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Research article

Research on discrete differential solution methods for derivatives of chaotic systems

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Abstract: The pivotal differential parameters inherent in chaotic systems hold paramount significance across diverse disciplines. This study delves into the distinctive features of discrete differential parameters within three typical chaotic systems: the logistic map, the henon map, and the tent map. A pivotal discovery emerges: both the mean value of the first-order continuous and discrete derivatives in the logistic map coincide, mirroring a similar behavior observed in the henon map. Leveraging the insights gained from the first derivative formulations, we introduce the discrete n-order derivative formulas for both logistic and henon maps. This revelation underscores a discernible mathematical correlation linking the mean value of the derivative, the respective chaotic parameters, and the mean of the chaotic sequence. However, due to the discontinuous points in the tent map, its continuous differential parameter cannot characterize its derivative properties, but its discrete differential has a clear functional relationship with the parameter μ. This paper proposes the use of discrete differential derivatives as an alternative to traditional derivatives, and demonstrates that the mean value of discrete derivatives has a clear mathematical relationship with chaotic map parameters in a statistical sense, providing a new direction for subsequent in-depth research and applications.

Keywords: chaotic systems; continuous differential derivative; discrete differential derivative
Mathematics Subject Classification: 39A12, 39B12, 65P20

1. Introduction

In recent years, with the rapid development of communication and signal processing theories as well as control theories, traditional chaotic signals have not only continued to deepen their fundamental theoretical research but also shown a diversified trend in applications across different fields.

Over the past few years, significant strides have been achieved in the foundational theory of chaotic systems, encompassing novel methodologies for calculating Lyapunov exponents [1,2], synchronization of chaotic signals [3], as well as advancements in separation techniques [4]. Furthermore, there have been remarkable accomplishments in the exploration of chaotic dynamics and topology [5–8]. Within the realm of chaos theory applications, research has delved into chaotic information entropy [9], and a plethora of groundbreaking results have emerged in the field of image encryption [10–16]. The investigation into the uncertainty and stability of chaotic systems has garnered attention [17,18], facilitating their seamless integration into control systems. Chaotic systems are also applied to being approximated by neural networks for nonlinear functions, as well as to being analyzed for the stability performance of neural networks [19–21]. Notably, some scholars have successfully applied these findings to the control of quadrotor aircraft, demonstrating their practical potential [22]. Concurrently, chaos theory has yielded numerous favorable outcomes in the investigation of memristor circuits, representing a significant milestone in this area of research [23–25]. Even in the field of social sciences, chaotic systems hold equally important application scenarios [26,27].

Drawing upon the current landscape of research in the fundamental theory of chaos and its diverse applications, it is evident that a vast majority of the research outcomes, either directly or indirectly, harness the differential characteristics of chaotic systems. A prime example of this is the utilization of these characteristics in solving for Lyapunov exponents ($\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left|\frac{df(x_n)}{dx_n}\right|$), which is a pivotal metric in understanding the stability and predictability of chaotic systems.

Taking the logistic map as an example, its chaotic expression is $f(x) = \mu x(1-x)$. According to the conventional method, the first derivative equation is $f'(x) = \frac{df(x)}{dx} = \mu - 2\mu x$. Then the second derivative can be obtained as $f''(x) = -2\mu$. Then the third derivative must be $f^3(x) = 0$. The aforementioned derivative solutions can be regarded as continuous in nature, standing in stark contrast to the derivatives of genuine the logistic chaotic sequences. Numerical evaluations reveal that, beyond the third order, the discrete derivatives of the logistic map do not converge to zero, underscoring the need for further analysis.

Through the examples provided above, it can be observed that there exists a deviation in the continuous derivative form for chaotic systems. Thus, the focal point of this paper lies in dissecting the continuous and discrete derivative formulations for three ubiquitous chaotic systems: the logistic map, the henon map, and the tent map. Taking the logistic map as an example, we present the continuous first derivative formula in Eq (1) and its discrete counterpart in Eq (2). Here, $E\{\cdot\}$ denotes taking the mean of a sequence.

$$\frac{df(x)}{dx} = \mu - 2\mu x \rightarrow \quad E\left\{\frac{df(x)}{dx}\right\} = \mu - 2\mu E\{x\}, \tag{1}$$

$$\Delta f(x_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \quad \rightarrow \quad E\{\Delta f(x_n)\} = E\left\{\frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}\right\}. \tag{2}$$

The main research findings of this paper include:

1-The use of discrete difference derivative forms as a substitute for continuous derivatives. And derive the formula for higher-order discrete derivatives. Thereby avoiding the situation where the continuous higher-order derivatives are zero.

2-Through rigorous mathematical formula derivation and numerical simulations, it is found that despite the notable discrepancies in the computational outcomes for discrete points of chaotic signals between the two differentiation methods, the average values of their respective derivatives coincide.

3-There exists a clear mathematical relationship between the average value of the discrete differential derivatives of chaotic maps and the parameters of those systems. In applications where the derivative of a chaotic sequence is required, the discrete difference derivative method proposed in this paper can be adopted.

2. Analysis of differential characteristic of several discrete chaotic system

2.1. Logistic map

The continuous expression of the logistic map is shown in Eq (3).

$$f(x) = \mu x(1 - x). \tag{3}$$

More often, it is described in the form of discretization, and its discrete expression is shown in Eq (4).

$$x_{n+1} = \mu x_n(1 - x_n). \tag{4}$$

According to Eq (3), the differential expression of the continuous function of the logistic chaotic system can be obtained as shown in Eq (5).

$$f'(x) = \frac{df(x)}{dx} = \mu - 2\mu x. \tag{5}$$

And the differential expression in the form of discretization according to Eq (4) is shown as Eq (6).

$$\Delta f(x_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} = \frac{\mu x_{n+1}(1 - x_{n+1}) - \mu x_n(1 - x_n)}{x_{n+1} - x_n}$$

$$= \frac{\mu x_{n+1} - \mu {x_{n+1}}^2 - \mu x_n + \mu {x_n}^2}{x_{n+1} - x_n} = \mu - \mu(x_{n+1} + x_n) = \mu - (\mu + \mu^2)x_n + \mu^2 {x_n}^2. \tag{6}$$

For each discrete chaotic sample point, Eqs (5) and (6) are not equal in form. However, the mean values of both could potentially be equal. So we assume that the mean values of Eqs (5) and (6) are equal, $E\left\{\frac{df(x)}{dx}\right\} = E\{\Delta f(x_n)\}$ and obtain Eq (7) below.

$$\mu - 2\mu E\{x\} = \mu - \mu E\{x\} - \mu^2 E\{x\} + \mu^2 E\{x^2\},$$

$$E\{x\} = \frac{\mu}{\mu - 1} E\{x^2\}. \tag{7}$$

According to Eq (2), the expression of the 2nd derivative can be derived, as shown in Eq (8).

$$\Delta^2 f(x_n) = \frac{\Delta f(x_{n+1}) - \Delta f(x_n)}{x_{n+1} - x_n}$$

$$= \frac{\mu - (\mu + \mu^2)x_{n+1} + \mu^2 {x_{n+1}}^2 - \mu + (\mu + \mu^2)x_n - \mu^2 {x_n}^2}{x_{n+1} - x_n}$$

$$
\begin{aligned}
&= \frac{-(\mu+\mu^2)x_{n+1} + \mu^2 {x_{n+1}}^2 + (\mu+\mu^2)x_n - \mu^2 {x_n}^2}{x_{n+1} - x_n} \\
&= \frac{-(\mu+\mu^2)(x_{n+1} - x_n) + \mu^2({x_{n+1}}^2 - {x_n}^2)}{x_{n+1} - x_n} \\
&= -(\mu+\mu^2) + \mu^2(x_{n+1} + x_n) \\
&= -\mu^3 {x_n}^2 + \mu^2(\mu+1)x_n - (\mu+\mu^2). \qquad (8)
\end{aligned}
$$

According to Eq (6) and Eq(8), the expression of the n-order derivative can be supposed as Eq(9).

$$
\Delta^n f(x_n) = \begin{cases} \mu^2 {x_n}^2 - (\mu+\mu^2)x_n + \mu, & n = 1, \\ (-1)^{n-1}\mu^{n+1}{x_n}^2 + (-1)^n \mu^n(\mu+1)x_n + (-1)^{n-1}\mu^{n-1}(\mu+1), & n \geq 2. \end{cases} \qquad (9)
$$

When $n \geq 2$, the hypothesis of Eq (9) can be proved by mathematical induction, as shown in Eq (10).

$$
\begin{aligned}
\Delta^{n+1} f(x_n) &= \frac{\Delta^n f(x_{n+1}) - \Delta^n f(x_n)}{x_{n+1} - x_n} \\
&= \frac{(-1)^{n-1}\mu^{n+1}{x_{n+1}}^2 + (-1)^n\mu^n(\mu+1)x_{n+1} - (-1)^{n-1}\mu^{n+1}{x_n}^2 - (-1)^n\mu^n(\mu+1)x_n}{x_{n+1} - x_n} \\
&= (-1)^n\mu^n(\mu+1) + (-1)^{n-1}\mu^{n+1}(x_{n+1} + x_n) \\
&= (-1)^n\mu^n(\mu+1) + (-1)^{n-1}\mu^{n+1}((\mu+1)x_n - \mu {x_n}^2) \\
&= (-1)^n\mu^{n+2}{x_n}^2 + (-1)^{n+1}\mu^{n+1}(\mu+1)x_n + (-1)^n\mu^n(\mu+1). \qquad (10)
\end{aligned}
$$

2.2. Henon map

The continuous expression for the henon map is $x = 1 - ax^2 + y$ and $y = bx$. The one-dimensional expression of the henon map, as shown in Eq (11), can be obtained by combining the two formulas.

$$f(x) = 1 + bx - ax^2. \qquad (11)$$

Its discrete expression is given by Eq (12).

$$x_{n+1} = 1 + bx_n - a{x_n}^2. \qquad (12)$$

According to Eq (11), the continuous differential derivative of the henon map is shown in Eq (13).

$$f'(x) = \frac{df(x)}{dx} = b - 2ax. \qquad (13)$$

According to Eq (12), the discretized differential expression is shown in Eq (14).

$$
\begin{aligned}
\Delta f(x_n) &= \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \\
&= \frac{1 + bx_{n+1} - a{x_{n+1}}^2 - 1 - bx_n + a{x_n}^2}{1 + bx_n - a{x_n}^2 - x_n}
\end{aligned}
$$

$$= \frac{b(x_{n+1} - x_n) - a({x_{n+1}}^2 - {x_n}^2)}{x_{n+1} - x_n}$$
$$= b - a(x_{n+1} + x_n)$$
$$= b - a(1 + bx_n - a{x_n}^2 + x_n)$$
$$= b - a - a(b+1)x_n + a^2{x_n}^2. \quad (14)$$

If the mean values of the two differential derivatives are assumed to be equal, that is, Eq (13) and Eq (14) are equal, then $E\left\{\frac{df(x)}{dx}\right\} = E\{\Delta f(x_n)\}$, and the Eq (15) is obtained.

$$E\{\Delta f(x_n)\} = b - 2aE\{x\} = b - a - a(b+1)E\{x\} + a^2E\{x^2\},$$
$$E\{x^2\} = \frac{(b-1)E\{x\}+1}{a}. \quad (15)$$

The variance relation Eq (16) is thus obtained.

$$Var\{x\} = E\{(x - \bar{x})^2\} = E\{x^2\} - E\{x\}^2 = \frac{(b-1)}{a}E\{x\} + \frac{1}{a} - E\{x\}^2. \quad (16)$$

According to Eq (14), the expression of the 2nd derivative can be derived, as shown in Eq (17).

$$\Delta^2 f(x_n) = \frac{\Delta f(x_{n+1}) - \Delta f(x_n)}{x_{n+1} - x_n}$$
$$= \frac{b - a - a(b+1)x_{n+1} + a^2{x_{n+1}}^2 - b + a + a(b+1)x_n - a^2{x_n}^2}{x_{n+1} - x_n}$$
$$= \frac{-a(b+1)(x_{n+1} - x_n) + a^2({x_{n+1}}^2 - {x_n}^2)}{x_{n+1} - x_n}$$
$$= -a(b+1) + a^2(x_{n+1} + x_n)$$
$$= -a^3{x_n}^2 + a^2(b+1)x_n - ab - a + a^2. \quad (17)$$

According to Eq (13) and Eq (14), the expression of the n-order derivative can be supposed to be Eq (18).

$$\Delta^n f(x_n) = \begin{cases} a^2{x_n}^2 - a(b+1)x_n + b - a, & n = 1, \\ (-1)^{n-1}a^{n+1}{x_n}^2 + (-1)^n a^n(b+1)x_n + (-1)^{n-1}a^{n-1}(b+1) + (-1)^n a^n, & n \geq 2. \end{cases} \quad (18)$$

When $n \geq 2$,the hypothesis of Eq (18) can be proved by mathematical induction, as shown in Eq (19).

$$\Delta^{n+1} f(x_n) = \frac{\Delta^n f(x_{n+1}) - \Delta^n f(x_n)}{x_{n+1} - x_n}$$
$$= \frac{(-1)^{n-1}a^{n+1}{x_{n+1}}^2 + (-1)^n a^n(b+1)x_{n+1} - (-1)^{n-1}a^{n+1}{x_n}^2 - (-1)^n a^n(b+1)x_n}{x_{n+1} - x_n}$$
$$= (-1)^n a^n(b+1) + (-1)^{n-1}a^{n+1}(x_{n+1} + x_n)$$
$$= (-1)^n a^n(b+1) + (-1)^{n-1}a^{n+1}(1 + (b+1)x_n - a{x_n}^2)$$

$$= (-1)^n a^{n+2} {x_n}^2 + (-1)^{n+1} a^{n+1} (b+1) x_n + (-1)^n a^n (b+1) + (-1)^{n+1} a^{n+1}. \tag{19}$$

2.3. Tent map

The continuous expression of the tent map is shown in Eq (20).

$$f(x) = \begin{cases} \frac{x}{\mu}, 0 < x \le \mu, \\ \frac{1-x}{1-\mu}, \mu < x < 1. \end{cases} \tag{20}$$

Its discretized form is shown in Eq (21).

$$x_{n+1} = \begin{cases} \frac{x_n}{\mu}, 0 < x_n \le \mu, \\ \frac{1-x_n}{1-\mu}, \mu < x_n < 1. \end{cases} \tag{21}$$

The value range of the tent map is $0 < x < 1$, and it satisfies uniform distribution.

The continuous differential form of the tent chaotic derivative obtained from Eq (20) is shown in Eq (22).

$$\frac{df(x)}{dx} = \begin{cases} \frac{1}{\mu}, 0 < x \le \mu, \\ \frac{-1}{1-\mu}, \mu < x < 1. \end{cases} \tag{22}$$

the tent chaotic system has discontinuous points, and the discrete differential expression of its derivative can be expressed by Eq (2) ($\Delta f(x_n) = \frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}$), but four different cases can be obtained according to the different value range of x_n.

The first case is shown in Eq (23):

$$E_1\{\Delta f(x_n)\} = \frac{\frac{x_n}{\mu^2} - \frac{x_n}{\mu}}{\frac{x_n}{\mu} - x_n} = \frac{1}{\mu}, \tag{23}$$

where $0 < x_n \le \mu$ and $0 < \frac{x_n}{\mu} \le \mu$, then $0 < x_n \le \mu^2$.

The second case is shown in Eq (24):

$$\Delta f(x_n) = \frac{\frac{1-\frac{x_n}{\mu}}{1-\mu} - \frac{x_n}{\mu}}{\frac{x_n}{\mu} - x_n} = \frac{\mu + (\mu-2)x_n}{(\mu-1)^2 x_n} = \frac{1}{(\mu-1)^2}\left(\frac{\mu}{x_n} + \mu - 2\right) = \frac{\mu-2}{(\mu-1)^2} + \frac{\mu}{(\mu-1)^2}\frac{1}{x_n}, \tag{24}$$

where $0 < x_n \le \mu$ and $\mu \le \frac{x_n}{\mu} < 1$, then $\mu^2 \le x_n \le \mu$.

According to the uniform distribution characteristics of the tent map and the value range of $\mu^2 \le x_n \le \mu$, the average value of the derivative of the chaotic system is as Eq (25).

$$E_2\{\Delta f(x_n)\} = \frac{\mu-2}{(\mu-1)^2} + \frac{\mu}{(\mu-1)^2} E\left\{\frac{1}{x_n}\right\}. \tag{25}$$

When the range of the tent map is $\mu^2 \le x_n \le \mu$, its probability density function is as in Eq (26).

$$f(x) = \frac{1}{\mu-\mu^2}. \tag{26}$$

Using the property of the probability density function of the random variable, the result of Eq (25) can be obtained as shown in Eq (27).

$$E_2\{\Delta f(x_n)\} = \frac{\mu-2}{(\mu-1)^2} + \frac{\mu}{(\mu-1)^2}\int_{\frac{1}{\mu}}^{\frac{1}{\mu^2}} \frac{1}{\mu-\mu^2} x \frac{1}{x^2} dx$$

$$= \frac{\mu-2}{(\mu-1)^2} + \frac{\mu}{(\mu-1)^2}\frac{-ln\mu}{\mu-\mu^2} = \frac{\mu-2}{(\mu-1)^2} - \frac{ln\mu}{(1-\mu)^3}. \tag{27}$$

The third case is shown in Eq (28):

$$E_3\{\Delta f(x_n)\} = \frac{\frac{1-\frac{1-x_n}{1-\mu}}{1-\mu} - \frac{1-x_n}{1-\mu}}{\frac{1-x_n}{1-\mu} - x_n} = \frac{-1}{1-\mu}, \tag{28}$$

where $\mu \le x_n < 1$ and $\mu \le \frac{1-x_n}{1-\mu} < 1$, then $\mu \le x_n \le \mu^2 - \mu + 1$.

The fourth case is shown in (29):

$$\Delta f(x_n) = \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} = \frac{\frac{x_{n+1}}{\mu} - x_{n+1}}{x_{n+1} - x_n}$$

$$= \frac{1-\mu}{\mu}\frac{x_{n+1}}{x_{n+1} - x_n} = \frac{1-\mu}{\mu}\frac{x_{n+1}}{x_{n+1} - 1 + (1-\mu)x_{n+1}}$$

$$= \frac{1-\mu}{\mu(2-\mu)}\frac{(2-\mu)x_{n+1} - 1 + 1}{(2-\mu)x_{n+1} - 1}$$

$$= \frac{1-\mu}{\mu(2-\mu)}\left(1 + \frac{1}{(2-\mu)x_{n+1} - 1}\right)$$

$$= \frac{1-\mu}{\mu(2-\mu)}\left(1 + \frac{1-\mu}{1-(2-\mu)x_n}\right), \tag{29}$$

where $\mu \le x_n < 1$ and $0 < \frac{1-x_n}{1-\mu} \le \mu$, then $\mu^2 - \mu + 1 \le x_n < 1$.

According to the uniform distribution characteristics of the tent map and the value range of $\mu^2 - \mu + 1 \le x_n < 1$, the average value of the derivative of this chaotic system is shown in (30).

$$E_4\{\Delta f(x_n)\} = \frac{1-\mu}{\mu(2-\mu)} + \frac{(1-\mu)^2}{\mu(2-\mu)} E\left\{\frac{1}{1-(2-\mu)x_n}\right\}. \tag{30}$$

When the tent map ranges from $\mu^2 - \mu + 1 \le x_n < 1$, its probability density function is shown in (31).

$$f(x) = \frac{1}{\mu-\mu^2}. \tag{31}$$

Similarly, the result of Eq (25) can be obtained by using the probability density function characteristics of random variables, as shown in Eq (32).

$$E_4\{\Delta f(x_n)\} = \frac{1-\mu}{\mu(2-\mu)} + \frac{(1-\mu)^2}{\mu(2-\mu)} \int_{\frac{1}{1-(2-\mu)(\mu^2-\mu+1)}}^{\frac{1}{\mu-1}} \frac{1}{\mu-\mu^2} x \frac{1}{2-\mu} \frac{1}{x^2} dx$$

$$= \frac{1-\mu}{\mu(2-\mu)} + \frac{(1-\mu)^2}{\mu(2-\mu)} \frac{1}{\mu-\mu^2} \frac{1}{2-\mu} \int_{\frac{1}{1-(2-\mu)(\mu^2-\mu+1)}}^{\frac{1}{\mu-1}} \frac{1}{x} dx$$

$$= \frac{1-\mu}{\mu(2-\mu)} + \frac{1-\mu}{\mu^2(2-\mu)^2} ln \frac{1-(2-\mu)(\mu^2-\mu+1)}{\mu-1}. \quad (32)$$

By combining the above four cases, the mean value of the discretized derivative of the tent map can be obtained as shown in Eq (33).

$$E\{\Delta f(x_n)\} = \sum_{i=1}^{4} E_i\{\Delta f(x_n)\} P_i, \quad (33)$$

where P_i is the probability of each segment. Since the tent map satisfies the uniform distribution and the value range is $0 < x < 1$, the value of P_i is equal to the value range of x. The final result of $E\{\Delta f(x_n)\}$ is shown in (34). Because of the discontinuity of the tent map, the expression of discrete differential derivative is more reasonable.

$$E\{\Delta f(x_n)\} = E_1\{\Delta f(x_n)\}(\mu^2 - 0) + E_2\{\Delta f(x_n)\}(\mu - \mu^2)$$

$$+ E_3\{\Delta f(x_n)\}(\mu^2 - \mu + 1 - \mu) + E_4\{\Delta f(x_n)\}(\mu - \mu^2)$$

$$= \frac{1}{\mu}(\mu^2 - 0) + \left(\frac{\mu-2}{(\mu-1)^2} - \frac{ln\mu}{(1-\mu)^3}\right)(\mu - \mu^2) + \frac{-1}{1-\mu}(\mu^2 - \mu + 1 - \mu)$$

$$+ \left(\frac{1-\mu}{\mu(2-\mu)} + \frac{1-\mu}{\mu^2(2-\mu)^2} ln \frac{1-(2-\mu)(\mu^2-\mu+1)}{\mu-1}\right)(\mu - \mu^2)$$

$$= 2\mu - 1 + \frac{\mu(\mu-2)}{1-\mu} - \frac{\mu ln\mu}{(1-\mu)^2} + \frac{(1-\mu)^2}{(2-\mu)} + \frac{(1-\mu)^2}{\mu(2-\mu)^2} ln \frac{1-(2-\mu)(\mu^2-\mu+1)}{\mu-1}. \quad (34)$$

3. Experimental results and analysis

3.1. Numerical simulation

Numerical simulation experiments were conducted to verify the equality in the average sense between continuous derivatives and discrete derivatives. Several numerical simulation experiments are as follows:

1-Using both continuous and discrete derivatives, the derivative values at each sample of the chaotic sequences were calculated separately.

For the logistic chaotic system, parameterized with μ = 3.71 and an initial value of 0.1, we derived the continuous differential derivative curves and the discrete differential derivative sequence using Eqs (5) and (6), respectively. These results are visually presented in Figure 1.

Similarly, for the henon chaotic system, with parameters a=1.6, b=1.6, and an initial value of 0.1, the continuous differential derivative curves and the discrete differential derivative sequence were obtained by applying Eq (13) and Eq (14), respectively. These outcomes are depicted in Figure 2.

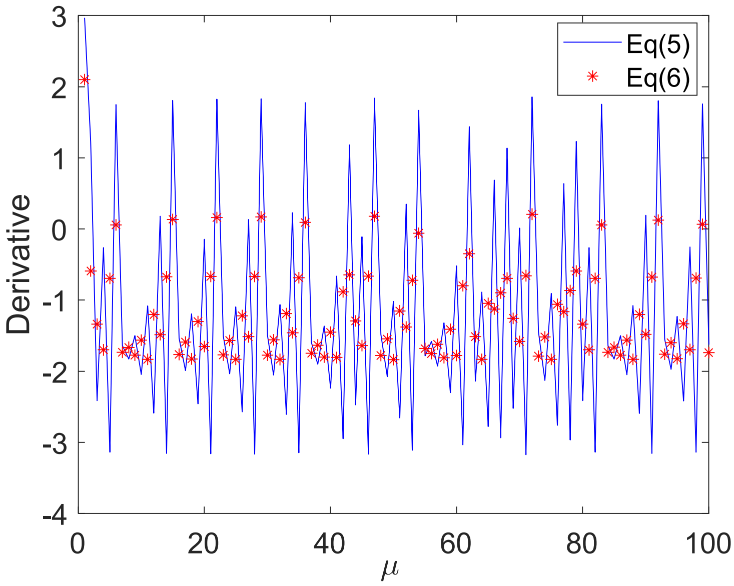

Figure 1. Comparison of derivative of the logistic map ($\mu = 3.71$, initial value is 0.1).

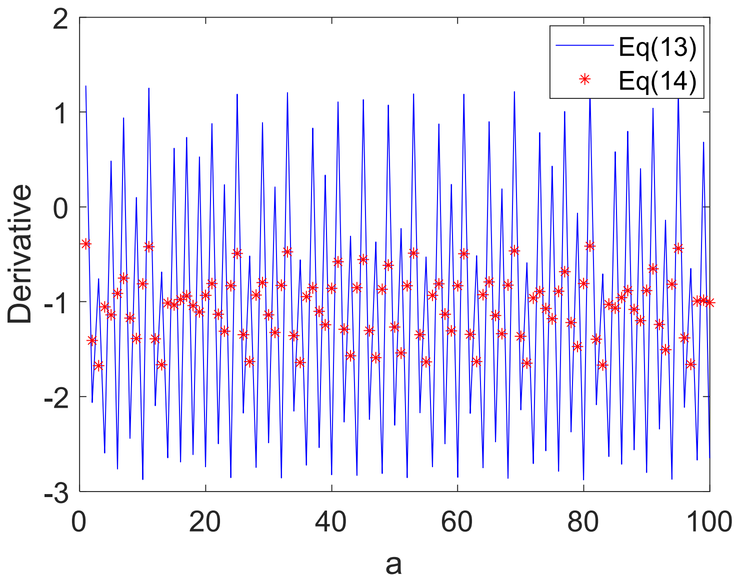

Figure 2. Comparison of derivative of the henon map ($a = 1.6$, $b = 1.6$, initial value is 0.1).

The results obtained by using continuous derivatives and discrete derivatives, respectively, for each discrete chaotic sampling point, exhibit differences.

2-The mean value graphs of continuous and discrete derivatives for three chaotic systems under different initial values and chaotic parameters are presented separately, as well as the mean value slice plots.

To comprehensively compare the mean values, Figures 3 and 4 showcase the relationship between the mean continuous and discrete differential derivatives of the logistic map system under varying μ values (ranging from 3.6 to 4) and initial conditions (x varying between 0.01 and 0.99), with increments of 0.001 for both parameters and a sequence length of 500,000. Figure 5 further illustrates the cross-sectional view of this comparison at x=0.1, where μ varies within the same range.

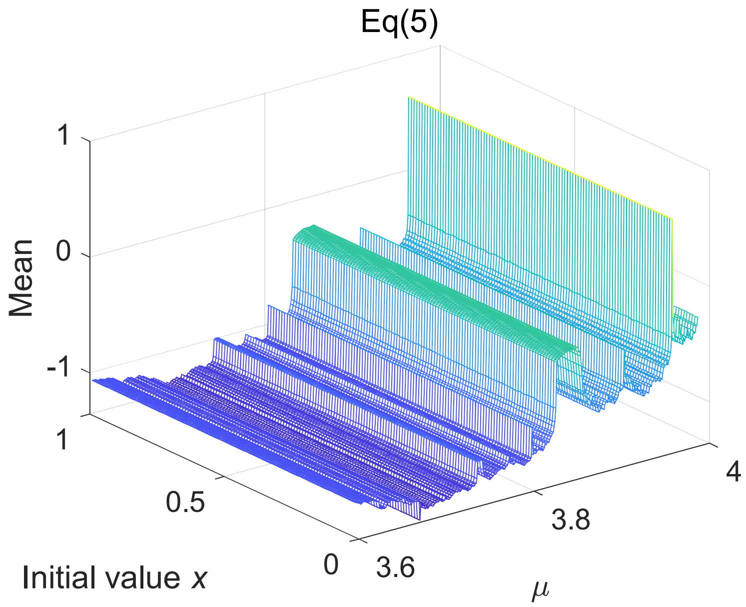

Figure 3. The mean for derivative of the logistic map calculated by Eq (5).

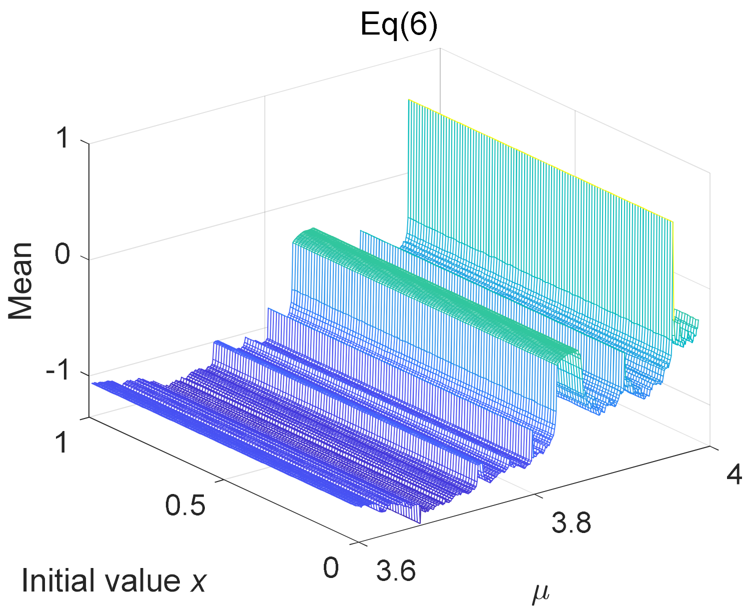

Figure 4. The mean for derivative of the logistic map calculated by Eq (6).

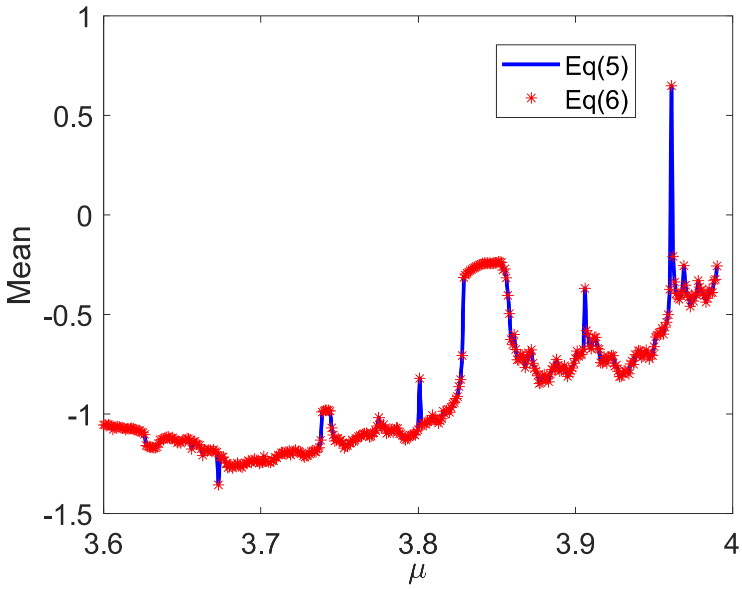

Figure 5. The curve of the cross-section of Figures 3 and 4 (3.6 < μ < 4, $x = 0.1$).

Analogously, Figures 6 and 7 compare the mean values of the continuous and discrete differential derivatives for the henon map system, exploring various values of the parameter a (from 1 to 2, with b set equal to a) and initial conditions (x ranging from 0.01 to 0.99), again with increments of 0.001 and a sequence length of 500,000. Figure 8 provides a cross-sectional view of this comparison, specifically for x=0.1 and a varying within the aforementioned range, with b=a.

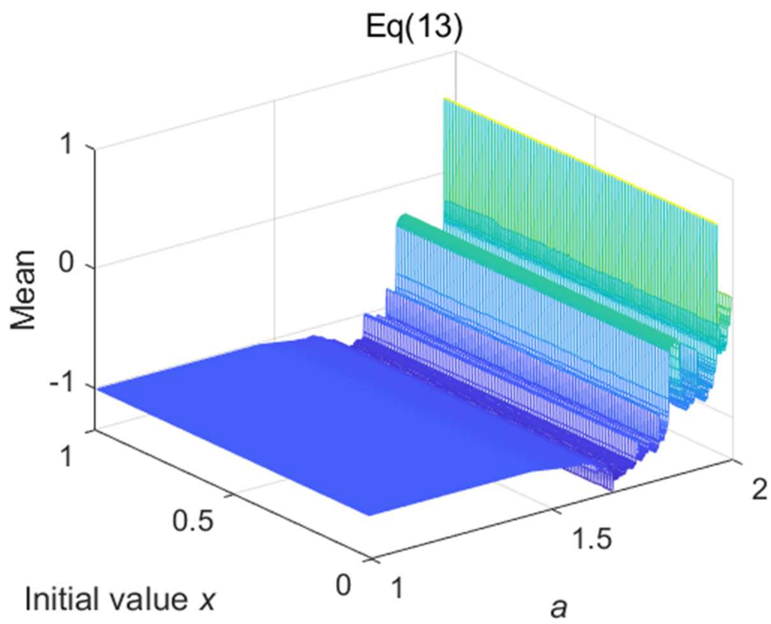

Figure 6. The mean for derivative of the henon map calculated by Eq (13).

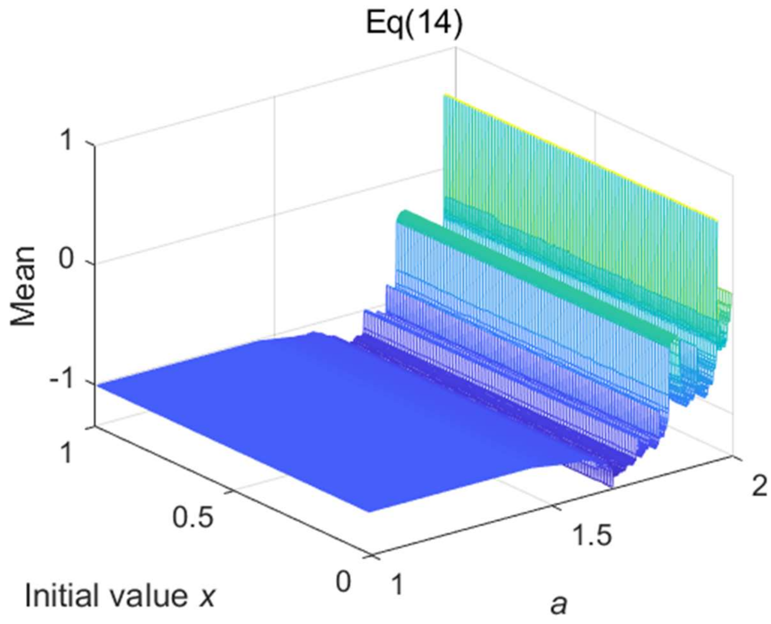

Figure 7. The mean for derivative of the henon map calculated by Eq (14).

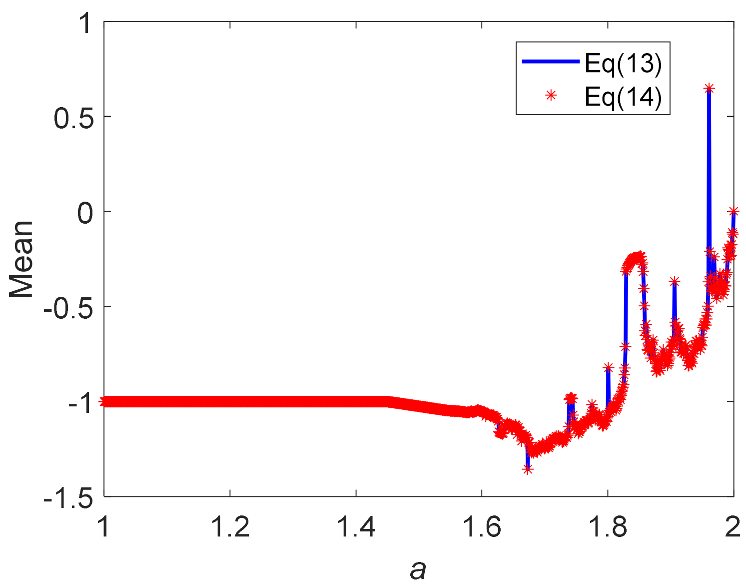

Figure 8. The curve of the cross-section of Figures 6 and 7 ($1 < a < 2,\ x = 0.1$).

The mean values of the discrete differential derivatives for the tent map chaotic system, under varying μ values within the range of 0 to 1 and initial conditions x spanning from 0.01 to 0.99, are presented in Figure 9. Here, the increments for μ and the initial value x are set to 0.001, and the sequence length is fixed at 500,000. For a closer inspection, Figure 10 showcases the comparison between the cross-sectional curve extracted from Figure 9 and the theoretical prediction based on Eq (34), specifically for μ ranging from 0 to 1 and an initial value of x=0.1.

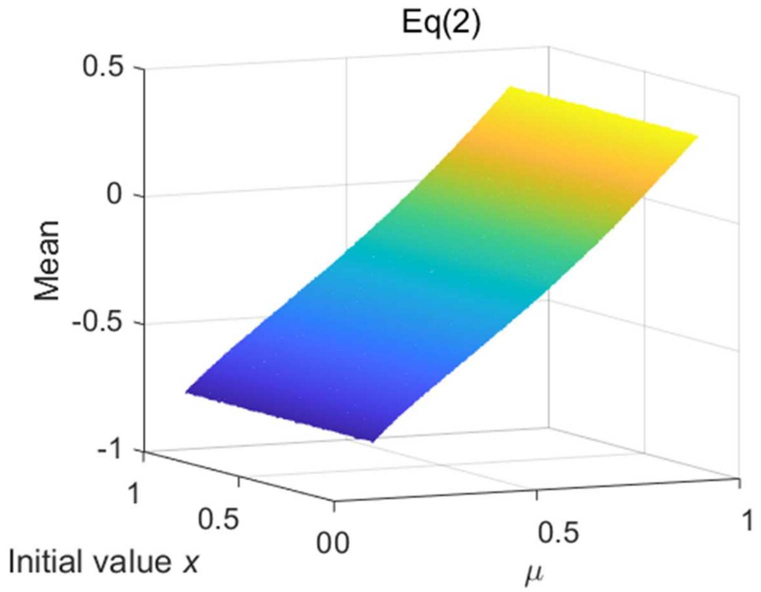

Figure 9. The mean for derivative of the tent map calculated by Eq (2).

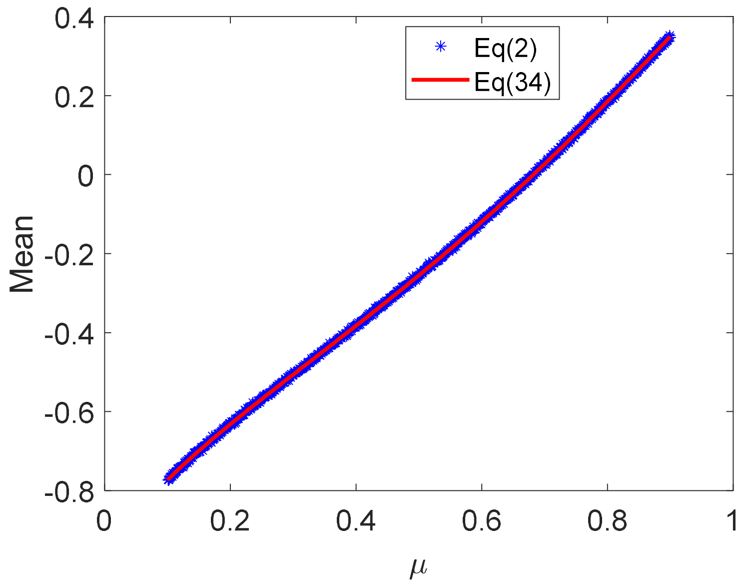

Figure 10. The comparison between the curve of the cross-section of Figure 9 and Eq (34).

3-Select representative chaotic parameters and present the differences in mean values between continuous and discrete derivatives.

To facilitate a more comprehensive understanding, Tables 1–3 have been compiled to systematically compare the mean values of partial derivatives for the logistic, henon, and tent chaotic systems, respectively. Table 1 details the mean values of partial derivatives for the logistic chaotic system, Table 2 presents the analogous data for the henon chaotic system, and Table 3 provides a similar comparison for the tent chaotic system. The results indicate that there is little or no difference between the mean values of continuous and discrete derivatives in most cases. These tables offer a concise and structured view of the key findings, enabling readers to easily compare and analyze the behavior of partial derivatives across different chaotic systems.

Table 1. Comparison the mean for derivative of the logistic map.

μ	3.6	3.7	3.8	3.9
Mean of Eq (5)	-1.054	-1.242	-1.079	-0.721
Mean of Eq (6)	-1.054	-1.242	-1.079	-0.721

Table 2. Comparison the mean for derivative of the henon map.

a	1.5	1.6	1.7	1.8	1.9
Mean of Eq (13)	-1.025	-1.054	-1.242	-1.080	-0.724
Mean of Eq (14)	-1.025	-1.054	-1.242	-1.080	-0.724

Table 3. Comparison the mean for derivative of the tent map.

μ	0.2	0.4	0.6	0.8
Mean of Eq (2)	-0.633	-0.383	-0.121	0.179
Mean of Eq (34)	-0.632	-0.383	-0.119	0.184

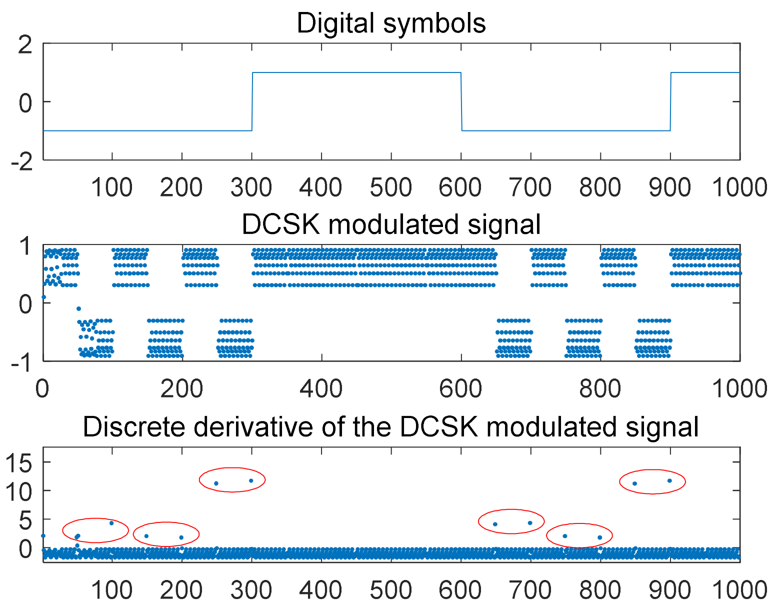

Figure 11. Application of discrete derivative in DCSK modulated signal.

4-To verify the application of the discrete derivative method in communication systems, a discrete derivative calculation was performed on the DCSK (differential chaos shift keying) modulated signal. The results are shown in Figure 11. In this context, the carrier signal employs the logistic map with a parameter μ =3.63 and an initial value of 0.1. The DCSK baseband signal consists of 10 randomly generated symbols (where symbols are either 1 or -1), with each symbol represented by 100 chaotic samples. The results indicate that after applying the discrete derivative, abrupt changes in the chaotic sequence can be clearly identified within the -1 symbols, facilitating the detection of differences between 1 and -1 symbols. Of course, due to the diverse types of chaotic maps and their variable parameters, the application of the discrete derivative method in communication systems warrants further in-depth research.

3.2. Analysis

Based on the equations detailed in the second part and the aforementioned chart, the following insights can be distilled:

1-As evident from Figures 1 and 2, the logistic map and the henon map exhibit distinct behaviors in terms of their continuous differential derivative curves and discrete differential derivative series, respectively. Notably, the continuous derivative's outcomes span a broader range, whereas the discrete derivatives cluster more tightly. This underscores the inherent difference between these two derivative series.

2-Analyzing the outcomes presented in Figures 3 through 10 reveals that the mean values of both continuous and discrete differential derivatives across three chaotic systems are largely equivalent. As can be seen from the figures, the mean value is only related to the parameter μ and is independent of the initial value. Further examination of Tables 1 to 3 highlights a subtle discrepancy solely in the thousandths place between the mean values of the continuous and discrete derivatives for the tent map.

3-The Eq (6) through (10) and (12) through (19) clearly illustrate a straightforward mathematical correlation between the mean discrete derivatives of the logistic map and the henon map, and the mean of their chaotic series. This relationship underscores the fundamental interdependence between these variables.

4-The discontinuous nature of the tent map precludes its direct representation through continuous function derivatives. Nevertheless, Eq (34) establishes a definitive mathematical link between the mean of the tent map's discretized derivative and the parameter μ, offering valuable insight into the system's behavior.

5-Numerical computations affirm that the mean values of the first-order continuous and discrete differentials for the three chaotic systems are indeed equivalent, thereby validating the accuracy of Eqs (6), (14), and (34). Consequently, the derived n-order discrete differential equations, namely Eq (9) and (18), can also be considered valid and trustworthy.

6-Both formula derivation and numerical simulation demonstrate that the first-order discrete derivative can serve as a substitute for the first-order continuous derivative, and it addresses the issue of some chaotic maps lacking higher-order continuous derivatives. In situations where derivative and differential calculations are required for chaotic maps, the discrete differentiation method proposed in this paper can be adopted.

4. Results

In this paper, the relationship between the continuous and discrete differential derivatives of three key chaotic systems (logistic, henon, and tent) is thoroughly investigated. Both formula derivation and numerical simulation have revealed a clear mathematical correlation between the mean values of these derivatives and the average values of the chaotic series or specific parameters. This discovery is embodied in Eqs (6), (14), and (34), providing valuable insights. Additionally, methodologies are presented for calculating the n-order discrete derivatives of both the logistic and the henon maps, as specified in Eqs (9) and (18), respectively. The issue that some chaotic maps lack higher-order derivatives is addressed. Despite the inherent unpredictability of chaotic systems, the results indicate that the discrete differential properties of these three chaotic systems can be statistically analyzed, suggesting their potential applicability to the other discrete chaotic systems. The outcomes of this study have broad implications.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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