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Research article

Construction of marginally coupled designs with mixed-level qualitative factors

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Abstract: Marginally coupled designs (MCDs) with more economical run sizes than sliced Latin hypercube designs were widely used in computer experiments with both quantitative and qualitative factors. However, the construction of MCDs with mixed-level qualitative factors was still very challenging. We developed five algorithms to generate MCDs with mixed-level qualitative factors, which were very easy to implement. In some of the MCDs constructed in this paper, the quantitative factor designs have two- or higher-dimensional space-filling properties compared to the existing MCDs, where the qualitative factors were mixed-level. Moreover, the resulting MCDs had more flexible run sizes than the existing MCDs with mixed-level qualitative factors.

Keywords: resolvable orthogonal array; sliced Latin hypercube design; space-filling; stratification; difference scheme
Mathematics Subject Classification: 62K05, 62K99

1. Introduction

Computer experiments, as a widely used method in scientific research, simulate complex real-world problems through complex computer codes [1–3]. It is very important to plan computer experiments efficiently. Latin hypercube designs (LHDs) introduced by McKay et al. [4] are very suitable to plan computer experiments involving only quanlitative factors. Numerous methods have been proposed to construct LHDs with good properties, such as low-dimensional projection property, orthogonality, and other uniform criteria (such as uniform discrepancies, maximin distance, etc.). Computer experiments with both qualitative and quantitative factors have also received a lot of attention (see, for example, [5–9]). Sliced space-filling designs and sliced LHDs (SLHDs) are efficient choices when both quantitative and qualitative factors are included in computer experiments [10,11]. However, such two types of designs are inefficient due to the increase in the number of runs as the number of level combinations of the qualitative factors increases. To solve this problem, Deng et al. [12] first proposed marginally coupled design (MCD), which is more cost-effective in terms of the number of runs, and possesses excellent space-filling properties, i.e., in which the design for the quantitative factors is an LHD, and such quantitative factor design is also an SLHD with respect to each qualitative factor. Some researchers have worked on improving the low-dimensional stratification of the design for the quantitative factors in MCDs; see, among others, [13, 14] and [15]. Other researchers have constructed orthogonal MCDs in which the designs for the quantitative factors are orthogonal [16]. In order to improve the stratification between qualitative and quantitative factors, Yang et al. [17] proposed doubly coupled design (DCD) which has the following attractive space-filling properties: (1) the whole design is an MCD, and (2) the design points for the quantitative factors form an SLHD with respect to the level combinations of any two qualitative factors. In the above improved MCDs and DCDs, the designs for the qualitative factors are all equal-level orthogonal arrays (OAs). However, there exist qualitative factors being mixed-level in real-world problems, and in MCDs the designs of the qualitative factors are often mixed-level OAs. In this paper, we aim to construct MCDs in which the designs for qualitative factors are mixed-level OAs.

For an MCD (D_1, D_2) where D_1 and D_2 are the designs for qualitative and quantitative factors, respectively, Deng et al. [12] investigated the existence and construction of an MCD for mixed-level qualitative factors. They gave the existence of an MCD (D_1, D_2) with D_1 being an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$, $s_1 = \beta s_2$, in terms of the structure of D_1. The existence is somewhat limited by the restriction $s_1 = \beta s_2$. To overcome this limitation, we provide a necessary and sufficient condition on both D_1 and D_2 to ensure the existence of an MCD with D_1 being an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$, $s_1 = \beta s_2$, or $s_1 \neq \beta s_2$. Given a small initial MCD with mixed-level qualitative factors, a large MCD with mixed-level qualitative factors can be constructed by Construction 3 of Deng et al. [12]. However, Deng et al. [12] did not address the question of how to construct the initial MCDs. Fortunately, the MCDs constructed in this paper can be used as initial MCDs for Construction 3 of [12]. Therefore, for the MCDs obtained in this paper, the run sizes are more flexible than for the MCDs constructed in Construction 3 of Deng et al. [12]. Based on the existence result of [12], for $s_1 = \beta s_2$, we give an algorithm to construct MCDs for D_2 with a large number of columns. By the necessary and sufficient condition in this paper, two algorithms are proposed to construct MCDs with D_1 being an $OA\left(n, 2^{k_1} s^{k_2}, 2\right)$, $s = 2\beta$, or $s \neq 2\beta$. For the D_2 constructed by Construction 3 of Deng et al. [12], the D_2 only has stratification property in one-dimensional projections. To enhance the space-filling property of D_2, we present two algorithms to construct MCDs with D_2 possessing stratification properties in two-, three-, or four- dimensional projections.

The paper is organized as follows: Section 2 introduces the basic definitions and notation. Section 3 gives five methods for constructing MCDs with mixed-level qualitative factors. Section 4 provides the conclusions. All proofs are deferred to Appendix A. Some tables are listed in Appendix B.

2. Definitions and notation

Let $GF(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}$, $\alpha_0 = 0$, $\alpha_1 = 1$, denote a Galois field of order s, which is simplified as $GF(s) = \{0, 1, \ldots, s-1\}$ if s is a prime. An $n \times p$ matrix is called a Latin hypercube design of n runs and p factors, denoted by $LHD\,(n, p)$, if each of its columns is a random permutation of $\{0, 1, \ldots, n-1\}$. An $n \times k$ array A is said to be a mixed-level OA of strength 2, denoted by $OA\left(n, {s_1}^{k_1}{s_2}^{k_2}, 2\right)$, if any $n \times 2$ sub-array of A contains all possible level combinations with equal frequency, where the entries in the first k_1 columns and the last k_2 columns are taken from $\{0, 1, \ldots, s_1 - 1\}$ and $\{0, 1, \ldots, s_2 - 1\}$, respectively. When $s_1 = s_2 = s$ and $k_1 + k_2 = k$, the orthogonal array A is equal-level, denoted by $OA\left(n, s^k, 2\right)$. An $OA\left(s^t, s^k, 2\right)$ with $v = (s^t - 1)/(s - 1)$ can be constructed using the Rao-Hamming construction, the details of which are described in Section 3.4 of [18]. For a prime power s, let η_1 and η_2 be two s-level independent columns of length s^2, where the entries of both η_1 and η_2 are taken from $GF(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}$, $\alpha_0 = 0$, $\alpha_1 = 1$. We apply the Rao-Hamming construction to create an $OA\left(s^2, s^{s+1}, 2\right)$ Φ as

$$\Phi = \{\eta_1, \eta_1 + \eta_2, \eta_1 + \alpha_2\eta_2, \eta_1 + \alpha_3\eta_2, \cdots, \eta_1 + \alpha_{s-1}\eta_2, \eta_2\},$$

where the addition and multiplication operations are based on $GF(s)$. An $OA\left(n, s_1^{k_1}s_2^{k_2}, 2\right)$ A is said to be a $(\beta_1{\times}\beta_2)$-resolvable OA, denoted by $(\beta_1{\times}\beta_2)$-$ROA\left(n, s_1^{k_1}s_2^{k_2}, 2\right)$, if for $i = 1, 2$, its rows can be divided into $n/(\beta_i s_i)$ sub-arrays $A_1, \ldots, A_{n/(\beta_i s_i)}$ of $\beta_i s_i$ rows each, where A_i is an $OA\left(\beta_i s_i, {s_1}^{k_1}{s_2}^{k_2}, 1\right)$ for $i = 1, 2$. In particular, when $s_1{=}s_2{=}s, \beta_1{=}\beta_2{=}\beta$, and $k_1 + k_2 = k$, then the array reduces to β-$ROA\left(n, s^k, 2\right)$. If $\beta{=}1$, the array A is called a completely resolvable OA (CROA), denoted by $CROA\left(n, s^k, 2\right)$.

Suppose D_1 is an $OA\left(n, s_1^{k_1}s_2^{k_2}, 2\right)$ and D_2 is an $LHD\,(n, p)$, then the design $D = (D_1, D_2)$ is called a MCD, denoted by $MCD\left(n, s_1^{k_1}s_2^{k_2}, p\right)$, where D_1 and D_2 are sub-designs for qualitative factors and quantitative factors, respectively, if for every level of any factor of D_1, the corresponding rows in D_2 form a small LHD. When $s_1 = s_2 = s$ and $k_1 + k_2 = k$, the MCD is denoted by $MCD\left(n, s^k, p\right)$.

Let $\mathbf{1}_s$ be an s-dimensional column vector whose entries are all ones. An $u \times r$ matrix A with entries from $GF(s)$ is called a difference scheme of strength 2 based on $GF(s)$, denoted by $\mathrm{D}(u, r, s)$, if for all i and k with $1 \le i, k \le r$, $i \ne k$, the vector difference between the ith and kth columns contains each element of $GF(s)$ exactly u/s times. Throughout, $D(u, r, s)$ is a u-row, r-column, and s-level difference schme (of strength 2). For an $n{\times}m$ matrix X and an $s{\times}p$ matrix Y, their Kronecker sum and Kronecker product are defined as $X \oplus Y{=}\left(x_{ij} + Y\right)$ and $X \otimes Y = \left(x_{ij}Y\right)$, respectively, where x_{ij} is the (i, j)th entry of X. For a matrix $X = \left(x_{ij}\right)_{n\times m}$, define an $n \times m$ matrix $f\,(X, s)$ as

$$f\,(X, s) = \left(\left\lfloor \frac{x_{ij}}{s} \right\rfloor\right). \tag{2.1}$$

He et al. [13] demonstrated a necessary and sufficient condition for the design (D_1, D_2) being an $MCD\left(n, s^k, p\right)$, as stated in Lemma 1.

Lemma 1 ([13]). *Suppose D_1 is an $OA\left(n, s^k, 2\right)$ and D_2 is an $LHD\,(n, p)$. Let d_i be the ith column of D_2 for $i = 1, 2, \ldots, p$. Then, $D = (D_1, D_2)$ is an $MCD\left(n, s^k, p\right)$ if, and only if, for $i = 1, 2, \ldots, p$, the $(D_1, f(d_i, s))$ is an $OA\left(n, s^k(n/s), 2\right)$, where $f(*, \cdot)$ can be obtained from Equation* (2.1).

Lemma 2 given by Deng et al. [12] presents a necessary and sufficient condition for the existence of an $MCD\left(n, s_1^{k_1}s_2^{k_2}, p\right)$ with $s_1 = \beta s_2$.

Lemma 2 ([12]). *Given that D_1 is an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$, an $MCD\left(n, s_1^{k_1} s_2^{k_2}, p\right)$ $D = (D_1, D_2)$ exists if, and only if, D_1 is a $(1 \times \beta)$-ROA $\left(n, s_1^{k_1} s_2^{k_2}, p\right)$ that can be expressed as*

$$\begin{pmatrix} A_{11}^T & \cdots & A_{m1}^T \\ A_{12}^T & \cdots & A_{m2}^T \end{pmatrix}^T$$

such that (A_{i1}, A_{i2}) is an $OA\left(s_1, s_1^{k_1} s_2^{k_2}, 1\right)$, where $m = n/s_1$, and the A_{i2} is a $CROA\left(s_1, s_2^{k_2}, 2\right)$, for $i = 1, 2, \ldots, m$.

The necessary and sufficient condition given by Lemma 2 is rather restricted by the restriction $s_1 = \beta s_2$. Similar to Lemma 1, we provide directly a necessary and sufficient condition to break this restriction, as shown in the following lemma.

Lemma 3. *Suppose $D_1 = (\Omega, \Lambda)$ is an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$, where Ω and Λ are an $OA\left(n, s_1^{k_1}, 2\right)$ and an $OA\left(n, s_2^{k_2}, 2\right)$, respectively, and D_2 is an $LHD(n, p)$. Let d_i be the ith column of D_2 for $i = 1, 2 \ldots, p$. Then, D=(D_1, D_2) is an $MCD\left(n, s_1^{k_1} s_2^{k_2}, p\right)$ if, and only if, for $i = 1, 2, \ldots, p$, $(\Omega, f(d_i, s_1))$ and $(\Lambda, f(d_i, s_2))$ are an $OA\left(n, s_1^{k_1}(n/s_1), 2\right)$ and an $OA\left(n, s_2^{k_2}(n/s_2), 2\right)$, respectively, where $f(*, \cdot)$ can be obtained from Equation* (2.1).

3. Construction of MCDs

3.1. Construction of MCDs for D_1 being an $OA\left(N, s_1^{k_1} s_2^{k_2}, 2\right)$ with $N = s_1^2$ ($s_1 = \beta s_2$), $N = 2s$ ($s \geq 2$), $N = 2\lambda s^2$ ($s \geq 2$)

This section presents three construction algorithms to construct MCDs. First, we construct an MCD $D = (D_1, D_2)$ via an $OA\left(s_1^2, s_1^{k_1+2}, 2\right)$ and a $CROA\left(s_1, s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$. Motivated by Lemma 2, we present the following algorithm.

Algorithm 1 Construction of MCDs via an $OA\left(s_1^2, s_1^{k_1+2}, 2\right)$ and a $CROA\left(s_1, s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$

Step 1. For $s_1 = \beta s_2$, given an $OA\left(s_1^2, s_1^{k_1+2}, 2\right)$, denote it as G. Rearrange the rows of G into $G = (l_1, l_2, A)$, so that $l_1 = (0, 1, \cdots, s_1 - 1)^T \otimes \mathbf{1}_{s_1}$ and $l_2 = \mathbf{1}_{s_1} \otimes (0, 1, \cdots, s_1 - 1)^T$. Then A can be expressed as $A = \left(A_1^T, A_2^T, \cdots, A_{s_1}^T\right)^T$, where A_i is an $OA\left(s_1, s_1^{k_1}, 1\right)$ and $i = 1, 2, \ldots, s_1$.

Step 2. Given a $CROA\left(s_1, s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$, denoted as B. B can be expressed as $B = \left(B_1^T, B_2^T, \cdots, B_\beta^T\right)^T$, where B_i is an $OA\left(s_2, s_2^{k_2}, 1\right)$, $i = 1, 2, \ldots, \beta$. Let $B^* = 1_{s_1} \otimes B$.

Step 3. Construct an $s_1^2 \times (k_1 + k_2)$ matrix D_1 as D_1=(A, B^*).

Step 4. For $1 \leq i \leq p$, let $e_i = \mu_i \otimes 1_{s_1}$, where μ_i is a random permutation of $(0, 1, \ldots, s_1 - 1)^T$. Stack the columns of e_i for $1 \leq i \leq p$ together to obtain $E = \left(e_1, e_2, \ldots, e_p\right)$.

Step 5. For $1 \leq i \leq p$ and $1 \leq j \leq \beta, c_i = 1_{s_1} \otimes w_i$, where w_i is a random permutation of $\left(c_{i,1}^T, c_{i,2}^T, \cdots, c_{i,\beta}^T\right)^T$ with $c_{i,j} = (j-1)s_2\mathbf{1}_{s_2} + \tau$, where τ is a random permutation of $(0, 1, \cdots, s_2 - 1)^T$. Stack the columns of c_i for $1 \leq i \leq p$ together to obtain C, i.e., $C = \left(c_1, c_2, \ldots, c_p\right)$.

Step 6. Construct an $s_1^2 \times p$ matrix D_2 as $D_2 = s_1 E + C$.

Step 7. Let $D = (D_1, D_2)$.

Theorem 1. *For $D = (D_1, D_2)$ constructed by Algorithm 1, we have*

(i) $D_1 = (A, B^*)$ *is an* $OA\left(s_1^2, s_1^{k_1} s_2^{k_2}, 2\right)$ *with* $s_1 = \beta s_2$*;*

(ii) D_2 *is an* $LHD\left(s_1^2, p\right)$*;*

(iii) $D = (D_1, D_2)$ *is an* $MCD\left(s_1^2, s_1^{k_1} s_2^{k_2}, p\right)$.

If we take G in Step 1 of Algorithm 1 to be a regular saturated $OA\left(s_1^2, s_1^{s_1+1}, 2\right)$, then $D = (D_1, D_2)$ is an $MCD\left(s_1^2, s_1^{s_1-1} s_2^{k_2}, p\right)$; alternatively, the G can also be a non-regular s_1-level OA. Hence, s_1 may or may not be a prime power. Furthermore, if B is a $CROA\left(s_1, s_2^{s_2}, 2\right)$ and $s_1 = {s_2}^2$ in Step 2, then $D = (D_1, D_2)$ is an $MCD\left(s_1^2, s_1^{s_1-1} s_2^{s_2}, p\right)$.

In Theorem 1, the MCDs with at most $\beta! s_1! s_2!$ distinct quantitative columns can be constructed from Steps 4 and 5 of Algorithm 1. Thus, Theorem 1 provides the MCD with a large number of quantitative factors. Let $\eta = \beta! s_1! s_2!$, and there can be as many as $(\eta!)/(p!(\eta - p)!)$ different MCDs. Thus, an optimal D_2 under maximin distance criterion [19] (or the centered L_2-discrepancy criterion [20, 21]) can be found by ranking the $(\eta!)/(p!(\eta - p)!)$ candidate MCDs or via the simulated annealing [22] or the threshold accepting algorithms [23] when the number of candidate MCDs is very large.

Example 1. Applying the Rao-Hamming construction to generate an $OA\left(16, 4^5, 2\right)$ G, then the A is obtained and listed in Table 1. The $CROA\left(4, 2^2, 2\right)$ B is obtained as $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^T$, then $B^* = \left(B^T, B^T, B^T, B^T\right)^T$. Thus, D_1 is constructed as $D_1 = (A, B^*)$. Consider the case $p = 3$. In Step 4, μ_1, μ_2, and μ_3 are obtained as $\mu_1 = (0, 1, 2, 3)^T$, $\mu_2 = (1, 0, 2, 3)^T$, $\mu_3 = (1, 2, 0, 3)^T$, then E can be obtained and listed in Table 1. In Step 5, w_1, w_2, and w_3 are obtained as $w_1 = (0, 1, 2, 3)^T$ and $w_2 = w_3 = (2, 3, 0, 1)^T$, then C can be obtained and listed in Table 1. By the matrix operation of $s_1 E + C$ in Step 6, D_2 can be generated. It is easy to check that $D = (D_1, D_2)$ is an $MCD\left(16, 4^3 2^2, 3\right)$, which is provided in Table 2.

Table 1. Matrices A, E, and C in Example 1.

Run	A			E			C			*Run*	A			E			C		
1	0	0	0	0	1	1	0	2	2	9	2	3	1	2	2	0	0	2	2
2	1	1	1	0	1	1	1	3	3	10	3	2	0	2	2	0	1	3	3
3	2	2	2	0	1	1	2	0	0	11	0	1	3	2	2	0	2	0	0
4	3	3	3	0	1	1	3	1	1	12	1	0	2	2	2	0	3	1	1
5	1	2	3	1	0	2	0	2	2	13	3	1	2	3	3	3	0	2	2
6	0	3	2	1	0	2	1	3	3	14	2	0	3	3	3	3	1	3	3
7	3	0	1	1	0	2	2	0	0	15	1	3	0	3	3	3	2	0	0
8	2	1	0	1	0	2	3	1	1	16	0	2	1	3	3	3	3	1	1

Table 2. $D = (D_1, D_2)$ in Example 1.

Run	$MCD(D_1,D_2)$								*Run*	$MCD(D_1,D_2)$							
	D_1					D_2				D_1					D_2		
1	0	0	0	0	0	0	6	6	9	2	3	1	0	0	8	10	2
2	1	1	1	1	1	1	7	7	10	3	2	0	1	1	9	11	3
3	2	2	2	0	1	2	4	4	11	0	1	3	0	1	10	8	0
4	3	3	3	1	0	3	5	5	12	1	0	2	1	0	11	9	1
5	1	2	3	0	0	4	2	10	13	3	1	2	0	0	12	14	14
6	0	3	2	1	1	5	3	11	14	2	0	3	1	1	13	15	15
7	3	0	1	0	1	6	0	8	15	1	3	0	0	1	14	12	12
8	2	1	0	1	0	7	1	9	16	0	2	1	1	0	15	13	13

For the $MCD\left(s_1^2, s_1^{k_1}s_2^{k_2}, p\right)$ constructed by Algorithm 1, the s_1 can be a non-prime power. The following example provides an illustration of the non-prime power case in Algorithm 1.

Example 2. The $OA\left(144, 12^7, 2\right)$ G listed in Table 17 of Appendix B and the $CROA\left(12, 2^2, 2\right)$ B listed in Table 3 are obtained from the library of orthogonal arrays maintained by Sloane (http://neilsloane.com/oadir/index.html). Then, divide G into $G = (l_1, l_2, A)$ listed in Table 17 of Appendix B. Moreover, $B^* = \left(B^T, B^T, B^T, B^T, B^T, B^T\right)^T$ is obtained and listed in Table 18. Thus, D_1 is constructed as $D_1 = (A, B^*)$. Consider the case $p = 3$. In Step 4, μ_1, μ_2, and μ_3 are obtained, which are listed in Table 3, and then the E can be obtained, which is listed in Table 17 of Appendix B. In Step 5, w_1, w_2, and w_3 are obtained, which are listed in Table 3, then the C can be obtained and listed in Table 17 of Appendix B. By the matrix operation of $s_1E + C$ in Step 6, D_2 can be generated. It is easy to check that $D = (D_1, D_2)$ is an $MCD\left(144, 12^5 2^2, 3\right)$, which is provided in Table 18.

Table 3. Matrix B, vectors u_i and w_i in Example 2, where $i = 1, 2, 3$.

Run	B		u_1	u_2	u_3	w_1	w_2	w_3	*Run*	B		u_1	u_2	u_3	w_1	w_2	w_3	*Run*	B		u_1	u_2	u_3	w_1	w_2	w_3
1	1	1	0	1	1	0	2	0	5	1	1	4	4	4	4	4	8	9	0	1	8	8	8	8	8	4
2	0	0	1	0	0	1	3	1	6	0	0	5	5	5	5	5	9	10	1	0	9	9	9	9	9	5
3	0	0	2	2	2	2	0	2	7	0	1	6	6	6	6	6	10	11	1	0	10	10	10	10	10	6
4	1	1	3	3	3	3	1	3	8	1	0	7	7	7	7	7	11	12	0	1	11	11	11	11	11	7

Algorithm 1 can produce some MCDs based on the above Theorem 1, as shown in Table 4.

Table 4. Some MCDs from Algorithm 1.

D_1	D_2	MCDs
$OA\left(16, 4^3 2^2, 2\right)$	$LHD(16, 2!4!2!)$	$MCD\left(16, 4^3 2^2, 2!4!2!\right)$
$OA\left(36, 6^1 2^2, 2\right)$	$LHD(36, 3!6!2!)$	$MCD\left(36, 6^1 2^2, 3!6!2!\right)$
$OA\left(64, 8^7 2^2, 2\right)$	$LHD(64, 4!8!2!)$	$MCD\left(64, 8^7 2^2, 4!8!2!\right)$
$OA\left(36, 6^1 3^3, 2\right)$	$LHD(36, 2!6!3!)$	$MCD\left(36, 6^1 3^3, 2!6!3!\right)$
$OA\left(81, 9^7 3^3, 2\right)$	$LHD(81, 3!9!3!)$	$MCD\left(81, 9^7 3^3, 3!9!3!\right)$
$OA\left(144, 12^1 3^3, 2\right)$	$LHD(144, 4!12!3!)$	$MCD\left(144, 12^1 3^3, 4!12!3!\right)$
$OA\left(64, 8^7 4^4, 2\right)$	$LHD(64, 2!8!4!)$	$MCD\left(64, 8^7 4^4, 2!8!4!\right)$
$OA\left(144, 12^1 4^4, 2\right)$	$LHD(144, 3!12!4!)$	$MCD\left(144, 12^1 4^4, 3!12!4!\right)$
$OA\left(256, 16^{15} 4^4, 2\right)$	$LHD(256, 4!16!4!)$	$MCD\left(256, 16^{15} 4^4, 4!16!4!\right)$

To employ Algorithm 1, we need several CROAs. Theorem 3 of He et al. [13] gives four types of CROAs, as shown in Lemma 4.

Lemma 4 ([13]). *For a prime h and three positive integers k, t ($t \geq 2$), and w, if $s = h^k$, the following four CROAs exist:* (i) $CROA\left(s^t, s^{s^{t-1}}, 2\right)$; (ii) $CROA\left(2s^t, s^{2s^{t-1}}, 2\right)$; (iii) $CROA\left(4s^t, s^{4s^{t-1}}, 2\right)$; and (iv) $CROA\left(h^w s^2, s^{h^w s}, 2\right)$.

According to Theorem 1 and Lemma 4, we can obtain a wealth of MCDs for mixed-level qualitative factors as follows.

Corollary 1. *For a prime h and three positive integers k, t ($t \geq 2$), and w, let $s = h^k$, then by Algorithm 1,*
(i) *if an $OA\left(s^{2t}, (s^t)^{k_1+2}, 2\right)$ exists, an $MCD\left(s^{2t}, (s^t)^{k_1}(s)^{s^{t-1}}, p\right)$ can be obtained;*
(ii) *if an $OA\left(4s^{2t}, (2s^t)^{k_1+2}, 2\right)$ exists, an $MCD\left(4s^{2t}, (2s^t)^{k_1}(s)^{2s^{t-1}}, p\right)$ can be obtained;*
(iii) *if an $OA\left(16s^{2t}, (4s^t)^{k_1+2}, 2\right)$ exists, an $MCD\left(16s^{2t}, (4s^t)^{k_1}(s)^{4s^{t-1}}, p\right)$ can be obtained;*
(iv) *if an $OA\left(h^{2w}s^4, (h^w s^2)^{k_1+2}, 2\right)$ exists, an $MCD\left(h^{2w}s^4, (h^w s^2)^{k_1}(s)^{h^w s}, p\right)$ can be obtained.*

If there exists a small initial MCD for mixed-level qualitative factors, then a series of large MCDs for mixed-level qualitative factors can be constructed by Construction 3 of Deng et al. [12], as shown in Lemma 5.

Lemma 5 ([12]). *Let $D_1^{(0)} = (\Phi, \Psi)$ and $D_2^{(0)}$ be an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$ and an $LHD(n, p)$, respectively, where Φ and Ψ are an $OA\left(n, s_1^{k_1}, 2\right)$ and an $OA\left(n, s_2^{k_2}, 2\right)$, respectively. For some u, there are two difference schemes $D(u, r_1, s_1)$ and $D(u, r_2, s_2)$ (of strength 2), denoted by $D(i)$ for $i = 1, 2$, respectively. Let $C = \left(c_{ij}\right)$ be an $u \times f$ matrix with $c_{ij} = 1$ and H be an $LHD(u, pf)$. Construct $D_1 = (D(1) \oplus \Phi, D(2) \oplus \Psi)$ and $D_2 = C \otimes D_2^{(0)} + nH \otimes \mathbf{1}_n$. If $D^{(0)} = \left(D_1^{(0)}, D_2^{(0)}\right)$ is an MCD, then $D = (D_1, D_2)$ is also an MCD, where D_1 and D_2 are an $OA\left(nu, s_1^{k_1 r_1} s_2^{k_2 r_2}, 2\right)$ and an $LHD(nu, pf)$, respectively.*

The key to constructing MCDs, $D = (D_1, D_2)$, using Lemma 5 is the existence of the initial MCD $D^{(0)} = \left(D_1^{(0)}, D_2^{(0)}\right)$. However, the construction method of $D^{(0)} = \left(D_1^{(0)}, D_2^{(0)}\right)$ is not mentioned in [12]. Excitingly, the MCDs obtained by Theorem 1 can be used as the initial MCDs. Based on Lemma 5, a large number of MCDs with more columns can be constructed from the initial MCDs obtained by Theorem 1 as follows.

Corollary 2. *For $D = (D_1, D_2)$ constructed by Algorithm 1 and Theorem 1, if there exist two difference schemes $D(u, r_1, s_1)$ and $D(u, r_2, s_2)$ (of strength 2) for some u, then for any integer f, an $MCD\left(us_1^2, (s_1)^{k_1 r_1}(s_2)^{k_2 r_2}, pf\right)$ can be obtained by Lemma 5.*

Based on Algorithm 1, Theorem 1 and Corollary 2 can generate a series of MCDs with D_1 being an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$, but they can be criticized for the $s_1 = \beta s_2$ restriction. However, when $s_1 \neq \beta s_2$, an MCD also exists, as in the following example.

Example 3. Given D_1 is an $OA\left(6, 2^1 3^1, 2\right)$ and D_2 is an $LHD(6, 6)$ as listed in Table 5, it is easy to verify that $D = (D_1, D_2)$ is an $MCD\left(6, 2^1 3^1, 6\right)$ according to Lemma 3.

Table 5. $D = (D_1, D_2)$ in Example 3.

Run	$MCD(D_1, D_2)$								*Run*	$MCD(D_1, D_2)$							
	D_1		D_2							D_1		D_2					
1	0	0	0	0	2	2	4	4	4	1	0	5	5	3	3	1	1
2	0	1	2	4	0	4	0	2	5	1	1	3	1	5	1	5	3
3	0	2	4	2	4	0	2	0	6	1	2	1	3	1	5	3	5

Obviously, the $MCD\left(6, 2^1 3^1, 6\right)$ listed in Table 5 cannot be constructed by Algorithm 1. Next, we propose a new algorithm for constructing $MCDs\left(2s, 2^1 s^1, s!\right)$.

Algorithm 2 Construction of MCDs based on $OA\left(2s, 2^1 s^1, 2\right)$

Step 1. Let $L_1 = (0, 1) \otimes \mathbf{1}_s$ and $L_2 = \mathbf{1}_2 \otimes e$, where $e = (0, 1, \dots, s-1)^T$. Obtain a $(2s) \times 2$ matrix $D_1 = (L_1, L_2)$.

Step 2. For $1 \leq i \leq s!$, $d_i = ((2u_i)^T, ((2s-1)\mathbf{1}_s - 2u_i)^T)^T$, where u_i is a random permutation of $(0, 1, 2, \dots, s-1)^T$, let $D_2 = (d_1, d_2, \dots, d_{s!})$.

Step 3. The resulting design is $D = (D_1, D_2)$.

Theorem 2. *The design $D = (D_1, D_2)$ constructed by Algorithm 2 is an $MCD\left(2s, 2^1 s^1, s!\right)$, where D_1 is an $OA\left(2s, 2^1 s^1, 2\right)$ and D_2 is an $LHD\,(2s, s!)$.*

If $p < s!$, there can be as many as $(s!)/(p!(s-p)!)$ different MCDs from Algorithm 2. Similar to Algorithm 1, an optimal D_2 under the maximin distance criterion or the centered L_2-discrepancy criterion can be obtained Hickernell [19, 20]. Next, we provide an example to illustrate Algorithm 2 and Theorem 2.

Example 4. Let $s = 4$, and an 8×2 matrix $D_1 = (L_1, L_2)$ is obtained from Step 1, as shown in Table 6. For $1 \leq i \leq 24$, $D_2 = (d_1, d_2, \cdots, d_{24})$ is constructed according to Step 2, as shown in Table 6. It is easy to verify that $D = (D_1, D_2)$ is an $MCD\left(8, 2^1 4^1, 24\right)$ from Lemma 3, which is provided in Table 6.

Table 6. $D = (D_1, D_2)$ in Example 4.

Run	$MCD(D_1, D_2)$																									
	D_1		D_2																							
1	0	0	0	0	0	0	0	0	2	2	2	2	2	2	4	4	4	4	4	4	6	6	6	6	6	6
2	0	1	2	2	4	4	6	6	0	0	4	4	6	6	0	0	2	2	6	6	0	0	2	2	4	4
3	0	2	4	6	2	6	2	4	4	6	0	6	0	4	2	6	0	6	0	2	2	4	0	4	0	2
4	0	3	6	4	6	2	4	2	6	4	6	0	4	0	6	2	6	0	2	0	4	2	4	0	2	0
5	1	0	7	7	7	7	7	7	5	5	5	5	5	5	3	3	3	3	3	3	1	1	1	1	1	1
6	1	1	5	5	3	3	1	1	7	7	3	3	1	1	7	7	5	5	1	1	7	7	5	5	3	3
7	1	2	3	1	5	1	5	3	3	1	7	1	7	3	5	1	7	1	7	5	5	3	7	3	7	5
8	1	3	1	3	1	5	3	5	1	3	1	7	3	7	1	5	1	7	5	7	3	5	3	7	5	7

Algorithm 2 can produce some MCDs based on the above Theorem 2, as shown in Table 7.

Table 7. Some MCDs from Algorithm 2.

D_1	D_2	MCDs
$OA\left(6,2^13^1,2\right)$	$LHD\,(6,3!)$	$MCD\left(6,2^13^1,3!\right)$
$OA\left(8,2^14^1,2\right)$	$LHD\,(8,4!)$	$MCD\left(8,2^14^1,4!\right)$
$OA\left(10,2^15^1,2\right)$	$LHD\,(10,5!)$	$MCD\left(10,2^15^1,5!\right)$
$OA\left(12,2^16^1,2\right)$	$LHD\,(12,6!)$	$MCD\left(12,2^16^1,6!\right)$
$OA\left(14,2^17^1,2\right)$	$LHD\,(14,7!)$	$MCD\left(14,2^17^1,7!\right)$
$OA\left(16,2^18^1,2\right)$	$LHD\,(16,8!)$	$MCD\left(16,2^18^1,8!\right)$
$OA\left(18,2^19^1,2\right)$	$LHD\,(18,9!)$	$MCD\left(18,2^19^1,9!\right)$
$OA\left(20,2^110^1,2\right)$	$LHD\,(20,10!)$	$MCD\left(20,2^110^1,10!\right)$
$OA\left(22,2^111^1,2\right)$	$LHD\,(22,11!)$	$MCD\left(22,2^111^1,11!\right)$

In Lemma 5, the MCDs constructed by Theorem 2 can also be used as the initial MCDs for Construction 3 of [12]. Based on Lemma 5, a large number of MCDs with D_1 being an $OA\,(2us, 2^{r_1}s^{r_2}, 2)$ can be obtained from the initial MCDs constructed by Theorem 2 as follows.

Corollary 3. *For $D = (D_1, D_2)$ constructed by Algorithm 2 and Theorem 2, if there exist two difference schemes $D\,(u, r_1, 2)$ and $D\,(u, r_2, s)$ (of strength 2) for some u, then for any integer f, an $MCD\,(2us, 2^{r_1}s^{r_2}, pf)$ can be obtained by Lemma 5.*

In the MCD (D_1, D_2) constructed by Algorithm 2 and Theorem 2, the D_1 has only two columns. In order to construct D_1 that can accommodate more qualitative factors, we present Algorithm 3 as follows.

Algorithm 3 Construction of MCDs via $MCD\,(n, s^m, p)$

Step 1. Given an $OA\,(n, s^m, 2)$ and $LHD\,(n, p)$, denoted as $D_1^{(0)}$ and $D_2^{(0)}$, respectively.
Step 2. Let $L_1 = (0, 1)^T \otimes \mathbf{1}_n$ and $L_2 = \mathbf{1}_2 \otimes D_1^{(0)}$. Obtain a $(2n) \times (m + 1)$ matrix $D_1 = (L_1, L_2)$.
Step 3. Construct a $(2n) \times p$ matrix D_2 as $D_2 = \left(\left(2D_2^{(0)}\right)^T, \left((2n-1)\mathbf{1}_n - 2D_2^{(0)}\right)^T\right)^T$.
Step 4. The resulting design is $D = (D_1, D_2)$.

Theorem 3. *For $D_1^{(0)}$ and $D_2^{(0)}$ in Algorithm 3, if $D^{(0)} = \left(D_1^{(0)}, D_2^{(0)}\right)$ is an $MCD\,(n, s^m, p)$, then the design $D = (D_1, D_2)$ constructed by Algorithm 3 is an $MCD\left(2n, 2^1s^m, p\right)$, where D_1 is an $OA\left(2n, 2^1s^m, 2\right)$, and D_2 is an $LHD\,(2n, p)$.*

Remark 1. *Note that Algorithm 2 and Algorithm 3 can construct MCDs with D_1 being an $OA\left(N, 2^1s^k, 2\right)$, $s = 2\beta$, or $s \neq 2\beta$, but the values of N in the two Algorithms are different. Algorithm 2 works for $k = 1$ and $N = 2s$, while Algorithm 3 works for $k \geq 2$ and $N = 2\lambda s^2$, where λ is a positive integer. Thus, Algorithm 3 is able to construct MCDs with more columns in D_1 than Algorithm 2, and Algorithm 2 is not a special case of Algorithm 3. For example, for $s = 3$, Algorithm 2 constructs an $MCD\left(6, 2^13^1, 6\right)$, where D_1 and D_2 are an $OA\left(6, 2^13^1, 2\right)$ and an $LHD\,(6, 6)$, respectively, while Algorithm 3 constructs an $MCD\left(18, 2^13^2, 2\right)$, where D_1 and D_2 are an $OA\left(18, 2^13^2, 2\right)$ and an*

$LHD(18, 2)$, respectively. This shows that Algorithm 2 and Algorithm 3 cannot be replaced by each other.

Next, we provide an example to illustrate Algorithm 3 and Theorem 3.

Example 5. Table 8 gives an $MCD\left(9, 3^2, 2\right)$ $D^{(0)} = \left(D_1^{(0)}, D_2^{(0)}\right)$, where $D_1^{(0)}$ is an $OA\left(9, 3^2, 2\right)$ and $D_2^{(0)}$ is an $LHD(9, 2)$. Then, an 18×3 matrix $D_1 = (L_1, L_2)$ is obtained by the operations $L_1 = (0, 1)^T \otimes \mathbf{1}_n$ and $L_2 = \mathbf{1}_2 \otimes D_1^{(0)}$ in Step 2, as shown in Table 9. An 18×2 matrix D_2 is obtained by the operations $D_2 = \left(\left(2D_2^{(0)}\right)^T, \left((2n-1)\mathbf{1}_n - 2D_2^{(0)}\right)^T\right)^T$ in Step 3, as shown in Table 9. It is easy to verify that $D = (D_1, D_2)$ is an $MCD\left(18, 2^1 3^2, 2\right)$ from Lemma 3, which is provided in Table 9.

Table 8. $D^{(0)} = \left(D_1^{(0)}, D_2^{(0)}\right)$ in Example 5.

Run	$MCD(D_1^{(0)}, D_2^{(0)})$				*Run*	$MCD(D_1^{(0)}, D_2^{(0)})$				*Run*	$MCD(D_1^{(0)}, D_2^{(0)})$			
	D_1		D_2			D_1		D_2			D_1		D_2	
1	0	0	0	2	4	1	0	4	4	7	2	0	8	7
2	0	1	3	8	5	1	1	7	0	8	2	1	2	3
3	0	2	6	5	6	1	2	1	6	9	2	2	5	1

Table 9. $D = (D_1, D_2)$ in Example 5.

Run	$MCD(D_1, D_2)$					*Run*	$MCD(D_1, D_2)$					*Run*	$MCD(D_1, D_2)$				
	D_1			D_2			D_1			D_2			D_1			D_2	
1	0	0	0	0	4	7	0	2	0	16	14	13	1	1	0	9	9
2	0	0	1	6	16	8	0	2	1	4	6	14	1	1	1	3	17
3	0	0	2	12	10	9	0	2	2	10	2	15	1	1	2	15	5
4	0	1	0	8	8	10	1	0	0	17	13	16	1	2	0	1	3
5	0	1	1	14	0	11	1	0	1	11	1	17	1	2	1	13	11
6	0	1	2	2	12	12	1	0	2	5	7	18	1	2	2	7	15

Algorithm 3 can produce some MCDs based on the above Theorem 3, as shown in Table 10.

Table 10. Some MCDs from Algorithm 3.

	$MCD\left(D_1^{(0)}, D_2^{(0)}\right)$		$MCD(D_1, D_2)$		
Source	$D_1^{(0)}$	$D_2^{(0)}$	D_1	D_2	MCDs
Table 5	$OA\left(9, 3^2, 2\right)$	$LHD(9, 2)$	$OA\left(18, 2^1 3^2, 2\right)$	$LHD(18, 2)$	$MCD\left(18, 2^1 3^2, 2\right)$
Table B1	$OA\left(27, 3^9, 2\right)$	$LHD(27, 4)$	$OA\left(54, 2^1 3^9, 2\right)$	$LHD(54, 4)$	$MCD\left(54, 2^1 3^9, 4\right)$
	$OA\left(32, 4^8, 2\right)$	$LHD(32, 7)$	$OA\left(64, 2^1 4^8, 2\right)$	$LHD(64, 7)$	$MCD\left(64, 2^1 4^8, 7\right)$
	$OA\left(32, 4^8, 2\right)$	$LHD(32, 7)$	$OA\left(64, 2^1 4^8, 2\right)$	$LHD(64, 7)$	$MCD\left(64, 2^1 4^8, 7\right)$
	$OA\left(100, 5^{20}, 2\right)$	$LHD(200, 19)$	$OA\left(100, 2^1 5^{20}, 2\right)$	$LHD(200, 19)$	$MCD\left(200, 2^1 5^{20}, 19\right)$
Example 2	$OA\left(49, 7^5, 2\right)$	$LHD(49, 3)$	$OA\left(98, 2^1 7^5, 2\right)$	$LHD(98, 3)$	$MCD\left(98, 2^1 7^5, 3\right)$
	$OA\left(64, 8^7, 2\right)$	$LHD(64, 2)$	$OA\left(128, 2^1 8^7, 2\right)$	$LHD(128, 2)$	$MCD\left(128, 2^1 8^7, 2\right)$
	$OA\left(81, 9^8, 2\right)$	$LHD(81, 2)$	$OA\left(162, 2^1 9^8, 2\right)$	$LHD(162, 2)$	$MCD\left(162, 2^1 9^8, 2\right)$

[1] Table 5, Table B1 and Example 2 come from [15], [13] and [12], respectively.

Similar to Corollary 2 and Corollary 3, we can obtain the following Corollary 4 for the initial MCDs constructed by Algorithm 3 and Theorem 3.

Corollary 4. *For $D = (D_1, D_2)$ constructed by Algorithm 3 and Theorem 3, if there exist two difference schemes $D(u, r_1, 2)$ and $D(u, r_2, s)$ (of strength 2) for some u, then for any integer f, an $MCD(2un, 2^{r_1}s^{r_2m}, pf)$ can be obtained by Lemma 5.*

Table 11 presents some designs D_1 for mixed-level qualitative factors in MCDs constructed via Algorithms 1, 2, and 3. In the fourth column of Table 11, the D_1's are obtained by Construction 3 of Deng et al. [12] from the initial designs listed in the first three columns.

Table 11. Some designs D_1 constructed by different algorithms.

Algorithm 1	Algorithm 2	Algorithm 3	Corollaries	
D_1	D_1	D_1	D_1	source
$OA(16, 4^3 2^2, 2)$	$OA(6, 2^1 3^1, 2)$	$OA(18, 2^1 3^3, 2)$	$OA(64, 4^{12} 2^8, 2)$	corollary 2
$OA(36, 6^1 2^2, 2)$	$OA(8, 2^1 4^1, 2)$	$OA(32, 2^1 4^4, 2)$	$OA(512, 8^{56} 2^{16}, 2)$	corollary 2
$OA(64, 8^7 2^2, 2)$	$OA(10, 2^1 5^1, 2)$	$OA(50, 2^1 5^5, 2)$	$OA(729, 9^{63} 3^{15}, 2)$	corollary 2
$OA(36, 6^1 3^3, 2)$	$OA(12, 2^1 6^1, 2)$	$OA(72, 2^1 6^2, 2)$	$OA(36, 2^2 3^3, 2)$	corollary 3
$OA(81, 9^7 3^3, 2)$	$OA(14, 2^1 7^1, 2)$	$OA(98, 2^1 7^7, 2)$	$OA(32, 2^4 4^4, 2)$	corollary 3
$OA(144, 12^1 3^3, 2)$	$OA(16, 2^1 8^1, 2)$	$OA(128, 2^1 8^8, 2)$	$OA(64, 2^8 4^4, 2)$	corollary 3
$OA(64, 8^7 4^4, 2)$	$OA(18, 2^1 9^1, 2)$	$OA(162, 2^1 9^9, 2)$	$OA(108, 2^2 3^3, 2)$	corollary 4
$OA(144, 12^1 4^4, 2)$	$OA(20, 2^1 10^1, 2)$	$OA(200, 2^1 10^2, 2)$	$OA(128, 2^4 4^{16}, 2)$	corollary 4
$OA(256, 16^{15} 4^4, 2)$	$OA(22, 2^1 11^1, 2)$	$OA(242, 2^1 11^{11}, 2)$	$OA(500, 2^2 5^{25}, 2)$	corollary 4

For the $MCD(s_1^2, s_1^{k_1} s_2^{k_2}, p)$ constructed by Algorithm 1, the relation $s_1 = \beta s_2$ is indispensable. When $s = 2\beta$, the $MCD(s^2, 2^2 s^{s-1}, p)$, $MCD(2s, 2^1 s^1, s!)$, and $MCD(2\lambda s^2, 2^1 s^m, p)$ $(\lambda \geq 1)$ can be constructed by Algorithms 1, 2, and 3, respectively. Clearly, the three MCDs have different numbers of run sizes. For $s \neq 2\beta$, the $MCD(2s, 2^1 s^1, s!)$ and $MCD(2\lambda s^2, 2^1 s^m, p)$ $(\lambda \geq 1)$ can also be obtained using Algorithms 2 and 3, respectively. Algorithm 2 is not a special case of Algorithm 3 due to the different number of run sizes for the constructed MCDs.

3.2. Construction of MCDs for D_1 being an $OA(s_1^2, s_1^{k_1} s_2^{k_2}, 2)$ with $s_1 = s_2^2$ and D_2 with the better space-filling property

In the MCDs (D_1, D_2) constructed by the above three algorithms, the space-filling property of D_2 is not considered. The space-filling property is very important for the quantitative factor design D_2. In this section, we introduce another algorithm to construct MCDs $D = (D_1, D_2)$ for D_2 with the better space-filling property.

Theorem 4. *For $s_1 = s_2^2$, D_1, and D_2 obtained in Algorithm 4, we have*

(i) *D_1 is an $OA(s_1^2, s_1^{s_1-1} s_2^{s_2}, 2)$;*

(ii) *D_2 is an $LHD(s_1^2, 2k)$, where if s_2 is odd, $k = (s_2 + 1)/2$; if s_2 is even, $k = s_2/2$;*

(iii) *(D_1, D_2) is an $MCD(s_1^2, s_1^{s_1-1} s_2^{s_2}, 2k)$;*

(iv) *any two distinct columns of D_2 achieve $s_2 \times s_2$ grids stratification.*

Algorithm 4 Construction of MCDs with the better space-filling property

Step 1. For $s_1 = s_2^2$, given an $OA\left(s_1^2, s_1^{s_1+1}, 2\right)$ F and an $OA\left(s_2^2, s_2^{s_2+1}, 2\right)$ H. Divide F as $F = (F_0, f_1, f_2)$, where F_0 is the first $s_1 - 1$ columns of F and f_1 and f_2 are the s_1th column and the $(s_1 + 1)$th column of F, respectively.

Step 2. Obtain an $s_1^2 \times (s_2 + 1)$ matrix U by replacing the levels $0, 1, \ldots, (s_1 - 1)$ of the f_1 with the 1st, 2nd, …, and the s_1th row of the H, respectively. Then partition U as $U = (U_0, u_1, u_2)$, where U_0 is the first $s_2 - 1$ columns of U and u_1 and u_2 are the s_2th column and the $(s_2 + 1)$th column of U, respectively.

Step 3. If s_2 is odd, let $H^* = H$ and $k = (s_2 + 1)/2$. If s_2 is even, let H^* be the first s_2 columns of H, $k = s_2/2$. Then, H^* is an $OA\left(s_2^2, s_2^{2k}, 2\right)$.

Step 4. Obtain an $s_1^2 \times (2k)$ matrix V by replacing the levels $0, 1, \ldots, (s_1 - 1)$ of the f_2 with the 1st, 2nd, …, and the s_1th row of the H^*, respectively. Denote V as $V = (v_1, v_2, \ldots, v_{2k})$, where v_i is the ith column of V for $i = 1, 2, \ldots, 2k$.

Step 5. Construct D_1 as $D_1 = (F_0, U_0, u_1)$.

Step 6. Let $W_1 = V$, $W_2 = (v_2, v_1, v_4, v_3, \ldots, v_{2k}, v_{2k-1})$, $W_3 = (u_2, u_2, \ldots, u_2)$, $W_4 = (u_1, u_1, \ldots, u_1)$. Construct D_2 as $D_2 = s_2^3 W_1 + s_2^2 W_2 + s_2 W_3 + W_4$.

Theorem 4 (iv) tells us that D_2 has two-dimensional projection property without considering D_1. For each level of any factor in D_1, and for each level combination of any two factors in some columns of D_1, the corresponding rows in D_2 can also achieve the two-dimensional space-filling property, as stated in the following corollary.

Corollary 5. *For $D = (D_1, D_2)$ ($D_1 = (F_0, U_0, u_1)$) constructed by Algorithm 4 and Theorem 4, we have*

(i) *the rows in D_2 corresponding to each level of any factor in D_1 can achieve stratification on the $s_2 \times s_2$ grids in any two-dimensional projection;*

(ii) *the rows in D_2 corresponding to each level combination of any two factors in (U_0, u_1) can achieve stratification on the $s_2 \times s_2$ grids in any two-dimensional projection.*

Next, we provide an example to illustrate Algorithm 4 and Theorem 4 .

Example 6. Consider the case $s_1 = 4$ and $s_2 = 2$. An $OA\left(16, 4^5, 2\right)$ F and an $OA\left(4, 2^3, 2\right)$ H are obtained from the Rao-Hamming construction. Divide F as $F = (F_0, f_1, f_2)$ listed in Table 12. For the H listed in Table 12, we obtain an 16×3 matrix U by replacing the levels $0, 1, 2, 3$ of the f_1 with the 1st, 2nd, 3rd, and the 4th row of the H, respectively. Then partition U as $U = (U_0, u_1, u_2)$ listed in Table 12. In Step 3 and Step 4, H^* is the first 2 columns of H and $k = s_2/2 = 1$, after replacing the levels $0, 1, 2, 3$ of the f_2 by the 1st, 2nd, 3rd, and the 4th row of the H^*, respectively. Then, V is obtained, and denote V as $V = (v_1, v_2)$ listed in Table 12.

From Step 5, $D_1 = (F_0, U_0, u_1)$, and it is easy to check that D_1 is an $OA\left(16, 4^3 2^2, 2\right)$. In Step 6, let $W_1 = V$, $W_2 = (v_2, v_1)$, $W_3 = (u_2, u_2)$, $W_4 = (u_1, u_1)$, then by matrix operation of $s_2^3 W_1 + s_2^2 W_2 + s_2 W_3 + W_4$, D_2 can be generated. It is easy to verify that (D_1, D_2) is an $MCD\left(16, 4^3 2^2, 2\right)$, which is provided in Table 13.

Table 12. Matrices H, F, U, and V in Example 6.

Run	H			F: F_1			f_1	f_2	U: U_0	u_1	u_2	V: v_1	v_2	*Run*	F: F_1			f_1	f_2	U: U_0	u_1	u_2	V: v_1	v_2
1	0	0	0	0	0	0	0	0	0	0	0	0	0	9	0	2	3	1	2	0	1	1	1	0
2	0	1	1	1	1	1	1	0	0	1	1	0	0	10	1	3	2	0	2	0	0	0	1	0
3	1	0	1	2	2	2	2	0	1	0	1	0	0	11	2	0	1	3	2	1	1	0	1	0
4	1	1	0	3	3	3	3	0	1	1	0	0	0	12	3	1	0	2	2	1	0	1	1	0
5				0	1	2	3	1	1	1	0	0	1	13	0	3	1	2	3	1	0	1	1	1
6				1	0	3	2	1	1	0	1	0	1	14	1	2	0	3	3	1	1	0	1	1
7				2	3	0	1	1	0	1	1	0	1	15	2	1	3	0	3	0	0	0	1	1
8				3	2	1	0	1	0	0	0	0	1	16	3	0	2	1	3	0	1	1	1	1

Table 13. $D = (D_1, D_2)$ in Example 6.

Run	$MCD(D_1, D_2)$: D_1					D_2		*Run*	$MCD(D_1, D_2)$: D_1					D_2	
1	0	0	0	0	0	0	0	9	0	2	3	0	1	11	7
2	1	1	1	0	1	3	3	10	1	3	2	0	0	8	4
3	2	2	2	1	0	2	2	11	2	0	1	1	1	9	5
4	3	3	3	1	1	1	1	12	3	1	0	1	0	10	6
5	0	1	2	1	1	5	9	13	0	3	1	1	0	14	14
6	1	0	3	1	0	6	10	14	1	2	0	1	1	13	13
7	2	3	0	0	1	7	11	15	2	1	3	0	0	12	12
8	3	2	1	0	0	4	8	16	3	0	2	0	1	15	15

Next, let $D_2 = (d_1, d_2)$. It is easy to see that d_1 and d_2 achieve stratification on 2×2 grids, as shown in Figure 1.

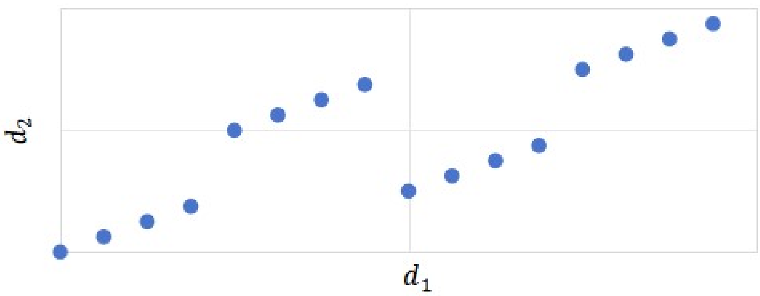

Figure 1. Stratification on 2×2 grids.

Algorithm 4 can produce some MCDs based on the above Theorem 4, as shown in Table 14.

Table 14. Some MCDs from Algorithm 4.

D_1	D_2	MCDs
$OA\left(16, 4^3 2^2, 2\right)$	$LHD\,(6, 2)$	$MCD\left(6, 4^3 2^2, 2\right)$
$OA\left(81, 9^8 3^3, 2\right)$	$LHD\,(8, 4)$	$MCD\left(8, 9^8 3^3, 4\right)$
$OA\left(256, 16^{15} 4^4, 2\right)$	$LHD\,(10, 4)$	$MCD\left(10, 16^{15} 4^4, 4\right)$
$OA\left(625, 25^{24} 5^5, 2\right)$	$LHD\,(12, 6)$	$MCD\left(12, 25^{24} 5^5, 6\right)$
$OA\left(1296, 36^{35} 6^6, 2\right)$	$LHD\,(14, 6)$	$MCD\left(14, 36^{35} 6^6, 6\right)$

Next, we introduce the following algorithm to generate MCDs by modifying Algorithm 4, that is, we rearrange the columns of F in Algorithm 4 and apply the idea of Step 4 in Algorithm 4 twice.

Algorithm 5 Modifying construction of MCDs

Step 1. For two OAs F and H in Algorithm 4, divide F as $F = (F_0{}^*, f_0, f_1, f_2)$, where $F_0{}^*$ is the first $s_1 - 2$ columns of F, and f_0, f_1, f_2 are the $(s_1 - 1)$th column, the s_1th column and the $(s_1 + 1)$th column of F, respectively.

Step 2. Let U as $U = (U_0, u_1, u_2)$, H^*, and V be obtained by Algorithm 4.

Step 3. Let D_1 as $D_1 = (F_0{}^*, U_0, u_1)$.

Step 4. Obtain an $s_1^2 \times (2k)$ matrix Z by replacing the levels $0, 1, \ldots, (s_1 - 1)$ of the f_0 with the 1st, 2nd, $\ldots$, and the s_1th row of the H^*, respectively. Denote Z as $Z = (z_1, z_2, \ldots, z_{2k})$, where z_i is the ith column of Z for $i = 1, 2, \ldots, 2k$.

Step 5. Let W_1, W_2, W_3, W_4 be obtained by Algorithm 4. Let $X_1 = Z$, $X_2 = (z_2, z_1, z_4, z_3, \ldots, z_{2k}, z_{2k-1})$. Construct two $s_1^2 \times 2k$ matrices D_{21} and D_{22} as $D_{21} = s_2^3 W_1 + s_2^2 W_2 + s_2 W_3 + W_4$ and $D_{22} = s_2^3 X_1 + s_2^2 X_2 + s_2 W_3 + W_4$.

Step 6. Let $D_2 = (D_{21}, D_{22})$.

Theorem 5. *For $s_1 = s_2^2$, D_1, and D_2 obtained in Algorithm 5, we have*

(i) *D_1 is an $OA\left(s_1^2, s_1^{s_1-2} s_2^{s_2}, 2\right)$;*

(ii) *D_2 is an $LHD\left(s_1^2, 4k\right)$, where if s_2 is odd, $k = (s_2 + 1)/2$; if s_2 is even, $k = s_2/2$;*

(iii) *(D_1, D_2) is an $MCD\left(s_1^2, s_1^{s_1-2} s_2^{s_2}, 4k\right)$.*

Theorem 6. *For D_1 and D_2 constructed by Algorithm 5 and Theorem 5, D_2 can be partitioned into two disjoint groups of $2k$ columns, i.e., $D_2 = (D_{21}, D_{22})$. For $i = 1, 2, \ldots, 2k$, let d_1^i and d_2^i be the ith columns of D_{21} and D_{22}, respectively. Then,*

(i) *any two distinct columns of D_2 achieve $s_2 \times s_2$ grids stratification;*

(ii) *any two columns from different groups, d_1^j and $d_2^{j'}$, achieve $s_2^2 \times s_2^2$ grids stratification, where $j, j' = 1, 2, \ldots, k$;*

(iii) *any three columns from two different groups, d_i^j, $d_{i'}^t$ and $d_{i'}^h$, achieve $s_2^2 \times s_2 \times s_2$ grids stratification, where $i, i' = 1, 2$, $i \neq i'$, $j, t, h = 1, 2, \ldots, 2k$, $t \neq h$;*

(iv) *any four columns from two different groups,* d_i^j, d_i^r, $d_{i'}^t$, $d_{i'}^h$, *achieve* $s_2 \times s_2 \times s_2 \times s_2$ *grids stratification, where* $i, i' = 1, 2$, $i \neq i'$, $j, r, t, h = 1, 2, \ldots, 2k$, $, j \neq r$, *and* $t \neq h$.

According to Theorem 6, there are $4k^2$ two-column groups achieving stratifications on $s_2^2 \times s_2^2$ grids, $2k^2(2k-1)$ three-column groups achieving stratifications on $s_2^2 \times s_2 \times s_2$ grids, and $k^2(2k-1)^2$ four-column groups achieving stratifications on $s_2 \times s_2 \times s_2 \times s_2$ grids, respectively. Theorem 6 shows that a large number of columns in D_2 have good two-, three-, or four-dimensional projections. Next, we provide an example to illustrate Algorithm 5, Theorem 5, and Theorem 6.

Example 7. Consider the case $s_1 = 9$ and $s_2 = 3$. An an $OA\left(9, 3^4, 2\right)$ H listed in Table 19 of Appendix B and an $OA\left(81, 9^{10}, 2\right)$ F listed in Table 20 of Appendix B are obtained from the library of orthogonal arrays maintained by Sloane (http://neilsloane.com/oadir/index.html). Divide F as $F = (F_0{}^*, f_0, f_1, f_2)$ listed in Table 20 of Appendix B. For the H, obtain an 81×4 matrix U by replacing the levels $0, 1, \ldots, 8$ of the f_1 with the 1st, 2nd, 3rd, $\ldots$, the 9th row of the H according to Step 2 of Algorithm 4, respectively. Then, partition U as $U = (U_0, u_1, u_2)$ listed in Table 20 of Appendix B. In Step 3 and Step 4 of Algorithm 4, due to $s_2 = 3$, let $H^* = H$ and $k = (s_2 + 1)/2 = 2$, after replacing the levels $0, 1, \ldots, 8$ of the f_2 by the 1st, 2nd, 3rd, $\ldots$, the 9th row of the H^*, respectively. Then, V is obtained, and denote V as $V = (v_1, v_2, v_3, v_4)$ listed in Table 20 of Appendix B. From Step 3, $D_1 = (F_0{}^*, U_0, u_1)$, and it is easy to check that D_1 is an $OA\left(81, 9^7 3^3, 2\right)$. In Step 4, obtain an 81×4 matrix Z by replacing the levels $0, 1, \ldots, 8$ of the f_0 with the 1st, 2nd, 3rd, $\ldots$, the 9th row of the H^*, respectively. Then, Z is obtained, and denote Z as $Z = (z_1, z_2, z_3, z_4)$ listed in Table 20 of Appendix B. In Step 5, let $W_1 = V$, $W_2 = (v_2, v_1, v_4, v_3)$, $W_3 = (u_2, u_2, u_2, u_2)$, $W_4 = (u_1, u_1, u_1, u_1)$ according to Step 6 of Algorithm 4 and let $X_1 = Z$, $X_2 = (z_2, z_1, z_4, z_3)$, then by matrix operation of $s_2^3 W_1 + s_2^2 W_2 + s_2 W_3 + W_4$ and $s_2^3 X_1 + s_2^2 X_2 + s_2 W_3 + W_4$, D_{21} and D_{22} can be generated, respectively. Then, $D_2 = (D_{21}, D_{22})$. It is easy to verify that (D_1, D_2) is an $MCD\left(81, 9^7 3^3, 8\right)$ listed in Table 21 of Appendix B. Next, let the first two columns of D_{21} be d_1 and d_2, and the first two columns of D_{22} be d_3, d_4. After collapsing the levels of d_1, d_2, d_3, d_4, it is easy to see that the d_1, d_2, d_3, d_4 satisfies the stratifications of (i) and (ii) in Theorem 6, as shown in Figure 2 and Figure 3.

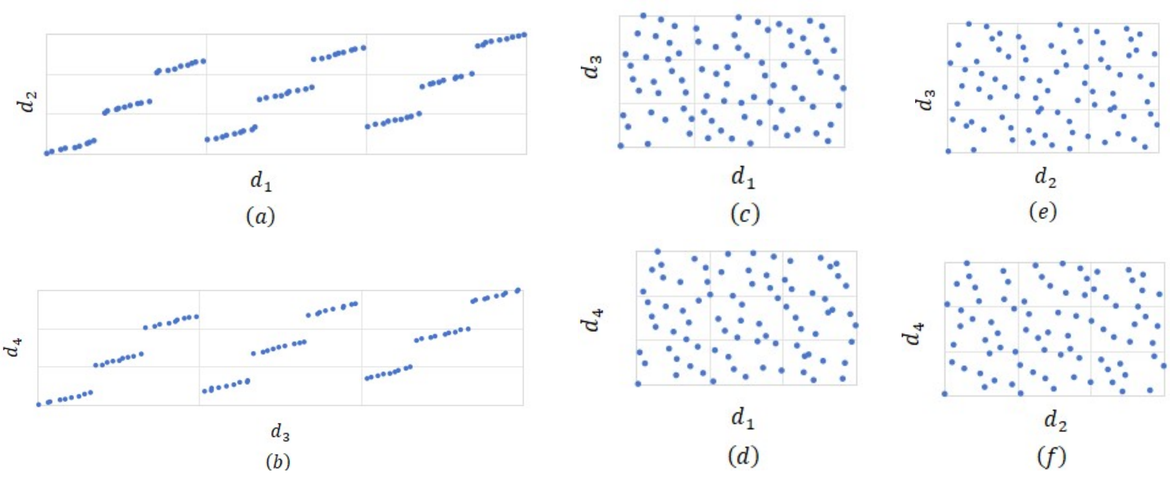

Figure 2. Stratification on 3×3 grids.

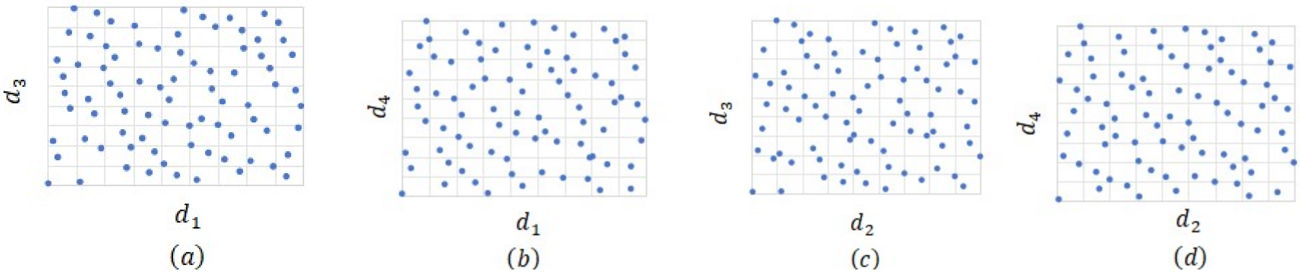

Figure 3. Stratification on 9×9 grids.

Inspired by Corollary 5, Corollary 6 is given as follows.

Corollary 6. *For $D = (D_1, D_2)$ ($D_1 = (F_0{}^*, U_0, u_1)$, $D_2 = (D_{21}, D_{22})$) constructed by Algorithm 5 and Theorem 5, we have*

(i) *the rows in D_{2i}, $i = 1, 2$, corresponding to each level of any factor in D_1 can achieve stratification on the $s_2 \times s_2$ grids in any two-dimensional projection;*

(ii) *the rows in D_{2i}, $i = 1, 2$, corresponding to each level combination of any two factors in (U_0, u_1) can achieve stratification on the $s_2 \times s_2$ grids in any two-dimensional projection.*

Algorithm 5 can produce some MCDs based on the above Theorem 5, as shown in Table 15.

Table 15. Some MCDs from Algorithm 5.

D_1	D_2	MCDs
$OA\left(16, 4^2 2^2, 2\right)$	$LHD\,(6, 4)$	$MCD\left(6, 4^3 2^2, 4\right)$
$OA\left(81, 9^7 3^3, 2\right)$	$LHD\,(8, 8)$	$MCD\left(8, 9^8 3^3, 8\right)$
$OA\left(256, 16^{14} 4^4, 2\right)$	$LHD\,(10, 8)$	$MCD\left(10, 16^{15} 4^4, 8\right)$
$OA\left(625, 25^{23} 5^5, 2\right)$	$LHD\,(12, 12)$	$MCD\left(12, 25^{24} 5^5, 12\right)$
$OA\left(1296, 36^{33} 6^6, 2\right)$	$LHD\,(14, 12)$	$MCD\left(14, 36^{35} 6^6, 12\right)$

4. Conclusions

Many researchers have constructed MCDs for equal-level qualitative factors. However, there has been less research on MCDs when the qualitative factors are mixed-level. Construction 3 of Deng et al. [12] generates large MCDs for mixed-level qualitative factors from small initial MCDs for mixed-level qualitative factors. Obviously, such a construction is not valid when the initial MCD does not exist. The key to Construction 3 of Deng et al. [12] is how to obtain a small initial MCD. However, they did not answer the question. Fortunately, the constructed MCDs in this paper can be considered as the initial MCDs for Construction 3 of [12].

In this paper, we propose five algorithms to construct MCDs where the designs for the qualitative factors are mixed-level. The construction of the first algorithm is characterized by the fact that it is based on an $OA\left(s_1^2, s_1^{k_1+2}, 2\right)$ and a $CROA\left(s_1, s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$. Clearly, its constructed MCD is limited by $s_1 = \beta s_2$. To break this limitation, Algorithms 2 and 3 employ a mirror-symmetric structure to construct D_2. Moreover, the D_1 constructed by Algorithm 3 can accommodate more columns than the one constructed by Algorithm 2, and the two algorithms construct different numbers of run sizes. The fourth and fifth algorithms construct the MCD using the level replacement method and the rotation method, where D_2 has stratification in two- or higher-dimensional projection. Finally, Table 16 lists some types and features of MCDs that can be constructed using our five algorithms. Obviously, compared to the MCDs constructed by Construction 3 of Deng et al. [12], our constructed MCDs have more flexible run sizes, and the more flexible fixed level D_1 -D_1 is an $OA\left(n, s_1^{k_1} s_2^{k_2}, 2\right)$, $s_1 = \beta s_2$, or $s_1 \neq \beta s_2$. Moreover, in contrast to Construction 3 of Deng et al. [12] , which does not consider the space-filling property of D_2, Algorithm 4 and Algorithm 5 construct D_2 with the space-filling property.

For future work, a direction is to introduce methods that can produce MCDs with three or more mixed-level qualitative factors, which deserves further investigation.

Table 16. Some of the MCDs (D_1, D_2) results.

Source	D_1	Constraints
Theorem 1	$OA\left(s_1^2, s_1^{k_1} s_2^{k_2}, 2\right)$	$s_1 = \beta s_2$, an $OA\left(s_1^2, s_1^{k_1+2}, 2\right)$ and a $CROA\left(s_1, s_2^{k_2}, 2\right)$ exist.
Corollary 1	$OA\left(s^{2t}, (s^t)^{k_1}(s)^{s^{t-1}}, 2\right)$	an $OA\left(s^{2t}, (s^t)^{k_1+2}, 2\right)$ exists.
	$OA\left(4s^{2t}, (2s^t)^{k_1}(s)^{2s^{t-1}}, 2\right)$	an $OA\left(4s^{2t}, (2s^t)^{k_1+2}, 2\right)$ exists.
	$OA\left(16s^{2t}, (4s^t)^{k_1}(s)^{4s^{t-1}}, 2\right)$	an $OA\left(16s^{2t}, (4s^t)^{k_1+2}, 2\right)$ exists.
	$OA\left(h^{2w}s^4, (h^w s^2)^{k_1}(s)^{h^w s}, 2\right)$	an $OA\left(h^{2w}s^4, (h^w s^2)^{k_1+2}, 2\right)$ exists.
Corollary 2	$OA\left(us_1^2, s_1^{r_1k_1} s_2^{r_2k_2}, 2\right)$	$s_1 = \beta s_2$, $D(u, r_1, s_1)$ and $D(u, r_2, s_2)$ exist.
Theorem 2	$OA\left(2s, 2^1 s^1, 2\right)$	$s \geq 2$.
Corollary 3	$OA\left(2su, 2^{r_1} s^{r_2}, 2\right)$	$s \geq 2$, $D(u, r_1, 2)$ and $D(u, r_2, s)$ exist.
Theorem 3	$OA\left(2n, 2^1 s^m, 2\right)$	$s \geq 2$.
Corollary 4	$OA\left(2nu, 2^{r_1} s^{r_2m}, 2\right)$	$s \geq 2$, $D(u, r_1, 2)$ and $D(u, r_2, s)$ exist.
Theorems 4	$OA\left(s_1^2, s_1^{s_1-1} s_2^{s_2}, 2\right)$	$s_1 = s_2^2$, s_2 is a prime or prime power.
Theorems 5	$OA\left(s_1^2, s_1^{s_1-2} s_2^{s_2}, 2\right)$	$s_1 = s_2^2$, s_2 is a prime or prime power.

Author contributions

Weiping Zhou: Algorithm, methodology, validation, investigation, resources, data curation, writing—original draft preparation, writing—review and editing; Wan He: Algorithm, software, validation, writing—original draft preparation; Wei Wang: Methodology, writing—review and editing, visualization, supervision, project administration; Shigui Huang: Software, writing—original draft preparation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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Appendix A. PROOFS

Proof of Lemma 3. From the definition of an MCD, it is clear that $D = (D_1, D_2)$ is an $MCD\left(n, {s_1}^{k_1}{s_2}^{k_2}, p\right)$ if, and only if, (Ω, D_2) and (Λ, D_2) are an $MCD\left(n, {s_1}^{k_1}, p\right)$ and an $MCD\left(n, {s_2}^{k_2}, p\right)$, respectively. Let d_i be the ith column of D_2, for $i = 1, \dots, p$. From Lemma 1, we have (i) (Ω, D_2) is an $MCD\left(n, {s_1}^{k_1}, p\right)$ if, and only if, $(\Omega, f(d_i, s_1))$ is an $OA\left(n, {s_1}^{k_1}(n/s_1), 2\right)$; (ii) (Λ, D_2) is an $MCD\left(n, {s_2}^{k_2}, p\right)$ if, and only if, $(\Lambda, f(d_i, s_2))$ is an $OA\left(n, {s_2}^{k_2}(n/s_2), 2\right)$ for $i = 1, \dots, p$. □

Proof of Theorem 1. (i) In the design (l_2, A, B^*), the levels $0, 1, \dots, s_1-1$ of the l_2 correspond to the 1st, 2nd, ..., s_1th rows of the B, respectively, where $B^* = 1_{s_1} \otimes B$. Thus, $D_1 = (A, B^*)$ is an $OA\left(s_1^2, s_1^{k_1} s_2^{k_2}, 2\right)$ with $s_1 = \beta s_2$.

(ii) From Steps 4 and 5 of Algorithm 1, it is clear that (e_i, c_i) is an $OA\left(s_1^2, s_1^2, 2\right)$ for $i = 1, 2, \dots, p$, where $s_1 = \beta s_2$. Thus, D_2 is an $LHD\left(s_1^2, p\right)$ from Step 6 of Algorithm 1.

(iii) Let a, b, and d be any columns of A, B^*, and D_2, respectively. From Steps 3, 4, 5, and 6, let e and c be the columns corresponding to d in E and C, respectively. From Step 6 and Equation (2.1), $(a, f(D_2, s)) = (a, e)$, thus $(a, f(D_2, s_1))$ is an $OA\left(s_1^2, s_1^2, 2\right)$. From Steps 4, 5, and 6 of Algorithm 1, it is clear that $f(D_2, s_2) = \beta e + c^*$, where $c^* = 1_{s_1} \otimes w$, w is a random permutation of $\left((c^*_{i,1})^T, (c^*_{i,2})^T, \cdots, (c^*_{i,\beta})^T\right)^T$ with $c^*_{i,j} = (j-1)\mathbf{1}_{s_2}$. Since (b, e, c^*) is an $OA\left(s_1^2, s_2^1 s_1^1 \beta^1, 3\right)$, $(b, f(D_2, s_2))$ is an $OA\left(s_1^2, s_2^1(\beta s_1)^1, 2\right)$. Thus, $D = (D_1, D_2)$ is an $MCD\left(s_1^2, s_1^{k_1} s_2^{k_2}, p\right)$ from Lemma 3. □

Proof of Theorem 2. From Steps 1 and 2 of Algorithm 2, it is easy to check that D_1 is an $OA\left(2s, 2^1 s^1, 2\right)$ and D_2 is an $LHD(2s, s!)$. By Step 2, we can see that d_i is the ith column of D_2 for $i = 1, 2, \dots, s!$. Since $f(d_i, 2)=\left((u_i)^T, ((s-1)\mathbf{1}_s - u_i)^T\right)^T$, $(L_1, f(d_i, 2))$ is an $OA\left(2s, 2^1 s^1, 2\right)$, where u_i is a random permutation of $(0, 1, 2, \dots, s-1)^T$, $i = 1, 2, \dots, s!$. For $1 \le i \le s!$, let $\xi_1 = 2u_i$ and $\xi_2 = (2s-1)\mathbf{1}_s - 2u_i$, then

$$(L_2, d_i) = \begin{pmatrix} e & \xi_1 \\ e & \xi_2 \end{pmatrix} \text{and} \, (L_2, f(d_i, s)) = \begin{pmatrix} e & f(\xi_1, s) \\ e & f(\xi_2, s) \end{pmatrix},$$

where $e = (0, 1, \dots, s-1)^T$. Obviously, the elements of $f(\xi_1, s)$ and $f(\xi_2, s)$ are all taken from $\{0, 1\}$. Since $f(\xi_2, s) = \mathbf{1}_s - f(\xi_1, s)$, $(L_2, f(d_i, s))$ is an $OA\left(2s, s^1 2^1, 2\right)$, $i = 1, 2, \dots, s!$. From Lemma 3, the design $D = (D_1, D_2)$ is an $MCD\left(2s, 2^1 s^1, s!\right)$ □

Proof of Theorem 3. The proof of Theorem 3 is similar to that of Theorem 2 and is therefore omitted here. □

Proof of Theorem 4. For $i = 1, 2, \dots, s_1 - 1$, $j = 1, 2, \dots, s_2 - 1$, let f_{0i} and u_{0j} be the ith and jth columns of F_0 and U_0, respectively.

(i) Since $F = (F_0, f_1, f_2)$ is an $OA\left(s_1^2, s_1^{s_1+1}, 2\right)$, $U = (U_0, u_1, u_2)$ is an $OA\left(s_1^2, s_2^{s_2+1}, 2\right)$, (f_{0i}, u_{0j}) is an $OA\left(s_1^2, s_1^1 s_2^1, 2\right)$, and (f_{0i}, u_1) is an $OA\left(s_1^2, s_1^1 s_2^1, 2\right)$, $i = 1, 2, \dots, s_1 - 1$, $j = 1, 2, \dots, s_2 - 1$, thus D_1 is an $OA\left(s_1^2, s_1^{s_1-1} s_2^{s_2}, 2\right)$.

(ii) According to Proposition 1 of [24], we can obtain that (v_i, v_j, u_1, u_2) is an $OA\left(s_1^2, s_2^4, 4\right)$, where $s_1 = s_2^2$, $i \neq j$, $i, j = 1, 2, \dots, 2k$. Thus, D_2 is an $LHD\left(s_1^2, 2k\right)$, where if s_2 is odd, $k = (s_2 + 1)/2$; if s_2 is even, $k = s_2/2$.

(iii) For $h = 1, 2, \ldots, k$, let d_{2h-1} and d_{2h} be the $(2h-1)$th and $2h$th columns of D_2, respectively, then $d_{2h-1} = s_2^3 v_{2h-1} + s_2^2 v_{2h} + s_2 u_2 + u_1$ and $d_{2h} = s_2^3 v_{2h} + s_2^2 v_{2h-1} + s_2 u_2 + u_1$. Obviously, for $i = 1, 2, \ldots, s_1 - 1$, $j = 1, 2, \ldots, s_2 - 1$, $h = 1, 2, \ldots, k$, $(f_{0i}, f(d_{2h-1}, s_1)) = (f_{0i}, s_2 v_{2h-1} + v_{2h})$, $(f_{0i}, f(d_{2h}, s_1)) = (f_{0i}, s_2 v_{2h} + v_{2h-1})$, $(u_{0j}, f(d_{2h-1}, s_1)) = (u_{0j}, s_2^2 v_{2h-1} + s_2 v_{2h} + u_2)$, $(u_{0j}, f(d_{2h}, s_1)) = (u_{0j}, s_2^2 v_{2h} + s_2 v_{2h-1} + u_2)$, $(u_1, f(d_{2h-1}, s_1)) = (u_1, s_2^2 v_{2h-1} + s_2 v_{2h} + u_2)$, and $(u_1, f(d_{2h}, s_1)) = (u_{0j}, s_2^2 v_{2h} + s_2 v_{2h-1} + u_2)$, where $s_1 = s_2^2$. According to Proposition 1 of [24], for $s_1 = s_2^2$, it is easy to obtain that $(f_{0i}, v_{2h-1}, v_{2h})$ is an $OA\left(s_1^2, s_1^1 s_2^2, 3\right)$, and both $(u_1, u_2, v_{2h-1}, v_{2h})$ and $(u_2, u_{0j}, v_{2h-1}, v_{2h})$ are $OA\left(s_2^4, s_2^4, 4\right)$'s, where $i = 1, 2, \ldots, s_1 - 1$, $j = 1, 2, \ldots, s_2 - 1$, $h = 1, 2, \ldots, k$. Therefore, both $(f_{0i}, f(d_{2h-1}, s_1))$ and $(f_{0i}, f(d_{2h}, s_1))$ are $OA\left(s_1^2, s_1^2, 2\right)$'s, and $(u_{0j}, f(d_{2h-1}, s_2))$, $(u_{0j}, f(d_{2h}, s_2))$, $(u_1, f(d_{2h-1}, s_2))$, and $(u_1, f(d_{2h}, s_2))$ are all $OA\left(s_2^4, s_2^1 (s_2^3)^1, 2\right)$'s, where $s_1 = s_2^2$, $i = 1, 2, \ldots, s_1 - 1$, $j = 1, 2, \ldots, s_2 - 1$, $h = 1, 2, \ldots, k$. From Lemma 3, the design (D_1, D_2) is an $MCD\left(s_1^2, s_1^{s_1-1} s_2^{s_2}, 2k\right)$.

(iv) Since $f\left(D_2, {s_2}^3\right) = W_1$ and W_1 is an $OA\left(s_1^2, s_2^{2k}, 2\right)$ with $s_1 = {s_2}^2$, thus any two distinct columns of D_2 achieve $s_2 \times s_2$ grids stratification. □

Proof of Theorem 5. The proof of Theorem 5 is similar to that of Theorem 4 and is therefore omitted here. □

Proof of Theorem 6. (i) Since $f(D_2, s_2^3) = (W_1, X_1)$ and (W_1, X_1) is an $OA\left(s_1^2, s_2^{4k}, 2\right)$, thus Theorem 6 (i) is true.

(ii) For $j, j' = 1, 2, \ldots, 2k$, it is easy to see that $f(d_1^j, s_2^2) = s_2 v_{2j-1} + v_{2j}$ or $f(d_1^j, s_2^2) = s_2 v_{2j} + v_{2j-1}$, and $f(d_2^{j'}, s_2^2) = s_2 z_{2j'-1} + z_{2j'}$ or $f(d_2^{j'}, s_2^2) = s_2 z_{2j'} + z_{2j'-1}$. According to Proposition 1 of [24], it is easy to obtain that $(v_{2j-1}, v_{2j}, z_{2j'-1}, z_{2j'})$ is an $OA\left(s_2^4, s_2^4, 4\right)$. Thus, Theorem 6 (ii) is true.

(iii-iv) From Proposition 1 of [24], it is known that any two columns of V in Algorithm 4 and any two columns of Z in Algorithm 5 form an $OA\left(s_1^4, s_2^4, 4\right)$ with $s_1 = {s_2}^2$. Similar to the proof of (ii), thus (iii) and (iv) are true. □

Appendix B. Tables

Table 17. Matrices G, E, and C in Example 2.

Run	G							E			C		
	l_1	l_2	A					e_1	e_2	e_3	c_1	c_2	c_3
1	0	0	0	0	0	0	0	0	1	1	0	2	0
2	0	1	1	6	3	8	4	0	1	1	1	3	1
3	0	2	2	8	6	1	11	0	1	1	2	0	2
4	0	3	3	2	1	11	10	0	1	1	3	1	3
5	0	4	4	7	9	5	2	0	1	1	4	4	8
6	0	5	5	1	11	9	7	0	1	1	5	5	9
7	0	6	6	9	2	3	8	0	1	1	6	6	10
8	0	7	7	11	8	10	6	0	1	1	7	7	11
9	0	8	8	4	5	2	9	0	1	1	8	8	4
10	0	9	9	10	4	7	1	0	1	1	9	9	5
11	0	10	10	5	7	6	3	0	1	1	10	10	6
12	0	11	11	3	10	4	5	0	1	1	11	11	7
13	1	0	1	1	1	1	1	1	0	0	0	2	0
14	1	1	2	7	4	9	5	1	0	0	1	3	1
15	1	2	3	9	7	2	6	1	0	0	2	0	2
16	1	3	4	3	2	6	11	1	0	0	3	1	3
17	1	4	5	8	10	0	3	1	0	0	4	4	8
18	1	5	0	2	6	10	8	1	0	0	5	5	9
19	1	6	7	10	3	4	9	1	0	0	6	6	10
20	1	7	8	6	9	11	7	1	0	0	7	7	11
21	1	8	9	5	0	3	10	1	0	0	8	8	4
22	1	9	10	11	5	8	2	1	0	0	9	9	5
23	1	10	11	0	8	7	4	1	0	0	10	10	6
24	1	11	6	4	11	5	0	1	0	0	11	11	7
25	2	0	2	2	2	2	2	2	2	2	0	2	0
26	2	1	3	8	5	10	0	2	2	2	1	3	1
27	2	2	4	10	8	3	7	2	2	2	2	0	2
28	2	3	5	4	3	7	6	2	2	2	3	1	3
29	2	4	0	9	11	1	4	2	2	2	4	4	8
30	2	5	1	3	7	11	9	2	2	2	5	5	9
31	2	6	8	11	4	5	10	2	2	2	6	6	10
32	2	7	9	7	10	6	8	2	2	2	7	7	11
33	2	8	10	0	1	4	11	2	2	2	8	8	4
34	2	9	11	6	0	9	3	2	2	2	9	9	5
35	2	10	6	1	9	8	5	2	2	2	10	10	6
36	2	11	7	5	6	0	1	2	2	2	11	11	7
37	3	0	3	3	3	3	3	3	3	3	0	2	0
38	3	1	4	9	0	11	1	3	3	3	1	3	1
39	3	2	5	11	9	4	8	3	3	3	2	0	2
40	3	3	0	5	4	8	7	3	3	3	3	1	3
41	3	4	1	10	6	2	5	3	3	3	4	4	8
42	3	5	2	4	8	6	10	3	3	3	5	5	9
43	3	6	9	6	5	0	11	3	3	3	6	6	10
44	3	7	10	8	11	7	9	3	3	3	7	7	11
45	3	8	11	1	2	5	6	3	3	3	8	8	4
46	3	9	6	7	1	10	4	3	3	3	9	9	5
47	3	10	7	2	10	9	0	3	3	3	10	10	6
48	3	11	8	0	7	1	2	3	3	3	11	11	7
49	4	0	4	4	4	4	4	4	4	4	0	2	0
50	4	1	5	10	1	6	2	4	4	4	1	3	1
51	4	2	0	6	10	5	9	4	4	4	2	0	2
52	4	3	1	0	5	9	8	4	4	4	3	1	3
53	4	4	2	11	7	3	0	4	4	4	4	4	8
54	4	5	3	5	9	7	11	4	4	4	5	5	9
55	4	6	10	7	0	1	6	4	4	4	6	6	10
56	4	7	11	9	6	8	10	4	4	4	7	7	11
57	4	8	6	2	3	0	7	4	4	4	8	8	4
58	4	9	7	8	2	11	5	4	4	4	9	9	5
59	4	10	8	3	11	10	1	4	4	4	10	10	6
60	4	11	9	1	8	2	3	4	4	4	11	11	7
61	5	0	5	5	5	5	5	5	5	5	0	2	0
62	5	1	0	11	2	7	3	5	5	5	1	3	1
63	5	2	1	7	11	0	10	5	5	5	2	0	2
64	5	3	2	1	0	10	9	5	5	5	3	1	3
65	5	4	3	6	8	4	1	5	5	5	4	4	8
66	5	5	4	0	10	8	6	5	5	5	5	5	9
67	5	6	11	8	1	2	7	5	5	5	6	6	10
68	5	7	6	10	7	9	11	5	5	5	7	7	11
69	5	8	7	3	4	1	8	5	5	5	8	8	4
70	5	9	8	9	3	6	0	5	5	5	9	9	5
71	5	10	9	4	6	11	2	5	5	5	10	10	6
72	5	11	10	2	9	3	4	5	5	5	11	11	7
73	6	0	6	6	6	6	6	6	6	6	0	2	0
74	6	1	7	0	9	2	10	6	6	6	1	3	1
75	6	2	8	2	0	7	5	6	6	6	2	0	2
76	6	3	9	8	7	5	4	6	6	6	3	1	3
77	6	4	10	1	3	11	8	6	6	6	4	4	8
78	6	5	11	7	5	3	1	6	6	6	5	5	9
79	6	6	0	3	8	9	2	6	6	6	6	6	10
80	6	7	1	5	2	4	0	6	6	6	7	7	11
81	6	8	2	10	11	8	3	6	6	6	8	8	4
82	6	9	3	4	10	1	7	6	6	6	9	9	5
83	6	10	4	11	1	0	9	6	6	6	10	10	6
84	6	11	5	9	4	10	11	6	6	6	11	11	7
85	7	0	7	7	7	7	7	7	7	7	0	2	0
86	7	1	8	1	10	3	11	7	7	7	1	3	1
87	7	2	9	3	1	8	0	7	7	7	2	0	2
88	7	3	10	9	8	0	5	7	7	7	3	1	3
89	7	4	11	2	4	6	9	7	7	7	4	4	8
90	7	5	6	8	0	4	2	7	7	7	5	5	9
91	7	6	1	4	9	10	3	7	7	7	6	6	10
92	7	7	2	0	3	5	1	7	7	7	7	7	11
93	7	8	3	11	6	9	4	7	7	7	8	8	4
94	7	9	4	5	11	2	8	7	7	7	9	9	5
95	7	10	5	6	2	1	10	7	7	7	10	10	6
96	7	11	0	10	5	11	6	7	7	7	11	11	7
97	8	0	8	8	8	8	8	8	8	8	0	2	0
98	8	1	9	2	11	4	6	8	8	8	1	3	1
99	8	2	10	4	2	9	1	8	8	8	2	0	2
100	8	3	11	10	9	1	0	8	8	8	3	1	3
101	8	4	6	3	5	7	10	8	8	8	4	4	8
102	8	5	7	9	1	5	3	8	8	8	5	5	9
103	8	6	2	5	10	11	4	8	8	8	6	6	10
104	8	7	3	1	4	0	2	8	8	8	7	7	11
105	8	8	4	6	7	10	5	8	8	8	8	8	4
106	8	9	5	0	6	3	9	8	8	8	9	9	5
107	8	10	0	7	3	2	11	8	8	8	10	10	6
108	8	11	1	11	0	6	7	8	8	8	11	11	7
109	9	0	9	9	9	9	9	9	9	9	0	2	0
110	9	1	10	3	6	5	7	9	9	9	1	3	1
111	9	2	11	5	3	10	2	9	9	9	2	0	2
112	9	3	6	11	10	2	1	9	9	9	3	1	3
113	9	4	7	4	0	8	11	9	9	9	4	4	8
114	9	5	8	10	2	0	4	9	9	9	5	5	9
115	9	6	3	0	11	6	5	9	9	9	6	6	10
116	9	7	4	2	5	1	3	9	9	9	7	7	11
117	9	8	5	7	8	11	0	9	9	9	8	8	4
118	9	9	0	1	7	4	10	9	9	9	9	9	5
119	9	10	1	8	4	3	6	9	9	9	10	10	6
120	9	11	2	6	1	7	8	9	9	9	11	11	7
121	10	0	10	10	10	10	10	10	10	10	0	2	0
122	10	1	11	4	7	0	8	10	10	10	1	3	1
123	10	2	6	0	4	11	3	10	10	10	2	0	2
124	10	3	7	6	11	3	2	10	10	10	3	1	3
125	10	4	8	5	1	9	6	10	10	10	4	4	8
126	10	5	9	11	3	1	5	10	10	10	5	5	9
127	10	6	4	1	6	7	0	10	10	10	6	6	10
128	10	7	5	3	0	2	4	10	10	10	7	7	11
129	10	8	0	8	9	6	1	10	10	10	8	8	4
130	10	9	1	2	8	5	11	10	10	10	9	9	5
131	10	10	2	9	5	4	7	10	10	10	10	10	6
132	10	11	3	7	2	8	9	10	10	10	11	11	7
133	11	0	11	11	11	11	11	11	11	11	0	2	0
134	11	1	6	5	8	1	9	11	11	11	1	3	1
135	11	2	7	1	5	6	4	11	11	11	2	0	2
136	11	3	8	7	6	4	3	11	11	11	3	1	3
137	11	4	9	0	2	10	7	11	11	11	4	4	8
138	11	5	10	6	4	2	0	11	11	11	5	5	9
139	11	6	5	2	7	8	1	11	11	11	6	6	10
140	11	7	0	4	1	3	5	11	11	11	7	7	11
141	11	8	1	9	10	7	2	11	11	11	8	8	4
142	11	9	2	3	9	0	6	11	11	11	9	9	5
143	11	10	3	10	0	5	8	11	11	11	10	10	6
144	11	11	4	8	3	9	10	11	11	11	11	11	7

Table 18. $D = (D_1, D_2)$ in Example 2.

Run	$MCD(D_1, D_2)$										Run	$MCD(D_1, D_2)$										Run	$MCD(D_1, D_2)$									
	D_1							D_2				D_1							D_2				D_1							D_2		
	A					B^*						A					B^*						A					B^*				
1	0	0	0	0	0	1	1	0	14	12	49	4	4	4	4	4	1	1	48	50	48	97	8	8	8	8	8	1	1	96	98	96
2	1	6	3	8	4	0	0	1	15	13	50	5	10	1	6	2	0	0	49	51	49	98	9	2	11	4	6	0	0	97	99	97
3	2	8	6	1	11	0	0	2	12	14	51	0	6	10	5	9	0	0	50	48	50	99	10	4	2	9	1	0	0	98	96	98
4	3	2	1	11	10	1	1	3	13	15	52	1	0	5	9	8	1	1	51	49	51	100	11	10	9	1	0	1	1	99	97	99
5	4	7	9	5	2	1	1	4	16	20	53	2	11	7	3	0	1	1	52	52	56	101	6	3	5	7	10	1	1	100	100	104
6	5	1	11	9	7	0	0	5	17	21	54	3	5	9	7	11	0	0	53	53	57	102	7	9	1	5	3	0	0	101	101	105
7	6	9	2	3	8	0	1	6	18	22	55	10	7	0	1	6	0	1	54	54	58	103	2	5	10	11	4	0	1	102	102	106
8	7	11	8	10	6	1	0	7	19	23	56	11	9	6	8	10	1	0	55	55	59	104	3	1	4	0	2	1	0	103	103	107
9	8	4	5	2	9	0	1	8	20	16	57	6	2	3	0	7	0	1	56	56	52	105	4	6	7	10	5	0	1	104	104	100
10	9	10	4	7	1	1	0	9	21	17	58	7	8	2	11	5	1	0	57	57	53	106	5	0	6	3	9	1	0	105	105	101
11	10	5	7	6	3	1	0	10	22	18	59	8	3	11	10	1	1	0	58	58	54	107	0	7	3	2	11	1	0	106	106	102
12	11	3	10	4	5	0	1	11	23	19	60	9	1	8	2	3	0	1	59	59	55	108	1	11	0	6	7	0	1	107	107	103
13	1	1	1	1	1	1	1	12	2	0	61	5	5	5	5	5	1	1	60	62	60	109	9	9	9	9	9	1	1	108	110	108
14	2	7	4	9	5	0	0	13	3	1	62	0	11	2	7	3	0	0	61	63	61	110	10	3	6	5	7	0	0	109	111	109
15	3	9	7	2	6	0	0	14	0	2	63	1	7	11	0	10	0	0	62	60	62	111	11	5	3	10	2	0	0	110	108	110
16	4	3	2	6	11	1	1	15	1	3	64	2	1	0	10	9	1	1	63	61	63	112	6	11	10	2	1	1	1	111	109	111
17	5	8	10	0	3	1	1	16	4	8	65	3	6	8	4	1	1	1	64	64	68	113	7	4	0	8	11	1	1	112	112	116
18	0	2	6	10	8	0	0	17	5	9	66	4	0	10	8	6	0	0	65	65	69	114	8	10	2	0	4	0	0	113	113	117
19	7	10	3	4	9	0	1	18	6	10	67	11	8	1	2	7	0	1	66	66	70	115	3	0	11	6	5	0	1	114	114	118
20	8	6	9	11	7	1	0	19	7	11	68	6	10	7	9	11	1	0	67	67	71	116	4	2	5	1	3	1	0	115	115	119
21	9	5	0	3	10	0	1	20	8	4	69	7	3	4	1	8	0	1	68	68	64	117	5	7	8	11	0	0	1	116	116	112
22	10	11	5	8	2	1	0	21	9	5	70	8	9	3	6	0	1	0	69	69	65	118	0	1	7	4	10	1	0	117	117	113
23	11	0	8	7	4	1	0	22	10	6	71	9	4	6	11	2	1	0	70	70	66	119	1	8	4	3	6	1	0	118	118	114
24	6	4	11	5	0	0	1	23	11	7	72	10	2	9	3	4	0	1	71	71	67	120	2	6	1	7	8	0	1	119	119	115
25	2	2	2	2	2	1	1	24	26	24	73	6	6	6	6	6	1	1	72	74	72	121	10	10	10	10	10	1	1	120	122	120
26	3	8	5	10	0	0	0	25	27	25	74	7	0	9	2	10	0	0	73	75	73	122	11	4	7	0	8	0	0	121	123	121
27	4	10	8	3	7	0	0	26	24	26	75	8	2	0	7	5	0	0	74	72	74	123	6	0	4	11	3	0	0	122	120	122
28	5	4	3	7	6	1	1	27	25	27	76	9	8	7	5	4	1	1	75	73	75	124	7	6	11	3	2	1	1	123	121	123
29	0	9	11	1	4	1	1	28	28	32	77	10	1	3	11	8	1	1	76	76	80	125	8	5	1	9	6	1	1	124	124	128
30	1	3	7	11	9	0	0	29	29	33	78	11	7	5	3	1	0	0	77	77	81	126	9	11	3	1	5	0	0	125	125	129
31	8	11	4	5	10	0	1	30	30	34	79	0	3	8	9	2	0	1	78	78	82	127	4	1	6	7	0	0	1	126	126	130
32	9	7	10	6	8	1	0	31	31	35	80	1	5	2	4	0	1	0	79	79	83	128	5	3	0	2	4	1	0	127	127	131
33	10	0	1	4	11	0	1	32	32	28	81	2	10	11	8	3	0	1	80	80	76	129	0	8	9	6	1	0	1	128	128	124
34	11	6	0	9	3	1	0	33	33	29	82	3	4	10	1	7	1	0	81	81	77	130	1	2	8	5	11	1	0	129	129	125
35	6	1	9	8	5	1	0	34	34	30	83	4	11	1	0	9	1	0	82	82	78	131	2	9	5	4	7	1	0	130	130	126
36	7	5	6	0	1	0	1	35	35	31	84	5	9	4	10	11	0	1	83	83	79	132	3	7	2	8	9	0	1	131	131	127
37	3	3	3	3	3	1	1	36	38	36	85	7	7	7	7	7	1	1	84	86	84	133	11	11	11	11	11	1	1	132	134	132
38	4	9	0	11	1	0	0	37	39	37	86	8	1	10	3	11	0	0	85	87	85	134	6	5	8	1	9	0	0	133	135	133
39	5	11	9	4	8	0	0	38	36	38	87	9	3	1	8	0	0	0	86	84	86	135	7	1	5	6	4	0	0	134	132	134
40	0	5	4	8	7	1	1	39	37	39	88	10	9	8	0	5	1	1	87	85	87	136	8	7	6	4	3	1	1	135	133	135
41	1	10	6	2	5	1	1	40	40	44	89	11	2	4	6	9	1	1	88	88	92	137	9	0	2	10	7	1	1	136	136	140
42	2	4	8	6	10	0	0	41	41	45	90	6	8	0	4	2	0	0	89	89	93	138	10	6	4	2	0	0	0	137	137	141
43	9	6	5	0	11	0	1	42	42	46	91	1	4	9	10	3	0	1	90	90	94	139	5	2	7	8	1	0	1	138	138	142
44	10	8	11	7	9	1	0	43	43	47	92	2	0	3	5	1	1	0	91	91	95	140	0	4	1	3	5	1	0	139	139	143
45	11	1	2	5	6	0	1	44	44	40	93	3	11	6	9	4	0	1	92	92	88	141	1	9	10	7	2	0	1	140	140	136
46	6	7	1	10	4	1	0	45	45	41	94	4	5	11	2	8	1	0	93	93	89	142	2	3	9	0	6	1	0	141	141	137
47	7	2	10	9	0	1	0	46	46	42	95	5	6	2	1	10	1	0	94	94	90	143	3	10	0	5	8	1	0	142	142	138
48	8	0	7	1	2	0	1	47	47	43	96	0	10	5	11	6	0	1	95	95	91	144	4	8	3	9	10	0	1	143	143	139

Table 19. Matrix H in Example 7.

Run	H				*Run*	H				*Run*	H			
1	0	0	0	0	4	1	0	1	1	7	2	0	2	2
2	0	1	1	2	5	1	1	2	0	8	2	1	0	1
3	0	2	2	1	6	1	2	0	2	9	2	2	1	0

Table 20. Matrices F, U, V, and Z in Example 7.

Run	F				U			V				Z			
	$F_0{}^*$	f_0	f_1	f_2	U_0	u_1	u_2	v_1	v_2	v_3	v_4	z_1	z_2	z_3	z_4
1	0 0 0 0 0 0 0	0	0	0	0 0	0	0	0	0	0	0	0	0	0	0
2	0 1 1 2 3 4 5	6	7	8	2 1	0	1	2	2	1	0	2	0	2	2
3	0 2 2 3 4 5 6	7	8	1	2 2	1	0	0	1	1	2	2	1	0	1
4	0 3 3 4 5 6 7	8	1	2	0 1	1	2	0	2	2	1	2	2	1	0
5	0 4 4 5 6 7 8	1	2	3	0 2	2	1	1	0	1	1	0	1	1	2
6	0 5 5 6 7 8 1	2	3	4	1 0	1	1	1	1	2	0	0	2	2	1
7	0 6 6 7 8 1 2	3	4	5	1 1	2	0	1	2	0	2	1	0	1	1
8	0 7 7 8 1 2 3	4	5	6	1 2	0	2	2	0	2	2	1	1	2	0
9	0 8 8 1 2 3 4	5	6	7	2 0	2	2	2	1	0	1	1	2	0	2
10	1 0 1 1 1 1 1	1	1	1	0 1	1	2	0	1	1	2	0	1	1	2
11	1 1 5 3 8 7 0	4	6	2	2 0	2	2	0	2	2	1	1	1	2	0
12	1 2 3 8 7 0 4	6	2	5	0 2	2	1	1	2	0	2	2	0	2	2
13	1 3 8 7 0 4 6	2	5	3	1 2	0	2	1	0	1	1	0	2	2	1
14	1 4 7 0 4 6 2	5	3	8	1 0	1	1	2	2	1	0	1	2	0	2
15	1 5 0 4 6 2 5	3	8	7	2 2	1	0	2	1	0	1	1	0	1	1
16	1 6 4 6 2 5 3	8	7	0	2 1	0	1	0	0	0	0	2	2	1	0
17	1 7 6 2 5 3 8	7	0	4	0 0	0	0	1	1	2	0	2	1	0	1
18	1 8 2 5 3 8 7	0	4	6	1 1	2	0	2	0	2	2	0	0	0	0
19	2 0 2 2 2 2 2	2	2	2	0 2	2	1	0	2	2	1	0	2	2	1
20	2 1 3 6 4 1 8	0	5	7	1 2	0	2	2	1	0	1	0	0	0	0
21	2 2 6 4 1 8 0	5	7	3	2 1	0	1	1	0	1	1	1	2	0	2
22	2 3 4 1 8 0 5	7	3	6	1 0	1	1	2	0	2	2	2	1	0	1
23	2 4 1 8 0 5 7	3	6	4	2 0	2	2	1	1	2	0	1	0	1	1
24	2 5 8 0 5 7 3	6	4	1	1 1	2	0	0	1	1	2	2	0	2	2
25	2 6 0 5 7 3 6	4	1	8	0 1	1	2	2	2	1	0	1	1	2	0
26	2 7 5 7 3 6 4	1	8	0	2 2	1	0	0	0	0	0	0	1	1	2
27	2 8 7 3 6 4 1	8	0	5	0 0	0	0	1	2	0	2	2	2	1	0
28	3 0 3 3 3 3 3	3	3	3	1 0	1	1	1	0	1	1	1	0	1	1
29	3 1 8 4 7 5 2	1	0	6	0 0	0	0	2	0	2	2	0	1	1	2
30	3 2 4 7 5 2 1	0	6	8	2 0	2	2	2	2	1	0	0	0	0	0
31	3 3 7 5 2 1 0	6	8	4	2 2	1	0	1	1	2	0	2	0	2	2
32	3 4 5 2 1 0 6	8	4	7	1 1	2	0	2	1	0	1	2	2	1	0
33	3 5 2 1 0 6 8	4	7	5	2 1	0	1	1	2	0	2	1	1	2	0
34	3 6 1 0 6 8 4	7	5	2	1 2	0	2	0	2	2	1	2	1	0	1
35	3 7 0 6 8 4 7	5	2	1	0 2	2	1	0	1	1	2	1	2	0	2
36	3 8 6 8 4 7 5	2	1	0	0 1	1	2	0	0	0	0	0	2	2	1
37	4 0 4 4 4 4 4	4	4	4	1 1	2	0	1	1	2	0	1	1	2	0
38	4 1 7 1 5 8 6	3	2	0	0 2	2	1	0	0	0	0	1	0	1	1
39	4 2 1 5 8 6 3	2	0	7	0 0	0	0	2	1	0	1	0	2	2	1
40	4 3 5 8 6 3 2	0	7	1	2 1	0	1	0	1	1	2	0	0	0	0
41	4 4 8 6 3 2 0	7	1	5	0 1	1	2	1	2	0	2	2	1	0	1
42	4 5 6 3 2 0 7	1	5	8	1 2	0	2	2	2	1	0	0	1	1	2
43	4 6 3 2 0 7 1	5	8	6	2 2	1	0	2	0	2	2	1	2	0	2
44	4 7 2 0 7 1 5	8	6	3	2 0	2	2	1	0	1	1	2	2	1	0
45	4 8 0 7 1 5 8	6	3	2	1 0	1	1	0	2	2	1	2	0	2	2
46	5 0 5 5 5 5 5	5	5	5	1 2	0	2	1	2	0	2	1	2	0	2
47	5 1 0 8 2 6 1	7	4	3	1 1	2	0	1	0	1	1	2	1	0	1
48	5 2 8 2 6 1 7	4	3	0	1 0	1	1	0	0	0	0	1	1	2	0
49	5 3 2 6 1 7 4	3	0	8	0 0	0	0	2	2	1	0	1	0	1	1
50	5 4 6 1 7 4 3	0	8	2	2 2	1	0	0	2	2	1	0	0	0	0
51	5 5 1 7 4 3 0	8	2	6	0 2	2	1	2	0	2	2	2	2	1	0
52	5 6 7 4 3 0 8	2	6	1	2 0	2	2	0	1	1	2	0	2	2	1
53	5 7 4 3 0 8 2	6	1	7	0 1	1	2	2	1	0	1	2	0	2	2
54	5 8 3 0 8 2 6	1	7	4	2 1	0	1	1	1	2	0	0	1	1	2
55	6 0 6 6 6 6 6	6	6	6	2 0	2	2	2	0	2	2	2	0	2	2
56	6 1 4 0 1 3 7	2	8	5	2 2	1	0	1	2	0	2	0	2	2	1
57	6 2 0 1 3 7 2	8	5	4	1 2	0	2	1	1	2	0	2	2	1	0
58	6 3 1 3 7 2 8	5	4	0	1 1	2	0	0	0	0	0	1	2	0	2
59	6 4 3 7 2 8 5	4	0	1	0 0	0	0	0	1	1	2	1	1	2	0
60	6 5 7 2 8 5 4	0	1	3	0 1	1	2	1	0	1	1	0	0	0	0
61	6 6 2 8 5 4 0	1	3	7	1 0	1	1	2	1	0	1	0	1	1	2
62	6 7 8 5 4 0 1	3	7	2	2 1	0	1	0	2	2	1	1	0	1	1
63	6 8 5 4 0 1 3	7	2	8	0 2	2	1	2	2	1	0	2	1	0	1
64	7 0 7 7 7 7 7	7	7	7	2 1	0	1	2	1	0	1	2	1	0	1
65	7 1 6 5 0 2 4	8	3	1	1 0	1	1	0	1	1	2	2	2	1	0
66	7 2 5 0 2 4 8	3	1	6	0 1	1	2	2	0	2	2	1	0	1	1
67	7 3 0 2 4 8 3	1	6	5	2 0	2	2	1	2	0	2	0	1	1	2
68	7 4 2 4 8 3 1	6	5	0	1 2	0	2	0	0	0	0	2	0	2	2
69	7 5 4 8 3 1 6	5	0	2	0 0	0	0	0	2	2	1	1	2	0	2
70	7 6 8 3 1 6 5	0	2	4	0 2	2	1	1	1	2	0	0	0	0	0
71	7 7 3 1 6 5 0	2	4	8	1 1	2	0	2	2	1	0	0	2	2	1
72	7 8 1 6 5 0 2	4	8	3	2 2	1	0	1	0	1	1	1	1	2	0
73	8 0 8 8 8 8 8	8	8	8	2 2	1	0	2	2	1	0	2	2	1	0
74	8 1 2 7 6 0 3	5	1	4	0 1	1	2	1	1	2	0	1	2	0	2
75	8 2 7 6 0 3 5	1	4	2	1 1	2	0	0	2	2	1	0	1	1	2
76	8 3 6 0 3 5 1	4	2	7	0 2	2	1	2	1	0	1	1	1	2	0
77	8 4 0 3 5 1 4	2	7	6	2 1	0	1	2	0	2	2	0	2	2	1
78	8 5 3 5 1 4 2	7	6	0	2 0	2	2	0	0	0	0	2	1	0	1
79	8 6 5 1 4 2 7	6	0	3	0 0	0	0	1	0	1	1	2	0	2	2
80	8 7 1 4 2 7 6	0	3	5	1 0	1	1	1	2	0	2	0	0	0	0
81	8 8 4 2 7 6 0	3	5	1	1 2	0	2	0	1	1	2	1	0	1	1

Table 21. $D = (D_1, D_2)$ in Example 7.

	$MCD(D_1, D_2)$			$MCD(D_1, D_2)$	
Run	D_1	D_2	*Run*	D_1	D_2
1	0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	42	4 5 6 3 2 0 7 1 2 0	78 78 33 15 15 33 51 69
2	0 1 1 2 3 4 5 2 1 0	75 75 30 12 57 21 75 75	43	4 6 3 2 0 7 1 2 2 1	55 19 73 73 46 64 19 55
3	0 2 2 3 4 5 6 2 2 1	10 28 46 64 64 46 10 28	44	4 7 2 0 7 1 5 2 0 2	35 17 44 44 80 80 35 17
4	0 3 3 4 5 6 7 0 1 1	25 61 70 52 79 79 34 16	45	4 8 0 7 1 5 8 1 0 1	22 58 67 49 58 22 76 76
5	0 4 4 5 6 7 8 0 2 2	32 14 41 41 14 32 50 68	46	5 0 5 5 5 5 5 1 2 0	51 69 24 60 51 69 24 60
6	0 5 5 6 7 8 1 1 0 1	40 40 58 22 22 58 67 49	47	5 1 0 8 2 6 1 1 1 2	29 11 38 38 65 47 11 29
7	0 6 6 7 8 1 2 1 1 2	47 65 20 56 29 11 38 38	48	5 2 8 2 6 1 7 1 0 1	4 4 4 4 40 40 58 22
8	0 7 7 8 1 2 3 1 2 0	60 24 78 78 42 42 60 24	49	5 3 2 6 1 7 4 0 0 0	72 72 27 9 27 9 36 36
9	0 8 8 1 2 3 4 2 0 2	71 53 17 35 53 71 26 62	50	5 4 6 1 7 4 3 2 2 1	19 55 64 46 1 1 1 1
10	1 1 1 1 1 1 1 0 1 1	16 34 52 70 16 34 52 70	51	5 5 1 7 4 3 0 0 2 2	59 23 77 77 77 77 32 14
11	1 1 5 3 8 7 0 2 0 2	26 62 71 53 44 44 62 26	52	5 6 7 4 3 0 8 2 0 2	17 35 53 71 26 62 71 53
12	1 2 3 8 7 0 4 0 2 2	50 68 23 59 59 23 77 77	53	5 7 4 3 0 8 2 0 1 1	70 52 16 34 61 25 79 79
13	1 3 8 7 0 4 6 1 2 0	33 15 42 42 24 60 69 51	54	5 8 3 0 8 2 6 2 1 0	39 39 57 21 12 30 48 66
14	1 4 7 0 4 6 2 1 0 1	76 76 31 13 49 67 22 58	55	6 0 6 6 6 6 6 2 0 2	62 26 80 80 62 26 80 80
15	1 5 0 4 6 2 5 2 2 1	64 46 10 28 28 10 37 37	56	6 1 4 0 1 3 7 2 2 1	46 64 19 55 19 55 64 46
16	1 6 4 6 2 5 3 2 1 0	3 3 3 3 75 75 30 12	57	6 2 0 1 3 7 2 1 2 0	42 42 60 24 78 78 33 15
17	1 7 6 2 5 3 8 0 0 0	36 36 54 18 63 45 9 27	58	6 3 1 3 7 2 8 1 1 2	2 2 2 2 47 65 20 56
18	1 8 2 5 3 8 7 1 1 2	56 20 74 74 2 2 2 2	59	6 4 3 7 2 8 5 0 0 0	9 27 45 63 36 36 54 18
19	2 0 2 2 2 2 2 0 2 2	23 59 68 50 23 59 68 50	60	6 5 7 2 8 5 4 0 1 1	34 16 43 43 7 7 7 7
20	2 1 3 6 4 1 8 1 2 0	69 51 15 33 6 6 6 6	61	6 6 2 8 5 4 0 1 0 1	67 49 13 31 13 31 49 67
21	2 2 6 4 1 8 0 2 1 0	30 12 39 39 48 66 21 57	62	6 7 8 5 4 0 1 2 1 0	21 57 66 48 30 12 39 39
22	2 3 4 1 8 0 5 1 0 1	58 22 76 76 67 49 13 31	63	6 8 5 4 0 1 3 0 2 2	77 77 32 14 68 50 14 32
23	2 4 1 8 0 5 7 2 0 2	44 44 62 26 35 17 44 44	64	7 0 7 7 7 7 7 2 1 0	66 48 12 30 66 48 12 30
24	2 5 8 0 5 7 3 1 1 2	11 29 47 65 56 20 74 74	65	7 1 6 5 0 2 4 1 0 1	13 31 49 67 76 76 31 13
25	2 6 0 5 7 3 6 0 1 1	79 79 34 16 43 43 61 25	66	7 2 5 0 2 4 8 0 1 1	61 25 79 79 34 16 43 43
26	2 7 5 7 3 6 4 2 2 1	1 1 1 1 10 28 46 64	67	7 3 0 2 4 8 3 2 0 2	53 71 26 62 17 35 53 71
27	2 8 7 3 6 4 1 0 0 0	45 63 18 54 72 72 27 9	68	7 4 2 4 8 3 1 1 2 0	6 6 6 6 60 24 78 78
28	3 0 3 3 3 3 3 1 0 1	31 13 40 40 31 13 40 40	69	7 5 4 8 3 1 6 0 0 0	18 54 63 45 45 63 18 54
29	3 1 8 4 7 5 2 0 0 0	54 18 72 72 9 27 45 63	70	7 6 8 3 1 6 5 0 2 2	41 41 59 23 5 5 5 5
30	3 2 4 7 5 2 1 2 0 2	80 80 35 17 8 8 8 8	71	7 7 3 1 6 5 0 1 1 2	74 74 29 11 20 56 65 47
31	3 3 7 5 2 1 0 2 2 1	37 37 55 19 55 19 73 73	72	7 8 1 6 5 0 2 2 2 1	28 10 37 37 37 37 55 19
32	3 4 5 2 1 0 6 1 1 2	65 47 11 29 74 74 29 11	73	8 0 8 8 8 8 8 2 2 1	73 73 28 10 73 73 28 10
33	3 5 2 1 0 6 8 2 1 0	48 66 21 57 39 39 57 21	74	8 1 2 7 6 0 3 0 1 1	43 43 61 25 52 70 25 61
34	3 6 1 0 6 8 4 1 2 0	24 60 69 51 69 51 15 33	75	8 2 7 6 0 3 5 1 1 2	20 56 65 47 11 29 47 65
35	3 7 0 6 8 4 7 0 2 2	14 32 50 68 50 68 23 59	76	8 3 6 0 3 5 1 0 2 2	68 50 14 32 41 41 59 23
36	3 8 6 8 4 7 5 0 1 1	7 7 7 7 25 61 70 52	77	8 4 0 3 5 1 4 2 1 0	57 21 75 75 21 57 66 48
37	4 0 4 4 4 4 4 1 1 2	38 38 56 20 38 38 56 20	78	8 5 3 5 1 4 2 2 0 2	8 8 8 8 71 53 17 35
38	4 1 7 1 5 8 6 0 2 2	5 5 5 5 32 14 41 41	79	8 6 5 1 4 2 7 0 0 0	27 9 36 36 54 18 72 72
39	4 2 1 5 8 6 3 0 0 0	63 45 9 27 18 54 63 45	80	8 7 1 4 2 7 6 1 0 1	49 67 22 58 4 4 4 4
40	4 3 5 8 6 3 2 2 1 0	12 30 48 66 3 3 3 3	81	8 8 4 2 7 6 0 1 2 0	15 33 51 69 33 15 42 42
41	4 4 8 6 3 2 0 0 1 1	52 70 25 61 70 52 16 34			