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Research article

Bifurcation and controller design in a 3D delayed predator-prey model

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Abstract: Delayed dynamical models demonstrate significant application value in depicting interactions and internal dynamics among different biological populations. Therefore, they have garnered significant interest from numerous scholars in both biology and mathematics. Based on previous studies, this article construct a novel delayed predator-prey model. By utilizing fixed point theory, inequality methods, and appropriate functions, this article examined the desirable properties of the solutions of the constructed delayed predator-prey system, including existence and uniqueness, boundedness, and non-negativity. This paper determines the parameter conditions for system stability and the occurrence of bifurcations by employing bifurcation theory and the stability theory of delayed differential equations. Using two control strategies, namely the mixed controller and the extended delay feedback controller, this paper effectively adjusts the stability domain of the delayed predator-prey systems and controls the time of bifurcation onset. The research explores how delays affect the stabilization of system and the adjustment of bifurcation. This paper provides computer simulation photos supporting the main obtained findings. The outcomes of this paper are groundbreaking and can provide critical guidance for the control and regulation of predator and prey population densities.

Keywords: predator-prey system; well-posedness of solution; Hopf bifurcation; stability; hybrid controller
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1. Introduction

Because of the universal existence of predation in nature, predator-prey models, especially delayed predator-prey models, have been a research topic in the field of population dynamics. In order to research mutual relationship between biological populations and regulating internal development, a great number of predator-prey models have been established. By studying the various dynamic behaviors of predator-prey models, we find that the parameters have a significant effect on the population density of organisms under certain environmental conditions. In recent years, there have been numerous research results on predator-prey models, and many excellent achievements have been presented. For instance, Enatsu et al. [1] explored the effects of cooperative hunting and maturation delay on the coexistence of predator-prey. Majumdar et al. [2] analyzed the equilibrium points and local bifurcations of the non-delayed prey-predator model, as well as the local and global stability of the interior equilibrium point in the presence of delay. Gao and Li. [3] discussed the Hopf bifurcation of a symbiotic predator-prey model with double time delay. Yan et al. [4] analyzed how nonlocal prey competition can trigger spatially inhomogeneous Hopf bifurcation and Turing instability, and its impacts on the amplitude of periodic solutions and the risk of species extinction. For more detailed studies, one can see [5–7].

In 2023, Pal et al. [8] put forward the following predator-prey model:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{1}{1+\varpi_1 u_2(t)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) - \dfrac{1}{1+\varpi_3 u_3}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}, \\ \dfrac{du_2}{dt} = \dfrac{1}{1+\varpi_3 u_3(t)}\dfrac{\bar{\lambda}_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t), \\ \dfrac{du_3}{dt} = \dfrac{\beta u_3(t)}{1+\beta_0 u_3(t)} + \bar{\lambda}_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t), \end{cases} \tag{1.1}$$

where $u_1(t)$, $u_2(t)$, $u_3(t)$ represent the densities of prey, intermediate predators, and top predators at any time $t > 0$. γ denotes the intrinsic growth rate of the prey species in the absence of predators, and d is the density-related mortality rate, a_1 is the capture rate of prey species by intermediate predators, m is the half saturation constant, $\bar{\lambda}_1$ represents the respective gain in intermediate predator density for prey and top predators, being $\bar{\lambda}_1$ is between 0 and 1, $\bar{\lambda}_2$ stands for the proportional constant between the growth rate of the top predator and the response of the function, ξ_1–ξ_3 represent the natural mortality of intermediate predators, the coefficient of intraspecies competition of intermediate predators, and the natural mortality of top predators, respectively, β and β_0 represent per capita regeneration rate and density-dependent intensity, ϖ_1–ϖ_4 are all fear parameters, ϖ_1 and ϖ_2 are the effects of fear on the reproduction ability and intraspecies competition of prey populations, ϖ_3 is the changes in the behavior of intermediate predators due to the presence of apex predators, and ϖ_4 is the impacts of predators' fear of top predators on intraspecies competition among intermediate predators. For a more detailed explanation of system (1.1), see [8]. Pal et al. [8] examined how fear responses influence population reproduction and foraging behavior in a three-species food chain model, and affect the system's stability of model (1.1). In numerous instances, the introduction of time delay in predator-prey models is necessary because the time delays in population development have an effect on the density of prey and predators. Based on this consideration, we can build a more suitable delayed predator-prey model, since the density of prey $u_1(t)$ not only depends on the current but also the past density of intermediate predators. In addition, the density of intermediate predator and top predators also depend on the current density and past density of top predators. Assuming that the density of prey is affected by the self-feedback time from u_1 to u_1, the density of intermediate predators is affected by the self-feedback time from u_2 to u_2, and the density of top predators is affected by the self-feedback time from u_3 to u_3. In many cases, the self-feedback time for u_1–u_3 is different. For the sake of mathematical simplification, we assume that all the self-feedback time for u_1–u_3 is similar, then, we can adjust the model (1.1) as follows:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}, \\ \dfrac{du_2}{dt} = \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\bar{\lambda}_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t), \\ \dfrac{du_3}{dt} = \dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \bar{\lambda}_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t), \end{cases} \tag{1.2}$$

where $\theta \geq 0$ represents the time delay, and all other parameters are positive real numbers. Many studies indicate that delay is a key factor affecting the dynamic characteristics of various differential systems. In a variety of examples, delay leads to loss of stability and appearance of bifurcation, and trigger chaotic behavior and so forth [9–11]. Among these phenomena, delay-induced Hopf bifurcation is a significant dynamic feature. In biology, delay-induced Hopf bifurcation effectively describes the equilibrium relationship between biological population densities. Therefore, we argue that the in-depth study of delay-induced Hopf bifurcation in predator-prey models holds crucial theoretical value. Inspired by the viewpoint mentioned above, we are going to examine the delay-induced Hopf bifurcation and its control for model (1.2). Specifically, we aim to assess the following essential issues: (1) Analyze the characteristics of the solution of system (1.2), such as non-negativity, existence, uniqueness, and boundedness. (2) Investigate the occurrence of the Hopf bifurcation phenomenon and stability of the system (1.2). (3) Design three disparate controllers to adjust the stability region of the system and control the time of bifurcation onset in model (1.2).

The crux prominent points in this study are described as follows: (i) Based on past research achievements, a new delay-independent bifurcation and stability criterion for system (1.2) is established. (ii) By employing various controllers, both the stability domain and the time of bifurcation onset in system (1.2) can be effectively managed. (iii) The study explores how time delay influences the control of Hopf bifurcation phenomena and the stabilization of predator and prey densities in model (1.2)

The structure of this article is arranged as follows: The properties of the solution for system (1.2), which include existence and uniqueness, non-negativity, and boundedness of the solution are presented in Section 2. Section 3 focuses on the bifurcation behavior and stability of the system (1.2). Section 4 addresses the control problem of the bifurcation phenomenon in system (1.2) by developing a practical hybrid delayed feedback controller that incorporates parameter perturbation with delay and state feedback. Section 5 selects a rational hybrid delayed feedback controller to solve the bifurcation control issue in the system (1.2) by means of combining parametric perturbation with delay feedback and state feedback. Section 6 manages the control problem of the bifurcation phenomenon in system (1.2) by devising a practical extended hybrid delayed feedback controller, which includes parameter perturbation with delay and state feedback. Section 7 presents simulation results obtained using Matlab software to confirm the critical findings. A conclusion is provided in Section 8.

2. Well-posedness of solution

In this section, we are about to construct a reasonable function to discuss the well-posedness of solutions to model (1.2) (including boundedness, existence, uniqueness, and non-negativity) by utilizing fixed point theory and inequality technique.

Theorem 2.1. *Let $\Omega = \{(u_1, u_2, u_3) \in R^3 : \max\{|u_1|, |u_2|, |u_3|\} \leq \mathcal{U}\}$, where $\mathcal{U} > 0$ indicates a constant. For $(u_{10}, u_{20}, u_{30}) \in \Omega$, system (1.2) under the initial value (u_{10}, u_{20}, u_{30}) owns a unique solution $U = (u_1, u_2, u_3) \in \Omega$.*

Proof. Set

$$f(U) = (f_1(U), f_2(U), f_3(U)), \tag{2.1}$$

where

$$\begin{cases} f_1(U) = \dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}, \\ f_2(U) = \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\bar{\lambda}_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t), \\ f_3(U) = \dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \bar{\lambda}_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t). \end{cases} \tag{2.2}$$

For arbitrary $U, \tilde{U} \in \Omega$, we obtain

$$\begin{aligned} &\|f(U) - f(\tilde{U})\| \\ &= \Bigg\|\left[\frac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_1(t)u_2(t)}{m+u_1(t)}\right] \\ &\quad - \left[\frac{1}{1+\varpi_1 \tilde{u}_2(t-\theta)}\gamma \tilde{u}_1(t) - d(1+\varpi_2 \tilde{u}_2(t))\tilde{u}_1^2(t) - \frac{1}{1+\varpi_3 \tilde{u}_3(t-\theta)}\frac{a_1 \tilde{u}_1(t)\tilde{u}_2(t)}{m+\tilde{u}_1(t)}\right]\Bigg\| \\ &\quad + \Bigg\|\left[\frac{1}{\varpi_3 u_3(t-\theta)}\frac{\bar{\lambda}_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3)u_2^2(t) - a_2 u_2(t)u_3(t)\right] \\ &\quad - \left[\frac{1}{\varpi_3 \tilde{u}_3(t-\theta)}\frac{\bar{\lambda}_1 a_1 \tilde{u}_1 \tilde{u}_2}{m+\tilde{u}_1} - \xi_1 \tilde{u}_2 - \xi_2(1+\varpi_4 \tilde{u}_3)\tilde{u}_2^2 - a_2 \tilde{u}_2 \tilde{u}_3\right]\Bigg\| \\ &\quad + \Bigg\|\left[\frac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \bar{\lambda}_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t)\right] \\ &\quad - \left[\frac{\beta \tilde{u}_3(t)}{1+\beta_0 \tilde{u}_3(t-\theta)} + \bar{\lambda}_2 a_2 \tilde{u}_2(t)\tilde{u}_3(t) - \xi_3 \tilde{u}_3(t)\right]\Bigg\| \\ &= \Bigg\|\left[\frac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - du_1^2(t) - du_1^2(t)\varpi_2 u_2(t) - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_1(t)u_2(t)}{m+u_1(t)}\right] \\ &\quad - \left[\frac{1}{1+\varpi_1 \tilde{u}_2(t-\theta)}\gamma \tilde{u}_1 - d\tilde{u}_1^2 - d\tilde{u}_1^2(t)\varpi_2 \tilde{u}_2(t) - \frac{1}{1+\varpi_3 \tilde{u}_3(t-\theta)}\frac{a_1 \tilde{u}_1 \tilde{u}_2}{m+\tilde{u}_1}\right]\Bigg\| \\ &\quad + \Bigg\|\left[\frac{1}{\varpi_3 u_3(t-\theta)}\frac{\bar{\lambda}_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2 u_2^2(t) - \xi_2\varpi_4 u_3(t)u_2^2(t) - a_2 u_2(t)u_3(t)\right] \\ &\quad - \left[\frac{1}{\varpi_3 \tilde{u}_3(t-\theta)}\frac{\bar{\lambda}_1 a_1 \tilde{u}_1(t)\tilde{u}_2(t)}{m+\tilde{u}_1(t)} - \xi_1 \tilde{u}_2(t) - \xi_2 \tilde{u}_2^2(t) - \xi_2\varpi_4 \tilde{u}_3(t)\tilde{u}_2^2 - a_2 \tilde{u}_2(t)\tilde{u}_3(t)\right]\Bigg\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|\left[\frac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)}+\bar{\lambda}_2 a_2 u_2(t) u_3(t)-\xi_3 u_3(t)\right]\right. \\
&\left.-\left[\frac{\beta \tilde{u}_3(t)}{1+\beta_0 \tilde{u}_3(t-\theta)}+\bar{\lambda}_2 a_2 \tilde{u}_2(t) \tilde{u}_3(t)-\xi_3 \tilde{u}_3(t)\right]\right\| \\
\leq\ & \gamma\left|u_1-\tilde{u}_1\right|+\gamma \varpi_1 \mathcal{U}\left|u_1-\tilde{u}_1\right|+2 d \mathcal{U}\left|u_1-\tilde{u}_1\right| \\
&+2 d \varpi_2 \mathcal{U}^2\left|u_1-\tilde{u}_1\right|+\frac{a_1}{m} \mathcal{U}\left|u_1-\tilde{u}_1\right|+\frac{\bar{\lambda}_1 a_1}{m} \mathcal{U}\left|u_1-\tilde{u}_1\right| \\
&+\mathcal{U}^2 d \varpi_2\left|u_2-\tilde{u}_2\right|+\frac{a_1}{m} \mathcal{U}\left|u_2-\tilde{u}_2\right|+\frac{a_1}{m^2} \mathcal{U}^2\left|u_2-\tilde{u}_2\right| \\
&+\frac{\bar{\lambda}_1 a_1}{m} \mathcal{U}\left|u_2-\tilde{u}_2\right|+\frac{\bar{\lambda}_1 a_1}{m^2} \mathcal{U}^2\left|u_2-\tilde{u}_2\right|+\xi_1\left|u_2-\tilde{u}_2\right| \\
&+2 \xi_2 \mathcal{U}\left|u_2-\tilde{u}_2\right|+2 \xi_2 \varpi_4 \mathcal{U}^2\left|u_2-\tilde{u}_2\right|+a_2 \mathcal{U}\left|u_2-\tilde{u}_2\right|+\bar{\lambda}_2 a_2 \mathcal{U}\left|u_2-\tilde{u}_2\right| \\
&+\xi_2 \varpi_4 \mathcal{U}^2\left|u_3-\tilde{u}_3\right|+a_2 \mathcal{U}\left|u_3-\tilde{u}_3\right|+\beta\left|u_3-\tilde{u}_3\right| \\
&+\bar{\lambda}_2 a_2 \mathcal{U}\left|u_3-\tilde{u}_3\right|+\xi_3\left|u_3-\tilde{u}_3\right| \\
\leq\ & \bar{\rho}_1\left|u_1-\bar{u}_1\right|+\bar{\rho}_2\left|u_2-\bar{u}_2\right|+\bar{\rho}_3\left|u_3-\bar{u}_3\right| \\
=\ & \left[\gamma(1+\varpi_1 \mathcal{U})+2 d \mathcal{U}(1+\varpi_2 \mathcal{U})+\frac{a_1 \mathcal{U}}{m}(1+\bar{\lambda}_1)\right]\left|u_1-\tilde{u}_1\right| \\
&+\left[(d \varpi_2 \mathcal{U}^2+\frac{a_1 \mathcal{U}}{m}(1+\frac{\mathcal{U}}{m}+\bar{\lambda}_1+\frac{\bar{\lambda}_1 \mathcal{U}}{m})+\xi_1\right. \\
&\left.+2 \xi_2 \mathcal{U}(1+\varpi_4 \mathcal{U})+a_2 \mathcal{U}(1+\bar{\lambda}_2)\right]\left|u_2-\tilde{u}_2\right| \\
&+\left[\xi_2 \varpi_4 \mathcal{U}^2+a_2 \mathcal{U}(1+\bar{\lambda}_2)+\beta+\xi_3\right]\left|u_2-\bar{u}_2\right| . \\
\leq\ & \rho(\left|u_1-\bar{u}_1\right|+\left|u_2-\bar{u}_2\right|+\left|u_3-\bar{u}_3\right|),
\end{aligned} \tag{2.3}
$$

where

$$
\left\{\begin{array}{l}
\bar{\rho}_1=\gamma(1+\varpi_1 \mathcal{U})+2 d \mathcal{U}(1+\varpi_2 \mathcal{U})+\frac{a_1 \mathcal{U}}{m}(1+\bar{\lambda}_1), \\
\bar{\rho}_2=d \varpi_2 \mathcal{U}^2+\frac{a_1 \mathcal{U}}{m}\left(1+\frac{\mathcal{U}}{m}+\bar{\lambda}_1+\frac{\bar{\lambda}_1 \mathcal{U}}{m}\right)+\xi_1+2 \xi_2 \mathcal{U}(1+\varpi_4 \mathcal{U})+a_2 \mathcal{U}(1+\bar{\lambda}_2), \\
\bar{\rho}_3=\xi_2 \varpi_4 \mathcal{U}^2+a_2 \mathcal{U}(1+\bar{\lambda}_2)+\beta+\xi_3 .
\end{array}\right. \tag{2.4}
$$

Let

$$
\rho=\max \{\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3\} . \tag{2.5}
$$

From (2.3), we get

$$
\|f(U)-f(\tilde{U})\| \leq \rho\|U-\tilde{U}\|. \tag{2.6}
$$

Thus, $f(U)$ satisfies the Lipschitz condition for U. Applying the fixed point theorem, one can readily conclude that Theorem 2.1 is accurate.

Theorem 2.2. *Each solution to model (1.2) beginning with R_+^3 is non-negative.*

Proof. Based on the first equation of system (1.2), we can obtain

$$
\frac{d u_1}{d t}=\frac{\gamma u_1}{1+\varpi_1 u_2(t-\theta)}-d(1+\varpi_2 u_2) u_1^2-\frac{1}{1+\varpi_3 u_3(t-\theta)} \frac{a_1 u_1 u_2}{m+u_1}, \tag{2.7}
$$

then

$$\frac{du_1}{u_1} = \left(\frac{\gamma}{1+\varpi_1 u_2} - d(1+\varpi_2 u_2)u_1 - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_2}{m+u_1}\right)dt, \tag{2.8}$$

which results in

$$\int_0^t \frac{du_1}{u_1} = \int_0^t \left[\frac{\gamma}{1+\varpi_1 u_2(t-\theta)} - d(1+\varpi_2 u_2(s))u_1(s) - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_2(s)}{m+u_1(s)}\right]ds. \tag{2.9}$$

Then one derives

$$\frac{u_1(t)}{u_1(0)} = \exp\left\{\int_0^t \left[\frac{\gamma}{1+\varpi_1 u_2(t-\theta)} - d(1+\varpi_2 u_2(s)) - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_1(s)}{m+u_1(s)}\right]ds\right\}. \tag{2.10}$$

Thus

$$\begin{aligned} u_1(t) &= u_1(0) \\ &\times \exp\left\{\int_0^t \left[\frac{\gamma}{1+\varpi_1 u_2(t-\theta)} - d(1+\varpi_2 u_2(s)) - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_1(s)}{m+u_1(s)}\right]ds\right\} > 0. \end{aligned} \tag{2.11}$$

By the same method, we know

$$\begin{aligned} u_2(t) &= u_2(0) \\ &\times \exp\left\{\int_0^t \left[\frac{1}{\varpi_3 u_3(t-\theta)}\frac{\bar{\lambda}_1 \xi_1 a_1 u_1(s)}{m+u_1(s)} - \xi_2(1+\varpi_4 u_3(s))u_2(s) - a_2 u_3(s)\right]ds\right\} > 0. \end{aligned} \tag{2.12}$$

$$u_3(t) = u_3(0)\exp\left\{\int_0^t \left[\frac{\eta}{1+\eta_0 u_3(t-\theta)} + \bar{\lambda}_2 a_2 u_2(s) - \xi_3\right]ds\right\} > 0. \tag{2.13}$$

Hence, Theorem 2.2 holds true.

Theorem 2.3. *If* $\theta = 0$ *and* $\bar{\lambda}_1 a_1 < \xi_1, \bar{\lambda}_2 < 1, \eta < \xi_3$*, therefore, all solutions of system (1.2) initialized at* R_+^3 *are uniformly bounded.*

Proof. Let

$$W(t) = u_1(t) + u_2(t) + u_3(t). \tag{2.14}$$

Then

$$\begin{aligned} \frac{dW(t)}{dt} &= \frac{du_1(t)}{dt} + \frac{du_2(t)}{dt} + \frac{du_3(t)}{dt} \\ &= \left[\frac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1 - d(1+\varpi_2 u_2)u_1^2 - \frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{a_1 u_1 u_2}{m+u_1}\right] \\ &\quad + \left[\frac{1}{1+\varpi_3 u_3(t-\theta)}\frac{\bar{\lambda}_1 a_1 u_1 u_2}{m+u_1} - \xi_2 u_2 - \xi_2(1+\varpi_4 u_3)u_2^2 - a_2 u_2 u_3\right] \\ &\quad + \left[\frac{\beta}{1+\beta_0 u_3(t-\theta)} + \bar{\lambda}_2 a_2 u_2 u_3 - \xi_3 u_3\right] \\ &\le \gamma u_1 + d(1+\varpi_2 u_2)u_1^2 + \bar{\lambda}_1 a_1 u_2 - \xi_1 u_2 - \xi_2(1+\varpi_4 u_3)u_2^2 - a_2 u_2 u_3 \end{aligned}$$

$$\begin{aligned} & +\eta u_3 + \bar{\lambda}_2 a_2 u_2 u_3 - \xi_3 u_3 \\ \le & \Big(-\gamma u_1 + (\bar{\lambda}_1 a_1 - \xi_1) u_2 + (\eta - \xi_3) u_3\Big) + 2\gamma u_1 - d u_1^2 - (a_2 - \bar{\lambda}_2 a_2) u_2 u_3 \\ = & -\gamma_0 W(t) + \frac{\gamma^2}{d}, \end{aligned} \tag{2.15}$$

where where

$$\gamma_0 = \min\{\gamma, \xi_1 - \theta_1 a_1, \xi_3 - \eta\}. \tag{2.16}$$

According to Eq (2.15), we obtain

$$W(t) \to \gamma^2/d\gamma_0, \text{when} \quad t \to \infty. \tag{2.17}$$

Consequently, the solutions to the system (1.2) are uniformly bounded.

3. Exploration of bifurcation of model (1.2)

In this section, we intend to explore the issues of bifurcation and stability of the model (1.2). To start, we suppose that $F(u_{1\star}, u_{2\star}, u_{3\star})$ is the equilibrium point of model (1.2), then $u_{1\star}$–$u_{3\star}$ adhere to the following requirement:

$$\begin{cases} \dfrac{1}{1+\varpi_1 u_{2\star}} \gamma u_{1*} - d(1+\varpi_2 u_{2\star}) u_{1\star}^2 - \dfrac{1}{1+\varpi_3 u_{3\star}} \dfrac{a_1 u_{1\star} u_{2\star}}{m + u_{1\star}} = 0, \\ \dfrac{1}{1+\varpi_3 u_{3\star}} \dfrac{\bar{\lambda}_1 a_1 u_{1\star} u_{2\star}}{m + u_{1\star}} - \xi_1 u_{2\star} - \xi_2 (1+\varpi_4 u_{3\star}) u_{2\star}^2 - a_2 u_{2\star} u_{3\star} = 0, \\ \dfrac{\beta u_{3\star}}{1+\beta_0 u_{3\star}} + \bar{\lambda}_2 a_2 u_{2\star} u_{3\star} - \xi_3 u_{3\star} = 0. \end{cases} \tag{3.1}$$

Let

$$\begin{cases} \bar{u}_1(t) = u_1(t) - u_{1\star}, \\ \bar{u}_2(t) = u_1(t) - u_{2\star}, \\ \bar{u}_3(t) = u_2(t) - u_{3\star}. \end{cases} \tag{3.2}$$

Insert system (3.2) into system (1.2) and we obtain the linear system of model (1.2) at $F(u_{1\star}, u_{2\star}, u_{3\star})$:

$$\begin{cases} \dfrac{d\bar{u}_1}{dt} = l_1 \bar{u}_1 + l_2 \bar{u}_2 - l_3 \bar{u}_2(t-\theta) - l_4 \bar{u}_3(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = l_5 \bar{u}_1 + l_6 \bar{u}_2 - l_7 \bar{u}_3 - l_8 \bar{u}_2(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = l_9 \bar{u}_2 + l_{10} \bar{u}_3 - l_{11} \bar{u}_3(t-\theta), \end{cases} \tag{3.3}$$

where

$$\begin{cases} l_1 = \dfrac{\gamma}{1+\varpi_1 u_{2\star}} - 2d\varpi_2 u_{1\star}u_{2\star} - \dfrac{a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})} + \dfrac{a_1 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})^2}, \\ l_2 = \dfrac{a_1 u_{1\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}, \\ l_3 = \dfrac{\gamma\varpi_1 u_{1\star}}{(1+\varpi_1 u_{2\star})^2}, \\ l_4 = \dfrac{a_1\varpi_3 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})^2(m+u_{1\star})}, \\ l_5 = \dfrac{\bar{\lambda}_1 a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})} - \dfrac{\bar{\lambda}_1 a_1 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})^2}, \\ l_6 = \dfrac{\bar{\lambda}_1 a_1 u_{1\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})} - \xi_1 - 2\xi_2(1+\varpi_4 u_{2\star}u_{3\star}) - a_2 u_{3\star}, \\ l_7 = \xi_2\varpi_4 u_{2\star}^2 + a_2 u_{2\star}, \\ l_8 = \dfrac{\bar{\lambda}_1 a_1\varpi_3 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})^2(m+u_{1\star})}, \\ l_9 = \bar{\lambda}_2 a_2 u_{3\star}, \\ l_{10} = \dfrac{\beta}{1+\beta_0 u_{3\star}} + \bar{\lambda}_2 a_2 u_{2\star} - \xi_3, \\ l_{11} = \dfrac{\beta\beta_0 u_{3\star}}{(1+\beta_0 u_{3\star})^2}. \end{cases} \tag{3.4}$$

The characteristic equation for system (3.3) takes the following expression:

$$\det\begin{bmatrix} \lambda - l_1 & -l_2 + l_3 e^{-\lambda\theta} & -l_4 e^{-\lambda\theta} \\ -l_5 & \lambda - l_6 & l_7 + l_8 e^{-\lambda\theta} \\ 0 & -l_9 & \lambda - l_{10} + l_{11}e^{-\lambda\theta} \end{bmatrix} = 0, \tag{3.5}$$

which results in

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 + (c_4\lambda^2 + c_5\lambda + c_6)e^{-\lambda\theta} + c_7 e^{-2\lambda\theta} = 0, \tag{3.6}$$

where

$$\begin{cases} c_1 = -l_{10} - l_6 - l_1, \\ c_2 = l_6 l_{10} + l_7 l_9 + l_1 l_{10} + l_1 l_6 - l_2 l_5, \\ c_3 = l_2 l_5 l_{10} - l_1 l_6 l_{10} - l_1 l_7 l_9, \\ c_4 = l_{11}, \\ c_5 = l_8 l_9 + l_3 l_5 - l_6 l_1 1 - l_1 l_1 1, \\ c_6 = l_1 l_6 l_{11} - l_1 b_8 l_9 - l_2 l_5 l_{11} - l_3 l_5 l_{10} - l_4 l_5 l_9, \\ c_7 = l_3 l_5 l_{11}. \end{cases} \tag{3.7}$$

If $\theta = 0$. Equation (3.6) changes to

$$\lambda^3 + (c_1 + c_4)\lambda^2 + (c_2 + c_5)\lambda + c_3 + c_6 + c_7 = 0. \tag{3.8}$$

If

$$(Q_1)\begin{cases} D_1 = c_1 + c_3 > 0, \\ D_2 = \begin{vmatrix} c_1 + c_4 & 1 \\ c_3 + c_6 + c_7 & c_2 + c_5 \end{vmatrix} > 0, \\ D_3 = (c_3 + c_6 + c_7)D_2 > 0. \end{cases}$$

is satisfied, then the three roots λ_1–λ_3 of Eq (3.8) have negative real parts. Thus the equilibrium point $F(u_{1\star}, u_{2\star}, u_{1\star})$ of system (1.2) under $\theta = 0$ is locally asymptotically stable.

Presume that $\lambda = i\nu$ is the root of Eq (3.6), then Eq (3.6) turns into

$$(i\nu)^3 + c_1(i\nu)^2 + c_2 i\nu + c_3 + [c_4(i\nu)^2 + c_5 i\nu + c_6]e^{-i\nu\theta} + c_7 e^{-2i\nu\theta} = 0, \tag{3.9}$$

which produces

$$\begin{cases} \Theta_1(\nu)\sin\nu\theta + \Theta_2(\nu)\cos\nu\theta = \Theta_3(\nu), \\ \Theta_4(\nu)\sin\nu\theta + \Theta_5(\nu)\cos\nu\theta = \Theta_6(\nu), \end{cases} \tag{3.10}$$

where

$$\begin{cases} \Theta_1 = \nu^3 - c_2\nu, \\ \Theta_2 = -c_1\nu^2 + e_1, \\ \Theta_3 = c_4\nu^2 - c_6, \\ \Theta_4 = -c_1\nu^2 + e_2, \\ \Theta_5 = -\nu^3 + c_2\nu, \\ \Theta_5 = -c_5\nu, \end{cases} \tag{3.11}$$

and

$$\begin{cases} e_1 = c_3 + c_7, \\ e_2 = c_3 - c_7. \end{cases} \tag{3.12}$$

It follows from (3.10) that

$$\begin{cases} \sin\nu\theta = \dfrac{\Theta_2\Theta_6 - \Theta_3\Theta_5}{\Theta_2\Theta_4 - \Theta_1\Theta_5}, \\ \cos\nu\theta = \dfrac{\Theta_3\Theta_4 - \Theta_1\Theta_6}{\Theta_2\Theta_4 - \Theta_1\Theta_5}. \end{cases} \tag{3.13}$$

Because of $\cos^2\nu\theta + \sin^2\nu\theta = 1$, we can get

$$\left[\frac{\Theta_2\Theta_6 - \Theta_3\Theta_5}{\Theta_2\Theta_4 - \Theta_1\Theta_5}\right]^2 + \left[\frac{\Theta_3\Theta_4 - \Theta_1\Theta_6}{\Theta_2\Theta_4 - \Theta_1\Theta_5}\right]^2 = 1. \tag{3.14}$$

So

$$\begin{aligned} &\Theta_1^2\Theta_6^2 + \Theta_2^2\Theta_6^2 + \Theta_3^2\Theta_4^2 + \Theta_3^2\Theta_5^2 - \Theta_1^2\Theta_5^2 - \Theta_2^2\Theta_4^2 \\ &-2\Theta_1\Theta_3\Theta_4\Theta_6 - 2\Theta_2\Theta_3\Theta_5\Theta_6 + 2\Theta_1\Theta_2\Theta_4\Theta_5 = 0. \end{aligned} \tag{3.15}$$

Hence, the result is as follows:

$$\nu^{12} - G_1\nu^{10} - G_2\nu^8 - G_3\nu^6 - G_4\nu^4 + G_5\nu^2 - G_6 = 0. \tag{3.16}$$

where

$$\begin{cases} G_1 = c_4^2 + 4c_2 - 2c_1^2, \\ G_2 = c_5^2 + c_1^2c_4^2 + 2c_1(e_1 + e_2 + 2c_1c_4) - c_1^4 - 6c_2^2 - 2c_2c_4^2 - 2c_4c_5, \\ G_3 = c_1^2c_5^2 + c_2^2c_4^2 + c_6^2 + 4c_2^3 + 4c_2c_4c_6 + 2c_5c_5e_2 + (e_1 + e_2)(2c_1^3 - 4c_1c_2) \\ \quad - 2c_1^2c_2^2 - 2c_1c_4^2e_2 - 2c_1^2c_4c_6 - 2c_2c_5^2 - 2c_4c_5e_1, \\ G_4 = c_2^2c_4^2 + c_1^2c_5^2 + c_3^2e_2^2 + 2c_2c_3c_4(1 - e_2) + 2c_4c_5(e_1 - e_2) + 4e_1e_2(c_2 - c_1^2) \\ \quad + 4c_1c_3c_5^2e_2 - c_1^2(e_1^2 + e_2^2) - c_2^4 - 2c_2c_5(c_2c_3 + c_5) - 2c_1c_4^2e_1, \\ G_5 = c_2^2c_5^2 + c_4^2e_1^2 + 2c_1(e_1e_2^2 + e_1^2e_2) + 2c_2c_4c_5(e_2 - e_1) \\ \quad - 2c_1c_5^2e_2 - 2c_2^2e_1e_2 - 2c_3c_5e_2^2, \\ G_6 = c_5^2e_2^2 - e_1^2e_2^2. \end{cases} \tag{3.17}$$

Let

$$\Upsilon_1(\nu) = \nu^{12} - G_1\nu^{10} - G_2\nu^8 - G_3\nu^6 - G_4\nu^4 + G_5\nu^2 - G_6. \tag{3.18}$$

Suppose that

$$(Q_2) \;\; |c_5| > |e_1|.$$

Due to (Q_2), we know $\Upsilon_1(0) = -G_6 < 0$, given that $\lim_{\nu\to\infty} \Upsilon_1(\nu) > 0$, we can conclude that Eq (3.17) has at least one positive real root. Therefore, Eq (3.6) has at least one pair of purely roots. Without loss of generality, we can postulate that Eq (3.17) has twelve positive real roots (say ν_j, $j = 1, 2, 3, \cdots, 12$). In light of (3.10), one gets

$$\theta_j^{(n)} = \frac{1}{\nu_j}\left[\arccos\left(\frac{\Theta_2(\nu_j)\Theta_6(\nu_j) - \Theta_3(\nu_j)\Theta_5(\nu_j)}{\Theta_2(\nu_j)(\Theta_4(\nu_j) - \Theta_1(\nu_j)\Theta_5(\nu_j)}\right) + 2n\pi\right], \tag{3.19}$$

where $j = 1, 2, 3 \cdots, 12; n = 0, 1, 2, \cdots;$

$$\begin{cases} \Theta_1(\nu_j) = \nu_j^3 - c_2\nu_j, \\ \Theta_2(\nu_j) = -c_1\nu_j^2 + e_1, \\ \Theta_3(\nu_j) = c_4\nu_j^2 - c_6, \\ \Theta_4(\nu_j) = -c_1\nu_j^2 + e_2, \\ \Theta_5(\nu_j) = -\nu_j^3 + c_2\nu_j, \\ \Theta_6(\nu_j) = -c_5\nu_j. \end{cases} \tag{3.20}$$

Assume $\theta_0 = \min_{\{j=1,2,3,\cdots,12;n=0,1,2,\cdots\}}\{\theta_j^{(n)}\}$ and suppose that when $\theta = \theta_0$, Eq (3.6) has a pair of imaginary roots $\pm i\nu_0$. In the next step, the assumption presented is:

$$(Q_3) \;\; J_{1R}J_{2R} + J_{1I}J_{2I} > 0,$$

where

$$\begin{cases} J_{1R} = (c_2 - 3\nu_0^2)\cos\nu_0\theta_0 - 2c_1\nu_0\sin\nu_0\theta_0 + 5c_4 + c_5, \\ J_{1I} = (c_2 - 3\nu_0^2)\sin\nu_0\theta_0 + 2c_1\nu_0\cos\nu_0\theta_0, \\ J_{2R} = (c_7\nu_0 + c_3\nu_0 - c_1\nu_0^3)\sin\nu_0\theta_0 + (c_2\nu_0^2 - \nu_0^4)\cos\nu_0\theta_0, \\ J_{2I} = (c_2\nu_0^2 - \nu_0^4)\sin\nu_0\theta_0 + (c_7\nu_0 + c_1\nu_0^3 - c_3\nu_0)\cos\nu_0\theta_0. \end{cases} \tag{3.21}$$

Suppose that $\lambda(\theta) = \varrho_1(\theta) + i\varrho_2(\theta)$ is the root of Eq (3.6) at $\theta = \theta_0$ such that $\epsilon_1(\theta_0) = 0$, $\epsilon_2(\theta_0) = \nu_0$, then $Re\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta=\theta_0,\nu=\nu_0} > 0$.

Proof. Using Eq (3.6), we can gain

$$[\left(3\lambda^2+2c_1\lambda+c_2)e^{\lambda\theta}\right]\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}+(\lambda^3+c_1\lambda^2+c_2\lambda+c_3)e^{\lambda\theta}(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta+\lambda)$$
$$+2c_4\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}+c_5\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}-c_7e^{-\lambda\theta}(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta+\lambda)=0. \tag{3.22}$$

It means that

$$\left(\frac{d\lambda}{d\theta}\right)^{-1}=\frac{J_1(\lambda)}{J_2(\lambda)}-\frac{\theta}{\lambda}, \tag{3.23}$$

where

$$\begin{cases} J_1(\lambda)=(3\lambda^2+c_1\lambda+c_2)e^{\lambda\theta}+2c_4+c_5,\\ J_2(\lambda)=c_7\lambda e^{-\lambda\theta}-(\lambda^3+c_1\lambda^2+c_2\lambda+c_3)\lambda e^{\lambda\theta}. \end{cases} \tag{3.24}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_0,\nu=\nu_0}=\mathrm{Re}\left[\frac{J_1(\lambda)}{J_2(\lambda)}\right]_{\theta=\theta_0,\nu=\nu_0}=\frac{J_{1R}J_{2R}+J_{1I}J_{2I}}{J_{2R}^2+J_{2I}^2}. \tag{3.25}$$

Through this assumption (Q_3), we obtain

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_0,\nu=\nu_0}>0. \tag{3.26}$$

The proof is concluded. Based on the discussion above, the following result can be easily obtained.

Theorem 3.1. *Assume that (Q_1)–(Q_3) are valid, then the equilibrium point $F(u_{1\star},u_{2\star},u_{3\star})$ of model (3.1) is locally asymptotically stable state if $\theta\in[0,\theta_0)$. Moreover, model (3.1) exhibits a cluster of Hopf bifurcations around the equilibrium point $F(u_{1*},u_{2*},u_{3*})$ when $\theta=\theta_0$.*

4. Regulation of bifurcation in model (1.2) through hybrid controller 1

In this portion, we will study the time of Hopf bifurcation onset in system (1.2) by utilizing a sensible hybrid controller, containing parameter perturbation with delay and state feedback. Building on the concepts from [13,14], we propose the following controlled predator-prey model:

$$\begin{cases} \dfrac{du_1}{dt}=\dfrac{1}{1+\varpi_1u_2(t-\theta)}\gamma u_1(t)-d(1+\varpi_2u_2(t))u_1^2(t)\\ \qquad-\dfrac{1}{1+\varpi_3u_3(t-\theta)}\dfrac{a_1u_1(t)u_2(t)}{m+u_1(t)},\\ \dfrac{du_2}{dt}=\dfrac{1}{1+\varpi_3u_3(t-\theta)}\dfrac{\lambda_1a_1u_1(t)u_2(t)}{m+u_1(t)}-\xi_1u_2(t)\\ \qquad-\xi_2(1+\varpi_4u_3(t))u_2^2(t)-a_2u_2(t)u_3(t),\\ \dfrac{du_3}{dt}=\eta_1\left[\dfrac{\beta u_3(t)}{1+\beta_0u_3(t-\theta)}+\bar{\lambda}_2a_2u_2(t)u_3(t)-\xi_3u_3(t)\right]+\eta_2[u_3(t)-u_3(t-\theta)], \end{cases} \tag{4.1}$$

where η_1,η_2 indicate control parameters. System (4.1) shares the same equilibrium point $F(u_{1\star},u_{2\star},u_{3\star})$ as that in system (1.2). Let

$$\begin{cases} u_{1\star}=u_1(t)-\bar{u}_1(t),\\ u_{2\star}=u_1(t)-\bar{u}_2(t),\\ u_{3\star}=u_2(t)-\bar{u}_1(t). \end{cases} \tag{4.2}$$

The linear system of system (4.1) near $F(u_{1\star}, u_{2\star}, u_{2\star})$ can be represented as follows:

$$\begin{cases} \dfrac{d\bar{u}_1}{dt} = k_1\bar{u}_1 + k_2\bar{u}_2 - k_3\bar{u}_2(t-\theta) + k_4\bar{u}_3(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = k_5\bar{u}_1 + 6_7\bar{u}_2 - k_7\bar{u}_3 - k_8\bar{u}_3(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = k_9\bar{u}_2 + k_{10}\bar{u}_3 - k_{11}\bar{u}_3(t-\theta), \end{cases} \tag{4.3}$$

where

$$\begin{cases} k_1 = \dfrac{\gamma}{1+\varpi_1 u_{2\star}} - 2du_{1\star}(1+\varpi_2 u_{2\star}) - \dfrac{a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}(1 - \dfrac{u_{1\star}}{m+u_{1\star}}), \\ k_2 = \dfrac{a_1 u_{1\star}}{(1+\varpi_3 u_{3*})(m+u_{1\star})}, \\ k_3 = \dfrac{\gamma\varpi_1 u_{1\star}}{(1+\varpi_1 u_{2\star})^2}, \\ k_4 = \dfrac{a_1\varpi_3 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})^2(m+u_{1\star})}, \\ k_5 = \dfrac{\bar{\lambda}_1 a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}(1 - \dfrac{u_{1\star}}{m+u_{1\star}}), \\ k_6 = \dfrac{\bar{\lambda}_1 a_1 u_{1\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})} - \xi_1 - 2\xi_2(1+\varpi_4 u_{2\star}u_{3\star}) - a_2 u_{3\star}, \\ k_7 = \xi_2\varpi_4 u_{2\star}^2 + a_2 u_{2\star}, \\ k_8 = \dfrac{\bar{\lambda}_1 a_1\varpi_3 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})^2(m+u_{1\star})}, \\ k_9 = \bar{\lambda}_2 a_2 u_{3\star}, \\ k_{10} = \eta_1\left[\dfrac{\beta}{1+\beta_0 u_{3\star}} + \bar{\lambda}_2 a_2 u_{2\star} - \xi_3\right], \\ k_{11} = \dfrac{\beta\beta_0 u_{3\star}}{(1+\beta_0 u_{3\star})^2} + \eta_2. \end{cases} \tag{4.4}$$

The characteristic equation of system (4.3) is given by the following expression:

$$\det\begin{bmatrix} \lambda - k_1 & -k_2 + k_3 e^{-\lambda\theta} & -k_4 e^{-\lambda\theta} \\ -k_5 & \lambda - k_6 & k_7 + k_8 e^{-\lambda\theta} \\ 0 & -k_9 & \lambda - k_{10} + k_{11}e^{-\lambda\theta} \end{bmatrix} = 0, \tag{4.5}$$

which leads to

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 + (h_4\lambda^2 + h_5\lambda + h_6)e^{-\lambda\theta} + h_7 e^{-2\lambda\theta} = 0, \tag{4.6}$$

where

$$\begin{cases} h_1 = -k_{10} - k_6 - k_1, \\ h_2 = k_6k_{10} + k_7k_9 + k_1k_{10} + k_1k_6 - k_2k_5, \\ h_3 = k_2k_5k_{10} - k_1k_6k_{10} - k_1k_7k_9, \\ h_4 = k_{11}, \\ h_5 = k_8k_9 + k_3k_5 - k_6k_{11} - k_1k_{11}, \\ h_6 = k_1k_6k_{11} - k_1k_8k_9 - k_2k_5k_{11} - k_3k_5k_{10}, \\ h_7 = k_3k_5k_{11}. \end{cases} \tag{4.7}$$

If $\theta = 0$, then Eq (4.6) simplifies to

$$\lambda^3 + (h_1 + h_4)\lambda^2 + (h_2 + h_5)\lambda + h_3 + h_6 + h_7 = 0. \tag{4.8}$$

Thus, all roots of Eq (4.8) have negative real parts if and only if

$$(Q_4)\begin{cases} \nabla_1 = h_1 + h_4 > 0, \\ \nabla_2 = \det\begin{bmatrix} h_1 + h_4 & 1 \\ h_3 + h_6 + h_7 & h_2 + h_5 \end{bmatrix} > 0, \\ \nabla_3 = (h_3 + h_6 + h_7)\nabla_2 > 0. \end{cases} \tag{4.9}$$

is satisfied, then the three roots λ_1–λ_3 of Eq (4.8) have negative real parts. Thus the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$ of system (4.1) with $\theta = 0$ is locally asymptotically stable.

Assume that $\lambda = i\nu^*$ is the root of Eq (4.6), then Eq (4.6) becomes:

$$((i\nu^*)^3 + h_1(i\nu^*)^2 + h_2 i\nu^* + h_3)e^{i\nu^*\theta} + h_4(i\nu^*)^2 + h_5 i\nu^* + h_6 + h_7 e^{-i\nu^*\theta} = 0, \tag{4.10}$$

From Eq (4.10), we can deduce that

$$\begin{cases} \Psi_1(\nu^*)\sin\nu^*\theta + \Psi_2(\nu^*)\cos\nu\theta = \Psi_3(\nu^*), \\ \Psi_4(\nu^*)\sin\nu^*\theta + \Psi_5\cos\nu^*\theta = \Psi_6(\nu^*), \end{cases} \tag{4.11}$$

where

$$\begin{cases} \Psi_1 = \nu^{*3} - h_2\nu^*, \\ \Psi_2 = -h_1\nu^{*2} + q_1, \\ \Psi_3 = h_4 nu^{*2} - h_6, \\ \Psi_4 = -h_1\nu^{*2} + q_2, \\ \Psi_5 = -\nu^{*3} + h_2\nu^*, \\ \Psi_5 = -h_5\nu^*, \end{cases} \tag{4.12}$$

and

$$\begin{cases} q_1 = h_3 + h_7, \\ q_2 = h_3 - h_7. \end{cases} \tag{4.13}$$

This means that

$$\begin{cases} \sin\nu^*\theta = \dfrac{\Psi_2\Psi_6 - \Psi_3\Psi_5}{\Psi_2\Psi_4 - \Psi_1\Psi_5}, \\ \cos\nu^*\theta = \dfrac{\Psi_3\Psi_4 - \Psi_1\Psi_6}{\Psi_2\Psi_4 - \Psi_1\Psi_5}. \end{cases} \tag{4.14}$$

On account of $\cos^2\nu^*\theta + \sin^2\nu^*\theta = 1$, from (4.14), it can be concluded that

$$\begin{aligned} &[\Psi_2(\nu^*)\Psi_4(\nu^*) - \Psi_3(\nu^*)\Psi_5(\nu^*)]^2 + [\Psi_3(\nu^*)\Psi_4(\nu^*) - \Psi_1(\nu^*)\Psi_6(\nu^*)]^2 \\ &= [\Psi_2(\nu^*)\Psi_4(\nu^*) - \Psi_1(\nu^*)\Psi_5(\nu^*)]^2. \end{aligned} \tag{4.15}$$

Thus, the following conclusions can be drawn

$$\nu^{*12} - V_1\nu^{*10} - V_2\nu^{*8} - V_3\nu^{*6} - V_4\nu^{*4} - V_5\nu^{*2} - V_6 = 0, \tag{4.16}$$

where

$$
\begin{cases}
V_1 = h_4^2 + 4h_2 - 2h_1^2, \\
V_2 = h_5^2 + h_1^2h_4^2 + 2h_1(q_1 + q_2 + 2h_1h_4) - h_1^4 - 6h_2^2 - 2h_2h_4^2 - 2h_4h_5, \\
V_3 = h_1^2h_5^2 + h_2^2h_4^2 + h_6^2 + 4h_2^3 + 4h_2h_4h_6 + 2h_5h_5q_2 + (q_1 + q_2)(2h_1^3 - 4h_1h_2) \\
\quad - 2h_1^2h_2^2 - 2h_1h_4^2e_2 - 2h_1^2h_4h_6 - 2h_2h_5^2 - 2c_4h_5q_1, \\
V_4 = h_2^2h_4^2 + h_1^2h_5^2 + h_3^2q_2^2 + 2h_2h_3h_4(1 - q_2) + 2h_4h_5(q_1 - q_2) + 4q_1q_2(h_2 - h_1^2) \\
\quad + 4h_1h_3h_5^2q_2 - h_1^2(q_1^2 + q_2^2) - h_2^4 - 2q_2q_5(q_2q_3 + h_5) - 2h_1h_4^2q_1, \\
V_5 = h_2^2h_5^2 + h_4^2q_1^2 + 2h_1(q_1q_2^2 + q_1^2q_2) + 2q_2q_4q_5(q_2 - q_1) \\
\quad - 2h_1h_5^2q_2 - 2h_2^2q_1q_2 - 2h_3h_5q_2^2, \\
V_6 = h_5^2q_2^2 - q_1^2q_2^2.
\end{cases}
\tag{4.17}
$$

Let

$$
\Upsilon_2(\nu^*) = \nu^{*12} - V_1\nu^{*10}V_2\nu^{*8} - V_3\nu^{*6} - V_4\nu^{*4} + V_5\nu^{*2} - V_6. \tag{4.18}
$$

Suppose that

$$
(Q_5)\ \ |h_5| > |q_1|.
$$

If (Q_5) remains, then $\Upsilon_2(0) = -V_6 < 0$, since $\lim_{\nu^*\to\infty}\Upsilon_2(\nu^*) > 0$, then the Eq (4.16) possesses at least one positive real root. Consequently, Eq (4.6) has at least one pair of purely imaginary roots. Without loss of generality, we can assume that Eq (4.16) has twelve positive real roots (say ν_j^*, $j = 1, 2, 3, \cdots, 12$). Relying on (4.14), we know

$$
\theta_j^{(k)} = \frac{1}{\nu_j^*}\left[\arccos\left(\frac{\Psi_2(\nu_j)\Psi_6(\nu_j^*) - \Psi_3(\nu_j^*)\Psi_5(\nu_j^*)}{\Psi_2(\nu_j^*)(\Psi_4(\nu_j^*) - \Psi_1(\nu_j^*)\Psi_5(\nu_j^*)} + 2k\pi\right)\right], \tag{4.19}
$$

where $j = 1, 2, 3, \cdots, 12; k = 0, 1, 2, \cdots$. Assume $\theta_* = \min_{\{j=1,2,3,\cdots,12;k=0,1,2,\cdots\}}\{\theta_j^{(k)}\}$ and suppose that when $\theta = \theta_*$, Eq (4.6) has a pair of imaginary roots $\pm i\nu_0^*$.

Next, we present the following assumption:

$$
(Q_6)\ \ M_{1R}M_{2R} + M_{1I}M_{2I} > 0,
$$

where

$$
\begin{cases}
M_{1R} = (h_2 - 3\nu_0^{*2})\cos\nu_0^*\theta_0 - 2h_1\nu_0^*\sin\nu_0^*\theta_0 + 2h_4 + h_5, \\
M_{1I} = (h_2 - 3\nu_0^{*2})\sin\nu_0^*\theta_0 + 2h_1\nu_0^*\cos\nu_0^*\theta_0, \\
M_{2R} = (h_7\nu_0^* + h_3\nu_0^* - h_1\nu_0^{*3})\sin\nu_0^*\theta_0 + (h_2{\nu^*}_0^2 - \nu_0^{*4})\cos\nu_0^*\theta_0, \\
M_{2I} = (h_2{\nu^*}_0^2 - \nu_0^{*4})\sin\nu_0^*\theta_0 + (h_7\nu_0^* + h_1\nu_0^{*3} - h_3\nu_0^*)\cos\nu_0^*\theta_0.
\end{cases}
\tag{4.20}
$$

Lemma 4.1. *Suppose that $\lambda(\theta) = \bar{\varrho}_1(\theta) + i\bar{\varrho}_2(\theta)$ is the root of Eq (4.6) at $\theta = \theta_*$ such that $\bar{\varrho}_1(\theta_*) = 0$, $\bar{\varrho}_2(\theta_*) = \nu_0^*$, then Re $\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_\star,\nu=\nu_0^*} > 0$.*

Proof. It follows from Eq (4.6) that

$$
\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 + (h_4\lambda^2 + h_5\lambda + h_6)e^{-\lambda\theta} + h_7e^{-2\lambda\theta} = 0, \tag{4.21}
$$

we can get

$$
[\big(3\lambda^2 + 2h_1\lambda + h_2)e^{\lambda\theta}\big]\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3)e^{\lambda\theta}(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda)
$$

$$+2h_4\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}+h_5\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}-h_7e^{-\lambda\theta}(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta+\lambda)=0. \tag{4.22}$$

It means that

$$\left(\frac{d\lambda}{d\theta}\right)^{-1}=\frac{M_1(\lambda)}{M_2(\lambda)}-\frac{\theta}{\lambda}, \tag{4.23}$$

where

$$\begin{cases} M_1(\lambda)=(3\lambda^2+h_1\lambda+h_2)e^{\lambda\theta}+2h_4+h_5,\\ M_2(\lambda)=h_7\lambda e^{-\lambda\theta}-(\lambda^3+h_1\lambda^2+h_2\lambda+h_3)\lambda e^{\lambda\theta}. \end{cases} \tag{4.24}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_*,\nu=\nu_0^*}=\mathrm{Re}\left[\frac{M_1(\lambda)}{M_2(\lambda)}\right]_{\theta=\theta_*,\nu=\nu_0^*}=\frac{M_{1R}M_{2R}+M_{1I}M_{2I}}{M_{2R}^2+M_{2I}^2}. \tag{4.25}$$

Through this assumption (Q_6), we obtain

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_*,\nu=\nu_0^*}>0. \tag{4.26}$$

The proof is concluded. Based on the discussion above, the following result can be easily obtained

Theorem 4.1. *Assuming that (Q_4)–(Q_6) is valid, then the equilibrium point $F(u_{1\star},u_{2\star},u_{3\star})$ of model (4.1) is locally asymptotically stable if $\theta\in[0,\theta_*)$; moreover, model (4.1) exhibits a cluster of Hopf bifurcations around the equilibrium point $F(u_{1\star},u_{2\star},u_{3\star})$ when $\theta=\theta_*$.*

Remark 4.1. *In this paper, the purpose of control is to adjust the stability domain and the time of onset of Hopf bifurcation. In this section, we enlarge (or reduce) some parameter values of model (1.2) and add (or reduce) a suitable parameter perturbation with delay to adjust the density of top predators and then achieve our control objectives. In the third equation, η_1 and η_2 are feedback gain parameters. If $\eta_1>0(<0)$, the density of top predators increases (decreases) and if $\eta_2>0(<0)$, parameter perturbation increases (decreases). In a biological sense, we realize the balance of biological populations (the densities of prey, intermediate predators, and top predators) via this hybrid controller. Of course, we can add this control to other equations of model (1.2), but whether the control goal can be achieved will be explored it via mathematical analysis and computer simulation.*

5. Regulation of bifurcation in model (1.2) through hybrid controller 2

In this segment, we will study the Hopf bifurcation problem of system (1.2) through the use of a reasonable hybrid controller that combines parameter perturbation with delay and state feedback. Based on the ideas presented in [13,14], we develop the following controlled predator-prey model:

$$\begin{cases} \frac{du_1}{dt}=\frac{1}{1+\varpi_1u_2(t-\theta)}\gamma u_1(t)-d(1+\varpi_2u_2(t))u_1^2(t)-\frac{1}{1+\varpi_3u_3(t-\theta)}\frac{a_1u_1(t)u_2(t)}{m+u_1(t)},\\ \frac{du_2}{dt}=\rho_1\left[\frac{1}{1+\varpi_3u_3(t-\theta)}\frac{\bar{\lambda}_1a_1u_1(t)u_2(t)}{m+u_1(t)}-\xi_1u_2(t)-\xi_2(1+\varpi_4u_3(t))u_2^2(t)-a_2u_2(t)u_3(t)\right]\\ \quad+\rho_2[u_2(t)-u_2(t-\theta)],\\ \frac{du_3}{dt}=\frac{\beta u_3(t)}{1+\beta_0u_3(t-\theta)}+\bar{\lambda}_2a_2u_2(t)u_3(t)-\xi_3u_3(t), \end{cases} \tag{5.1}$$

where ρ_1, ρ_2 indicate control parameters. System (5.1) shares the same equilibrium point as that in system (1.2). Let

$$\begin{cases} u_{1\star} = u_1(t) - \bar{u}_1(t), \\ u_{2\star} = u_1(t) - \bar{u}_2(t), \\ u_{3\star} = u_2(t) - \bar{u}_1(t). \end{cases} \tag{5.2}$$

The linear system of system (5.1) near $F(u_{1\star}, u_{2\star}, u_{2\star})$ can be described as:

$$\begin{cases} \dfrac{d\bar{u}_1}{dt} = m_1\bar{u}_1 + m_2\bar{u}_2 - m_3\bar{u}_2(t-\theta) + m_4\bar{u}_3(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = m_5\bar{u}_1 + m_6\bar{u}_2 - m_7\bar{u}_3 - m_8\bar{u}_2(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = m_9\bar{u}_2 + m_{10}\bar{u}_3 - m_{11}\bar{u}_3(t-\theta), \end{cases} \tag{5.3}$$

where

$$\begin{cases} m_1 = \dfrac{\gamma}{1+\varpi_1 u_{2\star}} - 2du_{1\star}(1+\varpi_2 u_{2\star}) - \dfrac{a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}(1 - \dfrac{u_{1\star}}{m+u_{1\star}}), \\ m_2 = \dfrac{a_1 u_{1\star}}{(1+\varpi_3 u_{3*})(m+u_{1\star})}, \\ m_3 = \dfrac{\gamma\varpi_1 u_{1\star}}{(1+\varpi_1 u_{2\star})^2}, \\ m_4 = \dfrac{a_1\varpi_3 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})^2(m+u_{1\star})}, \\ m_5 = \rho_1\left[\dfrac{\bar{\lambda}_1 a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}(1 - \dfrac{u_{1\star}}{m+u_{1\star}})\right], \\ m_6 = \rho_1\left[\dfrac{\bar{\lambda}_1 a_1 u_{1\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})} - \xi_1 - 2\xi_2(1+\varpi_4 u_{2\star}u_{3\star}) - a_2 u_{3\star}\right], \\ m_7 = \rho_1[\xi_2\varpi_4 u_{2\star}^2 + a_2 u_{2\star}] + \rho_2, \\ m_8 = \rho_1\left[\dfrac{\bar{\lambda}_1 a_1\varpi_3 u_{1\star}u_{2\star}}{(1+\varpi_3 u_{3\star})^2(m+u_{1\star})}\right] + \rho_2, \\ m_9 = \bar{\lambda}_2 a_2 u_{3\star}, \\ m_{10} = \dfrac{\beta}{1+\beta_0 u_{3\star}} + \lambda_2 a_2 u_{2\star} - \xi_3, \\ m_{11} = \dfrac{\beta\beta_0 u_{3\star}}{(1+\beta_0 u_{3\star})^2}. \end{cases} \tag{5.4}$$

The characteristic equation of system (5.3) owns the following expression:

$$\det\begin{bmatrix} \lambda - m_1 & -m_2 + m_3 e^{-\lambda\theta} & -m_4 e^{-\lambda\theta} \\ -m_5 & \lambda - m_6 & m_7 + m_8 e^{-\lambda\theta} \\ 0 & -m_9 & \lambda - m_{10} + m_{12}e^{-\lambda\theta} \end{bmatrix} = 0, \tag{5.5}$$

which leads to

$$\lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 + (r_4\lambda^2 + r_5\lambda + r_6)e^{-\lambda\theta} + r_7 e^{-2\lambda\theta} = 0, \tag{5.6}$$

where

$$\begin{cases} r_1 = -m_{11} - m_7 - m_1, \\ r_2 = m_7 m_{11} + m_8 m_{10} + m_1 m_{11} + m_1 m_7 - m_2 m_6, \\ r_3 = m_2 m_6 m_{11} - m_1 m_7 m_{11} - m_1 m_8 m_{10} - m_4 m_4 m_{10}, \\ r_4 = m_{12}, \\ r_5 = m_9 m_{10} + m_3 m_6 - m_1 m_{12} - m_7 m_{12}, \\ r_6 = m_1 m_7 m_{12} - m_1 m_9 m_{10} - m_2 m_6 m_{10} - m_3 m_6 m_{11} - m_5 m_6 m_{10}, \\ r_7 = m_3 m_6 m_{12}. \end{cases} \tag{5.7}$$

If $\theta = 0$, then Eq (5.6) becomes

$$\lambda^3 + (r_1 + r_4)\lambda^2 + (r_2 + r_5)\lambda + r_3 + r_6 + r_7 = 0. \tag{5.8}$$

Thus, all roots of Eq (5.8) have negative real parts if and only if

$$(Q_7)\begin{cases} \ell_1 = r_1 + r_4 > 0, \\ \ell_2 = \det\begin{bmatrix} r_1 + r_4 & 1 \\ r_3 + r_6 + r_7 & r_2 + r_5 \end{bmatrix} > 0, \\ \ell_3 = (r_3 + r_6 + r_7)\ell_2 > 0. \end{cases} \tag{5.9}$$

is fulfilled, then the three roots λ_1–λ_3 of Eq (5.8) have negative real parts. Thus the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$ of system (5.1) with $\theta = 0$ is locally asymptotically stable.

Assume that $\lambda = i\nu^\tau$ is the root of Eq (5.8), then Eq (5.6) becomes:

$$((i\nu^\tau)^3 + r_1(i\nu^\tau)^2 + r_2 i\nu^\tau + r_3)e^{i\nu^\tau\theta} + r_4(i\nu^\tau)^2 + r_5 i\nu^\tau + r_6 + r_7 e^{-i\nu^\tau\theta} = 0, \tag{5.10}$$

It follows from Eq (5.10) that

$$\begin{cases} \Phi_1(\nu^\tau)\sin\nu^\tau\theta + \Phi_2(\nu^\tau)\cos\nu\theta = \Phi_3(\nu^\tau), \\ \Phi_4(\nu^\tau)\sin\nu^\tau\theta + \Phi_5\cos\nu^\tau\theta = \Phi_6(\nu^\tau), \end{cases} \tag{5.11}$$

where

$$\begin{cases} \Phi_1 = \nu^{\tau 3} - r_2\nu^\tau, \\ \Phi_2 = -r_1\nu^{\tau 2} + p_1, \\ \Phi_3 = r_4 n u^{\tau 2} - r_6, \\ \Phi_4 = -r_1\nu^{\tau 2} + p_2, \\ \Phi_5 = -\nu^{\tau 3} + r_2\nu^\tau, \\ \Phi_5 = -r_5\nu^\tau, \end{cases} \tag{5.12}$$

and

$$\begin{cases} p_1 = r_3 + r_7, \\ p_2 = r_3 - r_7. \end{cases} \tag{5.13}$$

So there is

$$\begin{cases} \sin\nu^\tau\theta = \dfrac{\Phi_2\Phi_6 - \Phi_3\Phi_5}{\Phi_2\Phi_4 - \Phi_1\Phi_5}, \\ \cos\nu^\tau\theta = \dfrac{\Phi_3\Phi_4 - \Phi_1\Phi_6}{\Phi_2\Phi_4 - \Phi_1\Phi_5}. \end{cases} \tag{5.14}$$

In view of $\cos^2 \nu^\tau\theta + \sin^2 \nu^\tau\theta = 1$, it follows from (5.14) that

$$\begin{aligned}&[\Phi_2(\nu^\tau)\Phi_4(\nu^\tau) - \Phi_3(\nu^\tau)\Phi_5(\nu^\tau)]^2 + [\Phi_3(\nu^\tau)\Phi_4(\nu^\tau) - \Phi_1(\nu^\tau)\Phi_6(\nu^\tau)]^2\\ &= [\Phi_2(\nu^\tau)\Phi_4(\nu^\tau) - \Phi_1(\nu^\tau)\Phi_5(\nu^\tau)]^2,\end{aligned} \tag{5.15}$$

therefore, results can be obtained as follows:

$$\nu^{\tau 12} - R_1\nu^{\tau 10} - R_2\nu^{\tau 8} - R_3\nu^{\tau 6} - R_4\nu^{\tau 4} - R_5\nu^{\tau 2} - R_6 = 0, \tag{5.16}$$

where

$$\begin{cases} R_1 = r_4^2 + 4r_2 - 2r_1^2,\\ R_2 = r_5^2 + r_1^2r_4^2 + 2r_1(p_1 + p_2 + 2r_1r_4) - r_1^4 - 6r_2^2 - 2r_2r_4^2 - 2r_4r_5,\\ R_3 = r_1^2r_5^2 + r_2^2r_4^2 + r_6^2 + 4r_2^3 + 4r_2r_4r_6 + 2r_5r_5p_2 + (p_1 + p_2)(2r_1^3 - 4r_1r_2)\\ \quad - 2r_1^2r_2^2 - 2r_1r_4^2p_2 - 2r_1^2r_4r_6 - 2r_2r_5^2 - 2r_4r_5p_1,\\ R_4 = r_2^2r_4^2 + r_1^2r_5^2 + r_3^2p_2^2 + 2r_2r_3r_4(1 - p_2) + 2r_4r_5(p_1 - p_2) + 4p_1p_2(r_2 - r_1^2)\\ \quad + 4r_1r_3r_5^2p_2 - r_1^2(p_1^2 + p_2^2) - r_2^4 - 2r_2r_5(r_2r_3 + r_5) - 2r_1r_4^2p_1,\\ R_5 = r_2^2r_5^2 + r_4^2p_1^2 + 2r_1(p_1p_2^2 + p_1^2p_2) + 2r_2r_4r_5(p_2 - p_1)\\ \quad - 2r_1r_5^2p_2 - 2r_2^2p_1p_2 - 2r_3r_5p_2^2,\\ R_6 = r_5^2p_2^2 - p_1^2p_2^2. \end{cases} \tag{5.17}$$

Let

$$\Upsilon_3(\nu^\tau) = \nu^{\tau 12} - R_1\nu^{\tau 10}R_2\nu^{\tau 8} - R_3\nu^{\tau 6} - R_4\nu^{\tau 4} + R_5\nu^{\tau 2} - R_6. \tag{5.18}$$

Assume that

$$(Q_8)\ \ |r_5| > |p_1|.$$

If (Q_8) holds, then $\Upsilon_3(0) = -R_6 < 0$, since $\lim_{\nu^\tau\to\infty}\Upsilon_2(\nu^\tau) > 0$, then we know that Eq (5.16) has at least one positive real root. Therefore, Eq (5.6) has at least one pair of pure roots. Without loss of generality, we can assume that Eq (5.16) has twelve positive real roots (say $\nu_j^\tau, j = 1, 2, 3, \cdots, 12$). Relying on (5.14), we know

$$\theta_j^{(w)} = \frac{1}{\nu_j^\tau}\left[\arccos\left(\frac{\Phi_2(\nu_j)\Phi_6(\nu_j^\tau) - \Phi_3(\nu_j^\tau)\Phi_5(\nu_j^\tau)}{\Phi_2(\nu_j^\tau)(\Phi_4(\nu_j^\tau) - \Phi_1(\nu_j^\tau)\Phi_5(\nu_j^\tau)} + 2w\pi\right)\right], \tag{5.19}$$

where $j = 1, 2, 3, \cdots, 12; w = 0, 1, 2, \cdots$. Assume $\theta_\star = \min_{\{j=1,2,3,\cdots,12;w=0,1,2,\cdots\}}\{\theta_j^{(w)}\}$ and suppose that when $\theta = \theta_\star$, Eq (5.6) has a pair of imaginary roots $\pm i\nu_0^\tau$.

Next, we present the following assumption:

$$(Q_9)\ \ Y_{1R}Y_{2R} + Y_{1I}Y_{2I} > 0,$$

where

$$\begin{cases} Y_{1R} = (r_2 - 3\nu_0^{\tau 2})\cos\nu_0^\tau\theta_\star - 2r_1\nu_0^\tau\sin\nu_0^\tau\theta_\star + 2r_4 + r_5,\\ Y_{1I} = (r_2 - 3\nu_0^{\tau 2})\sin\nu_0^\tau\theta_\star + 2r_1\nu_0^\tau\cos\nu_0^\tau\theta_\star,\\ Y_{2R} = (r_7\nu_0^\tau + r_3\nu_0^\tau - r_1\nu_0^{\tau 3})\sin\nu_0^\tau\theta_\star + (r_2{\nu^\tau}_0^2 - \nu_0^{\tau 4})\cos\nu_0^\tau\theta_\star,\\ Y_{2I} = (r_2{\nu^\tau}_0^2 - \nu_0^{\tau 4})\sin\nu_0^\tau\theta_\star + (r_7\nu_0^\tau + r_1\nu_0^{\tau 3} - h_3\nu_0^\tau)\cos\nu_0^\tau\theta_\star. \end{cases} \tag{5.20}$$

Lemma 5.1. *Suppose that $\lambda(\theta) = \upsilon_1(\theta) + i\upsilon_2(\theta)$ is the root of Eq (4.6) at $\theta = \theta_\star$ such that $\upsilon_1(\theta_\star) = 0$, $\upsilon_2(\theta_\star) = \nu_0^\tau$, then $Re\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_\star,\nu=\nu_0^\tau} > 0$.*

Proof. It follows from Eq (5.6) that

$$\lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 + (r_4\lambda^2 + r_5\lambda + r_6)e^{-\lambda\theta} + r_7 e^{-2\lambda\theta} = 0, \tag{5.21}$$

we can get

$$\begin{aligned}&[\left(3\lambda^2 + 2r_1\lambda + r_2\right)e^{\lambda\theta}\Big]\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (\lambda^3 + r_1\lambda^2 + r_2\lambda + r_3)e^{\lambda\theta}(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda)\\ &+2r_4\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + r_5\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} - r_7e^{-\lambda\theta}(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda) = 0.\end{aligned} \tag{5.22}$$

It means that

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{Y_1(\lambda)}{Y_2(\lambda)} - \frac{\theta}{\lambda}, \tag{5.23}$$

where

$$\begin{cases} Y_1(\lambda) = (3\lambda^2 + r_1\lambda + r_2)e^{\lambda\theta} + 2r_4 + r_5, \\ Y_2(\lambda) = r_7\lambda e^{-\lambda\theta} - (\lambda^3 + r_1\lambda^2 + r_2\lambda + r_3)\lambda e^{\lambda\theta}. \end{cases} \tag{5.24}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_\star,\nu=\nu_0^\tau} = \mathrm{Re}\left[\frac{Y_1(\lambda)}{Y_2(\lambda)}\right]_{\theta=\theta_\star,\nu=\nu_0^\tau} = \frac{Y_{1R}Y_{2R} + Y_{1I}Y_{2I}}{Y_{2R}^2 + Y_{2I}^2}. \tag{5.25}$$

By the assumption (Q_9), we get

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_\star,\nu=\nu_0^\star} > 0, \tag{5.26}$$

which ends the proof. According to the above discussion, the following outcome is easily derived.

Theorem 5.1. *Suppose that (Q_7)–(Q_9) hold, then the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$ of model (5.1) is locally asymptotically stable if $\theta \in [0, \theta_\star)$, and model (5.1) generates a cluster of Hopf bifurcations around the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$ when $\theta = \theta_\star$.*

6. Controlling bifurcation in model (1.2) with an extended delayed feedback controller

In this segment, we will investigate the Hopf bifurcation issue of system (1.2) using a reasonable hybrid controller that encompasses parameter perturbation with delay and state feedback. In accordance with the notion in [15], we construct the controlled predator-prey model as follows:

$$\begin{cases} \dfrac{du_1}{dt} = \delta_1\left[\dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t)\right.\\ \qquad\left. - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}\right] + \delta_2[u_1(t) - u_1(t-\theta)],\\ \dfrac{du_2}{dt} = \delta_3\left[\dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\bar{\lambda}_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t)\right]\\ \qquad + \delta_4[u_2(t) - u_2(t-\theta)],\\ \dfrac{du_3}{dt} = \delta_5\left[\dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \bar{\lambda}_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t)\right] + \delta_6[u_3(t) - u_3(t-\theta)], \end{cases} \tag{6.1}$$

where δ_1–δ_6 represent control parameters. System (6.1) shares the same equilibrium point as that in system (1.2). Let

$$\begin{cases} u_{1\star} = u_1(t) - \bar{u}_1(t), \\ u_{2\star} = u_1(t) - \bar{u}_2(t), \\ u_{3\star} = u_2(t) - \bar{u}_1(t). \end{cases} \tag{6.2}$$

The linear system of system (1.3) near $E(u_{1\star}, u_{2\star}, u_{3\star})$ can be demonstrated as below:

$$\begin{cases} \dfrac{d\bar{u}_1}{dt} = n_1\bar{u}_1 + n_2\bar{u}_1(t-\theta) + n_3\bar{u}_2 - n_4\bar{u}_2(t-\theta) + n_5\bar{u}_3(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = n_6\bar{u}_1 + n_7\bar{u}_2 - n_8\bar{u}_2(t-\theta) - n_9\bar{u}_3 - n_{10}(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = n_{11}\bar{u}_2 + n_{12}\bar{u}_3 - n_{13}\bar{u}_3(t-\theta), \end{cases} \tag{6.3}$$

where

$$\begin{cases} n_1 = (\delta_1(\dfrac{\gamma}{1+\varpi_1 u_{2\star}} - 2du_{1\star}(1+\varpi_2 u_{2\star}) - \dfrac{a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}(1-\dfrac{u_{1\star}}{m+u_{1\star}}) + \delta_2), \\ n_2 = \delta_2, \\ n_3 = \dfrac{\delta_1 a_1 u_{1\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}, \\ n_4 = \dfrac{\delta_1 \gamma \varpi_1 u_{1\star}}{(1+\varpi_1 u_{2\star})^2}, \\ n_5 = \dfrac{\delta_1 a_1 \varpi_3 u_{1\star} u_{2\star}}{1+\varpi_3 u_{3\star}^2}, \\ n_6 = \delta_3 \left[\dfrac{\bar{\lambda}_1 a_1 u_{2\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})}(1-\dfrac{u_{1\star}}{m+u_{1\star}}) \right], \\ n_7 = \delta_3 \left[\dfrac{\bar{\lambda}_1 a_1 u_{1\star}}{(1+\varpi_3 u_{3\star})(m+u_{1\star})} - \xi_1 - 2\xi_2(1+\varpi_4 u_{2\star} u_{3\star}) - a_2 u_{3\star} \right] + \delta_4, \\ n_8 = \delta_4, \\ n_9 = \delta_3(\xi_2 \varpi_4 u_{2\star}^2 + a_2 u_{2\star}), \\ n_{10} = \dfrac{\delta_3 \bar{\lambda}_1 a_1 \varpi_3 u_{1\star} u_{2\star}}{(1+\varpi_3 u_{3\star})^2 (m+u_{1\star})}, \\ n_{11} = \delta_5 \bar{\lambda}_2 a_2 u_{3\star}, \\ n_{12} = \delta_5(\dfrac{\beta}{1+\beta_0 u_{3\star}} + \lambda_2 a_2 u_{2\star} - \xi_3) + \delta_6, \\ n_{13} = \dfrac{\delta_5 \beta \beta_0 u_{3\star}}{(1+\beta_0 u_{3\star})^2} + \delta_6. \end{cases} \tag{6.4}$$

The characteristic equation of system (6.3) takes the following form:

$$\det \begin{bmatrix} \lambda - n_1 + n_2 e^{-\lambda\theta} & -n_3 + n_4 e^{-\lambda\theta} & -n_5 e^{-\lambda\theta} \\ -n_6 & \lambda - n_7 + n_8 e^{-\lambda\theta} & n_9 + n_{10} e^{-\lambda\theta} \\ 0 & -n_{11} & \lambda - n_{12} + n_{13} e^{-\lambda\theta} \end{bmatrix} = 0, \tag{6.5}$$

which leads to:

$$\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 + (s_4\lambda^2 + s_5\lambda + s_6)e^{-\lambda\theta} + (s_7\lambda + s_8)e^{-2\lambda\theta} + s_9 e^{-3\lambda\theta} = 0, \tag{6.6}$$

where

$$\begin{cases} s_1 = -n_{12} - n_7 - n_1, \\ s_2 = n_7n_{12} + n_9n_{11} + n_1n_{12} + n_1n_7 - n_3k_6, \\ s_3 = n_3n_6n_{12} - n_1n_7n_{12} - n_1n_9n_{11}, \\ s_4 = n_{13} + n_8 + n_2, \\ s_5 = n_{10}n_{11} + n_4n_6 - n_7n_{10} - n_8n_{12} - n_1n_{13} - n_1n_8 - n_2n_{12} - n_2n_7, \\ s_6 = n_1n_7n_{13} + n_1n_8n_{12} + n_2n_7n_{12} + n_2n_9n_{11} - n_3n_6n_{13} - n_5n_6n_{11} - n_1n_10n_{11}, \\ s_7 = n_8n_{13} + n_2n_{13} + n_2n_8, \\ s_8 = n_2n_{10}n_{11} - n_1n_8n_{13} - n_2n_7n_{13} - n_2n_8n_{12} - n_4n_6n_{12} - n_4n_6n_{13}. \end{cases} \tag{6.7}$$

If $\theta = 0$, then Eq (6.6) becomes

$$\lambda^3 + (s_1 + s_4)\lambda^2 + (s_2 + s_5 + s_7)\lambda + s_3 + s_6 + s_8 + s_9 = 0. \tag{6.8}$$

Hence, the necessary and sufficient condition for all roots of Eq (6.8) to have negative real parts is

$$(Q_{10}) \begin{cases} \Delta_1 = s_1 + s_4 > 0, \\ \Delta_2 = \det \begin{bmatrix} s_1 + s_4 & 1 \\ s_3 + s_6 + s_8 + s_9 & s_2 + s_5 + s_7 \end{bmatrix} > 0. \\ \Delta_3 = (s_3 + s_6 + s_8 + s_9)\Delta_2 > 0. \end{cases} \tag{6.9}$$

Then, three roots λ_1–λ_3 of Eq (6.8) that have negative real parts. As a result, the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$ of system (1.2) with $\theta = 0$ is locally asymptotically stable.

It is assumed that $\lambda = i\nu^\mu$ is the root of Eq (6.6) and one obtains

$$(\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3)e^{2\lambda\theta} + (s_4\lambda^2 + s_5\lambda + s_6)e^{\lambda\theta} + s_9e^{\lambda\theta} + s_8 = 0. \tag{6.10}$$

Next, referring to Eq (6.10), we obtain

$$\begin{aligned} &[(i\nu^\mu)^3 + s_1(i\nu^\mu)^2 + s_2 i\nu^\mu + s_3](\cos 2\nu^\mu\theta + i\sin 2\nu^\mu\theta) \\ &+(s_4(i\nu^\mu)^2 + s_5 i\nu^\mu + s_6)(\cos \nu^\mu\theta + i\sin \nu^\mu\theta) \\ &+s_9(\cos \nu^\mu\theta - i\sin \nu^\mu\theta) + s_7 i\nu^\mu + s_8 = 0. \end{aligned} \tag{6.11}$$

From Eq (6.11), we are able to derive

$$\begin{cases} (\nu^{\mu 3} - s_2\nu^\mu)\sin 2\nu^\mu\theta + (s_3 - s_1\nu^{\mu 2})\cos 2\nu^\mu\theta \\ -s_5\nu^\mu \sin \nu^\mu\theta + (s_6 + s_9 - s_4)\nu^{\mu 2}\cos \nu^\mu\theta \\ = -s_8, \\ (s_2\nu^\mu - \nu^{\mu 3})\cos 2\nu^\mu\theta + (s_3 - s_1\nu^{\mu 2})\sin 2\nu^\mu\theta \\ +s_5\nu^\mu \cos \nu^\mu\theta + (s_6 - s_9 - s_4\nu^{\mu 2})\sin \nu^\mu\theta \\ = -s_7\nu^\mu. \end{cases} \tag{6.12}$$

Owing to $\sin^2 \nu^\mu\theta = \pm\sqrt{1 - \cos^2 \nu^\mu\theta}$, Eq (6.12) can be modified to

$$\begin{aligned} &[(\nu^{\mu 3} - s_2\nu^\mu)2(\pm\sqrt{1 - \cos^2 \nu^\mu\theta})cos\nu^\mu\theta) + (s_3 - s_1\nu^{\mu 2})(2\cos^2 \nu^\mu\theta - 1)] \\ &-r_5\nu^\mu(\pm\sqrt{1 - \cos^2 \nu^\mu\theta}) + (r_6 + r_9 - r_4\nu^{\mu 2})cos\nu^\mu\theta + s_8 = 0. \end{aligned} \tag{6.13}$$

Thus, the results can be summarized as

$$N_1 \cos^2 \nu^\mu\theta + N_2 cos\nu^\mu\theta + N_3 = (N_4 + N_5 cos\nu^\mu\theta) \pm \sqrt{1-\cos^2 \nu^\mu\theta}, \tag{6.14}$$

where

$$\begin{cases} N_1 = 2s_3 - 2s_1\nu^{\mu 2}, \\ N_2 = s_6 + h_9 - s_4\nu^{\mu 2}, \\ N_3 = r_1\nu^{\mu 2} + r_8 - r_3, \\ N_4 = r_5\nu^\mu, \\ N_5 = 2(r_2\nu^\mu - \nu^{\mu 3}). \end{cases} \tag{6.15}$$

Square each side of Eq (6.14) to achieve the result presented below.

$$T_1 cos^4\nu^\mu\theta + T_2 cos^3\nu^\mu\theta + T_3 cos^2\nu^\mu\theta + T_4 cos\nu^\mu\theta + T_5 = 0, \tag{6.16}$$

where

$$\begin{cases} T_1 = N_1^2 + N_5^5, \\ T_2 = 2(N_1N_2 + N_4N_5), \\ T_3 = 2N_1N_3 + N_2^2 + N_4^2 - N_5^2, \\ T_4 = 2(N_2N_3 - N_4N_5), \\ T_5 = N_5^2 - N_4^2. \end{cases} \tag{6.17}$$

Based on Eq (6.16), we can obtain the representation as presented below for $\cos \nu^\mu$. Suppose that

$$cos\nu^\mu\theta = f_1(\nu^\mu), \tag{6.18}$$

where $f_1(\nu^\mu)$ is a function that varies with ν^μ. We can derive the expression for $\sin \nu^\mu$. Assume that

$$sin\nu^\mu\theta = f_2(\nu^\mu), \tag{6.19}$$

where $f_2(\nu^\mu)$ denotes a function with respect to ν^μ. Referring to Eqs (6.18) and (6.19), we can determine

$$f_1^2(\nu^\mu) + f_2^2(\nu^\mu) = 1. \tag{6.20}$$

By using a computer, we can obtain the roots of Eq (6.20). We denote the root of (6.20) as $\nu^{\mu*}$, then

$$\theta_*^{(n)} = \frac{1}{\nu^{\mu*}}\left[\arccos\left(\frac{1}{f_1(\nu^{\mu*})} + 2n\pi\right)\right], \tag{6.21}$$

where $j = 1, 2, \cdots; n = 0, 1, 2, \cdots$.

Let $\theta_* = \min_{\{n=1,2,\cdots\}}\{\theta_*^{(n)}\}$ and suppose that when $\theta = \theta_*$, Eq (6.8) possesses a pair of imaginary roots $\pm i\nu_0^{\mu*}$.

As the next step, we put forward the following assumption:

$$(Q_{11})\ \ \bar{Y}_{1R}\bar{Y}_{2R} + \bar{Y}_{1I}\bar{Y}_{2I} > 0,$$

where

$$\begin{cases} \bar{Y}_{1R} = (s_2 - 3\nu_0^{\mu*2})\cos 2\nu_0^{\mu*}\theta_{0*} - 2s_1\nu_0^{\mu*}\sin 2\nu_0^{\mu*}\theta_{0*} \\ \qquad + (s_5 - 2s_4\nu_0^{\mu*2})\cos \nu_0^{\mu*}\theta_{0*} - 2s_4\nu_0^{\mu*2}\sin \nu_0^{\mu*}\theta_{0*}, \\ \bar{Y}_{1I} = (s_2 - 3\nu_0^{\mu*2})\sin 2\nu_0^{\mu*}\theta_{0*} + 2s_1\nu_0^{\mu*}\cos 2\nu_0^{\mu*}\theta_{0*} + s_5\sin \nu_0^{\mu*}\theta_{0*}, \\ \bar{Y}_{2R} = (2\nu_0^{\mu*2}(s_2 - \nu_0^{\mu*2}))\cos 2\nu_0^{\mu*}\theta_{0*} + (2s_3\nu_0^{\mu*} - 2s_1\nu_0^{\mu*3})\sin 2\nu_0^{\mu*}\theta_{0*} \\ \qquad + (\nu_0^{\mu*} + s_6\nu_0^{\mu*} + s_5\nu_0^{\mu*2}\theta_{0*} + s_4\nu_0^{\mu*3}\theta_{0*})\sin \nu_0^{\mu*}\theta_{0*}, \\ \bar{Y}_{2I} = (2\nu_0^{\mu*2}(s_2 - \nu_0^{\mu*2}))\sin 2\nu_0^{\mu*}\theta_{0*} + (2s_1\nu_0^{\mu*3} - 2s_3\nu_0^{\mu*})\cos 2\nu_0^{\mu*}\theta_{0*} \\ \qquad + (\nu_0^{\mu*} - s_6\nu_0^{\mu*} - s_5\nu_0^{\mu*2}\theta_{0*} - s_4\nu_0^{\mu*3}\theta_{0*})\sin \nu_0^{\mu*}\theta_{0*}. \end{cases} \tag{6.22}$$

Lemma 6.1. *Suppose that $\lambda(\theta) = \bar{\upsilon}_1(\theta) + i\bar{\upsilon}_2(\theta)$ is the root of Eq (6.6) at $\theta = \theta_{0*}$ such that $\bar{\upsilon}_1(\theta_{0*}) = 0$, $\bar{\upsilon}_2(\theta_{0*}) = \nu_0^{\mu*}$, then $Re\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_{0*},\nu=\nu_0^{\mu*}} > 0$.*

Proof. As derived from Eq (6.6), it follows that

$$\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 + (s_4\lambda^2 + s_5\lambda + s_6)e^{-\lambda\theta} + (s_7\lambda + s_8)e^{-2\lambda\theta} + s_9e^{-3\lambda\theta} = 0, \tag{6.23}$$

we can get

$$\begin{aligned} &(3\lambda^2 + 2s_1\lambda + s_2)e^{2\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3)2e^{2\lambda\theta}\left(\lambda + \theta\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\right) \\ &+(2s_4\lambda + s_5)e^{\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (s_4\lambda^2 + s_5\lambda + s_6)e^{\lambda\theta}\left(\lambda + \theta\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\right) \\ &+s_7\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} - e^{-\lambda\theta}\left(\lambda + \theta\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\right) = 0, \end{aligned} \tag{6.24}$$

which signifies

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{\bar{Y}_1(\lambda)}{\bar{Y}_2(\lambda)} - \frac{\theta}{\lambda}, \tag{6.25}$$

where

$$\begin{cases} \bar{Y}_1(\lambda) = (3\lambda^2 + 2s_1\lambda + s_2)e^{2\lambda\theta} + (2s_4\lambda + s_5)e^{\lambda\theta} + s_7, \\ \bar{Y}_1(\lambda) = \lambda e^{-\lambda\theta} - 2(\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3)\lambda e^{2\lambda\theta} - (s_4\lambda^2 + s_5\lambda + s_6)\lambda e^{\lambda\theta}. \end{cases} \tag{6.26}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_{0*},\nu=\nu_0^{\mu*}} = \mathrm{Re}\left[\frac{\bar{Y}_1(\lambda)}{\bar{Y}_2(\lambda)}\right]_{\theta=\theta_{0*},\nu=\nu_0^{\mu*}} = \frac{\bar{Y}_{1R}\bar{Y}_{2R} + \bar{Y}_{1I}\bar{Y}_{2I}}{\bar{Y}_{2R}^2 + \bar{Y}_{2I}^2}. \tag{6.27}$$

Under the assumption (Q_{11}), we derive

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_{0*},\nu=\nu_0^{\mu*}} > 0, \tag{6.28}$$

which wraps up the proof. Following the preceding analysis, the subsequent result is straightforwardly reached.

Theorem 6.1. *Assuming that conditions (Q_{10}) and (Q_{11}) hold, then the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$ of model (6.1) is locally asymptotically stable state for $\theta \in [0, \theta_{0*})$. Additionally,* *when $\theta = \theta_{0*}$, model (6.1) exhibits a cluster of Hopf bifurcations around the equilibrium point $F(u_{1\star}, u_{2\star}, u_{3\star})$.*

Remark 6.1. *Pal et al.[8] revealed the complex impact of fear-induced responses on the stability and behavior of multi-species food web systems. In this article, we develop a more realistic delayed predator-prey system model and investigate the bifurcation behavior and hybrid controller design based on this model. On a theoretical level, these research methods enhance the understanding of bifurcation theory for delayed differential equations. From a biological perspective, the findings of this study are significant for managing the densities of predator and prey populations. Consequently, we believe that this paper has a certain innovation to some degree.*

Remark 6.2. *Based on the existing literature methods and according to the biological implication of this predator-prey model, we can add the parameter perturbation with delay and state feedback artificially to adjust the densities of prey, intermediate predators, and top predators and achieve our control objectives. Thus, these control techniques have practical significance in biology.*

Remark 6.3. *Although the form of hybrid controller I and hybrid controller II are same, we add this controller to the third equation of model (1.2) and to the second equation of model (1.2), respectively. Then, we obtain two different delay critical valus to generate Hopf bifurcation. In order to illustrate the different control techniques, we give three different controllers in this paper. We apply these controllers to achieve the control of stability domain and Hopf bifurcation.*

Remark 6.4. *Based on the biological implication, we introduce one delay into model (1). Even though there is only one difference to model (1), the introduction of delay will lead to a great change to the model. In this paper, the Hopf bifurcation and its control issue are our research topic. By exploring the delay-induced Hopf bifurcation of this predator-prey, we can effectively control the balance among the densities of prey, intermediate predators, and top predators.*

Remark 6.5. *In this paper, we mainly show the effect of delay on the stability domain and the time of onset of Hopf bifurcation of model (1.2). For different controllers, we can choose different control parameter values to achieve our control objectives. For this problem, we can continue the discussion in the near future.*

Remark 6.6. *For example, corresponding to the predator-prey model (1.2), we can assume that $u_1(t)$ stands for the density of mice, $u_2(t)$ stands for the density of bobcats, and $u_3(t)$ represent the density of wolves. For controller I, we can adjust the value of per capita regeneration rate (β), the proportional constant between the growth rate of wolves and the response of the function ($\bar{\lambda}_2$), and the natural mortality of wolves (ξ_3) (for ($\beta, \bar{\lambda}_2, \xi_3$, the same multiple is enlarged) and add the small perturbation to change the density of wolves to obtain control. The small perturbation is also a function of time t, and the density of wolves will also change with delayed feedback term. Then we can obtain the control model (4.1) from model (1.2). Based on this viewpoint, we think that this designed controller is suitable. By choosing the suitable control parameters, we can effectively adjust the time of Hopf bifurcation onset to ensure the balance of biological populations (mice, bobcats, and wolves). Based on this viewpoint, we think that this control design has important biological significance. For other controllers, we can explain them in a similar way.*

Remark 6.7. *In the three controlled predator-prey models (4.1), (5.1), and (6.1), we use three different hybrid controllers that contain parameter perturbation with delay and state feedback, and then we obtain three controlled predator-prey models. For controlled model (4.1), we add this hybrid controller* *to the third equation of model (1.2) in order to change the density of top predators and then adjust the stability domain and onset of Hopf bifurcation of system (1.2). For the controlled model (5.1), we add this hybrid controller to the second equation of model (1.2) in order to change the density of intermediate predators, and then adjust the stability domain and onset of Hopf bifurcation of system (1.2). For the controlled model (6.1), we add this hybrid controller to the three equations of model (1.2) in order to change the densities of preys, intermediate predators, and top predators, and then adjust the stability domain and onset of Hopf bifurcation of system (1.2). In actual biological systems, this is possible (see* **Remark 6.6***). To evaluate whether these controllers are reasonable in the actual system and whether they are practical or not, we will carry out theoretical analysis via the stability and Hopf bifurcation theory and computer simulations. It is not feasible to add controllers at will.*

Remark 6.8. *For three different controllers, the conclusions of three controlled systems (4.1), (5.1) and (6.1) are different since the delay critical values $\delta_*, \theta_{*0}, \theta_0$ are different under different parameter conditions. When implementing control in ecosystems, we can control the growth rate and mortality rate of predators via killing predators to add the mortality rate of predators and foster predators to increase their growth rate artificially.*

7. Matlab simulation outcomes

In this section, we will use Matlab 2021 software to carry out numerical simulation.

Example 7.1. Take into account the predator-prey model, which contains a delay:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}, \\ \dfrac{du_2}{dt} = \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\lambda_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t), \\ \dfrac{du_3}{dt} = \dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \lambda_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t), \end{cases} \tag{7.1}$$

where $\gamma = 2, d = 0.3, m = 3, \varpi_1 = 0.5, \varpi_2 = 0.5, \varpi_3 = 0.1, \varpi_4 = 0.5, a_1 = 12, a_2 = 1, \lambda_1 = 0.7, \lambda_2 = 0.3, \xi_1 = 0.1, \xi_2 = 0.03, \xi_3 = 0.3, \beta_0 = 2, \beta_1 = 1$. Obviously, model (7.1) possesses a unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$. It is simple to demonstrate that the conditions (Q_1)–(Q_3) of Theorem 3.1 are satisfied. With the help of computational software, one can obtain that $\theta_0 \approx 5.1$. To assess the correctness of the results derived from Theorem 3.1, we pick two unequal delay values. One is $\theta = 4.8$ and the other is $\theta = 5.3$. If $\theta = 4.8 < \theta_0 \approx 5.1$, we derive the computer simulation diagrams depicted in Figure 1. As depicted in Figure 1, it is clear that $u_1 \to 2.0589, u_2 \to 0.4597, u_3 \to 2.5849$ when $t \to +\infty$. In other words, the unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$ of model (7.1) sustains a locally asymptotically stable status. As $\theta = 5.3 > \theta_0 \approx 5.1$, we acquire the computer simulation diagrams illustrated in Figure 2. As depicted in Figure 2, we can see that u_1 is going to maintain periodic quavering level around the value 2.0589, u_2 is about to keep periodic quavering level around the value 0.4597, and u_3 will hold periodic quavering level around the value 2.5849. In other words, a set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(2.0589, 0.4597, 2.5849)$.

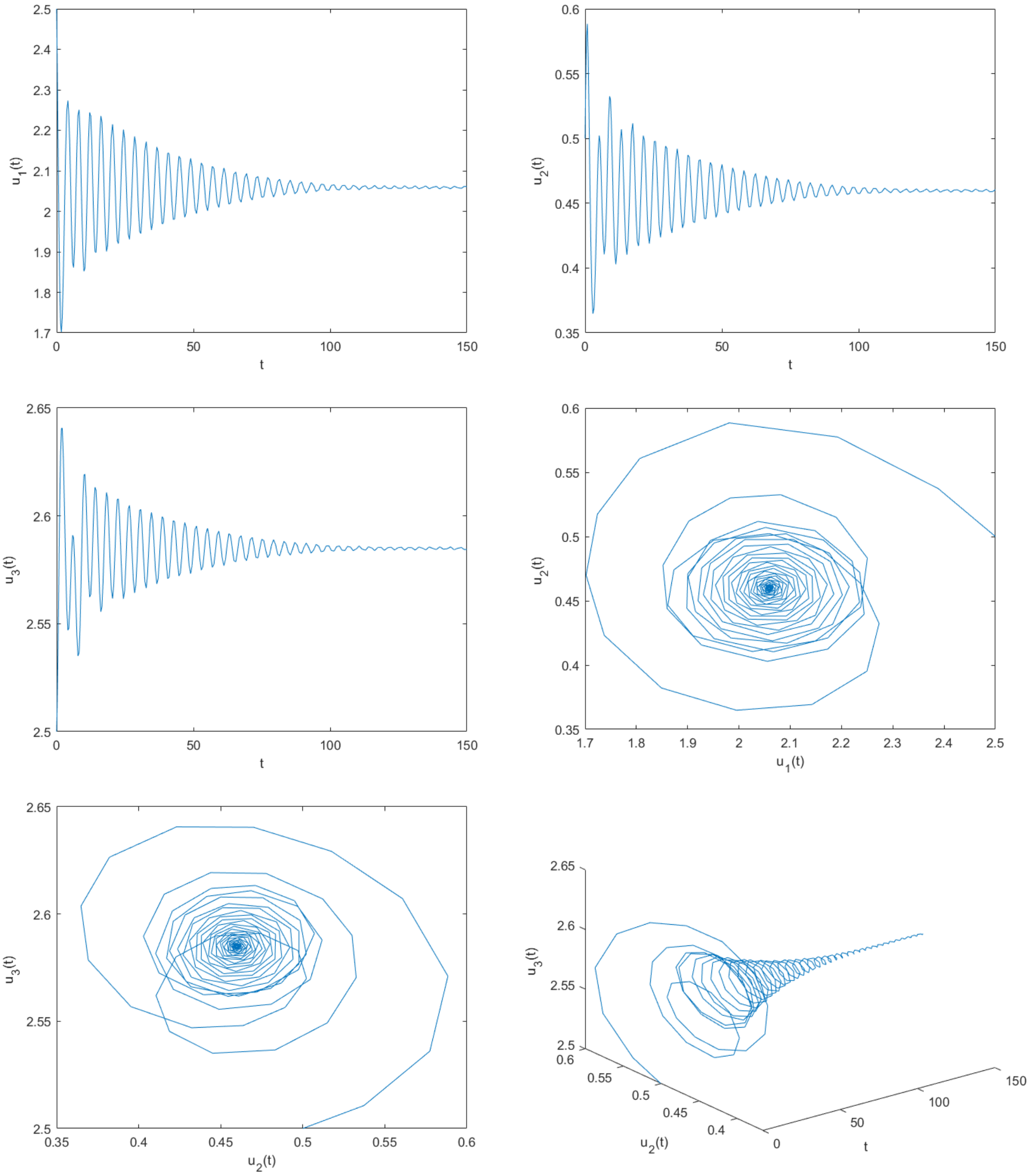

Figure 1. Matlab simulation figures of system (7.1) under the delay $\theta = 4.8 < \theta_0 = 5.1$. The equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$ holds a locally asymptotically stable level.

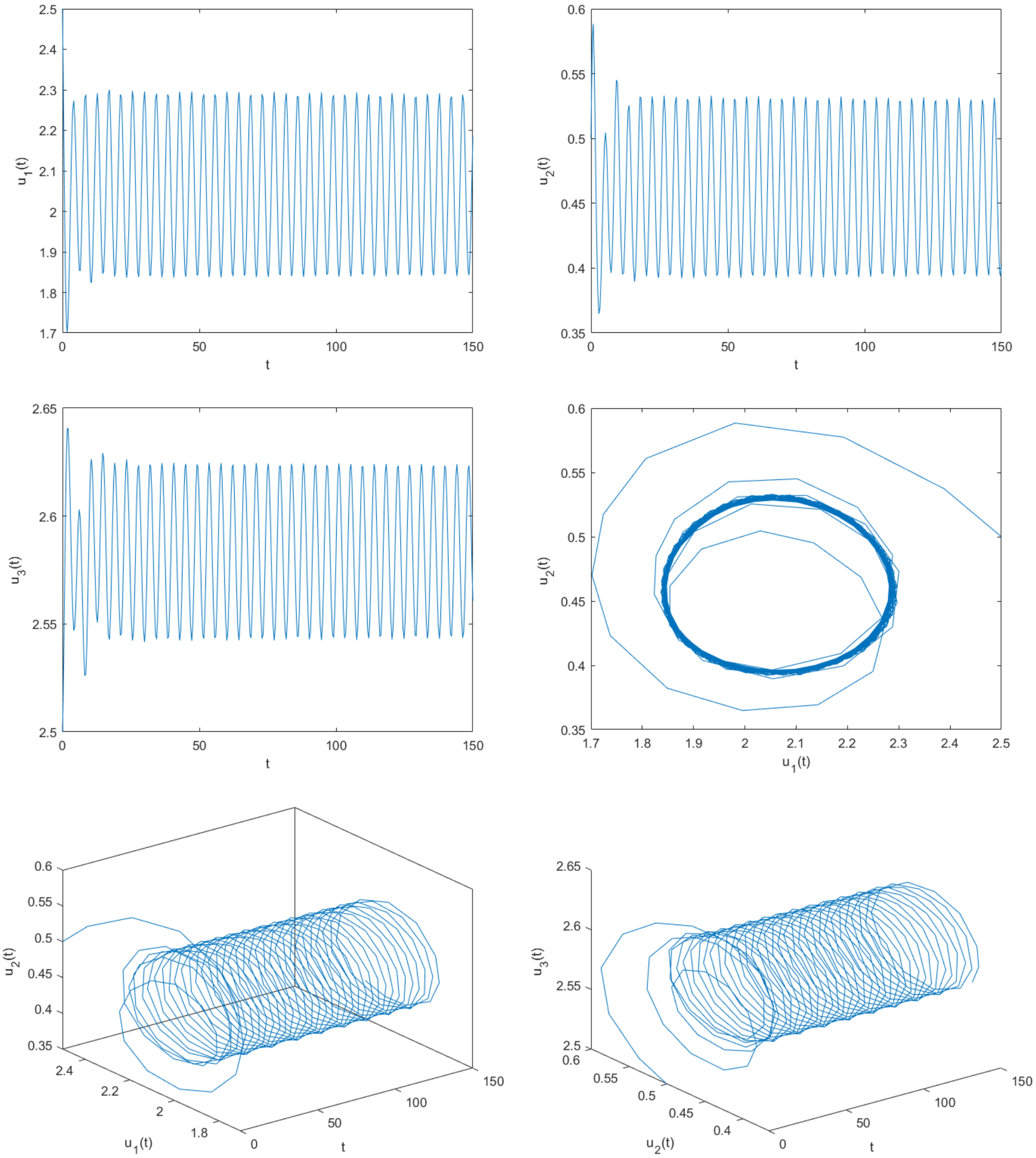

Figure 2. Matlab simulation figures of system (6.1) under the delay $\theta = 5.3 > \theta_0 = 5.1$. A set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$.

Example 7.2. Take into account the following controlled predator-prey model:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) \\ \qquad - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}, \\ \dfrac{du_2}{dt} = \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\lambda_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) \\ \qquad - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t), \\ \dfrac{du_3}{dt} = \eta_1\left[\dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \lambda_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t)\right] + \eta_2[u_3(t) - u_3(t-\theta)], \end{cases} \tag{7.2}$$

where $\gamma = 2, d = 0.3, m = 3, \varpi_1 = 0.5, \varpi_2 = 0.5, \varpi_3 = 0.1, \varpi_4 = 0.5, a_1 = 12, a_2 = 1, \lambda_1 = 0.7, \lambda_2 = 0.3, \xi_1 = 0.1, \xi_2 = 0.03, \xi_3 = 0.3, \beta_0 = 2$, and $\beta_1 = 1$. Let $\eta_1 = 0.4$ and $\eta_2 = 0.5$. Obviously, model (7.2) possesses a unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$. It is simple to demonstrate that the conditions (Q_4)–(Q_6) of Theorem 3.2 are satisfied. With the help of computational software, one can acquire that $\delta_\star \approx 0.9$. To assess the correctness of the results derived from Theorem 3.2, we pick two unequal delay values. One is $\theta = 0.8$ and the other is $\theta = 1.10$. If $\delta = 0.8 < \delta_* \approx 0.9$, we derive the computer simulation diagrams depicted in Figure 3. As depicted in Figure 3, it is clear that $u_1 \to 2.0589, u_2 \to 0.4597, u_3 \to 2.5849$ when $t \to +\infty$. In other words, the unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$ of model (7.2) sustains a locally asymptotically stable status. As $\theta = 5.3 > \theta_0 \approx 5.1$, we acquire the computer simulation diagrams illustrated in Figure 4. As depicted in Figure 4, we can see that u_1 is going to maintain a periodic quavering level around the value 2.0589, u_2 will keep a periodic quavering level around the value 0.4597, and u_3 will hold a periodic quavering level around the value 2.5849. In other words, a set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(2.0589, 0.4597, 2.5849)$.

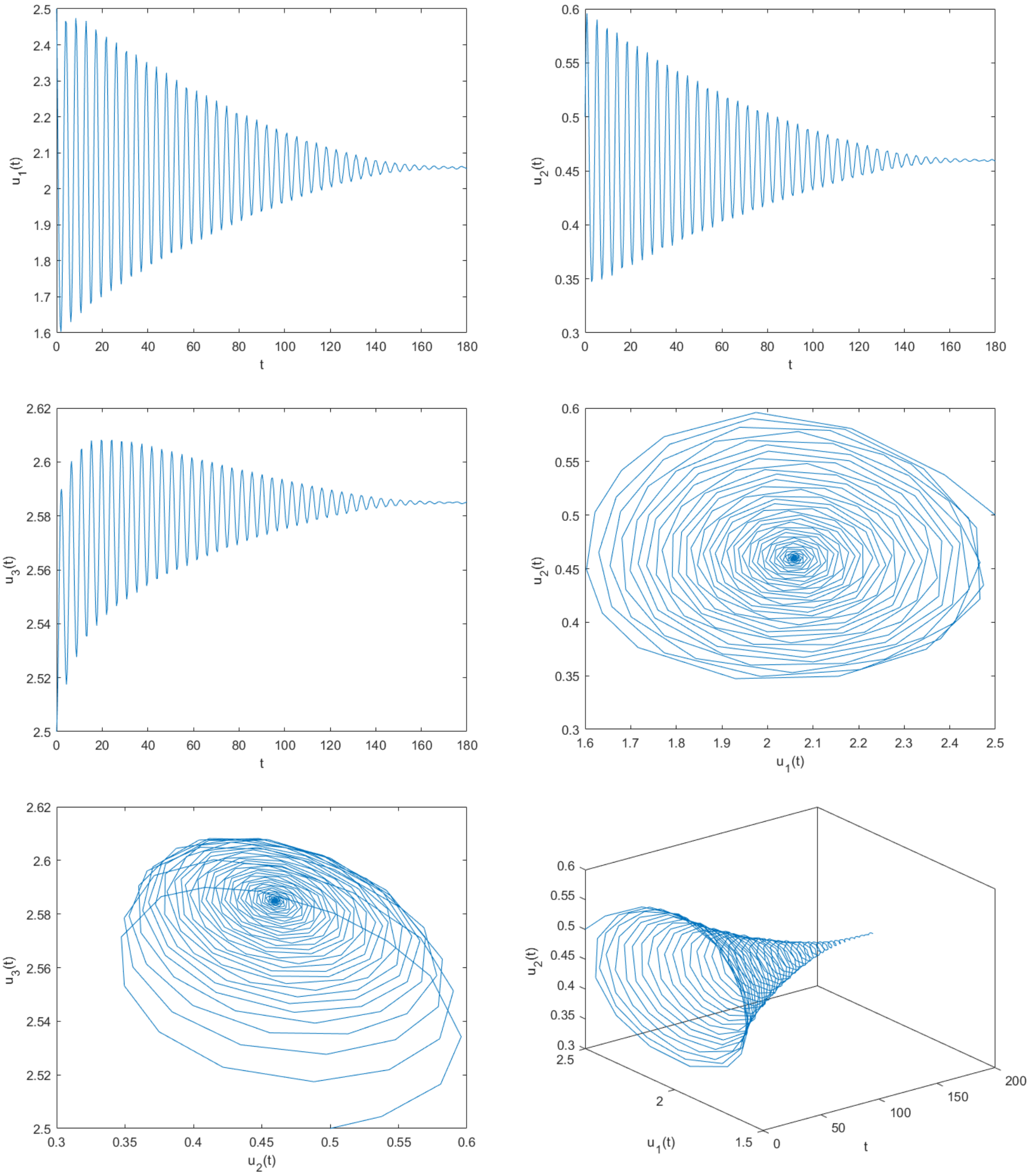

Figure 3. Matlab simulation figures of system (7.2) under the delay $\theta = 0.8 < \theta_* = 0.9$. The equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$ holds a locally asymptotically stable level.

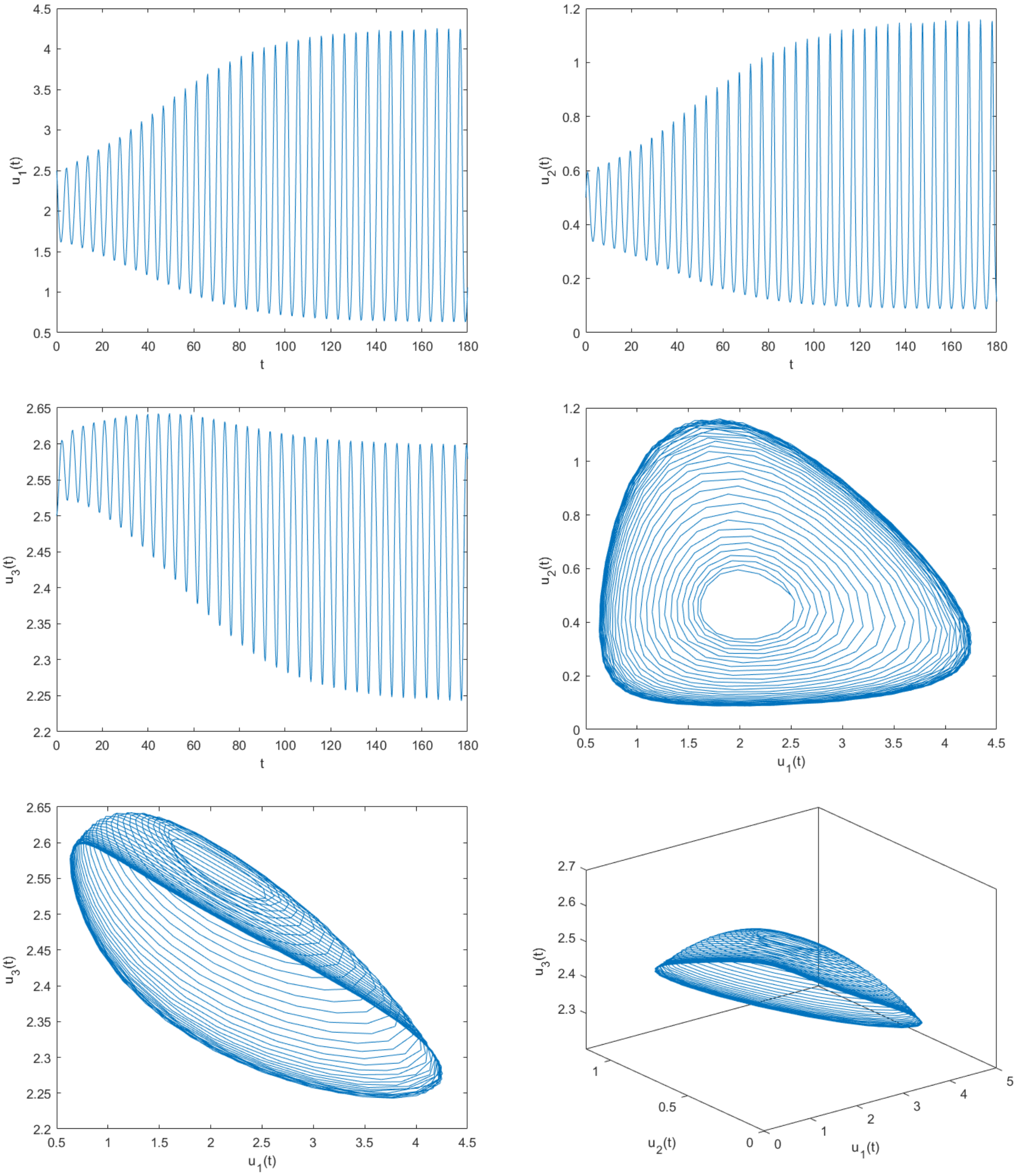

Figure 4. Matlab simulation figures of system (7.2) under the delay $\theta = 1.1 > \theta_* = 0.9$. A set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$.

Example 7.3. Take into account the following controlled predator-prey model:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}, \\ \dfrac{du_2}{dt} = \varsigma_1[\dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\lambda_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t)] \\ \quad + \varsigma_2[u_2(t) - u_2(t-\theta)], \\ \dfrac{du_3}{dt} = \dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \lambda_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t), \end{cases} \tag{7.3}$$

where $\gamma = 2, d = 0.3, m = 3, \varpi_1 = 0.5, \varpi_2 = 0.5, \varpi_3 = 0.1, \varpi_4 = 0.5, a_1 = 12, a_2 = 1, \lambda_1 = 0.7, \lambda_2 = 0.3, \xi_1 = 0.1, \xi_2 = 0.03, \xi_3 = 0.3, \beta_0 = 2$, and $\beta_1 = 1$. Let $\varsigma_1 = 0.4$ and $\varsigma_2 = 0.5$. Obviously, model (7.3) possesses a unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$. It is simple to demonstrate that the conditions (Q_7)–(Q_9) of Theorem 5.1 are satisfied. With the help of computational software, one can acquire that $\theta_{*0} \approx 3.3$. To assess the correctness of the results derived from Theorem 5.1, we pick two unequal delay values. One is $\theta = 3.1$ and the other is $\theta = 3.8$. If $\theta = 3.1 < \theta_{*0} \approx 3.3$, we derive the computer simulation diagrams depicted in Figure 5. As depicted in Figure 5, it is clear that $u_1 \to 2.0589, u_2 \to 0.4597, u_3 \to 2.5849$ when $t \to +\infty$. In other words, the unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$ of model (7.3) sustains a locally asymptotically stable status. As $\theta = 3.8 > \theta_{*0} \approx 3.3$, we acquire the computer simulation diagrams illustrated in Figure 6. As depicted in Figure 5, we can see that u_1 will maintain a periodic quavering level around the value 2.0589, u_2 will keep a periodic quavering level around the value 0.4597, and u_3 will hold a periodic quavering level around the value 2.5849. In other words, a set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(2.0589, 0.4597, 2.5849)$.

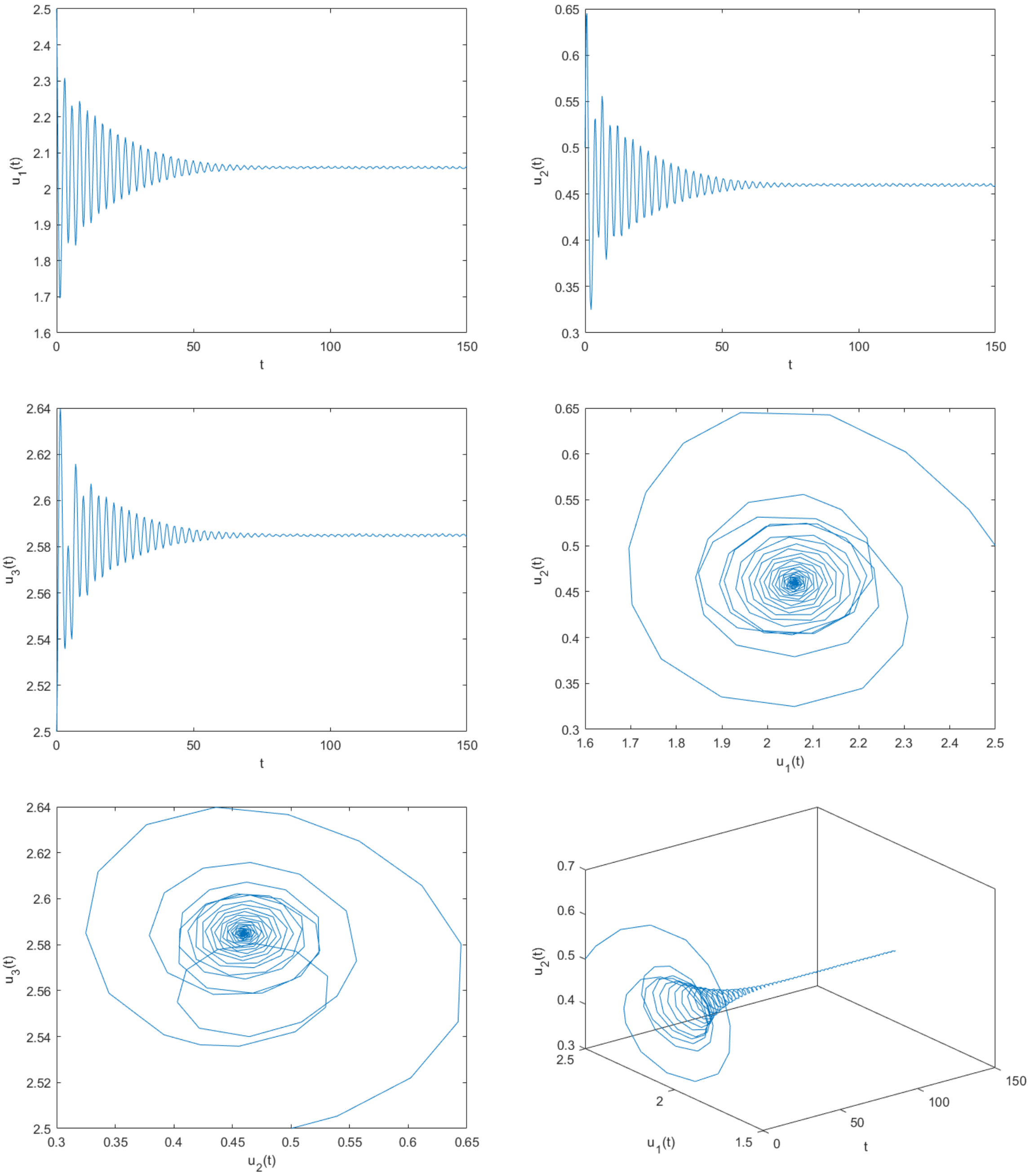

Figure 5. Matlab simulation figures of system (7.3) under the delay $\theta = 3.1 < \theta_{*0} = 3.3$. The equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$ holds a locally asymptotically stable level.

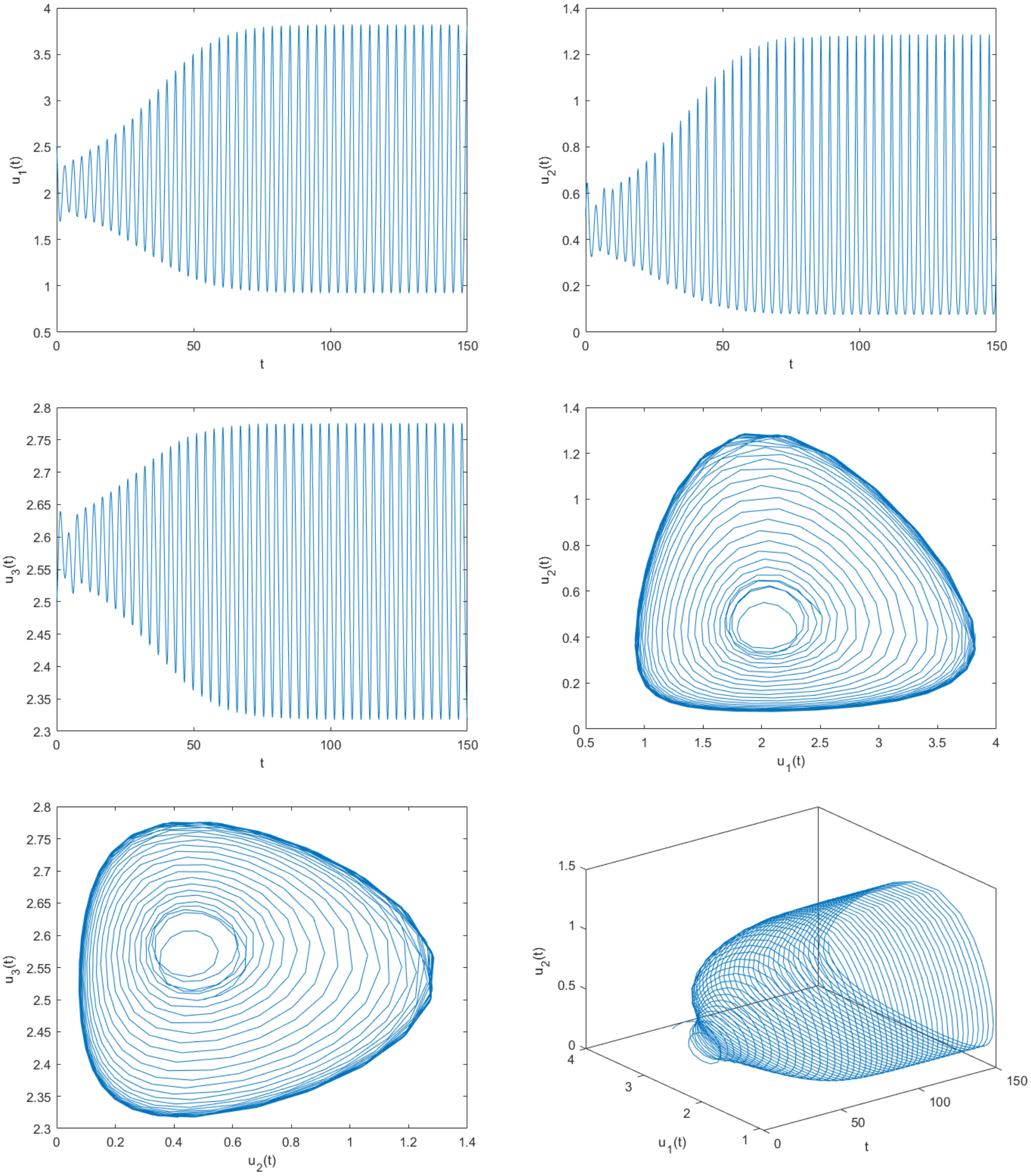

Figure 6. Matlab simulation figures of system (7.3) under the delay $\theta = 3.8 > \theta_{*0} = 3.3$. A set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$.

Example 7.4. Take into account the predator-prey model , which contains a delay:

$$
\begin{cases}
\dfrac{du_1}{dt} = \delta_1[\dfrac{1}{1+\varpi_1 u_2(t-\theta)}\gamma u_1(t) - d(1+\varpi_2 u_2(t))u_1^2(t) \\
\quad - \dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{a_1 u_1(t)u_2(t)}{m+u_1(t)}] + \delta_2[u_1(t) - u_1(t-\theta)], \\
\dfrac{du_2}{dt} = \delta_3[\dfrac{1}{1+\varpi_3 u_3(t-\theta)}\dfrac{\lambda_1 a_1 u_1(t)u_2(t)}{m+u_1(t)} - \xi_1 u_2(t) - \xi_2(1+\varpi_4 u_3(t))u_2^2(t) - a_2 u_2(t)u_3(t)] \\
\quad + \delta_4[u_2(t) - u_2(t-\theta)], \\
\dfrac{du_3}{dt} = \delta_5[\dfrac{\beta u_3(t)}{1+\beta_0 u_3(t-\theta)} + \lambda_2 a_2 u_2(t)u_3(t) - \xi_3 u_3(t)] + \delta_6[u_3(t) - u_3(t-\theta)],
\end{cases}
\tag{7.4}
$$

where $\gamma = 2, d = 0.3, m = 3, \varpi_1 = 0.5, \varpi_2 = 0.5, \varpi_3 = 0.1, \varpi_4 = 0.5, a_1 = 12, a_2 = 1, \lambda_1 = 0.7, \lambda_2 = 0.3, \xi_1 = 0.1, \xi_2 = 0.03, \xi_3 = 0.3, \beta_0 = 2$, and $\beta_1 = 1$. Let $\delta_1 = 0.6, \delta_2 = 0. - 0.3, \delta_3 = 0.3, \delta_4 = -0.3, \delta_5 = 0.4$, and $\delta_6 = 0.5$. Obviously, model (7.4) possesses a unique positive equilibrium point $E(2.0589, 0.4597, 2.5849)$. It is simple to demonstrate that the conditions (Q_{10}) and (Q_{11}) of Theorem 6.1 are satisfied. With the help of computational software, one can obtain that $\theta_0 \approx 1.450$. To assess the correctness of the results derived from Theorem 6.1, we pick two unequal delay values. One is $\theta = 1.4$ and the other is $\theta = 1.510$. If $\theta = 1.40 < \theta_0 \approx 1.450$, we derive the computer simulation diagrams depicted in Figure 7. As depicted in Figure 7, it is clear that $u_1 \to 2.0589, u_2 \to 0.4597, u_3 \to 2.5849$ when $t \to +\infty$. In other words, the unique positive equilibrium point $F(2.0589, 0.4597, 2.5849)$ of model (7.4) sustains a locally asymptotically stable status. As $\theta = 1.510 > \theta_0 \approx 1.450$, we obtain the computer simulation diagrams illustrated in Figure 8. As depicted in Figure 8, we can see that u_1 will maintain a periodic quavering level around the value 2.0589, u_2 will keep a periodic quavering level around the value 0.4597, and u_3 will hold a periodic quavering level around the value 2.5849. In other words, a set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(2.0589, 0.4597, 2.5849)$.

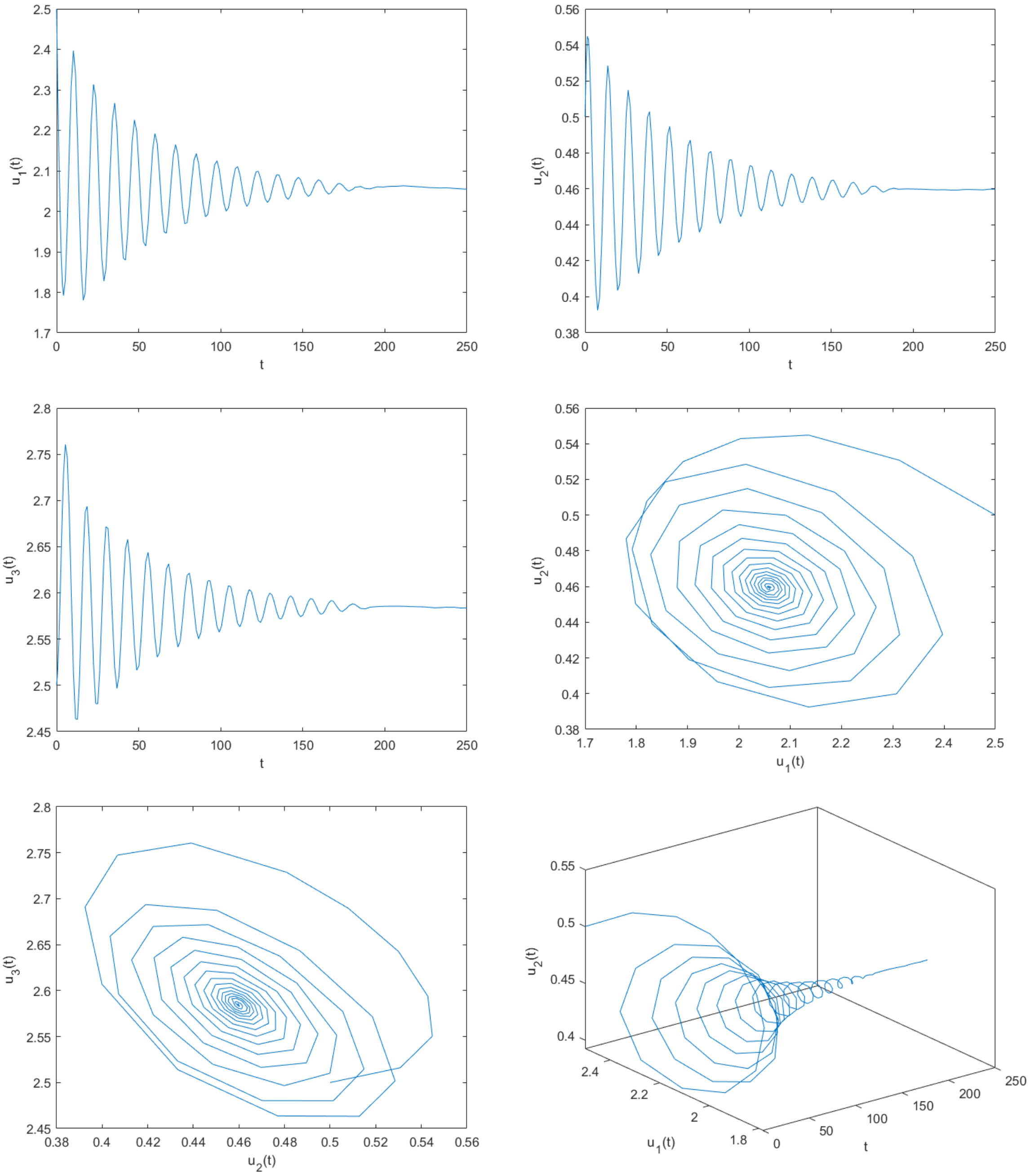

Figure 7. Matlab simulation figures of system (7.4) under the delay $\theta = 1.40 < \theta_{*0} = 1.450$ The equilibrium point $F(u_{1*}, u_{2*}) = F(2.0589, 0.4597, 2.5849)$ holds a locally asymptotically stable level.

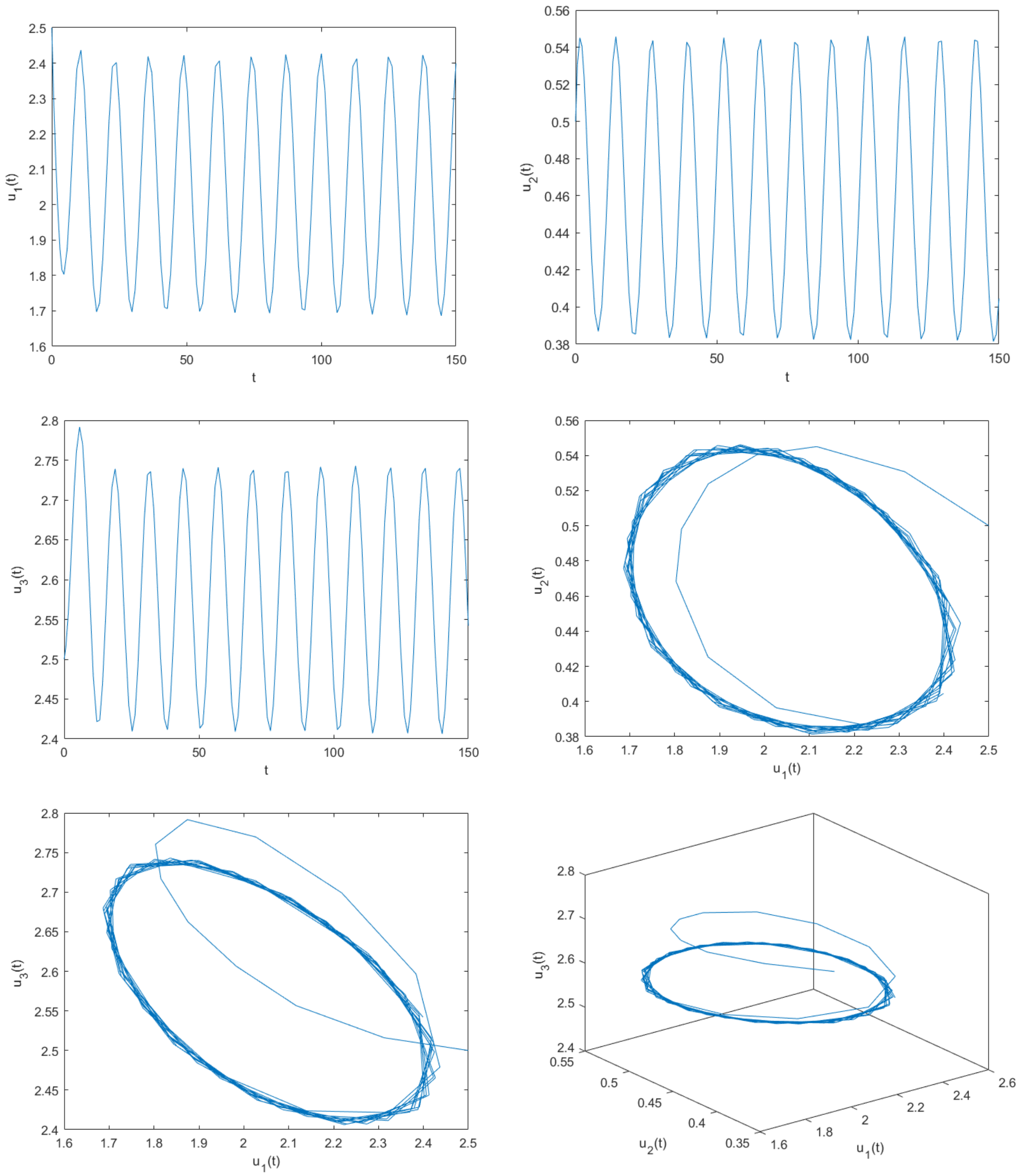

Figure 8. Matlab simulation figures of system (7.4) under the delay $\theta = 1.510 > \theta_{*0} = 1.450$. A set of periodic solutions (namely, Hopf bifurcations) arise in the vicinity of the positive equilibrium point $F(u_{1*}, u_{2*}, u_{3*}) = F(2.0589, 0.4597, 2.5849)$.

Remark 7.1. *On the basis of the computer simulation graphics in Examples 7.1 and 7.2, one can clearly observe that the bifurcation values of models (7.1) and (7.2) are* $\theta_0 \approx 5.1$ *and* $\theta_* \approx 0.9$*,which indicates that we can diminish the stability region and cut down the time to bifurcation in model (7.1) by implementing the formulated hybrid controller. On the basis of the computer simulation graphics* *in Examples 7.3 and 7.4, one can readily observe that the bifurcation values of models (7.3) and (7.4) are* $\theta_\star \approx 1.45, \theta_{0*} \approx 3.3$*, which indicates that we can enhance the stability region and prolong the time of bifurcation onset in model (7.1) by implementing the formulated hybrid controller.*

Remark 7.2. *Compared with the work of Zhao et al. [16], although the controller design in this paper is similar, since different controllers are added to different equations, then achieved control effects are different. For example, in [16], the stability domain is enlarged and the onset of Hopf bifurcation is postponed; in this paper, the stability domain is narrowed and the onset of Hopf bifurcation is advanced. In addition, the controller design in the Section 6 of this paper is also different from that in [16]. Also, the exploration methods on the boundedness of the solution in this paper are different from that in [16]. Based on this viewpoint, we think that this paper presents some novelties.*

8. Conclusions

For a long time, predator-prey models have played an important role in biology and attracted great interest from both mathematical and biological fields. From a mathematical standpoint, how to reveal the effect of time delay on the dynamics of predator-prey models is a vital theme. This paper introduces a new delayed predator-prey model and provides a detailed analysis on its peculiarities, including the non-negativity, existence, uniqueness, and boundedness of the solutions, as well as the Hopf bifurcation issue. We derive the sufficient conditions on the stability and bifurcation of the model. By applying various mixed delay feedback controllers, we have successfully adjusted the stability region and the time of bifurcation onset of this model. The findings of this study hold significant theoretical value in balancing the concentrations of predators and preys. Moreover, this research approach can also be applied to explore bifurcation control issues in other complex differential models. Recently, several works have focused on the stability and Hopf bifurcation control of fractional-order dynamical models. We also plan to explore Hopf bifurcation in a fractional-order version of this model in the near future.

Author contributions

Jinting Lin: Conceptualization, formal analysis, investigation, methodology, software, writing original draft, writing-review & editing; Changjin Xu: Conceptualization, formal analysis, investigation, methodology, software, writing-original draft, writing-review & editing; Yiya Xu: Formal analysis, investigation, software, writing-review & editing; Yingyan Zhao: Conceptualization, formal analysis, investigation, methodology, writing-original draft, writing-review & editing; Yicheng Pang: Conceptualization, investigation, methodology, writing-review & editing; Zixin Liu: Conceptualization, investigation, methodology, writing-review & editing; Jianwei Shen: Conceptualization, investigation, methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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