} = exp\left\{\frac{1}{\mathrm{v}_o-\mathrm{u}_o}\left(\int_{\mathrm{u}_o}^{\mathrm{v}_o}(\ln \mathrm{f}(\varkappa))\mathrm{d}\varkappa + \int_{\mathrm{u}_o}^{\mathrm{v}_o}(\ln \mathrm{g}(\varkappa))^{\mathrm{d}\varkappa}\right)\right\}\\
&= exp\left\{\int_0^1 \ln \mathrm{f}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)\mathrm{d}\mathrm{t}_o + \int_0^1 \ln \mathrm{g}(\mathrm{t}_o\mathrm{u}_o+(1-\mathrm{t}_o)\mathrm{v}_o)\mathrm{d}\mathrm{t}_o\right\}\\
&\le exp\left\{\int_0^1 \ln[\mathrm{t}_o\mathrm{f}(\mathrm{u}_o)+(1-\mathrm{t}_o)\mathrm{f}(\mathrm{v}_o)] + \int_0^1\left[\mathrm{t}_o \ln \mathrm{g}(\mathrm{u}_o)+(1-\mathrm{t}_o)\ln \mathrm{g}(\mathrm{v}_o) - c\mathrm{t}_o(1-\mathrm{t}_o)(\mathrm{v}_o-\mathrm{u}_o)^2\right]\right\}\\
&= \frac{\left(\frac{(\mathrm{f}(\mathrm{v}_o))^{\mathrm{f}(\mathrm{v}_o)}}{\mathrm{f}(\mathrm{u}_o)^{\mathrm{f}(\mathrm{u}_o)}}\right)^{\frac{1}{f(\mathrm{v}_o)-f(\mathrm{u}_o)}} G(\mathrm{g}(\mathrm{u}_o),\mathrm{g}(\mathrm{v}_o))}{exp\left\{\frac{c}{6}(\mathrm{v}_o-\mathrm{u}_o)^2\right\}}.
\end{aligned}$$

This ends the proof. □

Example 3.12. *The following Figure 6 describes the viability of Theorem 3.11 for* $\mathrm{f}(\varkappa) = exp\{\varkappa^2\}$, $\mathrm{g}(\varkappa) = exp\{5\left(1+\varkappa^2\right)^2\}$ *and* $c = 2$.

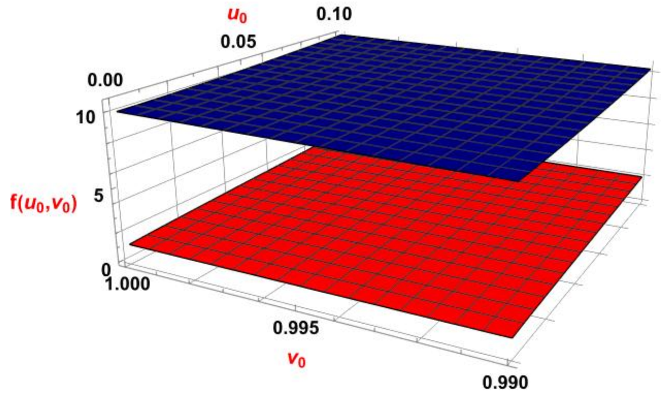

Figure 6. Graphical description for $u_o \in [0, 0.1]$ and $v_o \in [0.99, 1]$.

4. Milne type inequalities for strongly multiplicative convex function

In order to derive the Milne type inequalities, it is necessary to have a look on the proof of the below mentioned identity.

Lemma 4.1. *Let the function* $f : I \subset \Re^+ \to \Re$ *is possessing multiplicative differentiability on* I°, *where* I° *is the interior of the interval I, where* $u_o, v_o \in I^\circ$ *with* $u_o < v_o$ *and* $f^{**} \in L[u_o, v_o]$, *then the following identity holds*

$$I_1 \times I_2 = \frac{\left[[f^*(u_o)]^{\frac{2}{3}}\, [f^*(v_o)]^{\frac{2}{3}} \left(f^*\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{-1}{3}}\right]^{\frac{v_o-u_o}{4}}}{exp\left\{\frac{1}{(v_o-u_o)} \int\limits_{\frac{u_o+v_o}{2}}^{v_o} \left(\varkappa - \frac{u_o+v_o}{2}\right) \ln f^*(\varkappa)\, d\varkappa + \frac{1}{(v_o-u_o)} \int\limits_{u_o}^{\frac{u_o+v_o}{2}} \left(\frac{u_o+v_o}{2} - \varkappa\right) \ln f^*(\varkappa)\, d\varkappa\right\}},$$

where

$$I_1 = \left(\int_0^1 \left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2} + t_o v_o\right)^{t^2+\frac{1}{3}}\right)^{dt_o}\right)^{\frac{(v_o-u_o)^2}{2^4}},$$

$$I_2 = \left(\int_0^1 \left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2} + t_o u_o\right)^{-\left(t^2+\frac{1}{3}\right)}\right)^{dt_o}\right)^{\frac{(v_o-u_o)^2}{2^4}}.$$

Proof. Utilizing multiplicative integration by parts for I_1, we have

$$I_1 = \left(\int_0^1 \left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2} + t_o v_o\right)^{t^2+\frac{1}{3}}\right)^{dt_o}\right)^{\frac{(v_o-u_o)^2}{2^4}}$$

$$= \int_0^1 \left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2} + t_o v_o\right)^{\left(\frac{v_o-u_o}{2}\right)\left(\frac{(v_o-u_o)}{8}\left(t_o{}^2+\frac{1}{3}\right)\right)}\right)^{dt_o}$$

$$= \frac{\left(f^{*}(v_o)\right)^{\frac{(v_o-u_o)}{6}}}{\left(f^{*}\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{(v_o-u_o)}{24}}} \times \frac{1}{\int\limits_0^1 \left(f^{*}\left((1-t_o)\frac{u_o+v_o}{2}+t_o v_o\right)^{\frac{(v_o-u_o)}{4}t_o}\right)^{dt_o}}$$

$$\Rightarrow I_1 = \frac{\left[\left(f^{*}(v_o)\right)^{\frac{2}{3}}\left(f^{*}\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{-1}{6}}\right]^{\frac{(v_o-u_o)}{4}}}{exp\left\{\frac{1}{(v_o-u_o)}\int\limits_{\frac{u_o+v_o}{2}}^{v_o}\left(\varkappa-\frac{u_o+v_o}{2}\right)\ln f^{*}(\varkappa)\,d\varkappa\right\}}.$$

Similarly,

$$I_2 = \left(\int\limits_0^1 \left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2}+t_o u_o\right)^{-\left(t^2+\frac{1}{3}\right)}\right)^{dt_o}\right)^{\frac{(v_o-u_o)^2}{2^4}}$$

$$= \left[\left(f^{*}(u_o)\right)^{\frac{2}{3}}\left(f^{*}\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{-1}{6}}\right]^{\frac{(v_o-u_o)}{4}} \times \frac{1}{exp\left\{\frac{1}{(v_o-u_o)}\int\limits_{u_o}^{\frac{u_o+v_o}{2}}\left(\frac{u_o+v_o}{2}-\varkappa\right)\ln f^{*}(\varkappa)\,d\varkappa\right\}},$$

multiplying I_1 and I_2 side by side, we have

$$I_1 \times I_2 = \frac{\left[[f^{*}(u_o)]^{\frac{2}{3}}\,[f^{*}(v_o)]^{\frac{2}{3}}\left(f^{*}\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{-1}{3}}\right]^{\frac{v_o-u_o}{4}}}{exp\left\{\frac{1}{(v_o-u_o)}\int\limits_{\frac{u_o+v_o}{2}}^{v_o}\left(\varkappa-\frac{u_o+v_o}{2}\right)\ln f^{*}(\varkappa)\,d\varkappa+\frac{1}{(v_o-u_o)}\int\limits_{u_o}^{\frac{u_o+v_o}{2}}\left(\frac{u_o+v_o}{2}-\varkappa\right)\ln f^{*}(\varkappa)\,d\varkappa\right\}}.$$

□

Theorem 4.2. *Let the function* $f : I \subset \Re^{+} \to \Re$ *is possessing multiplicative differentiability on* $[u_o, v_o]$ *with* $u_o < v_o$ *and* f^{**} *is multiplicative convex on* $[u_o, v_o]$*, then the following inequality hold*

$$\left|\frac{\left[[f^{*}(u_o)]^{\frac{2}{3}}\,[f^{*}(v_o)]^{\frac{2}{3}}\left(f^{*}\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{-1}{3}}\right]^{\frac{v_o-u_o}{4}}}{exp\left\{\frac{1}{(v_o-u_o)}\int\limits_{\frac{u_o+v_o}{2}}^{v_o}\left(\varkappa-\frac{u_o+v_o}{2}\right)\ln f^{*}(\varkappa)\,d\varkappa+\frac{1}{(v_o-u_o)}\int\limits_{u_o}^{\frac{u_o+v_o}{2}}\left(\frac{u_o+v_o}{2}-\varkappa\right)\ln f^{*}(\varkappa)\,d\varkappa\right\}}\right|$$

$$\leq \left[\frac{\left(f^{**}(u_o)\right)^{\frac{5}{12}}\left(f^{**}\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{1}{2}}\left(f^{**}(v_o)\right)^{\frac{5}{12}}}{exp\left\{\frac{19c}{360}(v_o-u_o)^2\right\}}\right]^{\frac{(v_o-u_o)^2}{(3)^2}}.$$

Proof. Considering Lemma 4.1 as well as the strongly multiplicative convexity of f^{**}, we attain

$$\left| \frac{\left[[f^*(u_o)]^{\frac{2}{3}} [f^*(v_o)]^{\frac{2}{3}} \left(f^*\left(\frac{u_o+v_o}{2}\right)\right)^{\frac{-1}{3}} \right]^{\frac{v_o-u_o}{4}}}{exp\left\{ \frac{1}{(v_o-u_o)} \int\limits_{\frac{u_o+v_o}{2}}^{v_o} \left(\varkappa - \frac{u_o+v_o}{2}\right) \ln f^*(\varkappa)\, d\varkappa + \frac{1}{(v_o-u_o)} \int\limits_{u_o}^{\frac{u_o+v_o}{2}} \left(\frac{u_o+v_o}{2} - \varkappa\right) \ln f^*(\varkappa)\, d\varkappa \right\}} \right|$$

$$\leq exp\left\{ \frac{(v_o-u_o)^2}{2^4} \int\limits_0^1 \left| t_o{}^2 + \frac{1}{3} \right| \left(\ln\left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2} + t_o v_o \right)\right)\right) dt_o \right\}$$

$$\times\, exp\left\{ \frac{(v_o-u_o)^2}{2^4} \int\limits_0^1 \left| t_o{}^2 + \frac{1}{3} \right| \left(\ln\left(f^{**}\left((1-t_o)\frac{u_o+v_o}{2} + t_o u_o \right)\right)\right) dt_o \right\}$$

$$\leq exp\left\{ \frac{(v_o-u_o)^2}{2^4} \int\limits_0^1 \left| t_o{}^2 + \frac{1}{3} \right| \left(\ln\left(\frac{\left(f^*\left(\frac{u_o+v_o}{2}\right)\right)^{1-t_o} (f^*(v_o))^{t_o}}{e^{\frac{c}{4}t_o(1-t_o)(v_o-u_o)^2}} \right)\right) dt_o \right\}$$

$$\times\, exp\left\{ \frac{(v_o-u_o)^2}{2^4} \int\limits_0^1 \left| t_o{}^2 + \frac{1}{3} \right| \left(\ln\left(\frac{\left(f^*\left(\frac{u_o+v_o}{2}\right)\right)^{1-t_o} (f^*(u_o))^{t_o}}{exp\left\{\frac{c}{4}t_o(1-t_o)(v_o-u_o)^2\right\}} \right)\right) dt_o \right\}$$

$$\leq exp\left\{ \frac{(v_o-u_o)^2}{2^4} \int\limits_0^1 \left| t_o{}^2 + \frac{1}{3} \right| \left((1-t_o)\ln\left(f^*\left(\frac{u_o+v_o}{2}\right)\right) + t_o \ln(f^*(v_o)) - \frac{c}{4}t_o(1-t_o)(v_o-u_o)^2 \right) dt_o \right\}$$

$$\times\, exp\left\{ \frac{(v_o-u_o)^2}{2^4} \int\limits_0^1 \left| t_o{}^2 + \frac{1}{3} \right| \left((1-t_o)\ln\left(f^*\left(\frac{u_o+v_o}{2}\right)\right) + t_o \ln(f^*(u_o)) - \frac{c}{4}t_o(1-t_o)(v_o-u_o)^2 \right) dt_o \right\}$$

$$\leq exp\left\{ \frac{(v_o-u_o)^2}{2^4} \left(\frac{5}{12}\ln f^*(u_o) + \frac{1}{2}\ln f^*\left(\frac{u_o+v_o}{2}\right) + \frac{5}{12}\ln f^*(v_o) - \frac{19c}{360}(v_o-u_o)^2 \right) \right\},$$

after simplification, we get the required result. □

5. Conclusions

This paper has introduced the concept of strongly multiplicative convex functions within the framework of multiplicative calculus. The main contributions include the derivation of *H.H* type inequalities for these functions, as well as the extension of these inequalities to the product and quotient of strongly multiplicative convex and concave functions. The results have been validated through selected examples, ensuring both theoretical and practical consistency. Additionally, Milne-type inequalities for strongly multiplicative convex functions have been established, providing further insights for researchers. The findings in this paper open up several potential avenues for future research. One possible direction is the exploration of strongly multiplicative convex functions in other areas of mathematics, such as differential equations or optimization problems. For researchers and practitioners interested in multiplicative calculus, this paper provides a robust set of tools and inequalities that can be applied to a wide range of problems. The extension of *H.H* and Milne-type inequalities to strongly multiplicative convex and concave functions demonstrate the versatility of these concepts and their potential for further development. By building on the results presented here, readers are encouraged to explore new applications and generalizations, contributing to the ongoing advancement of this area of mathematical research.

Author contributions

Muhammad Umar: Conceptualization, formal analysis, investigation, writing-original draft, writing-review & editing, supervision; Saad Ihsan Butt: Conceptualization, formal analysis, investigation, methodology, writing-original draft, writing-review & editing, supervision; Youngsoo Seol: Conceptualization, formal analysis, methodology, software, writing-review & editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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