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Research article

Generalized Perron complements in diagonally dominant matrices

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Abstract: The concept of the generalized Perron complement concerning a nonnegative irreducible matrix was proposed by L. Z. Lu in 2002, and it was used to construct an algorithm for estimating the boundary of the spectral radius. In this study, we consider the properties of generalized Perron complements of nonnegative irreducible and diagonally dominant matrices. Moreover, we analyze the closure property of the generalized Perron complements of nonnegative irreducible H-matrices under certain conditions.

Keywords: irreducibility; nonnegative matrix; diagonally dominant; generalized Perron complement; H-matrix
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1. Introduction

Let $A = \left(a_{ij}\right) \in C^{n\times n}$ be an $n \times n$ complex matrix. We denote $N = \{1,\ 2, ...,\ n\}$ and define

$$J = \left\{ i \left| |a_{ii}| \geq \sum_{j\neq i}^{n} \left|a_{ij}\right|, i \in N \right. \right\}. \tag{1.1}$$

If $J = N$, then the matrix A is defined to be (row) diagonally dominant. In addition, A is defined as strictly diagonally dominant if

$$|a_{ii}| > \sum_{j\neq i}^{n} \left|a_{ij}\right|, i \in N.$$

Clearly, for strictly diagonally dominant matrices, the principal submatrices are strictly diagonally dominant.

The comparison matrix of A is denoted by $\mu(A) = (\mu_{ij})$, where

$$\mu_{ij} = \begin{cases} |a_{ii}|, i = j, \\ -|a_{ij}|, i \neq j. \end{cases}$$

The characterization of nonsingular M-matrices is an important topic in matrix theory. If there exists a nonnegative matrix P and a real number $s > \rho(P)$ such that $A = sI - P$, where $\rho(P)$ is the spectral radius of P, then A is defined as a nonsingular M-matrix.

Furthermore, it is well-known that a matrix A is a nonsingular H-matrix equivalent to its comparison matrix $\mu(A)$ is a nonsingular M-matrix, as stated in [1]. The connection between nonsingular M-matrices and H-matrices provides a deeper understanding of their properties and relationships, leading to the development of effective computational algorithms and methodologies for solving complex problems in diverse fields of engineering and science.

A matrix A is considered to be generalized diagonally dominant if there exists a positive diagonal matrix D such that the product AD is strictly diagonally dominant. Furthermore, it is well established that a matrix A is classified as an H-matrix if and only if it is generalized diagonally dominant, as indicated in [2]. H-matrices represent a special class of matrices that find wide-ranging applications in engineering and scientific computation, as noted in [3]. The properties of H-matrices make them particularly valuable in numerical analysis, optimization, finite element methods, and other areas of applied mathematics. Consequently, the development of direct algorithms for identifying H-matrices, as explored in [4–7], holds immense practical significance in various computational and engineering contexts.

Suppose that A is a complex matrix of order $n \geq 2$. Let N_1, N_2 be nonempty subsets of N with $N_1 \bigcup N_2 = N$ and $N_1 \bigcap N_2 = \emptyset$. The submatrix that lies in the rows of A indexed by N_1 and the columns of A indexed by N_2 is denoted as $A(N_1, N_2)$; in particular, $A(N_1, N_1)$ is abbreviated to $A(N_1)$.

Let $N_1 \subset N$ and $N_2 = N \backslash N_1$ be nonempty subsets. If $A(N_1)$ is invertible, then the Schur complement of A with respect to the principal submatrix $A(N_1)$ is defined as

$$A/N_1 = A(N_2) - A(N_2, N_1)[A(N_1)]^{-1} A(N_1, N_2).$$

Carlson and Markham established an important result in their work [8], which states that if the original matrix is strictly diagonally dominant, then its Schur complements will also possess the property of being diagonally dominant. This result has significant implications, because diagonally dominant matrices often have desirable properties and are frequently encountered in various areas of mathematics and engineering.

The Perron complement is indeed a smaller matrix which is derived naturally from a square matrix. The concept of the Perron complement in a nonnegative (irreducible) matrix was first introduced in [9] within the context of computing the Perron vectors of finite state Markov processes. The name Perron complement is derived from its analogy to the Schur complement.

Definition 1.1. *[9] Let A be an $n \times n$ nonnegative irreducible matrix with spectral radius $\rho(A)$ and nonempty subsets $N_1 \subset N, N_2 = N \backslash N_1$. The Perron complement of $A(N_1)$ in A is defined by the matrix*

$$P(A/N_1) = A(N_2) + A(N_2, N_1)[\rho(A) I - A(N_1)]^{-1} A(N_1, N_2). \tag{1.2}$$

Meyer [9] fully analyzed the properties of $P(A/N_1)$ in detail, providing some elegant results. For example, $P(A/N_1)$ inherits the irreducibility and nonnegativity of the matrix A. Moreover, $P(A/N_1)$ and A have the same spectral radius.

To improve the boundary of $\rho(A)$, L. Z. Lu [10] substituted λ for $\rho(A)$ in (1.2) and proposed the following definition.

Definition 1.2. *[10] Let A be an $n \times n$ nonnegative irreducible matrix and nonempty subsets $N_1 \subset N, N_2 = N \backslash N_1$. The generalized Perron complement of $A(N_1)$ in A is defined as*

$$P_\lambda(A/N_1) = A(N_2) + A(N_2, N_1)[\lambda I - A(N_1)]^{-1} A(N_1, N_2), \tag{1.3}$$

where $\lambda > \rho(A(N_1))$. It is clear that $P_\lambda(A/N_1)$ is well-defined for $\lambda > \rho(A(N_1))$.

The Perron complement is a valuable tool in many domains, including statistics, computational mathematics, and control theory, such as addressing limited optimization problems [11], achieving range-based threshold estimates for jump-diffusion models [12]. Research on the Perron complement of matrices has been ongoing since the 1980s, and experts and scholars worldwide have made significant contributions to this field. Over the years, continuous efforts have led to new findings and advancements in the understanding and utilization of the Perron complement. Moreover, for a special matrix family, the Perron complement can be applied to research properties by utilizing the relationship between the matrix and its Perron complement. Further discussions on the Perron complements of a special family of matrices are considered in [13–16].

Estimating the bounds of eigenvalues via Perron complement matrices is an important research direction that has practical applications in various fields. The use of Perron complement matrices has been extensively studied in the literature, as referenced in [17–20], and has yielded promising results. For example, researchers have utilized Perron complement matrices to optimize the bounds of the eigenvalues of Z-matrices [17]. Additionally, in the case of nonnegative irreducible matrices, G. X. Huang [18], S. M. Yang [19], and Z. M. Yang [20] significantly improved the bounds on the spectral radius by utilizing the Perron complement. This research direction holds significant potential for improving the estimation of eigenvalue bounds and has practical implications for various applications in mathematics, engineering, and other fields [21–23].

Inspired by the definition of the generalized Perron complement and the results in [8], we investigate the diagonal dominance of the generalized Perron complement of nonnegative irreducible matrices. The remainder of this paper is organized as follows.

In Section 2, we prove that the generalized Perron complements of nonnegative irreducible and diagonally dominant matrices concerning a given subset of indices are nonnegative irreducible and diagonally dominant matrices. On this basis, we acquire the closure property for the generalized Perron complements of nonnegative irreducible H-matrices under certain conditions in Section 3. Two numerical experiments are presented in Section 4 to support our findings. Finally, we summarize our conclusions in Section 5.

2. Generalized Perron complements of nonnegative irreducible and diagonally dominant matrices

Lemma 2.1. *[2] If A is an $n \times n$ nonnegative matrix, then*

$$\min_i \sum_{k=1}^{n} a_{ik} \leq \rho(A) \leq \max_i \sum_{k=1}^{n} a_{ik}.$$

Lemma 2.2. *[10] If A is an $n \times n$ nonnegative irreducible matrix and nonempty subsets $N_1 \subset N, N_2 = N \backslash N_1$, then for any $\lambda > \rho(A(N_1))$,*

$$P_\lambda(A/N_1) = A(N_2) + A(N_2, N_1)[\lambda I - A(N_1)]^{-1} A(N_1, N_2)$$

is also nonnegative irreducible.

Lemma 2.2 shows that nonnegative irreducible matrices are closed under generalized Perron complementation. The following theorem shows that nonnegative irreducible and diagonally dominant matrices are closed under generalized Perron complementation.

Theorem 2.1. *If A is an $n \times n$ nonnegative irreducible and diagonally dominant matrix and nonempty subsets $N_1 \subset N, N_2 = N \backslash N_1$, then for any $\lambda > \max\limits_{i \in N_1} \sum\limits_{k=1}^{n} |a_{ik}|$,*

$$P_\lambda(A/N_1) = A(N_2) + A(N_2, N_1)[\lambda I - A(N_1)]^{-1} A(N_1, N_2)$$

is a nonnegative irreducible and diagonally dominant matrix.

Proof. We suppose that $N_1 = \left\{i_1, i_2, \cdots, i_p\right\}, N_2 = \left\{j_1, j_2, \cdots, j_q\right\}$ with $p + q = n$. According to

$$\lambda > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| = \max_{i \in N_1} \left(\sum_{k \in N_1} |a_{ik}| + \sum_{k \in N_2} |a_{ik}| \right) \geq \max_{i \in N_1} \sum_{k \in N_1} |a_{ik}|,$$

we obtain

$$\lambda > \max_{i \in N_1} \sum_{k \in N_1} |a_{ik}|. \tag{2.1}$$

Here, $\max\limits_{i \in N_1} \sum\limits_{k \in N_1} |a_{ik}|$ is the maximum row sum of the matrix $A(N_1)$. Note that A is a nonnegative matrix, so is $A(N_1)$. By Lemma 2.1, we have

$$\max_{i \in N_1} \sum_{k \in N_1} |a_{ik}| \geq \rho(A(N_1)). \tag{2.2}$$

It follows from (2.1) and (2.2) that $\lambda > \rho(A(N_1))$. According to Lemma 2.2, the generalized Perron complement $P_\lambda(A/N_1)$ is nonnegative and irreducible.

In the following section, we consider the diagonal dominance of $P_\lambda(A/N_1)$. Because the matrix A is diagonally dominant, it holds that

$$|a_{ii}| \geq \sum_{k \neq i}^{n} |a_{ik}|, i \in N.$$

Specifically, for $j_k \in N_2$, we have

$$\left|a_{j_k j_k}\right| \geq \sum_{t=1, t \neq k}^{q} \left|a_{j_k j_t}\right| + \sum_{t=1}^{p} \left|a_{j_k i_t}\right|. \tag{2.3}$$

According to $\lambda > \max\limits_{i\in N_1} \sum\limits_{k=1}^{n} |a_{ik}|$, it holds that

$$\lambda > \max_{i\in N_1} \sum_{k=1}^{n} |a_{ik}| = \max_{i\in N_1} \left(\sum_{k\in N_1} |a_{ik}| + \sum_{k\in N_2} |a_{ik}| \right) = \max_{i_v\in N_1} \sum_{t=1}^{p} a_{i_v i_t} + \max_{i_v\in N_1} \sum_{t=1}^{q} a_{i_v j_t}. \tag{2.4}$$

We discuss this issue in two cases.

Case 1. $\max\limits_{i_v\in N_1} \sum\limits_{t=1}^{q} a_{i_v j_t} = 0$. This implies that $a_{i_v j_t} = 0$ for any $i_v \in N_1$, $j_t \in N_2$, that is, $A(N_1, N_2) = O$. In this case, we derive:

$$P_\lambda (A/N_1\ ) = A(N_2) + A(N_2, N_1)[\lambda I - A(N_1)]^{-1} A(N_1, N_2) = A(N_2).$$

Therefore, as a principal submatrix of the diagonally dominant matrix A, the matrix $P_\lambda (A/N_1\ )$ is clearly diagonally dominant.

Case 2. $\max\limits_{i_v\in N_1} \sum\limits_{t=1}^{q} a_{i_v j_t} > 0$. From (2.4), we obtain

$$0 < \max_{i_v\in N_1} \sum_{t=1}^{q} a_{i_v j_t} < \lambda - \max_{i_v\in N_1} \sum_{t=1}^{p} a_{i_v i_t} \le \lambda - \sum_{t=1}^{p} a_{i_v i_t}. \tag{2.5}$$

Thus, we further obtain

$$0 < \max_{i_v\in N_1} \frac{\sum\limits_{t=1}^{q} a_{i_v j_t}}{\lambda - \sum\limits_{t=1}^{p} a_{i_v i_t}} < 1. \tag{2.6}$$

Denote the column vector

$$X = \left(x_1, x_2, \cdots, x_p\right)^{\mathrm{T}} = [\lambda I - A(N_1)]^{-1} \left(\sum_{t=1}^{q} a_{i_1 j_t}, \sum_{t=1}^{q} a_{i_2 j_t}, \cdots, \sum_{t=1}^{q} a_{i_p j_t} \right)^{\mathrm{T}}, \tag{2.7}$$

or equivalently written as

$$[\lambda I - A(N_1)] X = \left(\sum_{t=1}^{q} a_{i_1 j_t}, \sum_{t=1}^{q} a_{i_2 j_t}, \cdots, \sum_{t=1}^{q} a_{i_p j_t} \right)^{\mathrm{T}}. \tag{2.8}$$

Let $x_v = \max\{x_t\}$, where x_t is the t-th coordinate of X. From (2.8), we obtain the following result:

$$\begin{aligned}
\sum_{t=1}^{q} a_{i_v j_t} &= (\lambda - a_{i_v i_v}) x_v + \sum_{t=1, t\ne v}^{p} (-a_{i_v i_t}) x_t \\
&\ge (\lambda - a_{i_v i_v}) x_v + \sum_{t=1, t\ne v}^{p} (-a_{i_v i_t}) x_v \\
&= \lambda x_v - \left(a_{i_v i_v} + \sum_{t=1, t\ne v}^{p} a_{i_v i_t} \right) x_v \\
&= \left(\lambda - \sum_{t=1}^{p} a_{i_v i_t} \right) x_v.
\end{aligned}$$

Therefore, we obtain

$$x_v \leq \frac{\sum_{t=1}^{q} a_{i_v j_t}}{\lambda - \sum_{t=1}^{p} a_{i_v i_t}} \leq \max_{i_v \in N_1} \frac{\sum_{t=1}^{q} a_{i_v j_t}}{\lambda - \sum_{t=1}^{p} a_{i_v i_t}}. \tag{2.9}$$

Considering inequalities (2.6) and (2.9), we obtain $x_v < 1$. Thus, it holds that

$$X = \left(x_1, x_2, \cdots, x_p\right)^{\mathrm{T}} < (1, 1, \cdots, 1)^{\mathrm{T}}.$$

Denote the element of $P_\lambda\left(A/N_1\right)$ located in the j_k-th row and j_t-th column as $\left(a'_{j_k j_t}\right)$. Thus, for any $j_k \in N_2$, we have

$$\begin{aligned}
&\left|a'_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a'_{j_k j_t}\right| \\
&= \left|a_{j_k j_k} + \left(a_{j_k i_1}, a_{j_k i_2}, \cdots, a_{j_k i_p}\right)[\lambda I - A\left(N_1\right)]^{-1}\left(a_{i_1 j_k}, a_{i_2 j_k}, \cdots, a_{i_p j_k}\right)^{\mathrm{T}}\right| \\
&- \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t} + \left(a_{j_k i_1}, a_{j_k i_2}, \cdots, a_{j_k i_p}\right)[\lambda I - A\left(N_1\right)]^{-1}\left(a_{i_1 j_t}, a_{i_2 j_t}, \cdots, a_{i_p j_t}\right)^{\mathrm{T}}\right| \\
&\geq \left|a_{j_k j_k}\right| - \left(\left|a_{j_k i_1}\right|, \left|a_{j_k i_2}\right|, \cdots, \left|a_{j_k i_p}\right|\right)[\lambda I - A\left(N_1\right)]^{-1}\left(a_{i_1 j_k}, a_{i_2 j_k}, \cdots, a_{i_p j_k}\right)^{\mathrm{T}} \\
&- \sum_{t=1, t\neq k}^{q} \left[\left|a_{j_k j_t}\right| + \left(\left|a_{j_k i_1}\right|, \left|a_{j_k i_2}\right|, \cdots, \left|a_{j_k i_p}\right|\right)[\lambda I - A\left(N_1\right)]^{-1}\left(a_{i_1 j_t}, a_{i_2 j_t}, \cdots, a_{i_p j_t}\right)^{\mathrm{T}}\right] \\
&= \left|a_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t}\right| - \left(\left|a_{j_k i_1}\right|, \left|a_{j_k i_2}\right|, \cdots, \left|a_{j_k i_p}\right|\right)[\lambda I - A\left(N_1\right)]^{-1}\left(\sum_{t=1}^{q} a_{i_1 j_t}, \sum_{t=1}^{q} a_{i_2 j_t}, \cdots, \sum_{t=1}^{q} a_{i_p j_t}\right)^{\mathrm{T}} \\
&= \left|a_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t}\right| - \left(\left|a_{j_k i_1}\right|, \left|a_{j_k i_2}\right|, \cdots, \left|a_{j_k i_p}\right|\right) X \quad (by\ Equation 2.7) \\
&\geq \left|a_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t}\right| - \left(\left|a_{j_k i_1}\right|, \left|a_{j_k i_2}\right|, \cdots, \left|a_{j_k i_p}\right|\right)(1, 1, \cdots, 1)^{\mathrm{T}} \quad (by\ X < (1, 1, \cdots, 1)^{\mathrm{T}}) \\
&= \left|a_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t}\right| - \sum_{t=1}^{p} \left|a_{j_k i_t}\right|.
\end{aligned}$$

On the other hand, according to inequality (2.3), we have

$$\left|a_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t}\right| - \sum_{t=1}^{p} \left|a_{j_k i_t}\right| \geq 0.$$

On the basis of the above derivations, we obtain

$$\left|a'_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a'_{j_k j_t}\right| \geq \left|a_{j_k j_k}\right| - \sum_{t=1, t\neq k}^{q} \left|a_{j_k j_t}\right| - \sum_{t=1}^{p} \left|a_{j_k i_t}\right| \geq 0,\ j_k \in N_2.$$

This shows that

$$\left|a'_{j_k j_k}\right| \geq \sum_{t=1,t\neq k}^{q} \left|a'_{j_k j_t}\right|, j_k \in N_2.$$

Therefore, we infer that $P_\lambda (A/N_1)$ is diagonally dominant.

From the analysis of the two cases, we conclude that $P_\lambda (A/N_1)$ is diagonally dominant. This completes the proof. □

Corollary 2.1. *If A is an $n \times n$ nonnegative irreducible and strictly diagonally dominant matrix and nonempty subsets $N_1 \subset N, N_2 = N \backslash N_1$, then for any $\lambda > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}|$,*

$$P_\lambda (A/N_1) = A(N_2) + A(N_2, N_1)[\lambda I - A(N_1)]^{-1} A(N_1, N_2)$$

is a nonnegative irreducible and strictly diagonally dominant matrix.

3. Generalized Perron complements of nonnegative irreducible H-matrices

Nonnegative irreducible H-matrices emerge in discretized partial differential equations derived from inverse scattering problems [24]. In this section, we discuss the generalized Perron complements of nonnegative irreducible H-matrices.

Theorem 3.1. *If A is an $n \times n$ nonnegative irreducible H-matrix and nonempty subsets $N_1 \subset N, N_2 = N \backslash N_1$, then for any $\lambda > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| \geq 2|a_{ii}|, i \in N_1$,*

$$P_\lambda (A/N_1) = A(N_2) + A(N_2, N_1)[\lambda I - A(N_1)]^{-1} A(N_1, N_2)$$

is a nonnegative irreducible H-matrix.

Proof. We suppose that $N_1 = \left\{i_1, i_2, \cdots, i_p\right\}, N_2 = \left\{j_1, j_2, \cdots, j_q\right\}$ with $p + q = n$. By Theorem 2.1, for any $\lambda > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}|, i \in N_1$, $P_\lambda (A/N_1)$ is nonnegative irreducible. Because A is an H-matrix, there is a positive diagonal matrix

$$D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$$

such that AD is strictly diagonally dominant, that is,

$$|a_{ii}| d_i > \sum_{k \neq i}^{n} |a_{ik}| d_k, i \in N,$$

or equivalently written as

$$|a_{ii}| > \sum_{k \neq i}^{n} \frac{|a_{ik}|}{d_i} d_k, i \in N. \tag{3.1}$$

Let $B = D^{-1}AD = \left(\frac{a_{ij}d_j}{d_i}\right) = \left(b_{ij}\right)$. Clearly, B is nonnegative and irreducible. Meanwhile, inequality (3.1) implies that B is strictly diagonally dominant. Since for any $i \in N_1$, $\lambda > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| \geq 2|a_{ii}|$ and

$$2|a_{ii}| = |b_{ii}| + |a_{ii}| > |b_{ii}| + \sum_{k \neq i}^{n} \frac{|a_{ik}|}{d_i} d_k = |b_{ii}| + \sum_{k \neq i}^{n} |b_{ik}| = \sum_{k=1}^{n} |b_{ik}|, i \in N_1,$$

we have $\lambda > \max_{i\in N_1} \sum_{k=1}^{n} |b_{ik}|$. According to Corollary 2.1, $P_\lambda(B/N_1)$ is nonnegative irreducible and strictly diagonally dominant. In addition, we have

$$\begin{aligned}
&P_\lambda(B/N_1) = B(N_2) + B(N_2,N_1)[\lambda I - B(N_1)]^{-1}B(N_1,N_2)\\
&= \begin{pmatrix} b_{j_1j_1} & \cdots & b_{j_1j_q}\\ \cdots & \cdots & \cdots\\ b_{j_qj_1} & \cdots & b_{j_qj_q}\end{pmatrix} + \begin{pmatrix} b_{j_1i_1} & \cdots & b_{j_1i_p}\\ \cdots & \cdots & \cdots\\ b_{j_qi_1} & \cdots & b_{j_qi_p}\end{pmatrix}\begin{pmatrix} \lambda - b_{i_1i_1} & \cdots & -b_{i_1i_p}\\ \cdots & \cdots & \cdots\\ -b_{i_pi_1} & \cdots & \lambda - b_{i_pi_p}\end{pmatrix}^{-1}\begin{pmatrix} b_{i_1j_1} & \cdots & b_{i_1j_q}\\ \cdots & \cdots & \cdots\\ b_{i_pj_1} & \cdots & b_{i_pj_q}\end{pmatrix}\\
&= \begin{pmatrix} a_{j_1j_1} & \cdots & a_{j_1j_q}\frac{d_{j_q}}{d_{j_1}}\\ \cdots & \cdots & \cdots\\ a_{j_qj_1}\frac{d_{j_1}}{d_{j_q}} & \cdots & a_{j_qj_q}\end{pmatrix} + \begin{pmatrix} a_{j_1i_1}\frac{d_{i_1}}{d_{j_1}} & \cdots & a_{j_1i_p}\frac{d_{i_p}}{d_{j_1}}\\ \cdots & \cdots & \cdots\\ a_{j_qi_1}\frac{d_{i_1}}{d_{j_q}} & \cdots & a_{j_qi_p}\frac{d_{i_p}}{d_{j_q}}\end{pmatrix}\\
&\times \begin{pmatrix} \lambda - a_{i_1i_1} & \cdots & -a_{i_1i_p}\frac{d_{i_p}}{d_{i_1}}\\ \cdots & \cdots & \cdots\\ -a_{i_pi_1}\frac{d_{i_1}}{d_{i_p}} & \cdots & \lambda - a_{i_pi_p}\end{pmatrix}^{-1}\begin{pmatrix} a_{i_1j_1}\frac{d_{j_1}}{d_{i_1}} & \cdots & a_{i_1j_q}\frac{d_{j_q}}{d_{i_1}}\\ \cdots & \cdots & \cdots\\ a_{i_pj_1}\frac{d_{j_1}}{d_{i_p}} & \cdots & a_{i_pj_q}\frac{d_{j_q}}{d_{i_p}}\end{pmatrix}\\
&= \operatorname{diag}\left(\frac{1}{d_{j_1}},\cdots,\frac{1}{d_{j_q}}\right)\begin{pmatrix} a_{j_1j_1} & \cdots & a_{j_1j_q}\\ \cdots & \cdots & \cdots\\ a_{j_qj_1} & \cdots & a_{j_qj_q}\end{pmatrix}\operatorname{diag}\left(d_{j_1},\cdots,d_{j_q}\right)\\
&+ \operatorname{diag}\left(\frac{1}{d_{j_1}},\cdots,\frac{1}{d_{j_q}}\right)\begin{pmatrix} a_{j_1i_1} & \cdots & a_{j_1i_p}\\ \cdots & \cdots & \cdots\\ a_{j_qi_1} & \cdots & a_{j_qi_p}\end{pmatrix}\operatorname{diag}\left(d_{i_1},\cdots,d_{i_p}\right)\\
&\times \operatorname{diag}\left(\frac{1}{d_{i_1}},\cdots,\frac{1}{d_{i_p}}\right)\begin{pmatrix} \lambda - a_{i_1i_1} & \cdots & -a_{i_1i_p}\\ \cdots & \cdots & \cdots\\ -a_{i_pi_1} & \cdots & \lambda - a_{i_pi_p}\end{pmatrix}^{-1}\operatorname{diag}\left(d_{i_1},\cdots,d_{i_p}\right)\\
&\times \operatorname{diag}\left(\frac{1}{d_{i_1}},\cdots,\frac{1}{d_{i_p}}\right)\begin{pmatrix} a_{i_1j_1} & \cdots & a_{i_1j_q}\\ \cdots & \cdots & \cdots\\ a_{i_pj_1} & \cdots & a_{i_pj_q}\end{pmatrix}\operatorname{diag}\left(d_{j_1},\cdots,d_{j_q}\right).
\end{aligned}$$

Denote $D(N_2) = \operatorname{diag}\left(d_{j_1},\cdots,d_{j_q}\right)$. We acquire

$$\begin{aligned}
P_\lambda(B/N_1) &= D^{-1}(N_2)\left\{A(N_2) + A(N_2,N_1)[\lambda I - A(N_1)]^{-1}A(N_1,N_2)\right\}D(N_2)\\
&= D^{-1}(N_2)\,P_\lambda(A/N_1)\,D(N_2).
\end{aligned} \tag{3.2}$$

Because $P_\lambda(B/N_1)$ is strictly diagonally dominant, from (3.2), we deduce that there exists a positive diagonal matrix $D(N_2)$ such that $P_\lambda(A/N_1)D(N_2)$ is strictly diagonally dominant. Therefore, we obtain that $P_\lambda(A/N_1)$ is an H-matrix. This completes the proof. □

4. Numerical examples

Example 4.1. *Let us consider a nonnegative irreducible matrix:*

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 4 & 0 & 1 \\ 2 & 1 & 6 & 2 \\ 3 & 1 & 1 & 6 \end{pmatrix}.$$

Obviously, A is diagonally dominant.

For $N_1 = \{1\}$, *set* $\lambda = 8 > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| = 7$. *We have*

$$\begin{aligned} P_8(A/N_1) &= \begin{pmatrix} 4 & 0 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} (8-4)^{-1} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4.5 & 0.5 & 1.5 \\ 1.5 & 6.5 & 2.5 \\ 1.5 & 1.5 & 6.5 \end{pmatrix}. \end{aligned}$$

Clearly, the generalized Perron complement $P_8(A/N_1)$ *is a nonnegative irreducible and diagonally dominant matrix.*

For $N_1 = \{1, 2\}$, *set* $\lambda = 8 > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| = 7$. *The generalized Perron complement of A at* λ *is*

$$\begin{aligned} P_8(A/N_1) &= \begin{pmatrix} 6 & 2 \\ 1 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \left[\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} - \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 10 \\ 11 & 17 \end{pmatrix}. \end{aligned}$$

It is obvious that $P_8(A/N_1)$ *is a nonnegative irreducible and diagonally dominant matrix. These results are consistent with Theorem 2.1.*

Example 4.2. *[25] Let us consider a nonnegative irreducible H-matrix:*

$$A = \begin{pmatrix} 3 & 0 & 4 & 2 \\ 100 & 99 & 1 & 1 \\ 1 & 1 & 100 & 1 \\ 1 & 1 & 1 & 100 \end{pmatrix}.$$

According to Theorem 3.1, $\max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| \geq 2|a_{ii}|, i \in N_1$, *so we can choose* $N_1 = \{1\}$, $N_1 = \{2\}$ *or* $N_1 = \{1, 2\}$.

For $N_1 = \{1\}$, *set* $\lambda = 11 > \max_{i \in N_1} \sum_{k=1}^{n} |a_{ik}| = 9 \geq 2|a_{ii}|$. *We obtain*

$$P_{11}(A/N_1) = \begin{pmatrix} 99 & 1 & 1 \\ 1 & 100 & 1 \\ 1 & 1 & 100 \end{pmatrix} + \begin{pmatrix} 100 \\ 1 \\ 1 \end{pmatrix} (11-3)^{-1} \begin{pmatrix} 0 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 99 & 51 & 26 \\ 1 & 100.5 & 1.25 \\ 1 & 1.5 & 100.25 \end{pmatrix},$$

whose comparison matrix

$$\begin{pmatrix} 99 & -51 & -26 \\ -1 & 100.5 & -1.25 \\ -1 & -1.5 & 100.25 \end{pmatrix}$$

is an M-matrix. Therefore, $P_{11}(A/N_1)$ is an H-matrix and is clearly nonnegative irreducible.

For $N_1 = \{2\}$, set $\lambda = 299 > \max_{i\in N_1} \sum_{k=1}^{n} |a_{ik}| = 201 \geq 2|a_{ii}|$. We obtain

$$P_{299}(A/N_1) = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 100 & 1 \\ 1 & 1 & 100 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} (299 - 99)^{-1} \begin{pmatrix} 100 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 4 & 2 \\ 1.5 & 100.005 & 1.005 \\ 1.5 & 1.005 & 100.005 \end{pmatrix},$$

whose comparison matrix

$$\begin{pmatrix} 3 & -4 & -2 \\ -1.5 & 100.005 & -1.005 \\ -1.5 & -1.005 & 100.005 \end{pmatrix}$$

is an M-matrix. Therefore, $P_{299}(A/N_1)$ is an H-matrix and is clearly nonnegative irreducible.

For $N_1 = \{1, 2\}$, set $\lambda = 203 > \max_{i\in N_1} \sum_{k=1}^{n} |a_{ik}| = 201 \geq 2|a_{ii}|$. The generalized Perron complement of A at λ is

$$P_{203}(A/N_1) = \begin{pmatrix} 100 & 1 \\ 1 & 100 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left[\begin{pmatrix} 203 & 0 \\ 0 & 203 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 100 & 99 \end{pmatrix} \right]^{-1} \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 100.0488 & 1.0292 \\ 1.0488 & 100.0292 \end{pmatrix},$$

whose comparison matrix

$$\begin{pmatrix} 100.0488 & -1.0292 \\ -1.0488 & 100.0292 \end{pmatrix}$$

is an M-matrix. Therefore, $P_{203}(A/N_1)$ is an H-matrix and is clearly nonnegative irreducible. These results are consistent with Theorem 3.1.

5. Conclusions

The generalized Perron complement is a smaller matrix derived from a square matrix. For a given matrix, the generalized Perron complement can be utilized to investigate the properties by exploiting the relationship between the matrix and its generalized Perron complement. Consequently, given a matrix family, further analysis of generalized Perron complements of the matrix families can be conducted. In this study, we examined the diagonal dominance of generalized Perron complements of the family of nonnegative irreducible matrices.

Initially, we demonstrate that the generalized Perron complements of nonnegative irreducible and diagonally dominant matrices concerning a given subset of indices are nonnegative irreducible and diagonally dominant matrices. Specifically, the set of nonnegative irreducible and strictly diagonally dominant matrices is closed under the process of generalized Perron complementation. Building upon this foundation, we established the closure property for the generalized Perron complements of nonnegative irreducible H-matrices under specific conditions.

Author contributions

Qin Zhong: Conceptualization, Data curation, Writing-original draft, Methodology; Na Li: Formal analysis, Writing-review & editing, Investigation, Data curation. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. M. Fiedler, V. Pták, On matrices with non-positive off-diagonal elements and positive principal minors, *Czech. Math. J.*, **12** (1962), 382–400.
2. A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Philadelphia: Society for Industrial and Applied Mathematics, 1994. https://doi.org/10.1137/1.9781611971262
3. K. R. James, W. Riha, Convergence criteria for successive overrelaxation, *SIAM J. Numer. Anal.*, **12** (1975), 137–143. https://doi.org/10.1137/0712013
4. W. W. Jiang, Q. Tuo, A set of new criteria for the iterative discrimination of subdivision of nonsingular H-Matrices, *Adv. Appl. Math.*, **9** (2020), 50–59. https://doi.org/10.12677/AAM.2020.91007
5. K. Ojiro, H. Niki, M. Usui, A new criterion for the H-matrix property, *J. Comput. Appl. Math.*, **150** (2003), 293–302. https://doi.org/10.1016/S0377-0427(02)00666-0
6. H. B. Li, T. Z. Huang, On a new criterion for the H-matrix property, *Appl. Math. Lett.*, **19** (2006), 1134–1142. https://doi.org/10.1016/j.aml.2005.12.005
7. Y. Li, X. Y. Chen, Y. Q. Wang, Some new criteria for identifying H-matrices, *Filomat*, **38** (2024), 1375–1387. https://doi.org/10.2298/FIL2404375L

8. D. Carlson, T. L. Markham, Schur complements of diagonally dominant matrices, *Czech. Math. J.*, **29** (1979), 246–251.
9. C. D. Meyer, Uncoupling the Perron eigenvector problem, *Linear Algebra Appl.*, **114** (1989), 69–94. https://doi.org/10.1016/0024-3795(89)90452-7
10. L. Z. Lu, Perron complement and Perron root, *Linear Algebra Appl.*, **341** (2002), 239–248. https://doi.org/10.1016/S0024-3795(01)00378-0
11. N. T. Binh, Smoothed lower order penalty function for constrained optimization problems, *IAENG Int. J. Appl. Math.*, **46** (2016), 76–81.
12. J. W. Cai, P. Chen, X. Mei, X. Ji, Realized range-based threshold estimation for jump-diffusion models, *IAENG Int. J. Appl. Math.*, **45** (2015), 293–299.
13. M. Neumann, Inverses of Perron complements of inverse M-matrices, *Linear Algebra Appl.*, **313** (2000), 163–171. https://doi.org/10.1016/S0024-3795(00)00128-2
14. S. M. Fallat, M. Neumann, On Perron complements of totally nonnegative matrices, *Linear Algebra Appl.*, **327** (2001), 85–94. https://doi.org/10.1016/S0024-3795(00)00312-8
15. S. W. Zhou, T. Z. Huang, On Perron complements of inverse N_0-matrices, *Linear Algebra Appl.*, **434** (2011), 2081–2088. https://doi.org/10.1016/j.laa.2010.12.004
16. Q. Zhong, C. Y. Zhao, Extended Perron complements of M-matrices, *AIMS Math.*, **8** (2023), 26372–26383. https://doi.org/10.3934/math.20231346
17. Z. G. Ren, T. Z. Huang, X. Y. Cheng, A note on generalized Perron complements of Z-matrices, *Electron. J. Linear Al.*, **15** (2006), 8–13. https://doi.org/10.13001/1081-3810.1217
18. G. X. Huang, F. Yin, K. Guo, The lower and upper bounds on Perron root of nonnegative irreducible matrices, *J. Comput. Appl. Math.*, **217** (2008), 259–267. https://doi.org/10.1016/j.cam.2007.06.034
19. S. M. Yang, T. Z. Huang, A note on estimates for the spectral radius of a nonnegative matrix, *Electron. J. Linear Algebra*, **13** (2005), 352–358. https://doi.org/10.13001/1081-3810.1168
20. Z. M. Yang, Some closer bounds of Perron root basing on generalized Perron complement, *J. Comput. Appl. Math.*, **235** (2010), 315–324. https://doi.org/10.1016/j.cam.2010.06.012
21. M. Adm, J. Garloff, Total nonnegativity of the extended Perron complement, *Linear Algebra Appl.*, **508** (2016), 214–224. https://doi.org/10.1016/j.laa.2016.07.002
22. H. Tanaka, Perturbed finite-state Markov systems with holes and Perron complements of Ruelle operators, *Isr. J. Math.*, **236** (2020), 91–131. https://doi.org/10.1007/s11856-020-1968-1
23. L. L. Wang, J. Z. Liu, S. Chu, Properties for the Perron complement of three known subclasses of H-matrices, *J. Inequal. Appl.*, **2015** (2015) , 1–10. https://doi.org/10.1186/s13660-014-0531-1
24. H. Diao, H. Liu, L. Tao, Stable determination of an impedance obstacle by a single far-field measurement, *Inverse Probl.*, **40** (2024), 055005. https://doi.org/10.1088/1361-6420/ad3087
25. T. B. Gan, T. Z. Huang, Simple criteria for nonsingular H-matrices, *Linear Algebra Appl.*, **374** (2003), 317–326. https://doi.org/10.1016/S0024-3795(03)00646-3

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