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Research article

Nearly Menger covering property via bitopological spaces

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Abstract: This paper is a continuation and complement for previous works on selective covering properties. We introduce the novel concept of the nearly Menger property in a bitopological context. We demonstrate its interrelations with existing covering properties and construct certain equivalences between those. We also investigate various properties of nearly Menger bitopological spaces by considering it under subspaces, products, and certain type of mappings.

Keywords: selection principles; covering properties; bitopological spaces; Menger spaces; generalized open sets
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1. Introduction and preliminaries

It is important to be able to define, characterize, and classify the properties of a topological space by using its covers. The theory known as selection principles in mathematics enables this. It can gather the properties of a topological space under a combinatorial roof and provide opportunities for methods that can combine topology with existing disciplines in the field of mathematics such as dimension theory, game theory, Ramsey theory, Karamata theory, function spaces, cardinal invariants, and algebraic structures. The core of this theory is based upon the diagonalization technique used by Cantor. With this technique, a mathematical object can be accessed as desired form from a sequence of mathematical objects with the same properties by performing a selection process associated with each object of the sequence according to a given rule. The theory itself has a long history, and even though its conventional notions have dated back to papers [1, 2], its systematic study was initiated by Scheepers in [3]. He implemented selection hypothesis to a varied kind of open covers of topological spaces (see also [4, 5]). Two well-known selection hypotheses are S_{fin} and S_1, formulated as follows:

Let X be an infinite set and $\mathcal{P}(X)$ denote the power set of X. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$.

(1) Selection Hypothesis $S_{fin}(\mathcal{A}, \mathcal{B})$: For every sequence $(A_n)_{n\in\omega}$ where $A_n \in \mathcal{A}$ for each $n \in \omega$, it

admits a sequence $(B_n)_{n\in\omega}$ such that $B_n \subset A_n$, B_n is finite for each $n \in \omega$ and $\bigcup_{n\in\omega} B_n$ is a member of $\mathcal{B}$.

(2) Selection Hypothesis $S_1(\mathcal{A}, \mathcal{B})$: For every sequence $(A_n)_{n\in\omega}$ such that $A_n \in \mathcal{A}$ for each $n \in \omega$, it admits a sequence $(a_n)_{n\in\omega}$ such that $a_n \in A_n$ for each $n \in \omega$ and $\{a_n : n \in \omega\}$ is a member of $\mathcal{B}$.

Varied topological notions can be identified and described by the selection hypotheses mentioned above. Selection principles and covering properties of a topological space are highly related to each other. In this paper, we will focus on a form of Menger covering property in bitopological spaces. A topological space (X, τ) is said to be Menger [2] if for each sequence $(\mathcal{O}_n)_{n\in\omega}$ of open covers of X admits a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that each $\mathcal{U}_n$ is a finite subfamily of $\mathcal{O}_n$ and X is covered by $\{\bigcup \mathcal{U}_n : n \in \omega\}$, i.e.,

$$X = \bigcup_{n\in\omega} \bigcup \mathcal{U}_n.$$

It is known that if $\mathcal{O}$ is the family of all open covers of a space (X, τ) and

$$\mathcal{A}, \mathcal{B} = \mathcal{O},$$

$S_{fin}(\mathcal{O}, \mathcal{O})$ denotes the Menger property while $S_1(\mathcal{O}, \mathcal{O})$ denotes the Rothberger property (see also [6]). The Menger covering property and its selective characterizations have extensively been studied by many authors [7–9]. Besides that, there are considerable works on the weaker forms of the Menger covering property and their selective characterizations, and it is still an active research field of mathematics. There are a few ways to obtain the weaker forms of Menger covering property. They can be formulated by using closure and interior operators, generalized open sets of the space, and ideal structures. Some of the weak versions of Menger property such almost Menger, nearly Menger, and weakly Menger properties were defined by using interior and closure operators and investigated [10–12]. With their studies in classical topology, Menger covering property and some of weaker forms of it such as nearly Menger and almost Menger covering properties were investigated in the context of soft structures, and a general framework was created [13–15].

We will focus on the nearly Menger property in bitopological spaces, and its definition is given as below in classical topology by Kočinac.

Definition 1.1. [12] A topological space (X, τ) is said to be *nearly Menger* if every sequence $(\mathcal{O}_n)_{n\in\omega}$ of open covers of X admits a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that each $\mathcal{U}_n \subset \mathcal{O}_n$ is finite and

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int(Cl(U)),$$

where Int and Cl denote the interior and closure operators, respectively.

A bitopological space (or a bispace) $(X, \lambda_1, \lambda_2)$ is simply a set X equipped with generally unrelated topological structures λ_1 and λ_2. There are several studies of selection principles and Menger type covering properties in the bitopological context. Study of some of weaker forms of the Menger property was initiated in [16] with the bitopological almost Menger property, in [17, 18] their continuation was provided. Also, selective properties in a bitopological context were studied. In [19, 20], the authors took the separability and selection principles into consideration. In [21,22], by considering the function spaces, the authors investigated the relationships between games and selection principles, selective separability and tightness. We also refer the reader to [4, 23] for more essentials and information about the Menger spaces and selective results, and also to [24, 25] for different kinds of covering properties and relative forms of Menger spaces in a bitopological setting. In this study, we mainly aim to extend the nearly Menger property to bitopological spaces. We aim to investigate its topological properties and the relationships between it and some other existing covering properties. And, by so doing, we aim to obtain a larger framework for this field. To this end, the study will both introduce a novel concept and open a new way for investigating other forms of selective covering properties. The paper also forms a ground for the future works in this context and field in the way it contains different and novel examples and results. By using closure and interior operators from both topologies, it is provided that both topologies are made more effective on the given set. A weaker form of the Menger covering property in the context of bitopology was game-theoretically characterized in [26]. With the new concept defined in the paper, examining game theoretic characterization of this form can be a core for future contributions to this field.

This paper is a chapter of the first author's PhD thesis that will be submitted to Erzurum Technical University, and is organized as follows: In Section 2, we introduce the nearly Menger property in a bitopological context and give some characterizations. By giving some counterexamples, we point to differences between it and some of the existing Menger covering properties in literature. We also consider the conditions which make them equivalent in bitopological spaces. In Section 3, we discuss some properties of nearly Menger bitopological spaces. We take it into consideration under subspaces, products, and some certain type of mappings. We end the paper with Section 4, in which we pose a new type of Menger covering properties worth investigating which will motivate the paper based upon generalized notions.

Throughout the paper, $(X, \lambda_1, \lambda_2)$ will denote a bispace on which no seperation axiom will be assumed unless explicitly stated. $Cl^{\lambda_j}(A)$ and $Int^{\lambda_i}(A)$ will denote the closure and interior of a given subset A of X with respect to the topology λ_i, $(i = 1, 2)$. If $\mathcal{P}$ is a topological property, for $(X, \lambda_1, \lambda_2)$, (i, j)-$\mathcal{P}$ will denote that the topology λ_i has $\mathcal{P}$ with respect to λ_j $(i, j \in \{1, 2\}, i \neq j)$. λ_i-$\mathcal{P}$ will denote that the set X has $\mathcal{P}$ with respect to λ_i $(i = 1, 2)$. Our terminology will follow [27] for topological spaces. For undefined notations and concepts for bitopological spaces, we refer readers to [28].

2. (i, j)-nearly Menger bitopological spaces

In this section, we will introduce the (i, j)-nearly Menger property and give some relations with existing covering properties. We will also consider the differences between those properties and the (i, j)-nearly Menger property and necessary condition(s) for their equivalences.

Definition 2.1. *A bispace $(X, \lambda_1, \lambda_2)$ is said to be (i, j)-nearly Menger ((i, j)-nM, for short) if for every sequence $(\mathcal{O}_n)_{n\in\omega}$ of covers of X by λ_i-open sets, it admits a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that each $\mathcal{U}_n$ is a finite subfamily of $\mathcal{O}_n$ and $\{Int^{\lambda_i}(Cl^{\lambda_j}(U)) : U \in \bigcup_{n\in\omega} \mathcal{U}_n\}$ covers X, i.e.,*

$$\bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int^{\lambda_i}(Cl^{\lambda_j}(U)) = X.$$

Example 2.1. *Let $\mathbb{R}$ be equipped with the λ_1 cocountable topology and λ_2 sorgenfrey topology. Then, the bispace $(\mathbb{R}, \lambda_1, \lambda_2)$ is (1,2)-nearly Menger. Indeed, if $(\mathcal{O}_n)_{n\in\omega}$ is any sequence of λ_1-open covers of $\mathbb{R}$, there exists an $O_x \in \mathcal{O}_1$ such that $x \in O_x$ for any fixed $x \in \mathbb{R}$. On the other hand, since $X \setminus O_1$ is* *countable, we can enumerate it as* $\{x_n : n \in \omega\}$*, and we can find* $O_n \in \mathcal{O}_n$ *such that* $x_n \in O_n$ *for each* $n \in \omega$*. If we set*

$$\mathcal{U}_1 = \{O_1, O_x\}$$

and

$$\mathcal{U}_n = \{O_n\}$$

for $n \neq 1$*, each* $\mathcal{U}_n$ *is a finite subfamily of* $\mathcal{O}_n$*, and since each member of* $\mathcal{U}_n$ *is* λ_i*-open,* $\mathbb{R}$ *is covered by* $\{Int^{\lambda_i}(Cl^{\lambda_j}(V)) : V \in \bigcup_{n\in\omega} \mathcal{U}_n\}$.

Example 2.2. *Let X be an uncountable set and* $p \in X$ *be fixed. Let X be equipped with the topologies*

$$\lambda_1 = \{O \subset X : p \notin O \text{ or } p \in O \Rightarrow \mid X \setminus O \mid < \omega\}$$

and

$$\lambda_2 = \{U \subset X : p \in U\} \cup \{\emptyset\}.$$

If $(\mathcal{O}_n)_{n\in\omega}$ *is any sequence of* λ_1*-open covers of X, there exists an* $O_n^p \in \mathcal{O}_n$ *for each* $n \in \omega$ *such that* $p \in O_n^p$*. It can easily be seen that each* O_n^p *is a* λ_2*-dense subset of X. Hence, the sequence* $(\mathcal{O}_n^{'})_{n\in\omega}$ *where each*

$$\mathcal{O}_n^{'} = \{O_n^p\}$$

shows that the bispace $(X, \lambda_1, \lambda_2)$ *is (1,2)-nearly Menger.*

By the definition, following proposition can be noted:

Proposition 2.1. *Let* $(X, \lambda_1, \lambda_2)$ *be a bitopological space. Then:*

(1) If (X, λ_i) *is Menger,* $(X, \lambda_1, \lambda_2)$ *is* (i, j)*-nM.*
(2) If (X, λ_i) *is nearly Menger and the topology* λ_j *is coarser than* λ_i*,* $(X, \lambda_1, \lambda_2)$ *is* (i, j)*-nM.*

Proof. Obvious from the corresponding definitions, and is thus omitted. □

The following examples ensure the inverse statement of Proposition 2.1(1) does not generally hold.

Example 2.3. *Let*

$$X = \mathbb{R}^2$$

and consider X where λ_2 *is the usual topology and* λ_1 *is the deleted diameter topology which admits a subbasis*

$$\mathcal{S} = \{O_\varepsilon(x_0, y_0) : x_0, y_0 \in \mathbb{R}, \varepsilon > 0\} \cup \{O^\varepsilon_{(x_0,y_0)} : x_0, y_0 \in \mathbb{R}, \varepsilon > 0\},$$

where

$$O^\varepsilon_{(x_0,y_0)} = (O_\varepsilon(x_0, y_0) \setminus \{(a, b) : y_0 = b\}) \cup \{(x_0, y_0)\}$$

and $O_\varepsilon(x_0, y_0)$ *is an open disc in* λ_2*. Then,* $(X, \lambda_1, \lambda_2)$ *is (1,2)-nearly Menger. Indeed, since* (X, λ_1) *is nearly Menger (see [29]) and* $\lambda_2 \subset \lambda_1$*, by Proposition 2.1(2),* $(X, \lambda_1, \lambda_2)$ *is (1,2)-nM. But,* (X, λ_1) *fails to be a Menger space since it is not Lindelöf (see also [30]).*

Example 2.4. *Let $\mathbb{R}$ and $\mathbb{P}$ be the set of real numbers and the set of all irrationals, respectively. Let $\mathbb{R}$ be equipped with the following topologies:*

$$\lambda_1 = \{U \subset \mathbb{R} : \mathbb{Q} \subset U\} \cup \{\emptyset\}$$

and

$$\lambda_2 = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \{p\}, \{p\}\},$$

where $\mathbb{Q}$ is set of all rationals and $p \in \mathbb{P}$ is fixed. Then, $(\mathbb{R}, \lambda_1, \lambda_2)$ is (1,2)-nearly Menger. If $(O_n)_{n\in\omega}$ is any sequence of λ_1-open covers of $\mathbb{R}$, there is $O_n^p \in O_n$ for each $n \in \omega$ such that $p \in O_n^p$. On the other hand, each O_n^p contains $\mathbb{Q}$, and hence each O_n^p is a λ_2-dense subset of $\mathbb{R}$. Thus, the sequence $(\mathcal{U}_n)_{n\in\omega}$ where

$$\mathcal{U}_n = \{O_n^p\}$$

is the desired sequence which witnesses for $(\mathbb{R}, \lambda_1, \lambda_2)$ is (1,2)-nearly Menger. But, the family

$$O = \{\mathbb{Q} \cup \{x\} : x \in \mathbb{P}\}$$

is a λ_1-open cover of $\mathbb{R}$ which fails to have a countable subcover. Hence, $(\mathbb{R}, \lambda_1)$ is not Lindelöf. Thus, $(\mathbb{R}, \lambda_1)$ can not be Menger.

Hence, the examples above lead us to a natural question under what condition(s) related properties are equivalent. To this end, we need the following definition.

Definition 2.2. *[31] A bispace $(X, \lambda_1, \lambda_2)$ is said to be (i, j)-regular if for every point $x \in X$ and each λ_i-closed $C \subset X$ such that $x \notin C$, there exists λ_i-open O and λ_j-open U such that $x \in O$, $C \subset U$, and $O \cap U = \emptyset$ $(i \neq j, i, j \in \{1, 2\})$.*

We also can state the following result:

Proposition 2.2. *[31] A bispace $(X, \lambda_1, \lambda_2)$ is (i, j)-regular iff for each $x \in X$ and $x \in O$ where O is a λ_i-open set, there exists a λ_i-open set U providing*

$$x \in U \subset Cl^{\lambda_j}(U) \subset O.$$

Theorem 2.1. *If $(X, \lambda_1, \lambda_2)$ is (i, j)-regular and $(i.j)$-nearly Menger, then (X, λ_i) is Menger.*

Proof. Let $(O_n)_{n\in\omega}$ be any sequence of λ_i-open covers of X. By Proposition 2.2, there exists a λ_i-open cover $\mathcal{U}_n$ for every $n \in \omega$ such that

$$Int^{\lambda_i}(Cl^{\lambda_j}(\mathcal{U}_n)) = \{Int^{\lambda_i}(Cl^{\lambda_j}(U)) : U \in \mathcal{U}_n\}$$

refines O_n. Hence, by the (i, j)-nearly Mengerness of $(X, \lambda_1, \lambda_2)$, there exists $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \omega$ such that each $\mathcal{V}_n$ is finite and

$$\bigcup_{n\in\omega} \bigcup_{V\in\mathcal{V}_n} Int^{\lambda_i}(Cl^{\lambda_j}(V)) = X.$$

Since $Int^{\lambda_i}(Cl^{\lambda_j}(\mathcal{U}_n))$ refines O_n for every $n \in \omega$, then for each $V \in \mathcal{V}_n$, there exists an $O_V \in O_n$ such that

$$Int^{\lambda_i}(Cl^{\lambda_j}(V)) \subset O_V.$$

Hence, the sequence $(O_n')_{n\in\omega}$, where

$$O_n' = \{O_V : V \in \mathcal{V}_n\}$$

shows that (X, λ_i) is Menger. □

Let $\mathcal{O}_{\mathcal{R}}$ denote the family of all (i, j)-regular open sets of a bispace $(X, \lambda_1, \lambda_2)$. For (i, j)-nearly Menger spaces, we have the following.

Theorem 2.2. *Let $(X, \lambda_1, \lambda_2)$ be a bispace. $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger iff X satisfies the selection principle $S_{fin}(\mathcal{O}_{\mathcal{R}}, \mathcal{O}_{\mathcal{R}})$.*

Proof. Let $(X, \lambda_1, \lambda_2)$ be (i, j)-nearly Menger and $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of covers of X such that each member of $\mathcal{O}_n$ is a (i, j)-regular open subset of X. Since $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger and each (i, j)-regular open subset of X is a λ_i-open set, then we can find a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that $\mathcal{U}_n$ is a finite subset of $\mathcal{O}_n$ and

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int^{\lambda_i}(Cl^{\lambda_j}(U)).$$

Since

$$Int^{\lambda_i}(Cl^{\lambda_j}(U)) = U$$

for all $n \in \omega$ and $U \in \mathcal{U}_n$, the sequence $(\mathcal{U}_n)_{n\in\omega}$ shows that X satisfies the selection principle $S_{fin}(\mathcal{O}_{\mathcal{R}}, \mathcal{O}_{\mathcal{R}})$.

Conversely, let X satisfy the selection principle $S_{fin}(\mathcal{O}_{\mathcal{R}}, \mathcal{O}_{\mathcal{R}})$. Let $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of λ_i-open covers of X. For every $n \in \omega$, define

$$\mathcal{U}_n = \{Int^{\lambda_i}(Cl^{\lambda_j}(O)) : O \in \mathcal{O}_n\}.$$

Then, each $\mathcal{U}_n \in \mathcal{O}_{\mathcal{R}}$ (see [28]). Then there is a sequence $(\mathcal{U}'_n)_{n\in\omega}$ such that each $\mathcal{U}'_n \subset \mathcal{U}_n$ is finite and

$$X = \bigcup_{n\in\omega} \bigcup_{V\in\mathcal{U}'_n} V.$$

Now, for each $n \in \omega$ and $V \in \mathcal{U}'_n$, find an $O_V \in \mathcal{O}_n$ such that

$$V = Int^{\lambda_i}(Cl^{\lambda_j}(O_V))$$

and put

$$\mathcal{O}'_n = \{O_V : V \in \mathcal{U}'_n\}.$$

Then, each $\mathcal{O}'_n \subset \mathcal{O}_n$ is finite and

$$X = \bigcup_{n\in\omega} \bigcup_{O_V\in\mathcal{O}'_n} Int^{\lambda_i}(Cl^{\lambda_j}(O_V)) = \bigcup_{n\in\omega} \bigcup_{V\in\mathcal{U}'_n} V$$

holds. Thus, $(X, \lambda_1, \lambda_2)$ is (i, j)-nM. □

Corollary 2.1. *A bitopological space $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger if and only if for every sequence $(\mathcal{O}_n)_{n\in\omega}$ of covers by (i, j)-regular open subsets of X, there is a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that for every $n \in \omega$, $\mathcal{U}_n$ is a finite subfamily of $\mathcal{O}_n$ and*

$$\bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int^{\lambda_i}(Cl^{\lambda_j}(U)) = X.$$

(i, j)-almost Menger and (i, j)-weakly Menger bitopological spaces were introduced [17, 20], respectively. A bitopological space $(X, \lambda_1, \lambda_2)$ is said to be (i, j)-almost Menger ((i, j)-aM) (resp. (i, j)-weakly Menger ((i, j)-wM) if each sequence $(\mathcal{O}_n)_{n\in\omega}$ of λ_i-open covers of X admits a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that each $\mathcal{U}_n \subset \mathcal{O}_n$ is finite and X is covered by

$$\Big\{ \bigcup_{U\in\mathcal{U}_n} Cl^{\lambda_j}(U) : n \in \omega \Big\},$$

resp.

$$X = Cl^{\lambda_j}\Big(\bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} U \Big).$$

In [17], the authors concluded that every (i, j)-aM bitopological space is (i, j)-wM. We can immediately note that every (i, j)-nearly Menger bispace is (i, j)-aM. However, the following examples show that neither of them implies the (i, j)-nearly Menger property.

Example 2.5. *A (1,2)-aM bitopological space which fails to be (1,2)-nearly Menger.*
Let ω_1 be the smallest uncountable ordinal. Let $i \in \omega_1$ and $j \in \omega$. Define

$$A_1 = \{x_{i,j} : j \in \omega, i < \omega_1\} \cup \{y_{i,j} : j \in \omega, i < \omega_1\}, \quad A_2 = \{c_i : i < \omega_1\}$$

and

$$A_3 = \{a, b\},$$

where each $x_{i,j}, y_{i,j}, c_i$, a, and b are distinct points. Let

$$X = A_1 \cup A_2 \cup A_3.$$

Consider X with the topology λ_1 such that the points $\{x_{i,j}\}$ and $\{y_{i,j}\}$ are isolated together with basic neighborhoods

$$N^n_{c_i} = \{c_i\} \cup \{x_{i,j} : j \geq n\} \cup \{y_{i,j} : j \geq n\}, \quad N^\phi_a = \{a\} \cup \{x_{i,j} : i \geq \phi, j \in \omega\}$$

and

$$N^\phi_b = \{b\} \cup \{y_{i,j} : i \geq \phi, j \in \omega\}$$

of the the points $\{c_i\}$, a, and b, respectively [32]. Take

$$\lambda_2 = \lambda_1(\{a\})$$

which is the simple extension of λ_1 such that

$$\lambda_2 = \{U \cup (V \cap \{a\}) : U, V \in \lambda_1\}.$$

According to the simple extension, the following can easily be obtained for each λ_1-open subset U of X:

$$Cl^{\lambda_2}(U) = \begin{cases} Cl^{\lambda_1}(U), & \text{if } a \in U, \\ Cl^{\lambda_1}(U) \setminus \{a\}, & \text{if } a \notin U \text{and } a \in Cl^{\lambda_1}(U), \\ Cl^{\lambda_1}(U), & \text{if } a \notin U \text{and } a \notin Cl^{\lambda_1}(U). \end{cases} \tag{2.1}$$

For seeing that $(X, \lambda_1, \lambda_2)$ is (1,2)-aM, let $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of λ_1-open covers of X. Since (X, λ_1) is almost Menger, there exists a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that for every $n \in \omega$, $\mathcal{U}_n$ is a finite subfamily of $\mathcal{O}_n$ and

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Cl^{\lambda_1}(U).$$

On the other hand, since each $\mathcal{O}_n$ is a λ_1-open cover of X, one can find $U_n^a \in \mathcal{O}_n$ for each $n \in \omega$ such that $a \in U_n^a$. Now, set

$$\mathcal{V}_n = \mathcal{U}_n \cup \{U_n^a\}$$

for each $n \in \omega$. By using Eq (2.1), the sequence $(\mathcal{V}_n)_{n\in\omega}$ guarantees $(X, \lambda_1, \lambda_2)$ is (1,2)-aM. Now, let

$$\mathcal{O} = \{N_{c_i}^0 : i < \omega_1\} \cup \{N_a^1, N_b^1\}$$

and

$$\mathcal{O}_n = \mathcal{O}$$

for each $n \in \omega$. Clearly, $(\mathcal{O}_n)_{n\in\omega}$ is a sequence of λ_1-open covers of X. If $(X, \lambda_1, \lambda_2)$ is (1,2)-nearly Menger, for each $n \in \omega$, there exists a finite subfamily $\mathcal{U}_n$ of $\mathcal{O}_n$ such that

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int^{\lambda_1}(Cl^{\lambda_2}(U)).$$

Without loss of generality, we may assume that each

$$\mathcal{U}_n = \{N_{c_{i_k^n}} : k = 1, 2, 3, ..., m_n\} \cup \{N_a^1, N_b^1\}$$

by the construction of fundamental neighborhoods of λ_1. Now, let

$$\alpha_n = max\{i_k^n : N_{c_{i_k^n}} \in \mathcal{U}_n\}$$

and

$$\beta_0 = sup\{\alpha_n : n \in \omega\}.$$

Then, $\beta_0 < \omega_1$ since each $\mathcal{U}_n$ is finite. If we choose $\beta^* > \beta_0$, then

$$c_{\beta^*} \notin \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int^{\lambda_1}(Cl^{\lambda_2}(U)),$$

see also Eq (2.1). Hence, $(X, \lambda_1, \lambda_2)$ is not (1,2)-nearly Menger.

Example 2.6. *A (1,2)-wM space that is not (1,2)-nearly Menger.*

Let X be set of real numbers and x be an arbitrary irrational point. We choose a sequence $(x_n)_{n\in\omega}$ where each x_n is rational and converging x with respect to the usual topology λ_2. The rational sequence topology λ_1 is defined as each singleton $\{q\}$ for $q \in \mathbb{Q}$ open and selecting the sets

$$B_n^x = \{x\} \cup \bigcup_{i=n}^{\infty} \{x_i\}$$

as a basis for x. In [17], it is shown that the bispace $(X, \lambda_1, \lambda_2)$ is (1,2)-wM. But, it is not (1,2)-nearly Menger. Indeed, consider the sequence $(\mathcal{O}_n)_{n\in\omega}$ of λ_1-open covers of X where

$$\mathcal{O}_n = \{B_n^x : x \in \mathbb{P}\} \cup \{\{q\} : q \in \mathbb{Q}\}$$

for each $n \in \omega$. Each member of $\mathcal{O}_n$ is closed with respect to λ_2 for all $n \in \omega$. Then $(\mathcal{O}_n)_{n\in\omega}$ is a sequence of covers by (1,2)-regular open sets of X. Since $\mathbb{P}$ is uncountable, X fails to satisfy the selection hypothesis $S_{fin}(\mathcal{O}_{\mathcal{R}}, \mathcal{O}_{\mathcal{R}})$, and hence by Theorem 2.2, $(X, \lambda_1, \lambda_2)$ is not (1,2)-nearly Menger.

Recall that a bitopological space $(X, \lambda_1, \lambda_2)$ is (i, j)-almost regular [33] if for every $x \in X$ and (i, j)-regular open set O such that $x \in O$ there is λ_i-open subset U of X such that

$$x \in U \subset Cl^{\lambda_j}(U) \subset O$$

and $(X, \lambda_1, \lambda_2)$ is an (i, j)-weakly P-space [34] if the equality

$$Cl^{\lambda_j}\Big(\bigcup_{n\in\omega} O_n\Big) = \bigcup_{n\in\omega} Cl^{\lambda_j}(O_n)$$

holds for each countable family $\{O_n : n \in \omega\}$ of λ_i-open subsets of X. In light of the given definitions, we have the following:

Theorem 2.3. *Let $(X, \lambda_1, \lambda_2)$ be (i, j)-almost regular and an (i, j)-weakly P-space. The following are equivalent:*

(1) $(X, \lambda_1, \lambda_2)$ is (i, j)-nM,
(2) $(X, \lambda_1, \lambda_2)$ is (i, j)-aM,
(3) $(X, \lambda_1, \lambda_2)$ is (i, j)-wM.

Proof. (3) ⇒ (1) Let $(X, \lambda_1, \lambda_2)$ be (i, j)-*wM* and $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of λ_i-open covers of X. Set

$$\mathcal{U}_n = \{Int^{\lambda_i}(Cl^{\lambda_j}(O)) : O \in \mathcal{O}_n\}$$

for each $n \in \omega$. Then, we obtain a sequence $(\mathcal{U}_n)_{n\in\omega}$ of covers by (i, j)-regular open sets of X. Let $x \in X$. There exists an $U_n^x \in \mathcal{U}_n$ for each $n \in \omega$ such that $x \in U_n^x$. Since $(X, \lambda_1, \lambda_2)$ is (i, j)-almost regular, we can find a $V_n^x \in \lambda_i$ such that

$$x \in V_n^x \subset Cl^{\lambda_j}(V_n^x) \subset U_n^x.$$

So, we get a sequence $(\mathcal{V}_n)_{n\in\omega}$ of λ_i-open covers of X where

$$\mathcal{V}_n = \{V_n^x : x \in X\}$$

for each $n \in \omega$. Hence, by (3) there is a sequence $(\mathcal{V}'_n)_{n\in\omega}$ such that each $\mathcal{V}'_n$ is a finite subfamily of $\mathcal{V}_n$ and

$$X = Cl^{\lambda_j}\Big(\bigcup_{n\in\omega} \bigcup_{V\in\mathcal{V}'_n} V\Big).$$

Since $\bigcup_{n\in\omega} \mathcal{V}'_n$ is countable and $(X, \lambda_1, \lambda_2)$ is an (i, j)-weakly P-space, we have

$$X = Cl^{\lambda_j}\Big(\bigcup_{n\in\omega} \bigcup_{V\in\mathcal{V}'_n} V\Big) = \bigcup_{n\in\omega} \bigcup_{V\in\mathcal{V}'_n} Cl^{\lambda_j}(V).$$

Now, select $O_V \in \mathcal{O}_n$ for each $n \in \omega$ and $V \in \mathcal{V}_n'$ such that

$$Cl^{\lambda_j}(V) \subset Int^{\lambda_i}(Cl^{\lambda_j}(O_V))$$

and set

$$\mathcal{O}_n' = \{O_V : V \in \mathcal{V}_n'\}.$$

Then, the sequence $(\mathcal{O}_n')_{n\in\omega}$ is the desired sequence which completes the proof. □

Corollary 2.2. *Let $(X, \lambda_1, \lambda_2)$ be (i, j)-regular and an (i, j)-weakly P-space. The following statements are equivalent:*

(1) (X, λ_i) is Menger,
(2) $(X, \lambda_1, \lambda_2)$ is (i, j)-nM,
(3) $(X, \lambda_1, \lambda_2)$ is (i, j)-aM,
(4) $(X, \lambda_1, \lambda_2)$ is (i, j)-wM.

3. Results related to (i, j)-nearly Menger bispaces

The section is for examining the properties of (i, j)-nearly Menger bispaces. We will present some preservation properties of this property under subspaces, products, and some certain type of mappings.

We first deal with subspaces of (i, j)-nearly Menger bispaces. We immediately note that a subset Y of $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger if it is (i, j)-nearly Menger with respect to subspace topologies λ_1^Y and λ_2^Y induced by λ_1 and λ_2, respectively.

We first state that the (i, j)-nearly Menger property is not invariant with respect to subspaces, and moreover to closed (closed with respect to both topologies) subspaces.

Example 3.1. *There exists a (1,2)-nearly Menger bitopological space $(X, \lambda_1, \lambda_2)$ whose λ_1- and λ_2 closed subspace fails to be (1,2)-nearly Menger.*

Let X be an uncountable set and $w \notin X$. Let

$$X^* = X \cup \{w\}.$$

Consider X^* with the topologies

$$\lambda_1 = \{U \subset X^* : \{p, w\} \subset U\} \cup \{\emptyset\},$$

where $p \in X$ is fixed and

$$\lambda_2 = \{U \subset X^* : w \in U\} \cup \{\emptyset\}.$$

$(X^*, \lambda_1, \lambda_2)$ is a (1,2)-nearly Menger bitopological space since each non-empty λ_1-open subset is dense in (X^*, λ_2). On the other hand, $(X, \lambda_1^X, \lambda_2^X)$ is not (1,2)-nearly Menger. Indeed,

$$\mathcal{O} = \{\{p, x\} : x \in X\}$$

is a λ_1^X-open cover of X which does not have a finite subcover, and if

$$\mathcal{O}_n = \mathcal{O}$$

for each $n \in \omega$, $(\mathcal{O}_n)_{n\in\omega}$ is a sequence of λ_1^X-open covers of X. But, since (X, λ_2^X) is discrete, hence $(X, \lambda_1^X, \lambda_2^X)$ is not (1,2)-nearly Menger.

Recall [35] that a subset Y of a bispace $(X, \lambda_1, \lambda_2)$ is called *open, (resp. closed)* if Y is open (resp. closed) with respect to both λ_1 and λ_2. Y is called (i, j)-clopen if it is λ_i-closed and λ_j-open. Y is said to be clopen if it is both (i, j)-clopen and (j, i)-clopen.

Proposition 3.1. *An open subspace $(Y, \lambda_1^Y, \lambda_2^Y)$ of a bitopological space $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger iff every sequence $(\mathcal{O}_n)_{n\in\omega}$ of covers of Y by λ_i-open subsets of X admits a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that $\mathcal{U}_n \subset \mathcal{O}_n$ is finite and Y is covered by the family*

$$\{Int^{\lambda_i}(Cl^{\lambda_j}(U)) : U \in \bigcup_{n\in\omega} \mathcal{U}_n\}.$$

Proof. $\Rightarrow$ Let $(\mathcal{O}_n)_{n\in\omega}$ be given as mentioned in the proposition. Define

$$\mathcal{O}_n' = \{Y \cap O : O \in \mathcal{O}_n\}$$

for every $n \in \omega$. Then, each $\mathcal{O}_n'$ is a λ_i^Y-open cover of Y. Hence, by the assumption, there exists a sequence $(\mathcal{U}_n')_{n\in\omega}$ such that $\mathcal{U}_n'$ is a finite subfamily of $\mathcal{O}_n'$ for every $n \in \omega$ and

$$Y = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n'} Int^{\lambda_i^Y}(Cl^{\lambda_j^Y}(U)). \tag{3.1}$$

On the other hand, there is $O_U \in \mathcal{O}_n$ for every $U \in \mathcal{U}_n'$ such that

$$U = O_U \cap Y.$$

Set

$$\mathcal{U}_n = \{O_U : U \in \mathcal{U}_n'\}$$

for each $n \in \omega$. Each $\mathcal{U}_n$ is finite, and since Y is both λ_i- and λ_j-open, we have

$$\begin{aligned} Int^{\lambda_i^Y}(Cl^{\lambda_j^Y}(U)) &= Int^{\lambda_i}\Big(Cl^{\lambda_j}(U) \cap Y\Big) \\ &\subset Int^{\lambda_i}(Cl^{\lambda_j}(Y \cap U)) \\ &= Int^{\lambda_i}(Cl^{\lambda_j}(Y \cap O_U)) \\ &\subset Int^{\lambda_i}(Cl^{\lambda_j}(O_U)). \end{aligned}$$

Hence, by (3.1), we have that

$$Y \subset \bigcup_{n\in\omega} \bigcup_{O_U\in\mathcal{U}_n} Int^{\lambda_i}(Cl^{\lambda_j}(O_U))$$

holds.

$\Leftarrow$ Conversely, let $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of λ_i^Y-open covers of Y. Then, for each $n \in \omega$ and $O \in \mathcal{O}_n$ there exists a $U_O \in \lambda_i$ such that

$$O = Y \cap U_O.$$

Then, $(\mathcal{U}_n)_{n\in\omega}$ is a sequence of covers of Y by λ_i-open subsets of X where

$$\mathcal{U}_n = \{U_O : O \in \mathcal{O}_n\}$$

for each $n \in \omega$. Then there exists a sequence $(\mathcal{U}_n')_{n\in\omega}$ such that each $\mathcal{U}_n' \subset \mathcal{U}_n$ is finite and

$$Y \subset \bigcup_{n\in\omega} \bigcup_{U_O\in\mathcal{U}_n} Int^{\lambda_i}(Cl^{\lambda_j}(U_O)) \tag{3.2}$$

holds. Set

$$O_n' = \{O : U_O \in \mathcal{U}_n'\}$$

for each $n \in \omega$. It is clear that each $O_n' \subset O_n$ is finite, and since Y is open,

$$\begin{aligned} Int^{\lambda_i^Y}(Cl^{\lambda_j^Y}(O)) &= Int^{\lambda_i}(Cl^{\lambda_j}(U_O \cap Y)) \\ &= Int^{\lambda_i}(Cl^{\lambda_j}(U_O)) \cap Y \end{aligned}$$

holds, and thus by Eq (3.2), $(Y, \lambda_1^Y, \lambda_2^Y)$ is (i, j)-nearly Menger. □

Theorem 3.1. *Open and λ_i-closed (hence clopen) subspaces of an (i, j)-nM bispace $(X, \lambda_1, \lambda_2)$ is (i, j)-nM.*

Proof. Let $(Y, \lambda_1^Y, \lambda_2^Y)$ be an open and λ_j-closed subspace of $(X, \lambda_1, \lambda_2)$. If $(O_n)_{n\in\omega}$ is a sequence of covers of Y by λ_i-open subsets of X, then since Y is λ_i-closed and

$$O_n' = O_n \cup \{X \setminus Y\}$$

is a λ_i-open cover for X. Hence, by assumption, there exists a subfamily $\mathcal{U}_n'$ of O_n' for each $n \in \omega$ such that

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n'} Int^{\lambda_i}(Cl^{\lambda_j}(U))$$

holds. Now, for each $n \in \omega$, set

$$\mathcal{U}_n = \{U : U \in \mathcal{U}_n', U \neq X \setminus Y\}.$$

It is clear that for every $n \in \omega$ $\mathcal{U}_n$ is a finite subfamily of O_n, and since $X \setminus Y$ is both λ_i-open and λ_j-closed, we have

$$Y \subset \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} Int^{\lambda_i}(Cl^{\lambda_j}(U)).$$

Thus, it follows by Proposition 3.1. □

We now consider the behavior of the (i, j)-nearly Menger property under certain types of mappings.

Definition 3.1. *[36] Let $(X, \lambda_1, \lambda_2)$ and (Y, ζ_1, ζ_2) be topological spaces and*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

be a mapping. f is called (i, j)-almost continuous if $f^{-1}(O) \in \lambda_i$ for each (i, j)-regular open subset O of Y. If f is both (1,2)- and (2,1)-almost continuous, then f is called a pairwise almost continuous (or p-almost continuous, for short) mapping.

Definition 3.2. *[28] A mapping*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

is d-open if the mappings

$$f_i : (X, \lambda_i) \to (Y, \zeta_i)$$

are open ($i = 1, 2$).

Theorem 3.2. *Let $(X, \lambda_1, \lambda_2)$ be (i, j)-nearly Menger and*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

be p-almost continuous and an i-open surjection. Then, (Y, ζ_1, ζ_2) is (i, j)-nearly Menger.

Proof. Let $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of covers by (i, j)-regular open sets of Y. Define

$$\mathcal{U}_n = \{f^{-1}(O) : O \in \mathcal{O}_n\}$$

for every $n \in \omega$. We obtain a sequence $(\mathcal{U}_n)_{n\in\omega}$ of λ_i-open covers of X, since f is (i, j)-almost continuous. Then, by the assumption, there is a sequence $(\mathcal{U}'_n)_{n\in\omega}$ such that each $\mathcal{U}'_n \subset \mathcal{U}_n$ is a finite subfamily, and the family

$$\{Int^{\lambda_i}(Cl^{\lambda_j}(O_{U'})) : U' \in \bigcup_{n\in\omega} \mathcal{U}'_n\}$$

covers X. On the other hand, we can find an $O_{U'} \in \mathcal{O}_n$ for each $n \in \omega$ and $U' \in \mathcal{U}'_n$ such that

$$U' = f^{-1}(O_{U'}).$$

Now set

$$\mathcal{V}_n = \{O_{U'} : U' \in \mathcal{U}'_n\}$$

for each $n \in \omega$. We now show that the family

$$\mathcal{O} = \left\{Int^{\zeta_i}(Cl^{\zeta_j}(O_{U'})) = O_{U'} : U' \in \bigcup_{n\in\omega} \mathcal{U}'_n\right\}$$

covers Y. For showing that $\mathcal{O}$ covers Y, Let

$$y = f(x) \in Y$$

be arbitrary. There is $n \in \omega$ and $U' \in \mathcal{U}'_n$ such that $x \in Int^{\lambda_i}(Cl^{\lambda_j}(U'))$, and thus

$$x \in Int^{\lambda_i}(Cl^{\lambda_j}(f^{-1}(O_{U'})).$$

On the other hand, $O_{U'}$ is an (i, j)-regular open set. Hence, we have

$$Cl^{\zeta_j}(O_{U'}) = Cl^{\zeta_j}(Int^{\zeta_i}\Big(Cl^{\zeta_j}(O_{U'})\Big))$$

thus $Cl^{\zeta_j}(O_{U'})$ is a (j, i)-regular closed set. $Y \setminus Cl^{\zeta_j}(O_{U'})$ is a (j, i)-regular open subset of Y, and hence, by the (j, i)-almost continuity of f, $f^{-1}(Cl^{\zeta_j}(O_{U'}))$ is a λ_j-closed subset of X. Then, since

$$Cl^{\lambda_j}(f^{-1}(O_{U'})) \subset f^{-1}(Cl^{\zeta_j}(O_{U'}))$$

and f is i-open, we have

$$\begin{aligned} y = f(x) &\in f\Big(Int^{\lambda_i}(Cl^{\lambda_j}(f^{-1}(O_{U'})))\Big) \\ &\subset Int^{\zeta_i}\Big(f(Cl^{\lambda_j}(f^{-1}(O_{U'})))\Big) \\ &\subset Int^{\zeta_i}\Big(f(f^{-1}(Cl^{\zeta_j}(O_{U'})))\Big) \\ &= Int^{\zeta_i}(Cl^{\zeta_j}(O_{U'})) \\ &= O_{U'}. \end{aligned}$$

And, since each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{O}_n$, then by Theorem 2.2, (Y, ζ_i, ζ_j) is (i, j)-nearly Menger.
The proof is complete. □

Recall that the mapping $(X, \lambda_1, \lambda_2) \to (Y, \zeta_i, \zeta_j)$ is called d-continuous if the mappings

$$f_i : (X, \lambda_i) \to (Y, \zeta_i)$$

are continuous ($i = 1, 2$). Since each d-continuous mapping is pairwise almost continuous, the following corollary is immediate.

Corollary 3.1. *(i, j)-nearly Menger property invariant under d-continuos and i-open surjections.*

Recall [37] a mapping

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

is said to be (i, j)-precontinuous if

$$f^{-1}(O) \subset Int^{\lambda_i}(Cl^{\lambda_j}(f^{-1}(O)))$$

holds for each ζ_i-open subset O of Y and is said to be (i, j)-contracontinuous if $f^{-1}(O)$ is λ_j-closed for each ζ_i-open subset O of Y.

Theorem 3.3. *If*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

is both (i, j)-precontinuous and (i, j)-contracontinuous, and $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger, then (Y, ζ_i) is Menger.

Proof. Let $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of ζ_i-open covers of Y. Since f is (i, j)-contracontinuous, $f^{-1}(O)$ is λ_j-closed for each $n \in \omega$ and $O \in \mathcal{O}_n$. Moreover, since f is (i, j)-precontinuous, we have

$$f^{-1}(O) \subset Int^{\lambda_i}(Cl^{\lambda_j}(f^{-1}(O))).$$

Thus, each $f^{-1}(O)$ is a λ_i-open subset of X. Set

$$\mathcal{U}_n = \{f^{-1}(O) : O \in \mathcal{O}_n\}$$

for each $n \in \omega$. Then, we obtain a sequence $(\mathcal{U}_n)_{n\in\omega}$ of λ_i-open covers of X. Since $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger, there exists a sequence $(\mathcal{U}'_n)_{n\in\omega}$ such that each $\mathcal{U}'_n$ is a finite subfamily of $\mathcal{U}_n$ and

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}'_n} Int^{\lambda_i}(Cl^{\lambda_j}(U)).$$

Now, for each $n \in \omega$ and $U \in \mathcal{U}_n'$, assign an $O_U \in \mathcal{O}_n$ such that

$$f^{-1}(O_U) = U.$$

Then, the sequence $(\mathcal{O}_n')_{n\in\omega}$ where each

$$\mathcal{O}_n' = \{O_U : U \in \mathcal{U}_n'\}$$

is the desired sequence, since

$$\begin{aligned} f(X) = Y &= f\Big(\bigcup_{n\in\omega}\bigcup_{U\in\mathcal{U}_n'} Int^{\lambda_i}(Cl^{\lambda_j}(U))\Big) \\ &= \bigcup_{n\in\omega}\bigcup_{U\in\mathcal{U}_n'} f(Int^{\lambda_i}(Cl^{\lambda_j}(U))) \\ &= \bigcup_{n\in\omega}\bigcup_{U\in\mathcal{U}_n'} f(Int^{\lambda_i}(Cl^{\lambda_j}(f^{-1}(O_U))) \\ &= \bigcup_{n\in\omega}\bigcup_{U\in\mathcal{U}_n'} f(f^{-1}(O_U)) \\ &\subset \bigcup_{n\in\omega}\bigcup_{U\in\mathcal{U}_n'} O_U. \end{aligned}$$

The proof is complete. □

However, the (i, j)-nearly Menger property is not inverse invariant under d-continuous and i-open surjective mappings. Indeed, let X be set of real numbers and $p \in \mathbb{P}$ be fixed. Let

$$\lambda_1 = \{U \subset X : p \in U\} \cup \{\emptyset\}, \quad \lambda_2 = \{U \subset X : p \notin U\} \cup \{X\}, \quad \zeta_1 = \{U \cap \mathbb{P} : U \in \lambda_1\} \cup \{X\}$$

and ζ_2 be indiscrete topology on X, respectively. It can easily be seen that the bispace (X, ζ_1, ζ_2) is (1,2)-nearly Menger. But, considering the identity function

$$f : (X, \lambda_1, \lambda_2) \to (X, \zeta_1, \zeta_2),$$

it is a 1-open and d-continuous surjection. On the other hand,

$$\mathcal{O} = \{\{p, x\} : x \in X\}$$

is a λ_1-open cover of X. But, the sequence $(\mathcal{O}_n)_{n\in\omega}$ where each

$$\mathcal{O}_n = \mathcal{O}$$

shows that $(X, \lambda_1, \lambda_2)$ is not (1,2)-nearly Menger.

To this end, we need following definitions:

Definition 3.3. *[28] A bispace $(X, \lambda_1, \lambda_2)$ is d-compact if (X, λ_i) is compact for $i = 1, 2$.*

Definition 3.4. *[28]*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

is d-closed if the mappings

$$f_i : (X, \lambda_i) \to (Y, \zeta_i)$$

are closed ($i = 1, 2$).

Definition 3.5. *[38] A d-closed and d-continuous mapping*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

is perfect if for every $y \in Y$, the fiber $f^{-1}(y)$ is a d-compact subset of X.

Definition 3.6. *[37] A mapping*

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

is called (i, j)-completely continuous if $f^{-1}(U)$ is (i, j)-regular open for each i-open subset U of Y.

We can now give the following result:

Theorem 3.4. *The (i, j)-nearly Menger property is inverse invariant under d-open, (i, j)-completely continuous perfect mappings.*

Proof. Let (Y, ζ_1, ζ_2) be (i, j)-nearly Menger and

$$f : (X, \lambda_1, \lambda_2) \to (Y, \zeta_1, \zeta_2)$$

be a d-open, (i, j)-completely continuous perfect mapping. Let $(\mathcal{O}_n)_{n\in\omega}$ be a sequence of covers of X by (i, j)-regular open sets. For each $y \in Y$, the fiber $f^{-1}(y)$ is a λ_i-compact subset of X, and hence we can find a finite $\mathcal{U}_n^y \subset \mathcal{O}_n$ such that

$$f^{-1}(y) \subset \bigcup \mathcal{U}_n^y = U_n^y.$$

On the other hand, since f is i-closed and U_n^y is λ_i-open, then

$$W_n^y = Y \setminus f(X \setminus U_n^y)$$

is a ζ_i-open subset of Y that contains y. Then, we obtain a sequence $(\mathcal{W}_n)_{n\in\omega}$ of ζ_i-open covers of Y, where for each $n \in \omega$,

$$\mathcal{W}_n = \{W_n^y : y \in Y\}.$$

Hence, by the assumption, there exists a finite $\mathcal{W}_n' \subset \mathcal{W}_n$ for every $n \in \omega$ such that the family

$$\{Int^{\zeta_i}(Cl^{\zeta_j}(W)) : W \in \bigcup_{n\in\omega} \mathcal{W}_n'\}$$

covers Y. Now let F_n be a finite subset of Y for each $n \in \omega$. We may assume that

$$\mathcal{W}_n' = \{W_n^y : y \in F_n\}.$$

Let

$$\mathcal{O}_n' = \bigcup_{y\in F_n} \mathcal{U}_n^y.$$

Clearly, $O_n^{'}$ is a finite subfamily of O_n for each $n \in \omega$, and with the fact that f is d-open and (i, j)-completely continuous, we have

$$\begin{aligned} f^{-1}(Y) = X &= f^{-1}\Big(\bigcup_{n\in\omega}\bigcup_{y\in F_n} Int^{\zeta_i}(Cl^{\zeta_j}(W_n^y))\Big) \\ &= \bigcup_{n\in\omega}\bigcup_{y\in F_n} f^{-1}\Big(Int^{\zeta_i}(Cl^{\zeta_j}(W_n^y))\Big) \\ &\subset \bigcup_{n\in\omega}\bigcup_{y\in F_n} Int^{\lambda_i}(Cl^{\lambda_j}(f^{-1}(W_n^y))) \\ &\subset \bigcup_{n\in\omega}\bigcup_{y\in F_n} Int^{\lambda_i}(Cl^{\lambda_j}(U_n^y)) \\ &= \bigcup_{n\in\omega}\bigcup_{y\in F_n} U_n^y \\ &= \bigcup_{n\in\omega}\bigcup_{y\in F_n} \Big(\cup\, \mathcal{U}_n^y\Big) \\ &= \bigcup_{n\in\omega}\bigcup_{U\in O_n^{'}} U. \end{aligned}$$

Hence, by Theorem 2.2, $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly Menger.

The proof is complete. □

We now will investigate the productivity of the (i, j)-nearly Menger property. We immediately note that the product of (i, j)-nearly Menger bitopological spaces need not be (i, j)-nearly Menger.

Example 3.2. *Let X be set of real numbers. Consider X with the Sorgenfrey topology λ_1 and discrete topology λ_2. Let τ denote the usual topology on X. Lelek showed in [39] that for every Lusin subset L in (X, τ), $i^{-1}(L)$ is a Menger (hence nearly Menger) subset in (X, λ_1), where*

$$i : (X, \lambda_1) \to (X, \tau)$$

is an identity map. And, he stated that $i^{-1}(L) \times i^{-1}(L)$ is not Menger whenever

$$(L \times L) \cap A$$

is uncountable where

$$A = \{(x, y) : x + y = 0\}.$$

Now, consider the subspace

$$(\mathcal{L} = i^{-1}(L) \times i^{-1}(L), (\lambda_1^{\mathcal{L}})^2, (\lambda_2^{\mathcal{L}})^2)$$

of $(X^2, \lambda_1^2, \lambda_2^2)$. It can easily be seen that $(\mathcal{L}, (\lambda_1^{\mathcal{L}})^2, (\lambda_2^{\mathcal{L}})^2)$ is a (1,2)-regular subspace of $(X^2, \lambda_1^2, \lambda_2^2)$. On the other hand, since $(\mathcal{L}, (\lambda_1^{\mathcal{L}})^2)$ is not a Menger space, then by the Theorem 2.1, $(\mathcal{L}, (\lambda_1^{\mathcal{L}})^2, (\lambda_2^{\mathcal{L}})^2)$ is not (1,2)-nearly Menger.

Recall [40] that $(X, \lambda_1, \lambda_2)$ is (i, j)-nearly compact if for each λ_i-open cover O of X there exists a finite subfamily $\mathcal{U} \subset O$ such that

$$X = \bigcup_{U\in\mathcal{U}} Int^{\lambda_i}(Cl^{\lambda_j}(U)).$$

$(X, \lambda_1, \lambda_2)$ is said to be pairwise nearly compact (p-nearly compact, for short) if it is both (1,2)- and (2,1)-nearly compact.

Before we give the behavior of the product of (i, j)-nearly Menger bispaces and (i, j)-nearly compact bispaces, it can easily be seen that every (i, j)-nearly compact bispace is (i, j)-nearly Menger. However, (i, j)-nearly Menger bispaces need not be (i, j)-nearly compact as the following example illustrates:

Example 3.3. *Let $\mathbb{Z}$ be the set of all integers. If we consider $\mathbb{Z}$ with the topologies*

$$\lambda_1 = \{O \subset \mathbb{Z} : 1 \in \mathbb{Z}\} \cup \{\emptyset\}$$

and

$$\lambda_2 = \{O \subset \mathbb{Z} : 1 \notin \mathbb{Z}\} \cup \{\mathbb{Z}\},$$

the bispace $(\mathbb{Z}, \lambda_1, \lambda_2)$ is (1,2)-nearly Menger, but it is not (1,2)-nearly compact. Regardless of the topologies λ_1 and λ_2, since $\mathbb{Z}$ is countable, it can easily be seen that it is (1,2)-nearly Menger. Now, consider the family

$$\mathcal{O} = \{\{1, n\} : n \in \mathbb{Z}\}.$$

$\mathcal{O}$ is a λ_1-open cover of $\mathbb{Z}$. On the other hand, the set $\{1, n\}$ is a λ_2-closed subset of $\mathbb{Z}$, and hence

$$Int^{\lambda_1}(Cl^{\lambda_2}(\{1, n\})) = \{1, n\}$$

holds for each $n \in \mathbb{Z}$, and thus one can not find a finite subfamily of $\mathcal{O}$ whose interiors of closures of its members cover $\mathbb{Z}$.

We now return to products. We have the following result:

Theorem 3.5. *The product of an (i, j)-nearly Menger bispace and an (i, j)-nearly compact bispace is (i, j)-nearly Menger.*

Proof. Let $(X, \lambda_1, \lambda_2)$ be (i, j)-nearly Menger and (Y, ζ_1, ζ_2) be (i, j)-nearly compact bitopological spaces. If $(\mathcal{O}_n)_{n\in\omega}$ is a sequence of $\lambda_i \times \zeta_i$-open covers of $X \times Y$, we may assume that for each $n \in \omega$,

$$\mathcal{O}_n = \mathcal{U}_n \times \mathcal{V}_n,$$

where $\mathcal{U}_n$ is a λ_i-open cover of X and $\mathcal{V}_n$ is a ζ_i-open cover of Y. Hence, by the assumptions, one can find a sequence $(\mathcal{U}_n')_{n\in\omega}$ such that $\mathcal{U}_n'$ is a finite subfamily of $\mathcal{U}_n$ for each $n \in \omega$ such that

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n'} Int^{\lambda_i}(Cl^{\lambda_j}(U))$$

and a finite subfamily $\mathcal{V}_n'$ for each $n \in \omega$ such that

$$Y = \bigcup_{V\in\mathcal{V}_n'} Int^{\zeta_i}(Cl^{\zeta_j}(V)).$$

Define

$$\mathcal{O}_n' = \mathcal{U}_n' \times \mathcal{V}_n'$$

for each $n \in \omega$. Then, the sequence $(\mathcal{O}_n^{'})_{n\in\omega}$ shows that the bitopological space $(X \times Y, \lambda_1 \times \zeta_1, \lambda_2 \times \zeta_2)$ is (i, j)-nearly Menger. Indeed, it is clear that each $\mathcal{O}_n^{'}$ is a finite subfamily of $\mathcal{O}_n$. Let

$$(x, y) \in X \times Y$$

be arbitrary. Then, we can find $n_0 \in \omega$ and $U_x \in \mathcal{U}_{n_0}^{'}$ such that

$$x \in Int^{\lambda_i}(Cl^{\lambda_j}(U_x)).$$

On the other hand, there exists a $V_y^{n_0} \in \mathcal{V}_{n_0}^{'}$ such that

$$y \in Int^{\zeta_i}(Cl^{\zeta_j}(V_y^{n_0})).$$

Hence,

$$(x, y) \in Int^{\lambda_i}(Cl^{\lambda_j}(U_x)) \times Int^{\zeta_i}(Cl^{\zeta_j}(V_y^{n_0})) = Int^{\lambda_i\times\zeta_i}(Cl^{\lambda_j\times\zeta_j}(U_x \times V_y^{n_0})).$$

Thus,

$$X \times Y = \bigcup_{n\in\omega} \bigcup_{O\in\mathcal{O}_n^{'}} Int^{\lambda_i\times\zeta_i}(Cl^{\lambda_j\times\zeta_j}(O)),$$

which completes the proof. □

Corollary 3.2. *The product of an (i, j)-nearly Menger bispace and d-compact bispace is (i, j)-nearly Menger.*

Recall [3] that an open cover $\mathcal{O}$ of a topological space (X, λ) is an ω-cover of X if $X \notin \mathcal{O}$ and there is a $U_F \in \mathcal{O}$ for each finite $F \subset X$ such that $F \subset U_F$. Let $\mathcal{O}_\Omega^{\lambda_i}$ denote the family of all λ_i-ω covers of the bispace $(X, \lambda_1, \lambda_2)$. Let $\mathcal{F}_\Omega$ denotes the family of all collections $\mathcal{O}$ of subsets of X such that, for each finite $F \subset X$, there exists an $O_F \in \mathcal{O}$ such that

$$F \subset Int^{\lambda_i}(Cl^{\lambda_j}(O_F)).$$

We have the following for finite powers of bitopological spaces.

Theorem 3.6. *Let $(X, \lambda_1, \lambda_2)$ be a bitopological space. If the product bitopological space $(X^n, \lambda_1^n, \lambda_2^n)$ $(n \in \omega)$ is (i, j)-nearly Menger, then X has selection hypothesis $S_{fin}(\mathcal{O}_\Omega^{\lambda_i}, \mathcal{F}_\Omega)$.*

Proof. Let $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of λ_i-ω-covers of X. If $\{P_m : m \in \omega\}$ is any partition of ω, let

$$\mathcal{O}_{k,m} = \{(O)^m : O \in \mathcal{O}_k\}$$

for each $m \in \omega$ and $k \in P_m$. Then, we obtain a sequence $(\mathcal{O}_{k,m})_{k\in P_m}$ of λ_i^m-open covers of the product space X^m. Hence, by the assumption, there exists a finite

$$\mathcal{U}_{k,m} \subset \mathcal{O}_{k,m}$$

for every $k \in P_m$ such that

$$X^m = \bigcup_{k\in P_m} \bigcup_{U\in\mathcal{U}_{k,m}} Int^{\lambda_i^m}(Cl^{\lambda_j^m}(U)).$$

For each $k \in P_m$ and $U \in \mathcal{U}_{k,m}$, we can find an $O_U \in \mathcal{O}_k$ such that

$$(O_U)^m = U.$$

Take

$$\mathcal{U}_k = \{O_U : U \in \mathcal{U}_{k,m}\}$$

for each $k \in \omega$. Then, the sequence $(\mathcal{U}_n)_{n\in\omega}$ is the desired sequence: It is clear that each $\mathcal{U}_n$ is a finite subfamily of $\mathcal{O}_n$. On the other hand, if

$$F = \{x_1, x_2, ..., x_t\} \subset X,$$

then

$$(x_1, x_2, ..., x_t) \in X^t.$$

Thus, we can find a $k \in P_t$ and $U \in \mathcal{U}_{k,t}$ such that

$$(x_1, x_2, ..., x_t) \in Int^{\lambda_i^t}(Cl^{\lambda_j^t}(O_U)) = \left(Int^{\lambda_i}(Cl^{\lambda_j}(O_U))\right)^t$$

for an $O_U \in \mathcal{O}_k$. Hence,

$$F \subset Int^{\lambda_i}(Cl^{\lambda_j}(O_U)).$$

The proof is complete. □

For the converse statement of the above theorem, recall [28] that a bitopological space is called (i, j)-extremally disconnected if

$$Cl^{\lambda_j}(O) = Int^{\lambda_i}(Cl^{\lambda_j}(O))$$

for each λ_i-open subset O of X.

Theorem 3.7. *Let $(X, \lambda_1, \lambda_2)$ be (i, j)-extremally disconnected. If X satisfies the selection principle $S_{fin}(\mathcal{O}_\Omega^{\lambda_i}, \mathcal{F}_\Omega)$, then $(X^n, \lambda_1^n, \lambda_2^n)$ is (i, j)-nearly Menger for every $n \in \omega$.*

Proof. Let $k \in \omega$ and $(\mathcal{O}_n)_{n\in\omega}$ be any sequence of λ_i^k-open covers of X^k. Let

$$\mathcal{O}_n = \{O_{n,s} : s \in S_n\}$$

for each $n \in \omega$ where S_n is an index set. If $F \subset X$ is any finite subset, then $(F)^k \subset X^k$ is finite and thus λ_i^k-compact. Hence, for each $n \in \omega$, there exists a finite subfamily $\mathcal{O}_{n,F} \subset \mathcal{O}_n$ for each $n \in \omega$ such that

$$F \subset \bigcup \mathcal{O}_{n,F}.$$

Let $J_{n,F} \subset S_n$ be finite for each $n \in \omega$ and

$$\mathcal{O}_{n,F} = \{O_{n,s} : s \in J_{n,F}\}.$$

On the other hand, there is a $U_F \in \lambda_i$ such that $F \subset U_F$ and

$$(U_F)^k \subset \bigcup_{s\in J_{n,F}} O_{n,s}.$$

Hence, the family

$$\mathcal{U}_n = \{U_F : F \subset X \text{ is finite }\}$$

is a λ_i-ω cover of X. Hence, by the assumption, there exists a finite $\mathcal{U}_n^{'} \subset \mathcal{U}_n$ for each $n \in \omega$ such that, for every finite $F \subset X$, there exists $n \in \omega$ and $U \in \mathcal{U}_n^{'}$ such that

$$F \subset Int^{\lambda_i}(Cl^{\lambda_j}(U)).$$

Let

$$\mathcal{U}_n^{'} = \{U_{F_t} : t \in T_n\},$$

where T_n is a finite index set for each $n \in \omega$. With the fact that $(X, \lambda_1, \lambda_2)$ is (i, j)-extremally disconnected,

$$X^k = \bigcup_{n\in\omega}\Big(\bigcup_{s\in \bigcup_{t\in T_n} J_{n,F_t}} Int^{\lambda_i^k}(Cl^{\lambda_j^k}(O_{n,s}))\Big) \tag{3.3}$$

holds. Indeed, for any k-tuples $(x_1, x_2, ..., x_k) \in X^k$,

$$A = \{x_1, x_2, ..., x_k\}$$

is a finite subset of X. Hence, there exists $n \in \omega$ and $V \in \mathcal{U}_n^{'}$ such that

$$A \subset Int^{\lambda_i}(Cl^{\lambda_j}(V)).$$

Let

$$V = U_{F_t}$$

for a $t \in T_n$. Hence,

$$\begin{aligned}(A)^k &\subset \Big(Int^{\lambda_i}(Cl^{\lambda_j}(U_{F_t}))\Big)^k \\ &= Int^{\lambda_i^k}(Cl^{\lambda_j^k}((U_{F_t})^k)) \\ &\subset Int^{\lambda_i^k}(Cl^{\lambda_j^k}(\bigcup_{s\in J_{n,F_t}} O_{n,s})) \\ &= \bigcup_{s\in J_{n,F_t}} Cl^{\lambda_j^k}(O_{n,s}) \\ &= \bigcup_{s\in J_{n,F_t}} Int^{\lambda_i^k}(Cl^{\lambda_j^k}(O_{n,s}))\end{aligned}$$

holds. Thus,

$$x \in Int^{\lambda_i^k}(Cl^{\lambda_j^k}(O_{n,s}))$$

for an $s \in J_{n,F_t}$. On the other hand, since

$$\{O_{n,s} : s \in \bigcup_{t\in T_n} J_{n,F_t}\} \subset O_n$$

is finite for each $n \in \omega$, and $(X^k, \lambda_1^k, \lambda_2^k)$ is (i, j)-nearly Menger.

The proof is complete. □

4. Conclusions and future works

This paper was composed for introducing nearly Menger spaces in a bitopological context. We investigated its various properties and relations with some existing covering properties. By giving some specific examples and establishing a few properties of the mentioned covering property, we concluded that it is different and stronger than some of the existing Menger type covering properties in bitopological spaces. We hope that the paper will enlighten those who are interested in this field of mathematics for future works. There are several papers related to Menger type covering properties by means of generalized notions [12,41,42]. Hence, we also believe that it would be interesting to extend and study the generalized type Menger covering and selective properties in bispaces which motivate this paper. This paper hereby can be a nice initiation for it. By using of two different topological structures on a set X, we define these versions in a similar way, as follows:

We first note that a subset of A of a bitopological space $(X, \lambda_1, \lambda_2)$ is said to be (i, j)-semi open [43] (resp. (i, j)-preopen [44], (i, j)-β-open [36], (i, j)-α-open [45]), if

$$A \subset Cl^{\lambda_j}(Int^{\lambda_i}(A)),$$

resp.

$$A \subset Int^{\lambda_i}(Cl^{\lambda_j}(A)), \quad A \subset Cl^{\lambda_j}(Int^{\lambda_i}(Cl^{\lambda_j}(A))), \quad A \subset Int^{\lambda_i}(Cl^{\lambda_j}(Int^{\lambda_i}(A))).$$

Definition 4.1. *A bitopological space $(X, \lambda_1, \lambda_2)$ is said to be (i, j)-semi Menger (resp. (i, j)-pre Menger, (i, j)-β Menger, (i, j)-α Menger) if for each sequence $(\mathcal{O}_n)_{n\in\omega}$ of covers of X by (i, j)-semi open (resp. (i, j)-preopen,(i, j)-β-open, (i, j)-α-open) sets of X, it admits a sequence $(\mathcal{U}_n)_{n\in\omega}$ such that each $\mathcal{U}_n \subset \mathcal{O}_n$ is finite and*

$$\Big\{\bigcup_{n\in\omega} \mathcal{U}_n : n \in \omega\Big\}$$

covers X.

Definition 4.2. *A bitopological space $(X, \lambda_1, \lambda_2)$ is (i, j)-almost semi Menger (resp. (i, j)-nearly semi Menger) if every sequence $(\mathcal{O}_n)_{n\in\omega}$ where each $\mathcal{O}_n$ is an (i, j)-semi open cover of X admits sequence $(\mathcal{U}_n)_{n\in\omega}$ such that every*

$$\mathcal{U}_n \subset \mathcal{O}_n$$

is finite and

$$X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} (i, j) - sCl(U) \ \Big(resp.\ X = \bigcup_{n\in\omega} \bigcup_{U\in\mathcal{U}_n} (i, j) - sInt((i, j) - sCl(U))\Big).$$

((i, j)-$sCl(U)$ and (i, j)-$sInt(U)$ are (i, j)-semi closure and (i, j)-semi interior of U, respectively [46] (see also [47] for more details.)

We also define (i, j)-almost (nearly) pre Menger, (i, j)-almost (nearly) β-Menger, and (i, j)-almost (nearly) α-Menger spaces in a similar way.

Author contributions

N. C. Açıkgöz: conceptualization, methodology, writing original draft, editing; C. S. Elmalı: supervision, validation. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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