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Uniform boundedness of $(SL_2(\mathbb{C}))^n$ and $(PSL_2(\mathbb{C}))^n$

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Abstract: Let G be a group and S be a subset of G. We say that S normally generates G if G is the normal closure of S in G. In this situation, every element $g \in G$ can be written as a product of conjugates of elements of S and their inverses. If $S \subseteq G$ normally generates G, then the length $\|g\|_S \in \mathbb{N}$ of $g \in G$ with respect to S is the shortest possible length of a word in $\mathrm{Conj}_G(S^{\pm 1}) := \{h^{-1}sh | h \in G, s \in S \text{ or } s^{-1} \in S\}$ expressing g. We write $\|G\|_S = \sup\{\|g\|_S \mid g \in G\}$ for any normally generating subset S of G. The conjugacy diameter of any group G is $\Delta(G) := \sup\{\|G\|_S \,| S$ is a finite normally generating subset of $G\}$. We say that G is uniformly bounded if $\Delta(G) < \infty$. This concept is a strengthening of boundedness. Motivated by previously known results approximating $\Delta(G)$ for any algebraic group G, we find the exact values of the conjugacy diameters of the direct product of finitely many copies of $SL_2(\mathbb{C})$ and the direct product of finitely many copies of $PSL_2(\mathbb{C})$. We also prove that if $G_1, \ldots, G_n$ be quasisimple groups such that G_i is uniformly bounded for each $i \in \{1, \ldots, n\}$, then $G_1 \times \cdots \times G_n$ is uniformly bounded. This is also a generalization of some previously known results in the literature.

Keywords: normally generating subsets; word norm; conjugacy diameter
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1. Introduction

1.1. Background on norms and boundedness

Let G be a group. A *norm* on G is a function $\nu : G \longrightarrow [0, \infty)$, which satisfies the following axioms:
(i) $\nu(g) = 0$ if and only if $g = 1$;
(ii) $\nu(g) = \nu(g^{-1})\ \forall\ g \in G$;
(iii) $\nu(gh) \leq \nu(g) + \nu(h)\ \forall\ g, h \in G$.
We call ν *conjugation-invariant* if in addition
(iv) $\nu(g^{-1}hg) = \nu(h)\ \forall\ g, h \in G$.

Examples of conjugation-invariant norms are word norms associated with generating sets invariant under conjugations [5], the commutator length [9], verbal norms [10], the Hofer norm on the group of Hamiltonian diffeomorphisms of a symplectic manifold [21], and others [7, 18, 22].

The *diameter* of a group G with respect to a conjugation-invariant norm ν on G, denoted $\mathrm{diam}_\nu(G)$ or $\mathrm{diam}(\nu)$, is $\sup\{\nu(g) | g \in G\}$. Burago et al. [8] introduced the concept of *bounded groups*, namely groups for which every conjugation invariant norm has a finite diameter. Examples of bounded groups include finite index subgroups in S-arithmetic Chevalley groups [14], the commutator subgroup of Thompson's group [13], the automorphism groups of regular trees [15], and others [12, 18, 25]. On the other hand, if G is not bounded, then G is said to be *unbounded*. Groups with infinite abelianization are examples of unbounded groups [8]. There are also other examples of unbounded groups, and for these examples, we refer the reader to [3, 4, 6].

A normally generating subset of a group is one of the most important sources of conjugation invariant norms. Assume that G is a group and S be a subset of G. The *normal closure of* S in G, denoted by $\langle\langle S \rangle\rangle$, is the smallest normal subgroup of G containing S. In other words, it is the subgroup of G that is generated by all conjugates of elements of S. We say that G is *normally generated* by S if $G = \langle\langle S \rangle\rangle$. In this situation, every element of G can be written as a product of elements of

$$\mathrm{Conj}_G(S^{\pm 1}) := \{h^{-1}sh | h \in G, s \in S \text{ or } s^{-1} \in S\}. \tag{1.1}$$

If $S \subseteq G$ normally generates G, then the *length* $\|g\|_S \in [0, \infty)$ of $g \in G$ with respect to S is defined to be

$$\|g\|_S = \inf\{n \in \mathbb{N} | g = s_1 \cdots s_n \text{ for some } s_1, \ldots, s_n \in \mathrm{Conj}_G(S^{\pm 1})\}.$$

In other words, it is the shortest possible length of a word in $\mathrm{Conj}_G(S^{\pm 1})$ expressing g.

The *word norm* is

$$\| \, . \, \|_S : G \to [0, \infty),$$

$$g \mapsto \|g\|_S.$$

It is easy to see that the word norm $\| \, . \, \|_S$ is a conjugation-invariant norm on G. The *diameter* of a group G with respect to the conjugation-invariant word norm $\| \, . \, \|_S$ is given by

$$\|G\|_S := \sup\{\|g\|_S | g \in G\}.$$

Let G be a group normally generated by a finite set S. It has been shown in [18, Corollary 2.5] that G is bounded if and only if $\|G\|_S < \infty$. This highlights the significance of using word norms for finite normally generating subsets and their diameters to study the boundedness of groups.

In light of [18, Corollary 2.5], there are some refinements of the notion of boundedness. We will introduce some notation to describe these refinements. For any group G and any $n \geq 1$, let

$$\Gamma_n(G) := \{S \subseteq G \mid |S| \leq n \text{ and } S \text{ normally generates } G\},$$

$$\Gamma(G) := \{S \subseteq G \mid |S| < \infty \text{ and } S \text{ normally generates } G\}.$$

Set

$$\Delta_n(G) := \sup\{\|G\|_S \; | S \in \Gamma_n(G)\},$$

$$\Delta(G) := \sup\{\|G\|_S \ |S \in \Gamma(G)\}.$$

The *conjugacy diameter* of any group G is denoted as $\Delta(G)$. The group G is said to be *strongly bounded* if $\Delta_n(G) < \infty$ for all $n \in \mathbb{N}$ (see [18, Definition 1.1]). The group G is said to be *uniformly bounded* if $\Delta(G) < \infty$ (see [18, Definition 1.1]).

1.2. Motivation and statements of results

Recently, conjugacy diameters were studied for several classes of finite and infinite groups. For the case of finite groups, Kaspczyk and the author of this paper studied the conjugacy diameters of non-abelian finite p-groups with cyclic maximal subgroups in [1]. They determined the conjugacy diameters of the semidihedral 2-groups (see [1, Theorem 1.4]), the generalized quaternion groups (see [1, Theorem 1.5]), and the modular p-groups (see [1, Theorem 1.6]). Kędra et al. also proved that $\Delta(PSL(n,q)) \leq 12(n-1)$ for any $n \geq 3$ and any prime power q (see [18, Example 7.2]). Also, Libman and Tarry showed that if G is a non-abelian group of order pq, where p and q are prime numbers with $p < q$, then $\Delta(G) = \max\{\frac{p-1}{2}, 2\}$ (see [20, Thereom 1.1]), and that $\Delta(S_n) = n-1$ for any $n \geq 2$ (see [20, Thereom 1.2]). For the case of infinite groups, we have $\Delta(D_\infty) \leq 4$ (see [18, Example 2.8]). Also, the conjugacy diameters of $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ have been studied, and we have:

Theorem 1.1. *([17, Theorem 3.3]) Let $G := SL_2(\mathbb{C})$. Then G is normally generated by any $g \in G \setminus Z(G)$, and moreover*

$$\|G\|_g = \begin{cases} 2, & \text{if trace}(g) = 0, \\ 3, & \text{otherwise.} \end{cases}$$

Hence, G is uniformly bounded and $\Delta(G) = 3$.

Theorem 1.2. *([17, Theorem 3.4]) Let $G := PSL_2(\mathbb{C})$. Then G is normally generated by any $g \in G \setminus Z(G)$, and moreover, $\|G\|_g = 2$. Hence G is uniformly bounded and $\Delta(G) = 2$.*

The next theorem provides a family of uniformly bounded groups.

Theorem 1.3. *([18, Theorem 4.3]) Let G be a finitely normally generated algebraic group over an algebraically closed field. Let G^0 denote the connected component of the identity. Then G is uniformly bounded and $\Delta(G) \leq 4\dim(G) + \Delta(G/G^0)$.*

Note that by [18, Proposition 4.2], a linear algebraic group over an algebraically closed field is finitely normally generated if and only if it has a finite abelianization.

In general, calculating $\Delta(G)$ for a group G is not easy, and the goal of this paper is to compute this value for some finitely normally generated algebraic groups over $\mathbb{C}$. We show that the bound for $\Delta(G)$ in Theorem 1.3 is not sharp, and we obtain the exact value for the conjugacy diameters of the direct product of finitely many copies of $SL_2(\mathbb{C})$ and the direct product of finitely many copies of $PSL_2(\mathbb{C})$. In particular, we prove the following:

Theorem 1.4. *Let $n \geq 1$ be a natural number and let $G := PSL_2(\mathbb{C})$. Then,*

$$\Delta(G^n) = \Delta_n(G^n) = 2n.$$

Theorem 1.5. *Let $n \geq 1$ be a natural number and let $G := SL_2(\mathbb{C})$. Then,*

$$\Delta(G^n) = \Delta_n(G^n) = 3n.$$

The bound in Theorem 1.3 is far from sharp. As an example, let $G := SL_m(\mathbb{C})$ where $m \geq 1$. We have that G is a linear algebraic group and $\dim(G) = m^2 - 1$ (see [16, Section 7.2]). By [16, Proposition 3.1]), we have $\dim(G^n) = n(m^2-1)$ for every $n \geq 1$. Now, Theorem 1.5 shows that $\Delta((SL_2(\mathbb{C}))^n) = 3n$, whereas Theorem 1.3 yields that $\Delta((SL_2(\mathbb{C}))^n) \leq 4n(2^2 - 1)$. Also, there is some improvement on the bound in Theorem 1.3 for some special cases (see [19, Theorem 1.3]). However, this improvement does not give exact values of $\Delta((SL_2(\mathbb{C}))^n)$ and $\Delta((PSL_2(\mathbb{C}))^n)$.

In order to prove Theorems 1.4 and 1.5, we need first to give an alternative method to prove Theorems 1.1 and 1.2. The new method we used there is based on the so-called *rational canonical form* and a new definition (see Definition 2.2). The importance of this definition will be mentioned in Remark 5.1.

The other main result of this paper is to generalise the following result:

Theorem 1.6. *[17, Lemma 2.20 (c)] Let $n \geq 1$ be a natural number, and let $G_1, \ldots, G_n$ be finitely normally generated groups. Then $G = G_1 \times \cdots \times G_n$ is finitely normally generated, and if the groups G_i are uniformly bounded and simple then G is uniformly bounded.*

Our main result generalises the above result as follows:

Theorem 1.7. *Let $n \geq 1$ be a natural number, and let $G_1, \ldots, G_n$ be quasisimple groups. Suppose that G_i is uniformly bounded for each $i \in \{1, \ldots, n\}$. Then $G = G_1 \times \cdots \times G_n$ is uniformly bounded.*

2. Preliminaries

In this section, we present some results and notation needed for the proofs of Theorems 1.4, 1.5, and 1.7.

Definition 2.1. *([18, Section 2]) Let X be a subset of a group G. For any $n \geq 0$, we define $B_X(n)$ to be the set of all elements of G that can be expressed as a product of at most n conjugates of elements of X and their inverses.*

By Definition 2.1, we have

$$\{1\} = B_X(0) \subseteq B_X(1) \subseteq B_X(2) \subseteq \ldots$$

The next lemma is [18, Lemma 2.3]. We prove the last two parts of this lemma since the proofs are not given in [18].

Lemma 2.1. *Let G be a group; let $X, Y \subseteq G$ and $n, m \in \mathbb{N}$. Then,*

(i) $B_X(n)^{-1} = B_X(n)$ *and* $B_X(n)$ *is invariant under conjugation in* G.

(ii) *If* $X \subseteq Y$ *then* $B_X(n) \subseteq B_Y(n)$.

(iii) $B_X(n)B_X(m) = B_X(n+m)$.

(iv) $Y \subseteq B_X(n) \implies B_Y(m) \subseteq B_X(mn)$.

(v) If $\pi\colon G \to H$ is a surjective group homomorphism, then $B^H_{\pi(X)}(n) = \pi(B^G_X(n))$ for any $X \subseteq G$.

(vi) If $\pi\colon G \to H$ is a surjective group homomorphism, then $B^G_{\pi^{-1}(Y)}(n) = \pi^{-1}(B^H_Y(n))$ for any $Y \subseteq H$.

Proof. (i)–(iv) follow from the definition.

We now prove (v). Let $\pi(g) \in \pi(B^G_X(n))$. Then $g \in B^G_X(n)$. So there exist $x_1, \dots, x_m \in X$ for some $0 \le m \le n$ such that

$$g = \prod_{i=1}^{m} h_i^{-1} x_i^{\epsilon_i} h_i,$$

where $h_i \in G$ and $\epsilon_i \in \{1, -1\}$ for each $1 \le i \le m$. We have

$$\pi(g) = \prod_{i=1}^{m} \pi(h_i)^{-1} \pi(x_i)^{\epsilon_i} \pi(h_i).$$

This shows that $\pi(g) \in B^H_{\pi(X)}(n)$. Therefore, $\pi(B^G_X(n)) \subseteq B^H_{\pi(X)}(n)$.

Now let $h \in B^H_{\pi(X)}(n)$. So there exist $\pi(x_1), \dots, \pi(x_m) \in \pi(X)$ for some $0 \le m \le n$ such that

$$h = \prod_{i=1}^{m} h_i^{-1} \pi(x_i)^{\epsilon_i} h_i,$$

where $h_i \in H$ and $\epsilon_i \in \{1, -1\}$ for each $1 \le i \le m$. For each $1 \le i \le m$, let $\tilde{h}_i \in G$ with $\pi(\tilde{h}_i) = h_i$. Then

$$h = \prod_{i=1}^{m} \pi(\tilde{h}_i)^{-1} \pi(x_i)^{\epsilon_i} \pi(\tilde{h}_i) = \prod_{i=1}^{m} \pi(\tilde{h}_i^{-1} x_i^{\epsilon_i} \tilde{h}_i).$$

This shows that $h \in \pi(B^G_X(n))$. Therefore, $B^H_{\pi(X)}(n) \subseteq \pi(B^G_X(n))$. The proof is complete.

Finally, the proof of (vi). Let $g \in B^G_{\pi^{-1}(Y)}(n)$. Then there are elements $x_1, \dots, x_m$ of $\pi^{-1}(Y)$ for some $0 \le m \le n$ such that

$$g = \prod_{i=1}^{m} h_i^{-1} x_i^{\epsilon_i} h_i,$$

where $h_i \in G$ and $\epsilon_i \in \{1, -1\}$ for each $1 \le i \le m$. We have

$$\pi(g) = \prod_{i=1}^{m} \pi(h_i)^{-1} \pi(x_i)^{\epsilon_i} \pi(h_i).$$

Since $\pi(x_i) \in Y$ for each $1 \le i \le m$, it follows that $\pi(g) \in B^H_Y(n)$. Therefore, $g \in \pi^{-1}(B^H_Y(n))$, and it follows that $B^G_{\pi^{-1}(Y)}(n) \subseteq \pi^{-1}(B^H_Y(n))$.

Suppose now that $g \in \pi^{-1}(B^H_Y(n))$. Then $\pi(g) \in B^H_Y(n)$. So there exist $y_1, \dots, y_m \in Y$ for some $0 \le m \le n$ such that

$$\pi(g) = \prod_{i=1}^{m} h_i^{-1} y_i^{\epsilon_i} h_i,$$

where $h_i \in H$ and $\epsilon_i \in \{1, -1\}$ for each $1 \le i \le m$. For each $1 \le i \le m$, let $\tilde{h}_i, \tilde{y}_i \in G$ with $\pi(\tilde{h}_i) = h_i, \pi(\tilde{y}_i) = y_i$. Then,

$$\pi(g) = \prod_{i=1}^{m} \pi(\tilde{h}_i)^{-1} \pi(\tilde{y}_i)^{\epsilon_i} \pi(\tilde{h}_i) = \prod_{i=1}^{m} \pi(\tilde{h}_i^{-1} \tilde{y}_i^{\epsilon_i} \tilde{h}_i) = \pi(\prod_{i=1}^{m} \tilde{h}_i^{-1} \tilde{y}_i^{\epsilon_i} \tilde{h}_i).$$

Setting $w := \prod_{i=1}^{m} \tilde{h}_i^{-1}\tilde{y}_i^{\epsilon_i}\tilde{h}_i$, it follows that $g = uw$ for some $u \in \ker(\pi)$. We have

$$g = uw = u(\prod_{i=1}^{m} \tilde{h}_i^{-1}\tilde{y}_i^{\epsilon_i}\tilde{h}_i) = \tilde{h}_1^{-1}\tilde{h}_1 u(\prod_{i=1}^{m} \tilde{h}_i^{-1}\tilde{y}_i^{\epsilon_i}\tilde{h}_i) = \tilde{h}_1^{-1}\,\tilde{h}_1\,u\,\tilde{h}_1^{-1}\,\tilde{y}_1^{\epsilon_1}\,\tilde{h}_1\;\cdot\prod_{i=2}^{m} \tilde{h}_i^{-1}\tilde{y}_i^{\epsilon_i}\tilde{h}_i.$$

With $w_0 := \tilde{h}_1\,u\,\tilde{h}_1^{-1}\,\tilde{y}_1^{\epsilon_1}$, we have

$$\pi(w_0) = \pi(\tilde{h}_1 u\tilde{h}_1^{-1}\tilde{y}_1^{\epsilon_1}) = \pi(\tilde{h}_1)\pi(u)\,\pi(\tilde{h}_1)^{-1}\,\pi(\tilde{y}_1)^{\epsilon_1} = \pi(\tilde{h}_1)\,\pi(\tilde{h}_1)^{-1}\,\pi(\tilde{y}_1)^{\epsilon_1} = \pi(\tilde{y}_1)^{\epsilon_1}.$$

It follows that $w_0^{\epsilon_1} \in \pi^{-1}(Y)$. So we have $w_0 \in B^G_{\pi^{-1}(Y)}(1)$. Now $g = \tilde{h}_1^{-1}\,w_0\,\tilde{h}_1\;\cdot\prod_{i=2}^{m} \tilde{h}_i^{-1}\tilde{y}_i^{\epsilon_i}\tilde{h}_i \in B^G_{\pi^{-1}(Y)}(n)$. Therefeore, $\pi^{-1}(B^H_Y(n)) \subseteq B^G_{\pi^{-1}(Y)}(n)$. □

Next, we will introduce Definition 2.2 to prove Theorems 1.1 and 1.2. The significance of this definition will be explained later (see Remark 5.1).

Definition 2.2. *Let X be a subset of a group G. For any $n \geq 0$, define $W_X(n)$ to be the set of all elements of G that can be written as a product of exactly n conjugates of elements of X and their inverses.*

The next result follows easily from the above definitions.

Lemma 2.2. *Let G be a group, let $X \subseteq G$, and let $n, m \in \mathbb{N}$. Then,*

(i) $W_X(n)^{-1} = W_X(n)$ *and* $W_X(n)$ *is invariant under conjugation in G.*

(ii) $W_X(n)W_X(m) = W_X(n + m)$.

(iii) $W_X(n) \subseteq B_X(n)$.

Remark 2.1. *Let G be a group, $X \subseteq G$, and $n \in \mathbb{N}$. In general, we do not have $B_X(n) \subseteq W_X(n)$. For example, let $G := (\mathbb{Z}, +)$, and let X be the set of odd integers. It is not hard to observe that $B_X(2) = \mathbb{Z}$, while $W_X(2) = 2\mathbb{Z}$. Also, $B_X(1) = X \cup \{0\}$, while $W_X(1) = X$.*

Lemma 2.3. *Let G be a group such that $[G, G] = G$ and $G/Z(G)$ is simple. Then every $g \in G \setminus Z(G)$ normally generates G.*

Proof. Let $g \in G \setminus Z(G)$. Set $S := \{g\}$ and $N := \langle\langle S\rangle\rangle$. By the definition, N is the smallest normal subgroup of G containing S. We have to show that $G = N$. Since $g \in N$ and $g \notin Z(G)$, we have $N \nleq Z(G)$. As $N \trianglelefteq G$, Lemma [24, Proposition 6.2(iii)] implies that $G = N$. □

Next, let F be an arbitrary field. We will discuss the rational canonical form over F. Most of the material presented here comes from [11]. Let A be an $n_1 \times n_2$ matrix and B be an $m_1 \times m_2$ matrix. Then their *direct sum*, denoted by $A \oplus B$, is the $(n_1 + m_1) \times (n_2 + m_2)$ matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let A be a square matrix over F. The *characteristic polynomial* of A, denoted by $CP_A(x)$, is the polynomial defined by $CP_A(x) = \det(xI - A)$, where x is a variable. A *monic polynomial* over F is a single variable polynomial with coefficients in F, whose highest order coefficient is equal to one. Therefore, a monic polynomial has the form $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in F[x]$. The *minimal* *polynomial* of A, denoted by $m_A(x)$, is the unique monic polynomial $f(x)$ of smallest degree such that $f(A) = 0$. The *companion matrix* of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ over F is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

We denote the companion matrix of $f(x)$ by $C_{f(x)}$.

A square matrix R over F is said to be a *rational canonical form* if R is a direct sum of companion matrices,

$$R = C_{f_1(x)} \oplus C_{f_2(x)} \oplus \cdots \oplus C_{f_m(x)} = \begin{pmatrix} C_{f_1(x)} & 0 & \cdots & 0 \\ 0 & C_{f_2(x)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & C_{f_m(x)} \end{pmatrix},$$

for monic polynomials $f_1(x), ..., f_m(x)$ of degree at least one such that

$$f_1(x)\big|f_2(x)\big|\ldots\big|f_m(x).$$

The next result is [11, Section 12.2, Theorem 16].

Theorem 2.1. *Let $A \in GL_n(F)$.*

(i) The matrix A is $GL_n(F)$-conjugate to a matrix R in rational canonical form.

(ii) The rational canonical form R for A is unique.

Let A be a square matrix over F. Then the matrix R from Theorem 2.1 is said to be the *rational canonical form* of A. If A is conjugate to a rational canonical form

$$R = C_{f_1(x)} \oplus C_{f_2(x)} \oplus \cdots \oplus C_{f_m(x)},$$

where $f_1(x)\big|f_2(x)\big|\ldots\big|f_m(x)$, then we say that the *invariant factors* of A are $f_1(x), f_2(x), \ldots, f_m(x)$. We call $f_m(x)$ the *largest invariant factor* of A.

The next result gives a link between the characteristic polynomial of a matrix and its invariant factors. This result is quite helpful for determining the invariant factors, especially for matrices of small size.

Proposition 2.1. *Let A be a square matrix over F.*

(i) The product of all the invariant factors of A is the characteristic polynomial of A.

(ii) The minimal polynomial $m_A(x)$ divides the characteristic polynomial $CP_A(x)$.

(iii) The minimal polynomial $m_A(x)$ is the largest invariant factor of A.

Proof. See [11, Section 12.2, Proposition 20] for (i) and (ii). (iii) follows from [11, Section 12.2, Proposition 13]. □

The finding of the characteristic and minimal polynomials determines all the invariant factors for 2×2 and 3×3 matrices. However, for $n \times n$ matrices with $n \geq 4$, the determination of the characteristic and minimal polynomials is generally insufficient for determining all of the invariant factors; see [11, Section 12.2, Examples (3) and (4)]. However, there are algorithms to obtain the invariant factors and convert a given $n \times n$ matrix to rational canonical form; see [11, p. 480] or [2]. We will be interested in 2×2 matrices, and so our main tool in determining invariant factors and rational canonical forms will be Proposition 2.1.

Let us now recall the conjugacy classes of $SL_2(\mathbb{C})$. For all $t \in \mathbb{C}$, $s \in \mathbb{C}^*$, define

$$D(s) := \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \quad U(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \text{ and } \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any element of $SL_2(\mathbb{C})$ is conjugate to one of the matrices defined above; see [23, Section 1.2, Example 9]. More precisely, the following holds for $g \in SL_2(\mathbb{C})$.

(i) If $\mathrm{trace}(g) = \pm 2$, then g is $SL_2(\mathbb{C})$-conjugate to $\pm I$ or $\pm U(1)$.

(ii) If $\mathrm{trace}(g) \neq \pm 2$, then g is $SL_2(\mathbb{C})$-conjugate to $D(t)$ for some $t \in \mathbb{C}^*$ with $\mathrm{trace}(g) = t + t^{-1}$, and t is unique up to replacing t with t^{-1}.

For $\ell \in \{-2, 2\}$, there are only two $SL_2(\mathbb{C})$-conjugacy classes of matrices of trace ℓ. In fact, $\pm U(1)$ and $\pm U(-1)$ are conjugate to each other by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Remark 2.2. *Observe that* $\mathrm{trace}(g) = \mathrm{trace}(g^{-1})$ *for all* $g \in SL_2(\mathbb{C})$. *Let* $g, h \in SL_2(\mathbb{C}) \setminus \{\pm I\}$ *and* $\mathrm{trace}(g) = \mathrm{trace}(h)$. *Then,* g *is* $SL_2(\mathbb{C})$*-conjugate to* h. *Therefore,* $g \sim g^{-1}$ *in* $SL_2(\mathbb{C})$.

Proposition 2.2. *Every non-scalar element* $g \in SL_2(\mathbb{C})$ *is* $GL_2(\mathbb{C})$*-conjugate to the companion matrix* $h := \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$, *where* $x = \mathrm{trace}(g)$. *Moreover,* $C(x) := \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$ *is* $SL_2(\mathbb{C})$*-conjugate to* $C'(x) := \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$.

Proof. Let g be a non-scalar element of $SL_2(\mathbb{C})$. Then $m_g(x)$ has degree 2. Using Proposition 2.1, we deduce that $m_g(x)$ is the only invariant factor of g and that $m_g(x)$ is equal to $CP_g(x)$. So the rational canonical form of g is equal to the companion matrix of $m_g(x) = CP_g(x)$. By a well-known formula for the characteristic polynomial of a 2×2 matrix, we have $CP_g(x) = x^2 - \mathrm{trace}(g)x + 1$. Its companion matrix and hence the rational canonical form of g are given by

$$h = \begin{pmatrix} 0 & -1 \\ 1 & \mathrm{trace}(g) \end{pmatrix}.$$

The first statement of the proposition now follows from Theorem 2.1.

One can show that $C(x)$ is $SL_2(\mathbb{C})$-conjugate to $C'(x)$ by $\begin{pmatrix} -xi & -i \\ -i & 0 \end{pmatrix}$. □

3. Boundedness of $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$

This section is based on [17, Section 3]. We will give a different approach based on the rational canonical form and Definition 2.2 to prove Theorems 1.1 and 1.2. Some results presented here will be needed in Section 5. We need first to introduce some notation and results:

Let $L \subseteq SL_2(\mathbb{C})$, and let us denote the subset $\{\text{trace}(A)|A \in L\}$ of $\mathbb{C}$ by trace(L). Recall the notations $C(x), C'(x), D(s), U(t)$, and I from Section 2. We can prove the following results.

Lemma 3.1. *Suppose that $L \subseteq SL_2(\mathbb{C})$ is conjugation invariant and $L = L^{-1}$. If* $\text{trace}(L) = \mathbb{C}$ *and* $\pm C(2) \in L$, *then* $L \supseteq SL_2(\mathbb{C}) \setminus \{\pm I\}$.

Proof. Let $\pm I \neq g \in SL_2(\mathbb{C})$. We want to show that $g \in L$. It follows from the conjugacy classes of $SL_2(\mathbb{C})$ that we have two cases for g (see the previous section). The first case, if $\text{trace}(g) = \pm 2$, then, g is $SL_2(\mathbb{C})$-conjugate to $\pm C(2)$ (see Proposition 2.2). By hypothesis, we have that $\pm C(2) \in L$ and L is conjugation invariant. So $g \in L$. The second case, if $\text{trace}(g) \neq \pm 2$, then g is $SL_2(\mathbb{C})$-conjugate to $C(t + t^{-1})$ for some $t \in \mathbb{C}^*$ and t is unique up to replacing t with t^{-1} (see Proposition 2.2). Since $\text{trace}(L) = \mathbb{C}$, there exists $\tilde{g} \in L$ such that $\text{trace}(\tilde{g}) = \text{trace}(g)$. So, by Remark 2.2, we have $\tilde{g}$ is $SL_2(\mathbb{C})$-conjugate to g. Since L is conjugation invariant, we have that $g \in L$. This completes the proof. □

Lemma 3.2. *([17, Lemma 3.6]) Let $g \in SL_2(\mathbb{C}) \setminus \{\pm 1\}$ and let $n \geq 1$. Then $-I \in B_g(n+1) \iff -I \in B_g(n)$ or there exists $h \in B_g(n)$ such that $h \neq \pm I$ and* $\text{trace}(h) = -\text{trace}(g)$.

Corollary 3.1. *([17, Corollary 3.7]) Let $g \in SL_2(\mathbb{C}) \setminus \{\pm 1\}$ Then $-I \in B_g(2) \iff$* $\text{trace}(g) = 0$.

Lemma 3.3. *Let $g \in SL_2(\mathbb{C}) \setminus \{\pm 1\}$ and let $n \geq 1$. Then $-I \in W_g(n+1) \iff -I \in W_g(n)$ or there exists $h \in W_g(n)$ such that $h \neq \pm I$ and* $\text{trace}(h) = -\text{trace}(g)$.

Proof. We follow arguments found in the proof of Lemma 3.2. Let $\pm I \neq g \in SL_2(\mathbb{C})$. If $-I \in W_g(n+1) \setminus W_g(n)$, then $-I = h \cdot m$ for some $h \in W_g(n)$ and some $\pm I \neq m \in W_g(1)$. This implies that $h = -m^{-1} \neq \pm I$. Thus, $\text{trace}(h) = -\text{trace}(m^{-1}) = -\text{trace}(m)$ (see Remark 2.2). Since $m \in W_g(1)$, we have $\text{trace}(m) = \text{trace}(k^{-1} g^{\pm 1} k)$ for some $k \in SL_2(\mathbb{C})$. Hence, $\text{trace}(h) = -\text{trace}(k^{-1} g^{\pm 1} k) = -\text{trace}(g)$. Conversely, if $h \in W_g(n)$ such that $h \neq \pm I$ and $\text{trace}(h) = -\text{trace}(g)$, then h is conjugate to $-g^{\pm 1}$. But $W_g(n) = W_g(n)^{-1}$. Hence, $-g^{-1} \in W_g(n)$ and $-I = -g^{-1} \cdot g \in W_g(n) \cdot W_g(1) = W_g(n+1)$. □

Corollary 3.2. *Let $g \in SL_2(\mathbb{C}) \setminus \{\pm I\}$. Then $-I \in W_g(2) \iff$* $\text{trace}(g) = 0$.

Proof. Let $\pm I \neq g \in SL_2(\mathbb{C})$. If $-I \in W_g(2)$, then there exist $m, k \in W_g(1)$ such that $m \cdot k = -I \in W_g(2)$. This implies that $k = -m^{-1}$ and that $\text{trace}(k) = \text{trace}(-m^{-1}) = -\text{trace}(m)$. In fact, we have $m = h^{-1} g^{\pm 1} h$ and $k = h'^{-1} g^{\pm 1} h'$ for some h and h' in $SL_2(\mathbb{C})$. Since the trace is invariant under conjugation and $\text{trace}(g) = \text{trace}(g^{-1})$ in $SL_2(\mathbb{C})$, we have $\text{trace}(m) = \text{trace}(g) = \text{trace}(k)$. But $k = -m^{-1}$. So $\text{trace}(g) = -\text{trace}(g)$. It follows that $\text{trace}(g) = 0$. Conversely, suppose that $\text{trace}(g) = 0$. We want to show that $-I \in W_g(2)$. Since $\text{trace}(g) = 0$, we have g is $SL_2(\mathbb{C})$-conjugate to $C(i + i^{-1})$ (see Proposition 2.2). So, $-I = C(i + i^{-1}) \cdot C(i + i^{-1}) \in W_g(2)$. □

Lemma 3.4. *Suppose that $g \in SL_2(\mathbb{C})$ is conjugate either to $\pm C(2)$ or to $C(t + t^{-1})$ for some $t \neq 0, \pm 1$. Then $W_g(2) \supseteq SL_2(\mathbb{C}) \setminus \{-I\}$.*

Proof. Let $a,\ b,\ v \in \mathbb{C}^*$. Assume that $X = \begin{pmatrix} v+v^{-1} & a^{-1} \\ -a & 0 \end{pmatrix} \in W_g(1)$ and $Y = \begin{pmatrix} v+v^{-1} & b^{-1} \\ -b & 0 \end{pmatrix} \in W_g(1)$. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL_2(\mathbb{C})$. We want to show that

$$\begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} = A^{-1}XAY \in W_g(1) \cdot W_g(1) = W_g(2), \text{where} k \in \mathbb{C}. \tag{3.1}$$

In particular, we want to show that $\operatorname{trace}(W_g(2)) = \mathbb{C}$.

Apply (3.1) with

$$a_1 = a_2 = \frac{(k-1)}{(k-2)},\ a_3 = \frac{1}{(k-1)},\ a_4 = 1,\ a = -\frac{v(k-2)}{(k-1)^2},\ \text{ and } b = v,$$

where $k \notin \{1,\ 2\}$ to see that

$$\operatorname{trace}(W_g(2)) = \mathbb{C} \setminus \{1, 2\}.$$

But

$$I = XX^{-1} \in W_g(2)$$

and

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} v+v^{-1} & v^{-1} \\ -v & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v+v^{-1} & v^{-1} \\ -v & 0 \end{pmatrix} \in W_g(2).$$

Thus $\operatorname{trace}(W_g(2)) = \mathbb{C}$.

We now want to show that $\pm C(2) \in W_g(2)$. Clearly, we have $-C(2) \in W_g(2)$ since $-2 \in \operatorname{trace}(W_g(2))$ and $\begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} \neq -I$. It remains to show that $C(2) \in W_g(2)$. Assume first that $g = C(t + t^{-1})$. Apply (3.1) with

$$a_1 = \frac{1}{(t^2-1)},\ a_2 = -a_3 = -1,\ a_4 = 0,\ v = t,\ a = -\frac{(t^2-1)}{t},\ \text{and } b = t^{-1},$$

where $t \neq 0,\ \pm 1$ obtains $C(2) \in W_g(2)$. Next, assume that $g = \pm C(2)$. Then

$$C(2) \sim \begin{pmatrix} v+v^{-1} & \frac{v^2}{b} \\ -\frac{b}{v^2} & 0 \end{pmatrix} \begin{pmatrix} v+v^{-1} & b^{-1} \\ -b & 0 \end{pmatrix} \in W_g(2)\ (v = \pm 1).$$

We showed that $\operatorname{trace}(W_g(2)) = \mathbb{C}$ and $\pm C(2) \in W_g(2)$. Lemma 3.1 implies that $W_g(2) \supseteq SL_2(\mathbb{C}) \setminus \{-I\}$. □

Corollary 3.3. *Suppose that $g \in SL_2(\mathbb{C})$ is conjugate either to $\pm C(2)$ or to $C(t+t^{-1})$ for some $t \neq 0, \pm 1$. Then $B_g(2) \supseteq SL_2(\mathbb{C}) \setminus \{-I\}$.*

Proof. It follows from Lemma 3.4 that $W_g(2) \supseteq SL_2(\mathbb{C}) \setminus \{-I\}$. Thus $B_g(2) \supseteq SL_2(\mathbb{C}) \setminus \{-I\}$ by Lemma 2.2(iii). □

Corollary 3.4. *Suppose that $g \in SL_2(\mathbb{C})$ is conjugate either to $\pm C(2)$ or to $C(t+t^{-1})$ for some $t \neq 0, \pm 1$. Then $W_g(2+n) = SL_2(\mathbb{C})$ for all $n \geq 1$.*

Proof. Let $\pm I \neq g \in SL_2(\mathbb{C})$. If trace(g) ≤ -2 or trace(g) ≥ 2 then up to conjugacy and taking inverses we assume that either $g = \pm C(2)$ or $g = C(t + t^{-1})$ for some $t \neq 0, \pm 1$.

First, we show that $W_g(3) = SL_2(\mathbb{C})$. It follows from Lemma 3.4 and Corollary 3.2 that $W_g(2) = SL_2(\mathbb{C}) \setminus \{-I\}$. In other words, let $-I \neq h \in SL_2(\mathbb{C})$. Then there exist $g_1, g_2 \in W_g(1)$ such that

$$h = g_1 \cdot g_2 \in W_g(1) \cdot W_g(1) = W_g(2).$$

Using Lemma 3.4 and Corollary 3.2 again, we have

$$g_1 = \tilde{g}_1 \cdot \tilde{\tilde{g}}_2 \in W_g(1) \cdot W_g(1) = W_g(2).$$

So,

$$h = \tilde{g}_1 \cdot \tilde{\tilde{g}}_1 \cdot g_2 \in W_g(3).$$

It remains to show that $-I \in W_g(3)$. This also follows from Lemma 3.4 and Corollary 3.2 since if $W_g(2) = SL_2(\mathbb{C}) \setminus \{-I\}$ then $-g^{-1} \in W_g(2)$ and therefore $g \cdot -g^{-1} = -I \in W_g(3)$. So we have $W_g(3) = SL_2(\mathbb{C})$.

We show next that $W_g(4) = SL_2(\mathbb{C})$. Since $W_g(3) = SL_2(\mathbb{C})$, there are $g_1, g_2, g_3 \in W_g(1)$ such that

$$\tilde{h} = g_1 \cdot g_2 \cdot g_3 \in W_g(3),$$

for all $\tilde{h} \in G$. But

$$g_1 = \tilde{g}_1 \cdot \tilde{\tilde{g}}_2 \in W_g(1) \cdot W_g(1).$$

So,

$$\tilde{h} = \tilde{g}_1 \cdot \tilde{\tilde{g}}_2 \cdot g_2 \cdot g_3 \in W_g(4).$$

This shows that $W_g(4) = SL_2(\mathbb{C})$. Similarly, one can show that $W_g(n+1) = SL_2(\mathbb{C})$ for all $n \geq 4$. This completes the proof. □

Proof of Theorem 1.1. Let $g \in SL_2(\mathbb{C}) \setminus Z(SL_2(\mathbb{C}))$. It follows from Lemma 2.3 that g normally generates $SL_2(\mathbb{C})$. We have $B_g(1) \neq B_h(1)$ for any $h \in SL_2(\mathbb{C})$ with trace$(g) \neq$ trace(h). Then $B_g(1) \neq SL_2(\mathbb{C})$. Now $g^{\pm 1}$ is conjugate either to $\pm C(2)$ or to $C(t + t^{-1})$ for some $t \in \mathbb{C}^*$ with $t \neq \pm 1$. It follows from Corollaries 3.1 and 3.3 that $B_g(2) = SL_2(\mathbb{C}) \setminus \{-I\}$. Then $-g^{-1} \in B_g(2)$ and $g \cdot -g^{-1} = -I \in B_g(3) = SL_2(\mathbb{C})$. Therefore, $\Delta_1(SL_2(\mathbb{C})) = 3$. Moreover, $\Delta(SL_2(\mathbb{C})) = 3$ (see [17, Lemma 2.11]). □

Let $G = SL_2(\mathbb{C})$ and $H = PSL_2(\mathbb{C})$. Let $\pi \colon G \to H$ be the natural projection. If $g \in G$, then it follows from Lemma 2.1(v) that $B_{\pi(g)}(n) = \pi(B_g(n))$ for any $n \in \mathbb{N}$. Since G is finitely normally generated, we have that H is finitely normally generated. Using similar arguments as in the proof of Lemma 2.1(vi), one can show that $\pi^{-1}(B_{\pi(g)}(n)) = B_g(n) \cup -B_g(n)$ for all $n \in \mathbb{N}$.

Proof of Theorem 1.2. Set $h := \pi(g) \in PSL_2(\mathbb{C})$, where $\pm I \neq g \in SL_2(\mathbb{C})$. Since $PSL_2(\mathbb{C})$ has infinitely many conjugacy classes, we have that $B_h(1) \neq PSL_2(\mathbb{C})$. It follows from Corollary 3.3 that $B_g(2) \supseteq SL_2(\mathbb{C}) \setminus \{-I\}$. As $B_h(2) = \pi(B_g(2))$, we have $\pi(SL_2(\mathbb{C}) \setminus \{-I\}) = B_h(2)$. Thus $\Delta_1(PSL_2(\mathbb{C})) = 2$. Moreover, $\Delta(PSL_2(\mathbb{C})) = 2$ (see [17, Lemma 2.11]). □

4. Proof of Theorem 1.7

The key to the proof of Theorem 1.7 is the following lemma.

Lemma 4.1. *Let $n \geq 1$ be a natural number and let $G_1, \ldots, G_n$ be groups. Let $G = G_1 \times \cdots \times G_n$, and let S be a normally generating subset of G. Let $1 \leq i \leq n$ and*

$$\pi_i \colon G \to G_i,$$
$$(g_1, \ldots, g_n) \mapsto g_i.$$

Then $\{\pi_i(s) | s \in S\}$ normally generates G_i.

Proof. Let $g \in G_i$ and let us identify g with $(1_{G_1}, \ldots, 1_{G_{i-1}}, g, 1_{G_{i+1}}, \ldots, 1_{G_n})$. Then g can be written as a product of conjugates of elements of S and their inverses. This implies that g (considered as an element of G_i) can be written as a product of conjugates of elements of $\{\pi_i(s) | s \in S\} \cup \{\pi_i(s)^{-1} | s \in S\}$. So G_i is normally generated by $\{\pi_i(s) | s \in S\}$. □

Proof of Theorem 1.7. We follow arguments found in the proof of Theorem 1.6. For each $1 \leq i \leq n$, let

$$\pi_i \colon G \to G_i,$$
$$(g_1, \ldots, g_n) \mapsto g_i,$$

be the projection of G onto G_i. Let S be an element of $\Gamma(G)$. Let $1 \leq i \leq n$. Since S normally generates G, we have that $\{\pi_i(s) | s \in S\}$ normally generates G_i by Lemma 4.1. As $Z(G_i) \neq G_i$, it follows that there is some $s_i \in S$ with $\pi_i(s_i) \notin Z(G_i)$. We claim that there exists some $y_i \in G_i$ with $[y_i, \pi_i(s_i)] \notin Z(G_i)$. Otherwise, we would have that $[y_i, \pi_i(s_i)] \in Z(G_i)$ for all $y_i \in G_i$. But this would imply that $\pi_i(s_i) \cdot Z(G_i)$ is central in $G_i/Z(G_i)$, which is not possible since $G_i/Z(G_i)$ is nonabelian simple. Now take some $y_i \in G_i$ with $[y_i, \pi_i(s_i)] \notin Z(G_i)$ and define $x_i := [y_i, \pi_i(s_i)]$. By Lemma 2.3, x_i normally generates G_i. Considering x_i and y_i as elements of G, we have $x_i = [y_i, s_i] \in B_S(2)$. So we have $\|x_i\|_S \leq 2$. As x_i normally generates G_i for all $1 \leq i \leq n$, we have that $\{x_1, \ldots, x_n\}$ normally generates G. Since $\|x_i\|_S \leq 2$ for all $1 \leq i \leq n$, it follows that

$$\|G\|_S \leq 2\|G\|_{\{x_1,\ldots,x_n\}}.$$

It is easy to see that

$$\|G\|_{\{x_1,\ldots,x_n\}} \leq \sum_{i=1}^{n} \|G_i\|_{x_i} \leq \sum_{i=1}^{n} \Delta(G_i).$$

It follows that

$$\|G\|_S \leq 2 \sum_{i=1}^{n} \Delta(G_i).$$

Since S was assumed to be an arbitrary element of $\Gamma(G)$, it follows that G is uniformly bounded. □

5. Proofs of Theorems 1.4 and 1.5

The keys to the proofs of Theorems 1.4 and 1.5 are Definition 2.2 and the following proposition.

Proposition 5.1. *Let $n \geq 1$ be a natural number. Let $G_1, \ldots, G_n$, be groups and let N_i be a proper normal subgroup of G_i for each $i \in \{1, \ldots, n\}$. Suppose that, for $1 \leq i \leq n, G_i$ is normally generated by each $g_i \in G_i \setminus N_i$. Suppose, moreover, that $G_i/N_i, 1 \leq i \leq n$, is not cyclic. Let $G = G_1 \times \cdots \times G_n$. Then $\Delta(G) = \Delta_n(G)$.*

Proof. Since $\Gamma_n(G) \subseteq \Gamma(G)$, we have $\Delta_n(G) \leq \Delta(G)$. So it suffices to show that $\Delta(G) \leq \Delta_n(G)$. Let $S \in \Gamma(G)$. Let $1 \leq i \leq n$. Since S normally generates G and since N_i is a proper normal subgroup of G_i, there is some $x_i = (g_{i,1}, \ldots, g_{i,n}) \in S$, such that $g_{i,i} \notin N_i$. Define $T := \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$. We need to show that T normally generates G. By hypothesis, G_i/N_i is not cyclic. In particular, $G_i \neq \langle N_i,\, g_{i,i}\rangle$. Take some $h_i \in G_i \setminus \langle N_i,\, g_{i,i}\rangle$. Since $g_{i,i} \notin N_i$, we have by hypothesis that G_i is normally generated by $g_{i,i}$. So h_i can be written as a product of conjugates of elements of $\{g_{i,i},\, g_{i,i}^{-1}\}$. Hence, there exist a natural number $k \geq 1$, elements $a_1, \ldots, a_k \in G_i$, and $\epsilon_1, \ldots, \epsilon_k \in \{1, -1\}$, such that

$$h_i = \prod_{j=1}^{k} a_j^{-1} g_{i,i}^{\epsilon_j} a_j.$$

Set $\mathrm{Conj}_G(T) := \{g^{-1}tg | g \in G, t \in T\}$. For each $1 \leq j \leq k$, let

$$\overline{a_j} := (1_{G_1}, \ldots, 1_{G_{i-1}}, a_j, 1_{G_{i+1}}, \ldots, 1_{G_n}).$$

Also, set

$$l_i = \sum_{j=1}^{k} \epsilon_j.$$

Then we have

$$((g_{i,1})^{l_i}, \ldots, (g_{i,i-1})^{l_i},\, h_i,\, (g_{i,i+1})^{l_i}, \ldots, (g_{i,n})^{l_i}) = \prod_{j=1}^{k} \overline{a_j}^{\,-1} x_i^{\epsilon_j}\, \overline{a_j} \in \langle \mathrm{Conj}_G(T)\rangle. \tag{5.1}$$

We also have

$$x_i^{-l_i} = ((g_{i,1})^{-l_i}, \ldots, (g_{i,n})^{-l_i}) \in \langle \mathrm{Conj}_G(T)\rangle. \tag{5.2}$$

Multiplying (5.1) with (5.2), we obtain

$$(1_{G_1}, \ldots, 1_{G_{i-1}},\, h_i(g_{i,i})^{-l_i},\, 1_{G_{i+1}}, \ldots, 1_{G_n}) \in \langle \mathrm{Conj}_G(T)\rangle. \tag{5.3}$$

Since $h_i \notin \langle N_i,\, g_{i,i}\rangle$, we have $\tilde{h}_i := h_i(g_{i,i})^{-l_i} \notin \langle N_i,\, g_{i,i}\rangle$ and hence $\tilde{h}_i \notin N_i$. By hypothesis, G_i is normally generated by $\tilde{h}_i$. Hence, each element of G_i can be written as a product of conjugates of $\{\tilde{h}_i,\, \tilde{h}_i^{-1}\}$. Now, since $\langle \mathrm{Conj}_G(T)\rangle \trianglelefteq G$, (5.3) implies that

$$(1_{G_1}, \ldots, 1_{G_{i-1}},\, u_i,\, 1_{G_{i+1}}, \ldots, 1_{G_n}) \in \langle \mathrm{Conj}_G(T)\rangle, \tag{5.4}$$

for any $u_i \in G_i$. It follows that $\langle \mathrm{Conj}_G(T)\rangle = G$. In other words, G is normally generated by T and hence also normally generated by $T_0 := \{x_1, \ldots, x_n\}$. Since $T_0 \subseteq S$, we have $\|G\|_S \leq \|G\|_{T_0}$. Thus,

$$\Delta(G) = \sup\{\|G\|_S \,|\, S \in \Gamma(G)\} \leq \sup\{\|G\|_{T_0} |\, T_0 \in \Gamma_n(G)\} = \Delta_n(G).$$

Since $S \in \Gamma(G)$ was arbitrarily chosen, it follows that $\Delta(G) \leq \Delta_n(G)$. □

Proof of Theorem 1.4. We have $\Delta(G^n) = \Delta_n(G^n)$ by Proposition 5.1 and $\Delta(G^n) \geq 2n$ by [18, Lemma 2.11(b)] and Theorem 1.2. Therefore, it suffices to show that $\Delta(G^n) \leq 2n$.

Let $X \in \Gamma(G^n)$. We are going to show that $G^n = B_X(2n)$. This clearly implies that $\|G^n\|_X \leq 2n$, and as X was arbitrarily chosen, it follows that $\Delta(G^n) \leq 2n$.

Define surjective group homomorphisms

$$\pi_i\colon G^n \to G,$$
$$(g_1, \ldots, g_n) \mapsto g_i,$$

for all $1 \leq i \leq n$. Since X normally generates G^n, there exists some $x_i = (g_{i,1}, \ldots, g_{i,n}) \in X$ with $\pi_i(x_i) \neq I$, for each $1 \leq i \leq n$. In other words, we have that $g_{i,i} \neq I$ for all $1 \leq i \leq n$.

Let $(w_1, w_2, \ldots, w_n) \in G^n$. By Theorem 1.2, $\pi_1(x_1)$ normally generates G, and we have that $\|w_1\|_{\pi_1(x_1)} \leq 2$. It follows from Lemma 2.1(v) that $\pi_1\colon G^n \to G$ maps $B_X(2)$ to $B_{\pi_1(X)}(2)$. Hence, there exists $v \in B_X(2)$ with $\pi_1(v) = w_1$. Clearly, $v = (w_1, v_2, v_3, \ldots, v_n)$ for some $v_2, v_3, \ldots, v_n \in G$. Therefore,

$$(w_1, w_2, \ldots, w_n) = v \cdot (I, v_2^{-1}w_2, v_3^{-1}w_3, \ldots, v_n^{-1}w_n).$$

We will show that $(I, v_2^{-1}w_2, I, \ldots, I) \in B_X(2)$. By Lemma 3.4, we can write $v_2^{-1}w_2$ as a product of exactly two conjugates of $g_{2,2}$ and $g_{2,2}^{-1}$. Also, by Remark 2.2, $g_{2,2}$ and $g_{2,2}^{-1}$ are conjugate. Therefore, $(I, v_2^{-1}w_2, I, \ldots, I)$ can be written as a product of the form

$$(I, v_2^{-1}w_2, I, \ldots, I) = (I, h, I, \ldots, I)^{-1}\, x_2\, (I, h, I, \ldots, I) \cdot (I, \tilde{h}, I, \ldots, I)^{-1}\, x_2^{-1}\, (I, \tilde{h}, I, \ldots, I),$$

for some h and $\tilde{h} \in G$. Hence, $(I, v_2^{-1}w_2, I, \ldots, I) \in W_{x_2}(2) \subseteq W_X(2) \subseteq B_X(2)$.

Similarly, one can see that $(I, I, \ldots, I, v_i^{-1}w_i, I, \ldots, I, I) \in B_{x_i}(2)$, where $3 \leq i \leq n$. This implies that $(I, v_2^{-1}w_2, v_3^{-1}w_3, \ldots, v_n^{-1}w_n) \in B_X(2n-2)$. As $v \in B_X(2)$, it follows that

$$(w_1, w_2, \ldots, w_n) \in B_X(2) \cdot B_X(2n-2) = B_X(2n).$$

So $\Delta(G^n) \leq 2n$. □

The next lemma will be needed to prove Theorem 1.5.

Lemma 5.1. *Let G be a group. Let $n \geq 1$ be a natural number, and let $G^n = G \times \cdots \times G$. Let $x = (g_1, \ldots, g_n) \in G^n$. Suppose that $g \sim g^{-1}$ for all $g \in G$. Then,*

$$W_x(m) = (W_{g_1}(m) \times \cdots \times W_{g_n}(m)) \textit{ for all } m \geq 0.$$

Proof. Pick an element $y \in W_x(m)$. Then there exist $u_1, \ldots, u_m \in G^n$ and $\epsilon_1, \ldots, \epsilon_m \in \{1, -1\}$ such that $y = \prod_{i=1}^m u_i^{-1}x^{\epsilon_i}u_i$. For $1 \leq i \leq m$, let $u_{i,1}, \ldots, u_{i,n} \in G$ such that $u_i = (u_{i,1}, \ldots, u_{i,n})$. Then

$$y = \prod_{i=1}^m (u_{i,1}, \ldots, u_{i,n})^{-1} \cdot (g_1, \ldots, g_n)^{\epsilon_i} \cdot (u_{i,1}, \ldots, u_{i,n}) = \prod_{i=1}^m (u_{i,1}^{-1}, \ldots, u_{i,n}^{-1}) \cdot (g_1^{\epsilon_i}, \ldots, g_n^{\epsilon_i}) \cdot (u_{i,1}, \ldots, u_{i,n}).$$

Hence $y \in (W_{g_1}(m) \times \cdots \times W_{g_n}(m))$. Take an element $z = (z_1, \ldots, z_n)$ of $(W_{g_1}(m) \times \cdots \times W_{g_n}(m))$. Then, for each $1 \leq i \leq n$, we have

$$z_i = h_{i,1}^{-1} g_i^{\epsilon_{i,1}} h_{i,1} \cdots h_{i,m}^{-1} g_i^{\epsilon_{i,m}} h_{i,m},$$

for some $h_{i,1}, \dots, h_{i,m} \in G$ and some $\epsilon_{i,1}, \dots, \epsilon_{i,m} \in \{1, -1\}$. Since $g_i \sim g_i^{-1}$ by hypothesis, we may assume that $\epsilon_{i,1}, \dots, \epsilon_{i,m} = 1$. So we have

$$z = \Big(\prod_{i=1}^{m} h_{1,i}^{-1} g_1 h_{1,i}, \ \dots, \ \prod_{i=1}^{m} h_{n,i}^{-1} g_n h_{n,i} \Big) = \prod_{i=1}^{m} (h_{1,i}, \ \dots, \ h_{n,i})^{-1} \cdot (g_1, \ \dots, \ g_n) \cdot (h_{1,i}, \ \dots, \ h_{n,i}).$$

Thus, z can be written as a product of exactly m conjugates of $x = (g_1, \dots, g_n)$. So $z \in W_x(m)$. □

Corollary 5.1. *Let $n \geq 1$ be a natural number. Let $x = (g_1, \dots, g_n) \in (SL_2(\mathbb{C}))^n$. We have*

$$W_x(m) = (W_{g_1}(m) \times \cdots \times W_{g_n}(m)) \textit{ for all } m \geq 0.$$

Proof. This follows directly from Remark 2.2 and Lemma 5.1. □

Remark 5.1. *Let $n \geq 1$ be a natural number, G be a group, and $x = (g_1, \dots, g_n) \in G^n$. Also, let $m \geq 1$ be a natural number. In general, we do not have*

$$B_x(m) = (B_{g_1}(m) \times \cdots \times B_{g_n}(m)).$$

This is demonstrated by the following example.

Set

$$g := -U(1) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{C}),$$

and

$$x := (g, g) \in (SL_2(\mathbb{C}))^2.$$

In view of the conjugacy classes of $SL_2(C)$ and Definition 2.1, we have

$$B_x(1) = \{(I, I)\} \cup \{(M_1, M_2) \big| M_1, M_2 \in SL_2(\mathbb{C}) \setminus \{-I\} \textit{ and } trace(M_1) = trace(M_2) = -2\},$$

and

$$B_g(1) = \{I\} \cup \{M \big| M \in SL_2(\mathbb{C}) \setminus \{-I\} \textit{ and } trace(M) = -2\}.$$

Now, we have $(I, g) \in (B_g(1) \times B_g(1))$, but $(I, g) \notin B_x(1)$. Hence,

$$B_x(1) \neq (B_g(1) \times B_g(1)).$$

However, we have seen in Corollary 5.1 that

$$W_x(1) = (W_g(1) \times W_g(1)).$$

This explains the importance of Definition 2.2.

Proof of Theorem 1.5. We have $\Delta(G^n) = \Delta_n(G^n)$ by Proposition 5.1 and $\Delta(G^n) \geq 3n$ by [18, Lemma 2.11(b)] and Theorem 1.1. Therefore, it suffices to show that $\Delta(G^n) \leq 3n$.

Let $X \in \Gamma(G^n)$. We are going to show that $G^n = B_X(3n)$. This clearly implies that $\|G^n\|_X \leq 3n$, and as X was arbitrarily chosen, it follows that $\Delta(G^n) \leq 3n$.

Define surjective group homomorphisms

$$\pi_i \colon G^n \to G,$$

$$(g_1, \ldots, g_n) \mapsto g_i,$$

for all $1 \leq i \leq n$. Since X normally generates G^n, there exists some $x_i = (g_{i,1}, \ldots, g_{i,n}) \in X$ with $\pi_i(x_i) \neq \pm I$ for each $1 \leq i \leq n$. In other words, we have that $g_{i,i} \neq \pm I$ for all $1 \leq i \leq n$.

In order to complete the proof, we need the following three claims:

Claim 5.1. *Let* $2 \leq j \leq n$. *Then* $G = W_{g_{2,j}}(3) \cdots W_{g_{n,j}}(3)$.

Proof. Let $2 \leq k \leq n$. As a consequence of Corollary 3.4, we have $W_{g_{k,j}}(3) = G$ or $\{I\}$ or $\{-I\}$. Also, $W_{g_{j,j}}(3) = G$. It follows that $G = W_{g_{2,j}}(3) \cdots W_{g_{n,j}}(3)$. □

Claim 5.2. *We have* $I \in W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)$ *or* $-I \in W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)$.

Proof. Suppose that $g_{j,1} \neq \pm I$ for some $2 \leq j \leq n$. Then we have $W_{g_{j,1}}(3) = G$ by Corollary 3.4. It easily follows that $W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3) = G$, whence the claim holds.

Assume now that $g_{j,1} = \pm I$ for all $2 \leq j \leq n$. Then it is clear that $W_{g_{j,1}}(3) = \{I\}$ or $\{-I\}$ for all $2 \leq j \leq n$. Hence $W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3) = \{I\}$ or $\{-I\}$, and the claim follows. □

Claim 5.3. $W_{x_2}(3) \cdots W_{x_n}(3) = (W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)) \times \cdots \times (W_{g_{2,n}}(3) \cdots W_{g_{n,n}}(3))$.

Proof. By Corollary 5.1, we have

$$\begin{aligned} W_{x_2}(3) \cdots W_{x_n}(3) &= (W_{g_{2,1}}(3) \times \cdots \times W_{g_{2,n}}(3)) \cdots (W_{g_{n,1}}(3) \times \cdots \times W_{g_{n,n}}(3)) \\ &= (W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)) \times \cdots \times (W_{g_{2,n}}(3) \cdots W_{g_{n,n}}(3)). \end{aligned}$$

□

Now let $(w_1, w_2, \ldots, w_n) \in G^n$. Our goal is to show that $(w_1, w_2, \ldots, w_n) \in B_X(3n)$. By Theorem 1.1, $\pi_1(x_1)$ normally generates G, and we have that $\|G\|_{\pi_1(x_1)} \leq 3$. So we have $\|w_1\|_{\pi_1(x_1)} \leq 3$ and $\|-w_1\|_{\pi_1(x_1)} \leq 3$. Hence, w_1 and $-w_1$ are elements of $B_{\pi_1(X)}(3)$.

By Claim 5.2, we have $I \in W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)$ or $-I \in W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)$. We now consider both cases.

Case 1. $I \in W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)$.

By Lemma 2.1(v), there is some $v \in B_X(3)$ with $\pi_1(v) = w_1$. Clearly, $v = (w_1, v_2, \ldots, v_n)$ for some $v_2, \ldots, v_n \in G$. By Claims 5.1 and 5.3, we have

$$(I,\ v_2^{-1}w_2,\ v_3^{-1}w_3,\ \ldots,\ v_n^{-1}w_n) \in W_{x_2}(3) \cdots W_{x_n}(3) \subseteq B_X(3n-3).$$

As $v \in B_X(3)$, it follows that

$$(w_1, w_2, \ldots, w_n) = v \cdot (I,\ v_2^{-1}w_2,\ v_3^{-1}w_3,\ \ldots,\ v_n^{-1}w_n) \in B_X(3n).$$

Case 2. $-I \in W_{g_{2,1}}(3) \cdots W_{g_{n,1}}(3)$.

By Lemma 2.1(v), there is some $v \in B_X(3)$ with $\pi_1(v) = -w_1$. Clearly, $v = (-w_1, v_2, \ldots, v_n)$ for some $v_2, \ldots, v_n \in G$. By Claims 5.1 and 5.3, we have

$$(-I,\ v_2^{-1}w_2,\ v_3^{-1}w_3,\ \ldots,\ v_n^{-1}w_n) \in W_{x_2}(3) \cdots W_{x_n}(3) \subseteq B_X(3n-3).$$

As $v \in B_X(3)$, it follows that

$$(w_1, w_2, \ldots, w_n) = v \cdot (-I,\ v_2^{-1}w_2,\ v_3^{-1}w_3,\ \ldots,\ v_n^{-1}w_n) \in B_X(3n).$$

This completes the proof of Theorem 1.5. □

6. Conclusions

In general, calculating $\Delta(G)$ for a group G is not easy, and the goal of this paper was to compute this value for some finitely normally generated algebraic groups over $\mathbb{C}$. We found the exact values of $\Delta((SL_2(\mathbb{C}))^n)$ and $\Delta((PSL_2(\mathbb{C}))^n)$ for every $n \in \mathbb{N}$. Kędra et al. in [18] showed that every finitely normally generated linear algebraic group G over an algebraically closed field is uniformly bounded and

$$\Delta(G) \leq 4\dim(G) + \Delta(G/G^0), \tag{6.1}$$

where G^0 is the identity component of G. Thus we have improved (6.1) for the particular cases of $(SL_2(\mathbb{C}))^n$ and $(PSL_2(\mathbb{C}))^n$. More precisely, we showed that the bounds in (6.1) are far from sharp. Our main results showed that for any $n \in \mathbb{N}$, we have $\Delta((SL_2(\mathbb{C}))^n) = 3n$ (see Theorem 1.5) and $\Delta((PSL_2(\mathbb{C}))^n) = 2n$ (see Theorem 1.4), whereas formula (6.1) implies that $\Delta((SL_2(\mathbb{C}))^n) \leq 12n$ and $\Delta((PSL_2(\mathbb{C}))^n) \leq 12n$.

We remark that $\Delta(SL_2(\mathbb{C}))$ and $\Delta(PSL_2(\mathbb{C}))$ have been found in [17, Theorem 3.3] and [17, Theorem 3.4]), respectively. However, we used a different approach based on the rational canonical form and Definition 2.2 to study them. This new method was needed to prove Theorems 1.4 and 1.5. Moreover, this approach could be applicable to study $\Delta((SL_2(\mathbb{F}))^n)$ and $\Delta((PSL_2(\mathbb{F}))^n)$, where $\mathbb{F}$ is an arbitrary field. For example, we have that $\Delta(SL_2(\mathbb{R})) = 4$ (see [17, Theorem 3.1]) and $\Delta(PSL_2(\mathbb{R})) = 3$ (see [17, Theorem 3.2]). If we copy the same arguments as in Theorems 1.4 and 1.5, then we conjecture that $\Delta((SL_2(\mathbb{R}))^n) = 4n$ and $\Delta((PSL_2(\mathbb{R}))^n) = 3n$ for every $n \in \mathbb{N}$. We leave this in a future paper and compare that with [19, Theorem 1.4].

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Conflict of interest

The author declares no conflicts of interest.

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