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Research article

Hamiltonian paths passing through matchings in hypercubes with faulty edges

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Abstract: Chen considered the existence of a Hamiltonian cycle containing a matching and avoiding some edges in an n-cube Q_n. In this paper, we considered the existence of a Hamiltonian path and obtained the following result. For $n \geq 4$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq 2n - 6$. Let x and y be two vertices of Q_n with different parities satisfying $xy \notin M$. If all vertices in $Q_n - F$ have a degree of at least 2, then there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$, with the exception of two cases: (1) there exist two neighbors v and t of x (or y) satisfying $d_{Q_n-F}(v) = 2$ and $xt \in M$ (or $yt \in M$); (2) there exists a path $xvuy$ of length 3 satisfying $d_{Q_n-F}(v) = 2$ and $uy \in M$ or $d_{Q_n-F}(u) = 2$ and $xv \in M$.

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1. Introduction

Define $[n]$ as the set $\{1, 2, \ldots, n\}$. An *n-dimensional hypercube* Q_n is a graph with vertex set $V(Q_n) = \{v : v = v^1 \cdots v^n$ and $v^i \in \{0, 1\}$ for every $i \in [n]\}$ and edge set $E(Q_n) = \{uv \mid |D(u, v)| = 1\}$, where $D(u, v) = \{i \in [n] \mid u^i \neq v^i\}$.

One of the most popular and efficient interconnection networks is the hypercube Q_n. It is widely recognized for being Hamiltonian for every $n \geq 2$ [12]. Havel [13] proved that a Hamiltonian path exists in Q_n that connects any two vertices belonging to distinct partite sets. Since then, considerable attention has been given to the research exploring Hamiltonian cycles and paths within hypercubes that exhibit certain additional properties [2, 6, 7, 9, 10, 11, 14, 16, 19, 21, 22].

The *parity* $p(u)$ of a vertex $u = u^1 \cdots u^n$ in Q_n is defined by $p(u) = \sum_{i=1}^{n} u^i (\mathrm{mod}\ 2)$. Then, there are 2^{n-1} vertices with parity 0 and 2^{n-1} vertices with parity 1 in Q_n. Observe that Q_n is bipartite and the parity form bipartite sets of Q_n. Consequently, $p(u) \neq p(v)$ if, and only if, $d(u, v)$ is odd. First, let us revisit the following well-established result, initially derived by Havel in [13].

Theorem 1.1. *[13] Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$. Then, there exists a Hamiltonian path joining x and y in Q_n.*

A forest is deemed *linear* whenever each of its components is a path. Dvořák [7, 8] investigated the problem of Hamiltonian cycles or Hamiltonian paths in hypercubes with given edges and obtained the following results.

Theorem 1.2. *[7] For $n \geq 2$, let $P \subseteq E(Q_n)$ be such that $|P| \leq 2n-3$. Then, there exists a Hamiltonian cycle of Q_n containing P if, and only if, the induced subgraph of P is a linear forest.*

Theorem 1.3. *[8] For $n \geq 2$, let $P \subseteq E(Q_n)$ be such that $|P| \leq 2n-4$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$. If (1) the induced subgraph of P is a linear forest, and (2) the induced subgraph of P does not contain any path joining x and y, and neither x nor y is incident with more than one edge of P, then there exists a Hamiltonian path joining x and y passing through P in Q_n, except the case when $n \in \{3, 4\}$, $d(x, y) = 3$, and P consists of $2n-4$ edges, all of which are of the same dimension and have a pair-wise distance of 2, and each of x, y is incident with one edge of P.*

Let $P \subseteq E(Q_n)$ and $x, y \in V(Q_n)$ satisfying $p(x) \neq p(y)$. We say that $\{x, y, P\}$ is *compatible* if $\{x, y, P\}$ satisfies the conditions (1) and (2) in Theorem 1.3.

A *faulty edge* implies a forbidden edge, which refers to an edge that cannot be traversed when searching for a Hamiltonian cycle or path structure. In [17], Tsai considered the existence of Hamiltonian paths in hypercubes with faulty edges.

Theorem 1.4. *[17] For $n \geq 3$, let $x, y \in V(Q_n)$ and $F \subseteq E(Q_n)$ be such that $p(x) \neq p(y)$ and $|F| \leq 2n-5$. If all vertices in $Q_n - F$ have a degree of at least 2, then there exists a Hamiltonian path joining x and y in $Q_n - F$.*

It is natural to draw the following conclusion from Theorem 1.4.

Corollary 1.5. *[17] For $n \geq 3$, let $F \subseteq E(Q_n)$ with $|F| \leq 2n-5$. If all vertices in $Q_n - F$ have a degree of at least 2, then there exists a Hamiltonian cycle in $Q_n - F$.*

Chen [5] explored the problem of Hamiltonian cycles containing matchings and avoiding some edges in an n-cube Q_n and obtained the following result.

Theorem 1.6. *[5] Let $n \geq 3$, $M \subseteq E(Q_n)$, and $F \subseteq E(Q_n) \setminus M$ with $1 \leq |M| \leq 2n-4-|F|$. If M is a matching and all vertices in $Q_n - F$ have a degree of at least 2, then there exists a Hamiltonian cycle containing M in $Q_n - F$.*

In [5], Chen pointed out that if $|M| = 1$ or $|M| = 2$, then the upper bound of the number of faulty edges tolerated is sharp.

In this paper, we investigate the existence of a Hamiltonian path in Q_n passing through a matching and avoiding faulty edges.

Let M be a matching of Q_n $(n \geq 4)$, and let F be a set of edges in $Q_n - M$ such that $|M \cup F| \leq 2n-6$ and all vertices in $Q_n - F$ have a degree of at least 2. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$ and $xy \notin M$. We show that the following results hold:

(1) If there exist two neighbors v and t of x (or y) satisfying $d_{Q_n-F}(v) = 2$ and $xt \in M$ (or $yt \in M$), then there exists no Hamiltonian path joining x and y passing through M in $Q_n - F$. We denote this deadlock structure by $x/y{-}DS$; see Figure 1(1).

(2) If there exists a path $xvuy$ of length 3 satisfying $d_{Q_n-F}(v) = 2$ and $uy \in M$ or $d_{Q_n-F}(u) = 2$ and $xv \in M$, then there exists no Hamiltonian path joining x and y passing through M in $Q_n - F$. We denote this deadlock structure by $(x, y){-}DS$; see Figure 1(2).

(3) If there is neither $x/y{-}DS$ nor $(x, y){-}DS$ in $Q_n - F$, then there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$.

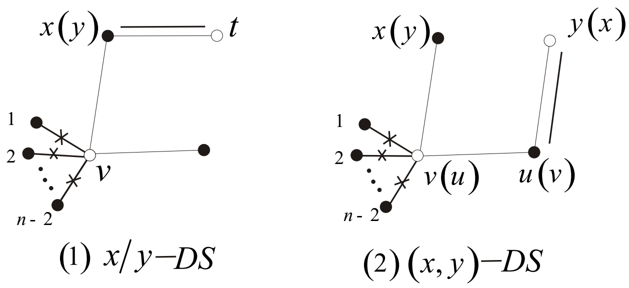

Figure 1. The two Deadlock Structures.

2. Preliminaries and lemmas

The terminology and notation employed in this paper, yet undefined within its scope, are referred in [1]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. In a simple graph G, the number of edges incident with a vertex v is referred to as its degree, denoted as $d_G(v)$ or simply $d(v)$. The minimum degree of a graph G, denoted by $\delta(G)$, is the minimum value of the degrees of all vertices.

Let H and H' denote two subgraphs of a graph G, and let F be a subset of $E(G)$. We use $H + H'$ to represent a graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. We define $H + F$ as the subgraph of G induced by the edge set $E(H) \cup F$. Also, we define $G - F$ as the subgraph of G obtained by removing all edges in F from G. Let $S \subseteq V(G)$, and we denote by $G - S$ the subgraph of G obtained by removing all vertices in S and all edges incident with vertices in S. When $S = \{s\}$ and $F = \{f\}$, we simply write $G - S$, $G - F$, and $H + F$ as $G - s$, $G - f$, and $H + f$.

The distance of vertices u and v in a graph G, denoted by $d_G(u, v)$, is the number of edges in a shortest path joining u and v. The distance $d_G(u, xy)$ of a vertex u and an edge xy in a graph G is defined by $d_G(u, xy) := \min\{d_G(u, x), d_G(u, y)\}$, and the distance $d_G(uv, xy)$ of edges uv and xy in a graph G is defined by $d_G(uv, xy) := \min\{d_G(u, xy), d_G(v, xy)\}$.

The dimension $dim(uv)$ of an edge $uv \in E(Q_n)$ is the integer j such that $D(u, v) = \{j\}$. We denote the set of all j-dimensional edges in Q_n by E_j. For every $j \in [n]$ and $\alpha \in \{0, 1\}$, let $Q^{\alpha}_{n-1,j}$, with the subscripts j being omitted when the context is clear and the $(n-1)$-dimensional sub-cube of Q_n induced by the vertex sets $\{u \in V(Q_n) : u^j = \alpha\}$. Thus, $Q_n - E_j = Q^0_{n-1} + Q^1_{n-1}$. We say that Q_n is *split* into two $(n - 1)$-cubes Q^0_{n-1} and Q^1_{n-1} at position j. Note that both Q^0_{n-1} and Q^1_{n-1} are isomorphic to Q_{n-1}. Given $M \subseteq E(Q_n)$, let $M_\alpha = M \cap E(Q^{\alpha}_{n-1})$ and $M_c = M \cap E_j$. Every vertex $u_\alpha \in V(Q^{\alpha}_{n-1})$ has in $Q^{1-\alpha}_{n-1}$

a unique neighbor, denoted by $u_{1-\alpha}$, and for every edge $e_\alpha = u_\alpha v_\alpha \in E(Q^{\alpha}_{n-1})$, $e_{1-\alpha}$ denotes the edge $u_{1-\alpha}v_{1-\alpha} \in E(Q^{1-\alpha}_{n-1})$.

A set of vertex-disjoint paths of a graph G is a *spanning k-path* if it covers all vertices of G. For a path P in G, we say that *P passes through E* if $E \subseteq E(P)$. Similarly, if $E \subseteq \bigcup_{i=1}^{k} E(P_i)$, then $P_1+\cdots+P_k$ *passes through E*. We use P_{xy} to denote a path joining vertices x and y. Let $P_{xy} = x\cdots u\cdots v\cdots y$, and the sub-path $u\cdots v$ of P_{xy} is denoted by $P_{xy}[u, v]$.

Next, we present some lemmas.

Lemma 2.1. *[18] For $n \geq 2$, let e and f be two disjoint edges in Q_n. Then, Q_n can be split into two $(n-1)$-cubes such that one contains e and the other contains f.*

Lemma 2.2. *[7] For $n \geq 2$, let $x, y \in V(Q_n)$ and $e \in E(Q_n)$ be such that $p(x) \neq p(y)$ and $e \neq xy$. Then, there exists a Hamiltonian path joining x and y passing through e in Q_n.*

Theorem 2.3. *[23] For $n \geq 2$, let $F \subseteq E(Q_n)$ and $L \subseteq E(Q_n) \setminus F$ be such that $|L| + |F| \leq n-2$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$. If $\{x, y, L\}$ is compatible, then there exists a Hamiltonian path joining x and y passing through L in $Q_n - F$.*

A matching is a special type of a linear forest. Thus, we can easily draw the following corollary.

Corollary 2.4. *[23] For $n \geq 2$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq n-2$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$ and $xy \notin M$. Then, there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$.*

3. Spanning 2-paths

In this section, we present some conclusions about spanning 2-paths. When the path P_{vy} is an edge vy, we abbreviate $P_{ux} + P_{vy}$ to $P_{ux} + vy$ for simplicity.

Theorem 3.1. *[7] For $n \geq 2$, let $x, y, u, v \in V(Q_n)$ be pair-wise distinct vertices such that $p(x) \neq p(y)$ and $p(u) \neq p(v)$. Then, (i) there exists a spanning 2-path $P_{xy}+P_{uv}$ in Q_n; (ii) moreover, in the case when $d(u, v) = 1$, path P_{uv} can be chosen such that $P_{uv} = uv$, unless $n = 3$, $d(x, y) = 1$, and $d(xy, uv) = 2$.*

Theorem 3.2. *[3] For $n \geq 4$, let $x, y, u, v \in V(Q_n)$ be pair-wise distinct vertices such that $p(x) = p(y) \neq p(u) = p(v)$. Then, there exists a spanning 2-path $P_{xy} + P_{uv}$ in Q_n.*

Theorem 3.3. *[15] For $n \geq 2$, let $x, y, u \in V(Q_n)$ be pair-wise distinct vertices such that $p(x) = p(y) \neq p(u)$. Then, there exists a Hamiltonian path joining x and y in $Q_n - u$.*

When $u = v$, the notation P_{uv} denotes the path of a single vertex u. Thus, in Theorem 3.2, when $u = v$, the conclusion still holds.

Corollary 3.4. *[3, 15] For $n \geq 4$, let $x, y, u, v \in V(Q_n)$ be such that $x \neq y$ and $p(x) = p(y) \neq p(u) = p(v)$. Then, there exists a spanning 2-path $P_{xy} + P_{uv}$ in Q_n.*

A set $\{u_1, u_2, \ldots, u_{2k-1}, u_{2k}\}$ of distinct vertices of Q_n is *balanced* if the number of odd vertices equals to the number of even vertices.

Lemma 3.5. *[4] For $n \geq 4$, let $\{x, y, u, v\}$ be a balanced vertex set in Q_n and $F \subseteq E(Q_n)$ with $|F| \leq 1$. Then, there exists a spanning 2-path $P_{xy} + P_{uv}$ in $Q_n - F$.*

Theorem 3.6. *[4] For $n \geq 4$, let $F \subseteq E(Q_n)$ be such that $|F| \leq 2n-7$ and the degree of every vertex in $Q_n - F$ is at least 3. Assume that $\{u, x, v, y\}$ is a balanced vertex set in Q_n. Then, there exists a spanning 2-path $P_{ux} + P_{vy}$ in $Q_n - F$.*

Lemma 3.7. *[20] Let ux and vy be two disjoint edges in Q_4 and $e \in E(Q_4)$. If $\{u, v\} \cap V(e) = \emptyset$ and $xy \neq e$, then there exists a spanning 2-path $P_{ux} + P_{vy}$ passing through e in Q_4.*

Lemma 3.8. *For $n \geq 4$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq n-3$. Let ux, vy be two disjoint edges in Q_n such that $\{u, v\} \cap V(M) = \emptyset$ and $xy \notin M$. Then, there exists a spanning 2-path $P_{ux} + P_{vy}$ passing through M in $Q_n - F$.*

Proof. We prove the conclusion by induction on n. When $n = 4$, we have $|M \cup F| \leq 1$. By Lemma 3.5 when $|M| = 0$ and Lemma 3.7 when $|M| = 1$, there exists a spanning 2-path $P_{ux} + P_{vy}$ passing through M in $Q_4 - F$. Assume that the conclusion holds for $n \geq 4$, and we are to show that it holds for $n+1$. Now, $|M \cup F| \leq (n+1) - 3 = n-2$.

Select $j \in [n+1] \setminus (D(u, x) \cup D(v, y))$ such that $|(M \cup F) \cap E_j| = 0$. Split Q_{n+1} into Q_n^0 and Q_n^1 at position j. Without loss of generality, assume that $ux \in E(Q_n^0)$. Let $M_\alpha = M \cap E(Q_n^\alpha)$ and $F_\alpha = F \cap E(Q_n^\alpha)$ for $\alpha \in \{0, 1\}$.

When $vy \in E(Q_n^1)$, we have $|M_0 \cup F_0| \leq n-2$ and $|M_1 \cup F_1| \leq n-2$. By Corollary 2.4, there exist Hamiltonian paths P_{ux} and P_{vy} passing through M_0 and M_1 in $Q_n^0 - F_0$ and $Q_n^1 - F_1$, respectively. Hence, $P_{ux} + P_{vy}$ is the desired spanning 2-path. So, we only need to consider the case that $vy \in E(Q_n^0)$.

Case 1. $|M_0 \cup F_0| \leq n-3$.

By induction hypothesis, there exists a spanning 2-path $P_{ux}^0 + P_{vy}^0$ passing through M_0 in $Q_n^0 - F_0$. Choose an edge $r_0w_0 \in E(P_{ux}^0 + P_{vy}^0) \setminus M_0$ satisfying $\{r_1, w_1\} \cap V(M_1) = \emptyset$. Since $|E(P_{ux}^0 + P_{vy}^0) \setminus M_0| - 4|M_1| = |E(P_{ux}^0 + P_{vy}^0)| - (|M_0| + 4|M_1|) \geq (2^n - 2) - 4(n-2) = 2^n - 4n + 6 > 1$ for $n \geq 4$, such an edge r_0w_0 exists. Without loss of generality, assume that $r_0w_0 \in E(P_{ux}^0)$. Since $M_1 \cup \{r_1w_1\}$ is a matching and $|(M_1 \cup \{r_1w_1\}) \cup F_1| \leq n-1 < 2n-4$ for $n \geq 4$, by Theorem 1.6 there exists a Hamiltonian cycle C_1 containing $M_1 \cup \{r_1w_1\}$ in $Q_n^1 - F_1$. Let $P_{ux} = P_{ux}^0 + C_1 + \{r_0r_1, w_0w_1\} - \{r_0w_0, r_1w_1\}$ and $P_{vy} = P_{vy}^0$. Hence, $P_{ux} + P_{vy}$ is a spanning 2-path passing through M in $Q_{n+1} - F$.

Case 2. $|M_0 \cup F_0| = n-2$. Now, $|M_1| = |F_1| = 0$.

When $|F_0| \geq 1$, choose an edge $r_0w_0 \in F_0$. Since $|M_0 \cup (F_0 \setminus \{r_0w_0\})| = n-3$, by induction hypothesis there exists a spanning 2-path $P_{ux}^0 + P_{vy}^0$ passing through M_0 in $Q_n^0 - (F_0 \setminus \{r_0w_0\})$. If $r_0w_0 \notin E(P_{ux}^0 + P_{vy}^0)$, then choose an edge $s_0t_0 \in E(P_{ux}^0 + P_{vy}^0) \setminus M_0$. If $r_0w_0 \in E(P_{ux}^0 + P_{vy}^0)$, then let $s_0t_0 = r_0w_0$. In the above two cases, without loss of generality, assume that $s_0t_0 \in E(P_{ux}^0)$. In Q_n^1, by Theorem 1.1, there exists a Hamiltonian path $P_{s_1t_1}^1$. Let $P_{ux} = P_{ux}^0 + P_{s_1t_1}^1 + \{s_0s_1, t_0t_1\} - s_0t_0$ and $P_{vy} = P_{vy}^0$. Hence, $P_{ux} + P_{vy}$ is a spanning 2-path passing through M in $Q_{n+1} - F$.

When $|F_0| = 0$, now $|M_0| = n-2$. Since $M_0 \cup \{ux, vy\}$ is a linear forest with $|M_0 \cup \{ux, vy\}| = n < 2n-3$ for $n \geq 4$, by Theorem 1.2 there exists a Hamiltonian cycle C_0 containing $M_0 \cup \{ux, vy\}$ in Q_n^0. Note that $C_0 - ux - vy$ consists of two paths, having endpoints u and x, respectively. We denote them by P_u and P_x. Since $\{u, v\} \cap V(M) = \emptyset$ and $xy \notin M$, we can choose two edges $r_0w_0 \in E(P_u) \setminus M_0$ and $s_0t_0 \in E(P_x) \setminus M_0$. Without loss of generality, assume that r_0 is closer to u than w_0 on P_u and s_0 is closer to x than t_0 on P_x. In Q_n^1, by Theorems 3.1 and 3.2, there exists a spanning 2-path $P_{r_1s_1}^1 + P_{w_1t_1}^1$. Hence, $P_{ux} + P_{vy} = C_0 + P_{r_1s_1}^1 + P_{w_1t_1}^1 + \{r_0r_1, w_0w_1, s_0s_1, t_0t_1\} - \{ux, vy, r_0w_0, s_0t_0\}$ is the desired spanning 2-path. □

Lemma 3.9. *For $n \geq 4$, let ux, vy be two disjoint edges in Q_n, and let $f \in E(Q_n)$ with $vy \neq f$, then there exists a spanning 2-path $P_{ux} + vy$ in $Q_n - f$.*

Proof. By Lemma 2.1, we can split Q_n into Q^0_{n-1} and Q^1_{n-1} such that $ux \in E(Q^0_{n-1})$ and $vy \in E(Q^1_{n-1})$. Next, we distinguish two cases to consider.

If $f \notin E(Q^1_{n-1})$, then by Theorem 1.4 there exists a Hamiltonian path P^0_{ux} in $Q^0_{n-1} - f$. Choose an edge $s_0t_0 \in E(P^0_{ux})$ such that $f \notin \{s_0s_1, t_0t_1\}$ and $d(s_1t_1, vy) \neq 2$ when $n = 4$. By Theorem 3.1 there exists a spanning 2-path $P^1_{s_1t_1} + vy$ in Q^1_{n-1}. Let $P_{ux} = P^0_{ux} + P^1_{s_1t_1} + \{s_0s_1, t_0t_1\} - s_0t_0$. Hence, $P_{ux} + vy$ is a spanning 2-path in $Q_n - f$.

If $f \in E(Q^1_{n-1})$, then by Theorem 1.4 there exists a Hamiltonian path P^1_{vy} in $Q^1_{n-1} - f$. Choose two neighbors r_1, w_1 of v, y on P^1_{vy}. If $r_0w_0 \neq ux$, then by Lemma 2.2 there exists a Hamiltonian path $P^0_{r_0w_0}$ passing through ux in Q^0_{n-1}. Let $P_{ux} = P^0_{r_0w_0} + P^1_{vy}[r_1, w_1] + \{r_0r_1, w_0w_1\} - ux$. Hence, $P_{ux} + vy$ is the desired spanning 2-path. If $r_0w_0 = ux$, then choose an edge $s_1t_1 \in E(P^1_{vy})$ such that $\{s_0, t_0\} \cap \{r_0, w_0\} = \emptyset$ and $d(s_0t_0, r_0w_0) \neq 2$ when $n = 4$. Since $|E(P^1_{vy})| - 4 - 1 > 1$ for $n \geq 4$, such an edge s_1t_1 exists. In Q^0_{n-1}, by Theorem 3.1 there exists a spanning 2-path $P^0_{s_0t_0} + ux$. Let $P_{ux} = P^0_{s_0t_0} + P^1_{vy}[r_1, w_1] + \{uu_1, xx_1, s_0s_1, t_0t_1\} - s_1t_1$. Hence, $P_{ux} + vy$ is the desired spanning 2-path. □

Lemma 3.10. *[20] For $n \geq 5$, let uv and e be two disjoint edges in Q_n, and let $x, y \in V(Q_n) \setminus \{u, v\}$ be such that $p(x) \neq p(y)$ and $xy \neq e$. Then, there exists a spanning 2-path $P_{xy} + uv$ passing through e in Q_n.*

Lemma 3.11. *For $n \geq 4$, let uv and e be two disjoint edges in Q_n, and let $x, y \in V(Q_n) \setminus \{u, v\}$ be such that $p(x) \neq p(y)$ and $xy \neq e$. Then, there exists a spanning 2-path $P_{xy} + uv$ passing through e in Q_n.*

Proof. When $n \geq 5$, the conclusion holds by Lemma 3.10. So, we only need to consider the case that $n = 4$. By Lemma 2.1, we can split Q_4 into Q^0_3 and Q^1_3 such that $e \in E(Q^0_3)$ and $uv \in E(Q^1_3)$.

Case 1. $x, y \in V(Q^0_3)$.

By Lemma 2.2, there exists a Hamiltonian path P^0_{xy} passing through e in Q^0_3. Choose an edge $s_0t_0 \in E(P^0_{xy}) \setminus \{e\}$ such that $\{s_1, t_1\} \cap \{u, v\} = \emptyset$ and $d(uv, s_1t_1) \neq 2$. Since $|E(P^0_{xy}) \setminus \{e\}| - 4 - 1 = 1$, such an edge s_0t_0 exists. In Q^1_3, by Theorem 3.1 there exists a spanning 2-path $P^1_{s_1t_1} + uv$. Let $P_{xy} = P^0_{xy} + P^1_{s_1t_1} + \{s_0s_1, t_0t_1\} - s_0t_0$. Hence, $P_{xy} + uv$ is a spanning 2-path passing through e in Q_4.

Case 2. $x \in V(Q^0_3)$, $y \in V(Q^1_3)$ (or $x \in V(Q^1_3)$, $y \in V(Q^0_3)$).

Since $2^2 > 2$, we can choose a vertex $t_0 \in V(Q^0_3)$ such that $p(t_0) \neq p(x)$, $t_1 \notin \{u, v\}$, and $d(t_1y, uv) \neq 2$. In Q^0_3, by Lemma 2.2 there exists a Hamiltonian path $P^0_{xt_0}$ passing through e. In Q^1_3, by Theorem 3.1 there exists a spanning 2-path $P^1_{t_1y} + uv$. Let $P_{xy} = P^0_{xt_0} + P^1_{t_1y} + t_0t_1$. Hence, $P_{xy} + uv$ is the desired spanning 2-path.

Case 3. $x, y \in V(Q^1_3)$.

By Lemma 2.2 there exists a Hamiltonian path P^1_{xy} passing through uv in Q^1_3. Without loss of generality, assume that u is closer to x than v on P^0_{xy}. Choose two neighbors r_1, w_1 of u, v on $P^1_{xy}[x, u]$ and $P^1_{xy}[y, v]$, respectively. If $r_0w_0 \neq e$, then by Lemma 2.2 there exists a Hamiltonian path $P^0_{r_0w_0}$ passing through e in Q^0_3. Let $P_{xy} = P^1_{xy}[x, r_1] + P^1_{xy}[y, w_1] + P^0_{r_0w_0} + \{r_0r_1, w_0w_1\}$. Hence, $P_{xy} + uv$ is the desired spanning 2-path. If $r_0w_0 = e$, then choose an edge $s_1t_1 \in E(P^1_{xy}) \setminus \{uv\}$ such that $\{s_0, t_0\} \cap \{r_0, w_0\} = \emptyset$ and $d(s_0t_0, r_0w_0) \neq 2$. Since $|E(P^1_{xy}) \setminus \{uv\}| - 4 - 1 = 1$, such an edge s_1t_1 exists. In Q^0_3, by Theorem 3.1 there exists a spanning 2-path $P^0_{s_0t_0} + e$. Let $P_{xy} = P^1_{xy}[x, r_1] + P^1_{xy}[y, w_1] + P^0_{s_0t_0} + e + \{r_0r_1, w_0w_1, s_0s_1, t_0t_1\} - s_1t_1$. Hence, $P_{xy} + uv$ is the desired spanning 2-path. □

Lemma 3.12. *For $n \geq 4$, let swrys be a cycle of length four in Q_n. Let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ such that $\{s, w, r, y\} \cap V(M) = \emptyset$, $ry \notin F$, and $|M \cup F| \leq n - 3$. Then, there exists a spanning 2-path $P_{sw} + ry$ passing through M in $Q_n - F$.*

Proof. Since $swrys$ is a cycle of length four, then $dim(sw) = dim(ry)$ and $dim(rw) = dim(sy)$. We prove the conclusion by induction on n. When $n = 4$, we have $|M \cup F| \leq 1$. Thus, the conclusion holds by Theorem 3.1 when $|M \cup F| = 0$, Lemma 3.9 when $|F| = 1$, and Lemma 3.11 when $|M| = 1$. Assume that the conclusion holds for $n \geq 4$, and we are to show that it holds for $n + 1$. Now, $|M \cup F| \leq (n + 1) - 3 = n - 2$.

Since $n + 1 > n - 2 + 2$, we can select $j \in [n + 1] \setminus (D(s, w) \cup D(r, w))$ such that $|(M \cup F) \cap E_j| = 0$. Then, $|\{sw, ry, rw, sy\} \cap E_j| = 0$. Split Q_{n+1} into Q_n^0 and Q_n^1 at position j. Without loss of generality, assume that $sw, ry \in E(Q_n^0)$.

Case 1. $|M_0 \cup F_0| \leq n - 3$.

By induction hypothesis, there exists a spanning 2-path $P_{sw}^0 + ry$ passing through M_0 in $Q_n^0 - F_0$. Choose an edge $u_0v_0 \in E(P_{sw}^0) \setminus M_0$ such that $\{u_1, v_1\} \cap V(M_1) = \emptyset$. Since $|E(P_{sw}^0 \setminus M_0)| - 4|M_1| = |E(P_{sw}^0)| - |M_0| - 4|M_1| \geq (2^n - 2 - 1) - 4(n-2) = 2^n - 4n + 5 \geq 1$ for $n \geq 4$, such an edge u_0v_0 exists. Since $|(M_1 \cup \{u_1v_1\}) \cup F_1| \leq n - 1 < 2n - 4$ for $n \geq 4$ and $M_1 \cup \{u_1v_1\}$ is a matching, by Theorem 1.6 there exists a Hamiltonian cycle C_1 containing $M_1 \cup \{u_1v_1\}$ in $Q_n^1 - F_1$. Let $P_{sw} = P_{sw}^0 + C_1 + \{u_0u_1, v_0v_1\} - \{u_0v_0, u_1v_1\}$. Hence, $P_{sw}^0 + ry$ is a spanning 2-path passing through M in $Q_{n+1} - F$.

Case 2. $|M_0 \cup F_0| = n - 2$. Now, $|M_1 \cup F_1| = 0$.

When $|M_0| \geq 1$, choose an edge $u_0v_0 \in M_0$. Since $|(M_0 \setminus \{u_0v_0\}) \cup F_0| = n - 3$, by induction hypothesis there exists a spanning 2-path $P_{sw}^0 + ry$ passing through $M_0 \setminus \{u_0v_0\}$ in $Q_n^0 - F_0$. If $u_0v_0 \in E(P_{sw}^0)$, then choose an edge $x_0t_0 \in E(P_{sw}^0) \setminus M_0$. In Q_n^1, by Theorem 1.1 there exists a Hamiltonian path $P_{x_1t_1}^1$. Let $P_{sw} = P_{sw}^0 + P_{x_1t_1}^1 + \{x_0x_1, t_0t_1\} - x_0t_0$. Hence, $P_{sw}^0 + ry$ is the desired spanning 2-path. If $u_0v_0 \notin E(P_{sw}^0)$, then without loss of generality assume that u_0 is closer to s than v_0 on P_{sw}^0. Since $\{s, w\} \cap V(M_0) = \emptyset$, we can choose clockwise neighbors x_0, t_0 of u_0, v_0 on P_{sw}^0. In Q_n^1, by Theorem 1.1 there exists a Hamiltonian path $P_{x_1t_1}^1$. Let $P_{sw} = P_{sw}^0 + P_{x_1t_1}^1 + \{u_0v_0, x_0x_1, t_0t_1\} - \{u_0x_0, v_0t_0\}$. Hence, $P_{sw}^0 + ry$ is the desired spanning 2-path.

When $|M_0| = 0$, we have $|F_0| = n - 2$. Choose an edge $u_0v_0 \in F_0$. Since $|F_0 \setminus \{u_0v_0\}| = n - 3$, by induction hypothesis there exists a spanning 2-path $P_{sw}^0 + ry$ in $Q_n^0 - (F_0 \setminus \{u_0v_0\})$. If $u_0v_0 \notin E(P_{sw}^0)$, then choose an edge $x_0t_0 \in E(P_{sw}^0)$. If $u_0v_0 \in E(P_{sw}^0)$, then let $u_0v_0 = x_0t_0$. In Q_n^1, by Theorem 1.1 there exists a Hamiltonian path $P_{x_1t_1}^1$. Let $P_{sw} = P_{sw}^0 + P_{x_1t_1}^1 + \{x_0x_1, t_0t_1\} - x_0t_0$. Hence, $P_{sw}^0 + ry$ is the desired spanning 2-path. □

4. Main results

Lemma 4.1. *For $n \geq 4$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq 2n - 6$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$ and $xy \notin M$. If $\delta(Q_n - F) = 2$ and if there is neither $x/y-DS$ nor $(x, y)-DS$ in $Q_n - F$, then there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$.*

Proof. We prove the conclusion by induction on n. When $n = 4$, the conclusion holds by Corollary 2.4. Assume that the conclusion holds for $n \geq 4$, and we are to show that it holds for $n + 1$. Now, $|M \cup F| \leq 2(n + 1) - 6 = 2n - 4$.

When $|M| = 0$, the conclusion holds by Theorem 1.4. So, we only need to consider the case that $|M| \geq 1$. Now, $|F| \leq 2n-5$. Thus, there is exactly one vertex, denoted by v, of degree 2 in $Q_{n+1} - F$, and all the other vertices of $Q_{n+1} - F$ are of degree of at least 4. If there is another vertex of degree 2 or 3 in $Q_{n+1} - F$, then $|F| \geq (n+1-2) + (n+1-3) - 1 = 2n-4$. A contradiction occurs.

Let $F_v = \{e \in F \mid e \text{ is incident with } v\}$. Note that $|F| \geq |F_v| = (n+1) - 2 = n-1$. Since $|M| \leq 2n-4-|F| \leq n-3 < |F_v|$, there exists a position j such that $F_v \cap E_j \neq \emptyset$ and $M \cap E_j = \emptyset$. Split Q_{n+1} into Q_n^0 and Q_n^1 at position j. We may assume $v \in V(Q_n^0)$, and denote v by v_0. Let $M_\alpha = M \cap E(Q_n^\alpha)$ and $F_\alpha = F \cap E(Q_n^\alpha)$ for $\alpha \in \{0,1\}$, and let $M_c = M \cap E_j$ and $F_c = F \cap E_j$. Hence, $d_{Q_n^0 - F_0}(v_0) = 2$, all the other vertices of $Q_n^0 - F_0$ are of degree of at least 3, and $\delta(Q_n^1 - F_1) \geq 3$. Note that $|F_0| \geq n-2$, $|M_0 \cup F_0| \leq 2n-5$, and $|M_1 \cup F_1| \leq n-3$.

We claim that Q_n^0 does not contain $x/y{-}DS$ and $(x,y){-}DS$. If not, then Q_{n+1} contains $x/y{-}DS$ or $(x,y){-}DS$. A contradiction occurs.

Case 1. $|M_0 \cup F_0| \leq 2n-6$.

Sub-case 1.1 $x, y \in V(Q_n^0)$. Note that $|F_c| \leq 2n-4-(n-2)-1 = n-3$.

By induction hypothesis, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - F_0$. Choose an edge $s_0t_0 \in E(P_{xy}^0) \setminus M_0$ such that $\{s_0s_1, t_0t_1\} \cap F_c = \emptyset$ and $s_1t_1 \notin M_1$. Since $|E(P_{xy}^0) \setminus M_0| - |M_1| - 2|F_c| = |E(P_{xy}^0)| - (|M_0| + |M_1| + |F_c|) - |F_c| \geq 2^n - 1 - (n-2) - (n-3) = 2^n - 2n + 4 > 1$ for $n \geq 4$, such an edge s_0t_0 exists. Since $|M_1 \cup F_1| \leq n-3 < n-2$, by Corollary 2.4 there exists a Hamiltonian path $P_{s_1t_1}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xy}^0 + P_{s_1t_1}^1 + \{s_0s_1, t_0t_1\} - s_0t_0$ is a Hamiltonian path joining x and y passing through M in $Q_{n+1} - F$.

Sub-case 1.2. $x \in V(Q_n^0)$, $y \in V(Q_n^1)$ (or $x \in V(Q_n^1)$, $y \in V(Q_n^0)$).

Choose a vertex z_0 in Q_n^0 such that $p(x) \neq p(z_0)$, $z_0 \notin V(M_0)$, $z_0z_1 \notin F_c$, $d_{Q_n^0 - F_0}(z_0, v_0) \neq 1$, and $z_1y \notin M_1$. Since $2^{(n+1)-2} - (|M_0| + |F_c|) - n - 1 \geq 2^{n-1} - (n-2) - n - 1 = 2^{n-1} - 2n + 1 \geq 1$ for $n \geq 4$, such a vertex z_0 exists. Note that Q_n^0 does not contain $x{-}DS$, and there is only one vertex v_0 of degree 2 in $Q_n^0 - F_0$. If Q_n^0 contains $z_0{-}DS$, then $z_0 \in V(M_0)$ and $d(z_0, v_0) = 1$. If Q_n^0 contains $(x, z_0){-}DS$, then $z_0 \in V(M_0)$ or $d(z_0, v_0) = 1$. The above two cases both contradict with the choice of z_0. Thus, Q_n^0 does not contain $x/z_0{-}DS$ and $(x, z_0){-}DS$. By induction hypothesis, there exists a Hamiltonian path $P_{xz_0}^0$ passing through M_0 in $Q_n^0 - F_0$. By Corollary 2.4, there exists a Hamiltonian path $P_{z_1y}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xz_0}^0 + P_{z_1y}^1 + z_0z_1$ is a Hamiltonian path joining x and y passing through M in $Q_{n+1} - F$.

Sub-case 1.3 $x, y \in V(Q_n^1)$.

By Corollary 2.4, there exists a Hamiltonian path P_{xy}^1 passing through M_1 in $Q_n^1 - F_1$. Choose an edge $s_1t_1 \in E(P_{xy}^1) \setminus M_1$ such that $\{s_0s_1, t_0t_1\} \cap F_c = \emptyset$ and $\{s_0, t_0\} \cap V(M_0) = \emptyset$. Since $|E(P_{xy}^1) \setminus M_1| - 2|F_c| - 4|M_0| \geq (2^n - 1) - 2 \times 2 - 4(n-4) = 2^n - 4n + 11 > 1$ for $n \geq 4$, such an edge s_1t_1 exists. Since $\{s_0, t_0\} \cap V(M_0) = \emptyset$, we have that Q_n^0 does not contain $s_0/t_0{-}DS$ and $(s_0, t_0){-}DS$. Hence, by induction hypothesis there exists a Hamiltonian path $P_{s_0t_0}^0$ passing through M_0 in $Q_n^0 - F_0$. Hence, $P_{xy} = P_{xy}^1 + P_{s_0t_0}^0 + \{s_0s_1, t_0t_1\} - s_1t_1$ is the desired Hamiltonian path.

Case 2. $|M_0 \cup F_0| = 2n-5$. Now, $|F_c| = 1$ and $|M_c| = |M_1| = |F_1| = 0$. Note that $|F_0| \geq n-2$, $1 \leq |M_0| \leq n-3$.

Sub-case 2.1 $x, y \in V(Q_n^0)$.

If $|F_0| \geq n-1$, then let $r_0w_0 \in F_0 \setminus F_v$. Note that $d_{Q_n^0 - (F_0 \setminus \{r_0w_0\})}(v_0) = 2$ and all the other vertices of $Q_n^0 - (F_0 \setminus \{r_0w_0\})$ are of degree of at least 3. Since $|M_0 \cup (F_0 \setminus \{r_0w_0\})| = 2n-6$, by induction hypothesis there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - (F_0 \setminus \{r_0w_0\})$. If $r_0w_0 \notin E(P_{xy}^0)$, then choose an edge $s_0t_0 \in E(P^0_{xy}) \setminus M_0$ satisfying $v_0 \notin \{s_0, t_0\}$. If $r_0w_0 \in E(P^0_{xy})$, then let $s_0t_0 = r_0w_0$. Since $r_0w_0 \notin F_v$, we also have $v_0 \notin \{s_0, t_0\}$. In Q^1_n, by Theorem 1.1 there exists a Hamiltonian path $P^1_{s_1t_1}$. Hence, $P_{xy} = P^0_{xy} + P^1_{s_1t_1} + \{s_0s_1, t_0t_1\} - s_0t_0$ is the desired Hamiltonian path.

If $|F_0| = n-2$, then $F_0 = F_v \setminus \{v_0v_1\}$. Denote the two edges incident with v_0 in $Q^0_n - F_0$ by u_0v_0 and v_0s_0.

When $v_0 \notin \{x, y\}$, first we claim that $\{x, y, M_0 \cup \{u_0v_0, v_0s_0\}\}$ is compatible. If $v_0 \in V(M_0)$, then let $v_0s_0 \in M_0$. Now, $u_0v_0 \notin M_0$. There are two possibilities for u_0 up to isomorphism; see Figure 2(1)(2). If $v_0 \notin V(M_0)$, then there are three possibilities for $\{u_0, s_0\}$ up to isomorphism; see Figure 2(3)-(5). Thus, $M_0 \cup \{u_0v_0, v_0s_0\}\}$ is a linear forest. Next, we will verify that $\{x, y, M_0 \cup \{u_0v_0, v_0s_0\}\}$ satisfies the condition (2) in Theorem 1.3.

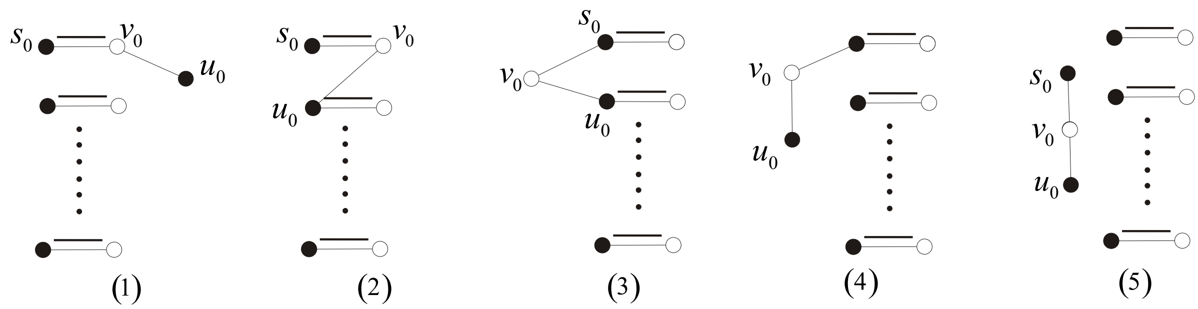

Figure 2. All possibilities for $M_0 \cup \{u_0v_0, v_0s_0\}$ up to isomorphism.

For the case (1) in Figure 2, if x or y is an internal vertex, then $x = v_0$ or $y = v_0$. A contradiction occurs. If x, y are endpoints of a path, then since $p(x) \neq p(y)$, we have $xy \in M_0$. A contradiction occurs. For the case (2) in Figure 2, if x or y is an internal vertex, then since $v_0 \notin \{x, y\}$, we have that x or y is the vertex u_0. Now, x (or y) is incident with a matching edge, and $d_{Q^0_n - F_0}(v_0) = 2$. Thus, Q^0_n contains $x/y{-}DS$. A contradiction occurs. If x, y are endpoints of a path, then since $xy \notin M_0$, we have that x, y are the two endpoints of the path of length 3 in Figure 2(2). Note that $d_{Q^0_n - F_0}(v_0) = 2$, thus, Q^0_n contains $(x, y){-}DS$. A contradiction occurs. For the case (3) in Figure 2, if x or y is an internal vertex, then since $v_0 \notin \{x, y\}$, we have that x (or y) is s_0 or u_0. Now, Q^0_n contains $x/y{-}DS$. A contradiction occurs. If x, y are endpoints of a path, then $xy \in M_0$. A contradiction occurs. For the case (4) in Figure 2, if x or y is an internal vertex, then since $v_0 \notin \{x, y\}$, we have that x (or y) is the vertex s_0 in Figure 2(4). Now, Q^0_n contains $x/y{-}DS$. A contradiction occurs. If x, y are endpoints of a path, then since $xy \notin M_0$, we have that x, y are endpoints of the path of length 3 in Figure 2(4). Thus, Q^0_n contains $(x, y){-}DS$. A contradiction occurs. For the case (5) in Figure 2, since $v_0 \notin \{x, y\}$, $p(x) \neq p(y)$, and $xy \notin M_0$, we have that $\{x, y, M_0 \cup \{u_0v_0, v_0s_0\}\}$ also satisfies the condition (2). In conclusion, $\{x, y, M_0 \cup \{u_0v_0, v_0s_0\}\}$ satisfies the condition (2). The claim is proved.

Since $|M_0 \cup \{u_0v_0, v_0s_0\}| \leq n-1 < 2n-4$ for $n \geq 4$, by Theorem 1.3 there exists a Hamiltonian path P^0_{xy} passing through $M_0 \cup \{u_0v_0, v_0s_0\}$ in Q^0_n. Note that all faulty edges incident with v_0 do not lie on the path P^0_{xy}. Thus, we have $E(P^0_{xy}) \cap F_0 = \emptyset$.

When $v_0 \in \{x, y\}$, we may assume $v_0 = x$. If $u_0v_0 \in M_0$ or $v_0s_0 \in M_0$, then without loss of generality assume that $u_0v_0 \in M_0$, and now $v_0s_0 \notin M_0$. Thus, $M_0 \cup \{u_0v_0\} = M_0$ and $\{x, y, M_0 \cup \{u_0v_0\}\}$ are compatible. If $u_0v_0 \notin M_0$ and $v_0s_0 \notin M_0$, then $v_0 \notin V(M_0)$. Since $xy \neq u_0v_0$ or $xy \neq v_0s_0$, without loss of generality, assume that $xy \neq u_0v_0$. Thus, $M_0 \cup \{u_0v_0\}$ is a linear forest with $d(v_0) = 1$ in it. Since $xy \notin M_0$, $xy \neq u_0v_0$, and $p(x) \neq p(y)$, we have that $\{x, y, M_0 \cup \{u_0v_0\}\}$ is compatible. Since $|M_0 \cup \{u_0v_0\}| \leq n-2 < 2n-4$ for $n \geq 4$, by Theorem 1.3 there exists a Hamiltonian path P^0_{xy} passing through $M_0 \cup \{u_0v_0\}$ in Q^0_n. Note that $v_0 = x$ and u_0v_0 lies on the path P^0_{xy}, and we also have $E(P^0_{xy}) \cap F_0 = \emptyset$.

In the above two cases, choose an edge $s_0t_0 \in E(P^0_{xy}) \setminus M_0$ satisfying $v_0 \notin \{s_0, t_0\}$. In Q^1_n, by Theorem 1.1 there exists a Hamiltonian path $P^1_{s_1t_1}$. Hence, $P_{xy} = P^0_{xy} + P^1_{s_1t_1} + \{s_0s_1, t_0t_1\} - s_0t_0$ is the desired Hamiltonian path.

Sub-case 2.2. $x \in V(Q^0_n)$, $y \in V(Q^1_n)$ (or $x \in V(Q^1_n)$, $y \in V(Q^0_n)$).

Since $|M_0 \cup F_0| = 2n-5 < 2n-4$, $|M_0| \geq 1$, and $\delta(Q^0_n - F_0) \geq 2$, by Theorem 1.6 there exists a Hamiltonian cycle C_0 containing M_0 in $Q^0_n - F_0$. Choose a neighbor s_0 of x on C_0 such that $xs_0 \notin M_0$ and $s_0 \neq v_0$. We claim that the vertex s_0 exists. If s_0 does not exist, then in one direction along the cycle C_0, x is adjacent to v_0, and in the other direction, x is incident with a matching edge. Thus, Q^0_n contains x–DS. A contradiction occurs. Since $p(s_1) \neq p(y)$, by Theorem 1.1 there exists a Hamiltonian path $P^1_{s_1y}$ in Q^1_n. Hence, $P_{xy} = C_0 + P^1_{s_1y} + s_0s_1 - xs_0$ is the desired Hamiltonian path.

Sub-case 2.3. $x, y \in V(Q^1_n)$.

Since $|M_0 \cup F_0| = 2n-5 < 2n-4$, $|M_0| \geq 1$, and $\delta(Q^0_n - F_0) \geq 2$, by Theorem 1.6 there exists a Hamiltonian cycle C_0 containing M_0 in $Q^0_n - F_0$. Choose an edge $s_0t_0 \in E(C_0) \setminus M_0$ such that $v_0 \notin \{s_0, t_0\}$ and $s_1t_1 \neq xy$. Since $|E(C_0) \setminus M_0| - 2 - 1 \geq 2^n - |M_0| - 3 \geq 2^n - (n-3) - 3 = 2^n - n > 1$ for $n \geq 4$, such an edge s_0t_0 exists. By Lemma 2.2, there exists a Hamiltonian path P^1_{xy} passing through s_1t_1 in Q^1_n. Hence, $P_{xy} = P^1_{xy} + C_0 + \{s_0s_1, t_0t_1\} - \{s_0t_0, s_1t_1\}$ is the desired Hamiltonian path. □

Lemma 4.2. *For $n \geq 4$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq 2n-6$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$ and $xy \notin M$. If $\delta(Q_n - F) = 3$, then there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$.*

Proof. We prove the conclusion by induction on n. When $n = 4$, the conclusion holds by Corollary 2.4. Assume that the conclusion holds for $n \geq 4$, and we are to show that it holds for $n+1$. Now, $|M \cup F| \leq 2(n+1) - 6 = 2n-4$.

By Theorem 1.4, we only need to consider the case that $|M| \geq 1$. Now, $|F| \leq 2n-5$. Thus, there are at most two vertices of degree 3 in $Q_{n+1} - F$. If there are three vertices of degree 3 in $Q_{n+1} - F$, then $|F| \geq 3(n+1-3) - 2 = 3n-8 > 2n-5$ for $n \geq 4$. A contradiction occurs.

Case A. There are two vertices of degree 3, denoted by v and v', in $Q_{n+1} - F$.

If $vv' \notin F$, then $|F| \geq 2(n+1-3) = 2n-4$. A contradiction occurs. Thus, $vv' \in F$. Now, $|F| = 2n-5$, $|M| = 1$, and all the other vertices (except v and v') of $Q_{n+1} - F$ are of degree of at least 4. Let $D(v, v') = \{j\}$. Split Q_{n+1} into Q^0_n and Q^1_n at position j. We may assume $v \in V(Q^0_n)$, and denote v by v_0. Then, $v' = v_1$, $\delta(Q^0_n - F_0) = 3$, and $\delta(Q^1_n - F_1) = 3$. Note that $|F_0| = |F_1| = n-3$ and $|M| = 1$. Let $M = \{e\}$.

Case 1. $e \notin E_j$. Without loss of generality, assume that $e \in E(Q^0_n)$.

Sub-case 1.1 $x, y \in V(Q^0_n)$.

Since $|M_0 \cup F_0| = n-2$, by Corollary 2.4 there exists a Hamiltonian path P^0_{xy} passing through M_0 in $Q^0_n - F_0$. Choose an edge $s_0t_0 \in E(P^0_{xy}) \setminus M_0$ such that $v_0 \notin \{s_0, t_0\}$. Since $|E(P^0_{xy}) \setminus M_0| - 2 = 2^n - 4 > 1$ for $n \geq 4$, such an edge s_0t_0 exists. Since $|F_1| = n-3 < 2n-5$ for $n \geq 4$, by Theorem 1.4 there exists a Hamiltonian path $P^1_{s_1t_1}$ in $Q^1_n - F_1$. Hence, $P_{xy} = P^0_{xy} + P^1_{s_1t_1} + \{s_0s_1, t_0t_1\} - s_0t_0$ is a Hamiltonian path joining x and y passing through M in $Q_{n+1} - F$.

Sub-case 1.2. $x \in V(Q^0_n)$, $y \in V(Q^1_n)$ (or $x \in V(Q^1_n)$, $y \in V(Q^0_n)$).

Since $2^{n-1} > 2$, we can choose a vertex z_0 in Q^0_n such that $p(x) \neq p(z_0)$, $z_0 \neq v_0$, and $xz_0 \neq e$. By Corollary 2.4, there exists a Hamiltonian path $P^0_{xz_0}$ passing through M_0 in $Q^0_n - F_0$. By Theorem 1.4, there exists a Hamiltonian path $P^1_{z_1y}$ in $Q^1_n - F_1$. Hence, $P_{xy} = P^0_{xz_0} + P^1_{z_1y} + z_0z_1$ is the desired Hamiltonian path.

Sub-case 1.3. $x, y \in V(Q^1_n)$.

By Theorem 1.4, there exists a Hamiltonian path P^1_{xy} in $Q^1_n - F_1$. Choose an edge $s_1t_1 \in E(P^1_{xy})$ such that $v_0 \notin \{s_0, t_0\}$ and $s_0t_0 \neq e$. Since $|E(P^1_{xy})| - 3 > 1$ for $n \geq 4$, such an edge s_1t_1 exists. By Corollary 2.4, there exists a Hamiltonian path $P^0_{s_0t_0}$ passing through M_0 in $Q^0_n - F_0$. Hence, $P_{xy} = P^1_{xy} + P^0_{s_0t_0} + \{s_0s_1, t_0t_1\} - s_1t_1$ is the desired Hamiltonian path.

Case 2. $e \in E_j$. Let $e = r_0r_1$. Note that $r_0 \neq v_0$.

Sub-case 2.1 $x, y \in V(Q^0_n)$ (or $x, y \in V(Q^1_n)$).

Since $n > 2$, we can choose a neighbor w_0 of r_0 in Q^0_n such that $w_0 \neq v_0$ and $r_0w_0 \neq xy$. Since $|\{r_0w_0\} \cup F_0| = n - 2$, by Corollary 2.4 there exists a Hamiltonian path P^0_{xy} passing through r_0w_0 in $Q^0_n - F_0$. By Theorem 1.4, there exists a Hamiltonian path $P^1_{r_1w_1}$ in $Q^1_n - F_1$. Hence, $P_{xy} = P^0_{xy} + P^1_{r_1w_1} + \{r_0r_1, w_0w_1\} - r_0w_0$ is the desired Hamiltonian path.

Sub-case 2.2. $x \in V(Q^0_n)$, $y \in V(Q^1_n)$ (or $x \in V(Q^1_n)$, $y \in V(Q^0_n)$).

When $p(x) \neq p(r_0)$, by Theorem 1.4 there exist Hamiltonian paths $P^0_{xr_0}$ and $P^1_{r_1y}$ in $Q^0_n - F_0$ and $Q^1_n - F_1$, respectively. Hence, $P_{xy} = P^0_{xr_0} + P^1_{r_1y} + r_0r_1$ is the desired Hamiltonian path.

When $p(x) = p(r_0)$, since $xy \neq r_0r_1$, by symmetry we may assume $y \neq r_1$. Since $n > 1$, we can choose a neighbor w_1 of r_1 in Q^1_n satisfying $w_1 \neq v_1$. Since $n > 2$, we can choose a neighbor s_1 of y in Q^1_n satisfying $s_1 \notin \{v_1, w_1\}$. Since $|\{r_0w_0\} \cup F_0| = n - 2$, by Corollary 2.4 there exists a Hamiltonian path $P^0_{xs_0}$ passing through r_0w_0 in $Q^0_n - F_0$. Note that $\{r_1, w_1, s_1, y\}$ is a balanced vertex set. Since $|F_1| = n - 3 \leq 2n - 7$ for $n \geq 4$ and $\delta(Q^1_n - F_1) \geq 3$, by Theorem 3.6 there exists a spanning 2-path $P^1_{r_1w_1} + P^1_{s_1y}$ in $Q^1_n - F_1$. Hence, $P_{xy} = P^0_{xs_0} + P^1_{r_1w_1} + P^1_{s_1y} + \{s_0s_1, r_0r_1, w_0w_1\} - r_0w_0$ is the desired Hamiltonian path.

Case B. There is exactly a vertex, denoted by v, of degree 3 in $Q_{n+1} - F$.

Now, all the other vertices in $Q_n - F$ are of degree of at least 4. Let $F_v = \{e \in F \mid e$ is incident with $v\}$. Note that $|F| \geq |F_v| = (n + 1) - 3 = n - 2$ and $|M| \leq 2n - 4 - |F| \leq n - 2$.

Case 1. $|M| = n - 2$. Now, $|F| = |F_v| = n - 2$.

Denote the three edges incident with v in $Q_{n+1} - F$ by uv, vr, and vs. If $v \in V(M)$, then let $uv \in M$.

When $v \notin \{x, y\}$, first we claim that $\{x, y, M \cup \{uv, vs\}\}$ is compatible or $\{x, y, M \cup \{uv, vr\}\}$ is compatible. Note that $M \cup \{uv, vs\}$ and $M \cup \{uv, vr\}$ are both linear forest. If $\{x, y, M \cup \{uv, vs\}\}$ is not compatible, then $\{x, y, M \cup \{uv, vs\}\}$ does not satisfy the condition (2) in Theorem 1.3. Now there are three possibilities for $\{M \cup \{uv, vs\}\}$ up to isomorphism; see Figure 3(1)-(3). Next, we will verify that $\{x, y, M \cup \{uv, vr\}\}$ satisfies the condition (2) in Theorem 1.3. Note that there are two cases for r, uncovered by M or covered by M. We denote the two cases by r' and r''. If $r \notin V(M)$, then let $r = r'$; if $r \in V(M)$, then let $r = r''$; see Figure 3.

If x or y is an internal vertex of the induced subgraph of $\{M \cup \{uv, vs\}\}$, then for the case (1)(2) in Figure 3, $x = s$ or $y = s$; for the case (3) in Figure 3, $x = s$ or u, or $y = s$ or u. Note that x and y cannot be s and u simultaneously because $p(x) \neq p(y)$. For the above three cases, without loss of generality, assume that $x = s$. Since $p(x) \neq p(y)$, we have $y \notin \{u, r\}$. Note that $y \neq v$ and $xy \notin M$. Thus, $\{x, y, M \cup \{uv, vr\}\}$ satisfies the condition (2) in Theorem 1.3. If x, y are endpoints of a path in the induced subgraph of $\{M \cup \{uv, vs\}\}$, then since $p(x) \neq p(y)$ and $xy \notin M$, we have that $\{M \cup \{uv, vs\}\}$ must be the case (1) or (2) in Figure 3 and x, y exactly are the two endpoints of the path of length 3. Note that the two internal vertices of this path are v and s, and $r \notin \{x, y, v, s\}$. Hence, $\{x, y, M \cup \{uv, vr\}\}$ satisfies the condition (2) in Theorem 1.3. The claim is proved. Without loss of generality, assume that $\{x, y, M \cup \{uv, vs\}\}$ is compatible.

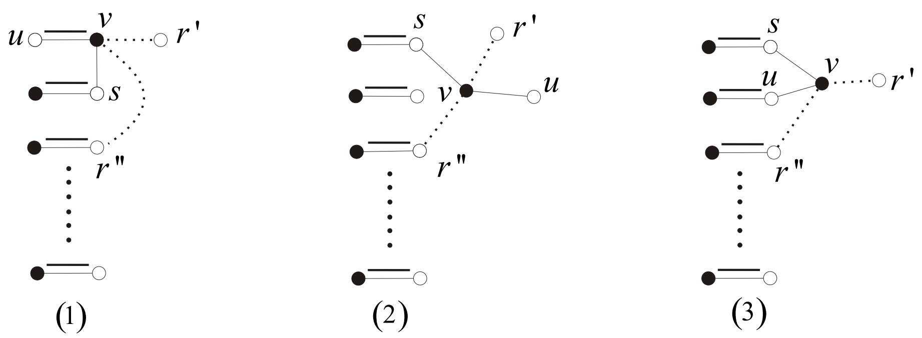

Figure 3. All possibilities for $M \cup \{uv, vs, vr\}$ up to isomorphism.

Since $|M \cup \{uv, vs\}| \leq n < 2(n + 1) - 4$ for $n \geq 4$, by Theorem 1.3 there exists a Hamiltonian path P_{xy} passing through $M \cup \{uv, vs\}$ in Q_{n+1}. Note that all faulty edges incident with v do not lie on the path P_{xy}. Thus, we have $E(P_{xy}) \cap F = \emptyset$.

When $v \in \{x, y\}$, we may assume $v = x$. If $v \in V(M)$, then now $uv \in M$ and $vr, vs \notin M$. Thus, $M \cup \{uv\} = M$ and $\{x, y, M\}$ is compatible. If $v \notin V(M)$, then since $xy \neq uv$ or $xy \neq vr$ or $xy \neq vs$, without loss of generality, assume that $xy \neq uv$. So, $\{x, y, M \cup \{uv\}\}$ is compatible. Since $|M \cup \{uv\}| \leq n - 1 < 2(n + 1) - 4$ for $n \geq 4$, by Theorem 1.3 there exists a Hamiltonian path P_{xy} passing through M in $Q_{n+1} - F$.

Case 2. $|M| \leq n - 3$.

Since $|M| \leq n - 3 < n - 2 = |F_v|$, there exists a position $j \in [n + 1]$ such that $|F_v \cap E_j| = 1$ and $|M \cap E_j| = 0$. Split Q_{n+1} into Q_n^0 and Q_n^1 at position j. Assume $v \in V(Q_n^0)$, and denote v by v_0. Hence, $\delta(Q_n^0 - F_0) = 3$ and $\delta(Q_n^1 - F_1) \geq 3$. Note that $|F_0| \geq n - 3$, $|M_0 \cup F_0| \leq 2n - 5$, and $|M_1 \cup F_1| \leq n - 2$.

Sub-case 2.1. $|M_0 \cup F_0| \leq 2n - 6$. Note that $|F_c| \leq 2n - 4 - (n - 3) - 1 = n - 2$.

Sub-case 2.1.1 $x, y \in V(Q_n^0)$ (or $x, y \in V(Q_n^1)$)

By induction hypothesis, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - F_0$. Choose an edge $s_0t_0 \in E(P_{xy}^0) \backslash M_0$ such that $\{s_0s_1, t_0t_1\} \cap F_c = \emptyset$ and $s_1t_1 \notin M_1$. Since $|E(P_{xy}^0) \backslash M_0| - 2|F_c| - |M_1| = |E(P_{xy}^0)| - (|M_0| + |F_c| + |M_1|) - |F_c| \geq 2^n - 1 - (n - 1) - (n - 2) = 2^n - 2n + 2 > 1$ for $n \geq 4$, such an edge s_0t_0 exists. By Corollary 2.4, there exists a Hamiltonian path $P_{s_1t_1}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xy}^0 + P_{s_1t_1}^1 + \{s_0s_1, t_0t_1\} - s_0t_0$ is the desired Hamiltonian path.

Sub-case 2.1.2. $x \in V(Q_n^0)$, $y \in V(Q_n^1)$ (or $x \in V(Q_n^1)$, $y \in V(Q_n^0)$).

Choose a vertex z_0 in Q_n^0 such that $p(x) \neq p(z_0)$, $z_0z_1 \notin F_c$, and $\{xz_0, z_1y\} \cap M = \emptyset$. Note that $|M| + |F_c| \leq 2n - 4 - (|F_v| - 1) = n - 1$. Since $2^{n-1} - |M| - |F_c| \geq 2^{n-1} - (n - 1) = 2^{n-1} - n + 1 > 1$ for $n \geq 4$, such a vertex z_0 exists. By induction hypothesis, there exists a Hamiltonian path $P_{xz_0}^0$ passing through M_0 in $Q_n^0 - F_0$. By Corollary 2.4, there exists a Hamiltonian path $P^1_{z_1y}$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P^0_{xz_0} + P^1_{z_1y} + z_0z_1$ is the desired Hamiltonian path.

Sub-case 2.2. $|M_0 \cup F_0| = 2n - 5$. Now, $|M_0| \geq 1$, $F_c = \{v_0v_1\}$, and $|M_1 \cup F_1| = 0$.

Sub-case 2.2.1. $x, y \in V(Q_n^0)$.

Since $|M_0| = |M| \leq n - 3$, we have $|F_0| \geq n - 2$. So we can choose an edge $r_0w_0 \in F_0 \setminus F_v$. Thus, $v_0 \notin \{r_0, w_0\}$. Note that $d_{Q_n^0-(F_0\setminus\{r_0w_0\})}(v_0) = 3$. So, we have $\delta(Q_n^0 - (F_0 \setminus \{r_0w_0\})) = 3$. Since $|M_0 \cup (F_0 \setminus \{r_0w_0\})| = 2n - 6$, by induction hypothesis there exists a Hamiltonian path P^0_{xy} passing through M_0 in $Q_n^0 - (F_0 \setminus \{r_0w_0\})$. If $r_0w_0 \notin E(P^0_{xy})$, then choose an edge $s_0t_0 \in E(P^0_{xy}) \setminus M_0$ satisfying $v_0 \notin \{s_0, t_0\}$. If $r_0w_0 \in E(P^0_{xy})$, then let $s_0t_0 = r_0w_0$. In Q_n^1, by Theorem 1.1 there exists a Hamiltonian path $P^1_{s_1t_1}$. Hence, $P_{xy} = P^0_{xy} + P^1_{s_1t_1} + \{s_0s_1, t_0t_1\} - s_0t_0$ is the desired Hamiltonian path.

Sub-case 2.2.2. $x \in V(Q_n^0)$, $y \in V(Q_n^1)$ (or $x \in V(Q_n^1)$, $y \in V(Q_n^0)$).

Since $|M_0 \cup F_0| = 2n - 5 < 2n - 4$, $|M_0| \geq 1$, and $\delta(Q_n^0 - F_0) \geq 3$, by Theorem 1.6 there exists a Hamiltonian cycle C_0 containing M_0 in $Q_n^0 - F_0$. Choose a neighbor s_0 of x on C_0 satisfying $xs_0 \notin M_0$. If $s_0 \neq v_0$, then by Theorem 1.1 there exists a Hamiltonian path $P^1_{s_1y}$ in Q_n^1. Hence, $P_{xy} = C_0 + P^1_{s_1y} + s_0s_1 - xs_0$ is the desired Hamiltonian path. If $s_0 = v_0$, then since $d_{Q_n^0-F_0}(v_0) = 3$, let u_0 be the neighbor of v_0 in Q_n^0 satisfying $d_{C_0}(u_0, v_0) > 1$ and $u_0v_0 \notin F_0$. Note that there are two neighbors of u_0 on C_0. One neighbor lies on the path joining v_0 and u_0 on C_0 which contains x. We denote it as w_0. The other neighbor is denoted as t_0.

If $u_0t_0 \notin M_0$, then by Theorem 1.1 there exists a Hamiltonian path $P^1_{t_1y}$ in Q_n^1. Hence, $P_{xy} = C_0 + P^1_{t_1y} + \{u_0v_0, t_0t_1\} - \{xv_0, u_0t_0\}$ is the desired Hamiltonian path. If $u_0t_0 \in M_0$, then $u_0w_0 \notin M_0$. Let r_0 be the neighbor of t_0 on C_0 which is not u_0. Now, $r_0t_0 \notin M_0$ and $p(w_1) = p(t_1) \neq p(r_1) = p(y)$. By Corollary 3.4, there exists a spanning 2-path $P^1_{t_1w_1} + P^1_{r_1y}$ in Q_n^1. Hence, $P_{xy} = C_0 + P^1_{t_1w_1} + P^1_{r_1y} + \{u_0v_0, t_0t_1, r_0r_1, w_0w_1\} - \{xv_0, u_0w_0, r_0t_0\}$ is the desired Hamiltonian path.

Sub-case 2.2.3. $x, y \in V(Q_n^1)$.

Since $|M_0 \cup F_0| = 2n - 5 < 2n - 4$, $|M_0| \geq 1$, and $\delta(Q_n^0 - F_0) \geq 3$, by Theorem 1.6 there exists a Hamiltonian cycle C_0 containing M_0 in $Q_n^0 - F_0$. Choose an edge $s_0t_0 \in E(C_0) \setminus M_0$ such that $v_0 \notin \{s_0, t_0\}$ and $s_1t_1 \neq xy$. Since $|E(C_0) \setminus M_0| - 3 > 1$ for $n \geq 4$, such an edge s_0t_0 exists. In Q_n^1, by Theorem 2.2 there exists a Hamiltonian path P^1_{xy} passing through s_1t_1. Hence, $P_{xy} = P^1_{xy} + C_0 + \{s_0s_1, t_0t_1\} - \{s_0t_0, s_1t_1\}$ is the desired Hamiltonian path. □

Lemma 4.3. *For $n \geq 4$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq 2n - 6$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$ and $xy \notin M$. If $\delta(Q_n - F) \geq 4$, then there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$.*

Proof. We prove the conclusion by induction on n. When $n = 4$, the conclusion is by Corollary 2.4. Assume that the conclusion holds for $n \geq 4$, and we are to show that it holds for $n + 1$. Now, $|M \cup F| \leq 2(n + 1) - 6 = 2n - 4$. By Theorems 1.3 and 1.4, we only need to consider the case that $|F| \geq 1$ and $|M| \geq 1$.

Select $j \in [n+1]$ such that $|(M \cup F) \cap E_j|$ is as small as possible. Since $|M \cup F| \leq 2n - 4$, there exists a position $j \in [n+1]$ such that $|(M \cup F) \cap E_j| \leq 1$. When $|(M \cup F) \cap E_j| = 1$, let $(M \cup F) \cap E_j = \{w_0w_1\}$, and now there are at least six possibilities of such j. We choose the j such that the edge in $(M \cup F) \cap E_j$ lies in F if possible. Otherwise, the edge in $(M \cup F) \cap E_j$ is a matching edge for every j of the above six possibilities. In this case, since x and y are incident with at most two edges in M, we can choose j such that the matching edge in $(M \cup F) \cap E_j$ is not incident with x and y, i.e., $\{x, y\} \cap \{w_0, w_1\} = \emptyset$.

Split Q_{n+1} into Q_n^0 and Q_n^1 at position j. Without loss of generality, assume that $|M_0 \cup F_0| \geq |M_1 \cup F_1|$. Now, $\delta(Q_n^0 - F_0) \geq 3$ and $\delta(Q_n^1 - F_1) \geq 3$.

Case 1. $|M \cap E_j| = 0$. Now, $|M_1 \cup F_1| \leq n-2$.

Sub-case 1.1. $|M_0 \cup F_0| \leq 2n-6$.

Sub-case 1.1.1. $x, y \in V(Q_n^0)$ (or $x, y \in V(Q_n^1)$).

By induction hypothesis when $\delta(Q_n^0 - F_0) \geq 4$ and Lemma 4.2 when $\delta(Q_n^0 - F_0) = 3$, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - F_0$. Choose an edge $u_0v_0 \in E(P_{xy}^0) \setminus M_0$ such that $u_1v_1 \notin M_1$ and $\{u_0u_1, v_0v_1\} \cap F_c = \emptyset$. Since $|E(P_{xy}^0) \setminus M_0| - |M_1| - 2|F_c| = |E(P_{xy}^0)| - |M| - 2|F_c| \geq 2^n - 1 - (2n-5) - 2 = 2^n - 2n + 2 > 1$ for $n \geq 4$, such an edge u_0v_0 exists. By Corollary 2.4, there exists a Hamiltonian path $P_{u_1v_1}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xy}^0 + P_{u_1v_1}^1 + \{u_0u_1, v_0v_1\} - u_0v_0$ is a Hamiltonian path joining x and y passing through M in $Q_{n+1} - F$.

Sub-case 1.1.2. $x \in V(Q_n^0)$, $y \in V(Q_n^1)$ (or $x \in V(Q_n^1)$, $y \in V(Q_n^0)$).

Since $2^{n-1} > 3$ for $n \geq 4$, we can choose a vertex $z_0 \in V(Q_n^0)$ such that $p(x) \neq p(z_0)$, $\{xz_0, z_1y\} \cap M = \emptyset$, and $z_0z_1 \notin F_c$. By induction hypothesis when $\delta(Q_n^0 - F_0) \geq 4$ and Lemma 4.2 when $\delta(Q_n^0 - F_0) = 3$, there exists a Hamiltonian path $P_{xz_0}^0$ passing through M_0 in $Q_n^0 - F_0$. By Corollary 2.4, there exists a Hamiltonian path $P_{z_1y}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xz_0}^0 + P_{z_1y}^1 + z_0z_1$ is the desired Hamiltonian path.

Sub-case 1.2. $|M_0 \cup F_0| = 2n-5$. Now, $|F_c| + |M_1 \cup F_1| \leq 1$.

Sub-case 1.2.1. $x, y \in V(Q_n^0)$.

Sub-case 1.2.1.1. $F_0 = \emptyset$. Now, $|F_c| + |F_1| = 1$, $|M_1| = 0$, $|M_0| = 2n-5 < 2n-4$.

By Theorem 1.3, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in Q_n^0. Choose an edge $u_0v_0 \in E(P_{xy}^0) \setminus M_0$ such that $\{u_0u_1, v_0v_1\} \cap F_c = \emptyset$. Since $|E(P_{xy}^0) \setminus M_0| - 2|F_c| \geq 2^n - 1 - (2n-5) - 2 = 2^n - 2n + 2 > 1$ for $n \geq 4$, such an edge u_0v_0 exists. Since $|F_1| \leq 1 < 2n-5$, by Theorem 1.4 there exists a Hamiltonian path $P_{u_1v_1}^1$ in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xy}^0 + P_{u_1v_1}^1 + \{u_0u_1, v_0v_1\} - u_0v_0$ is the desired Hamiltonian path.

Sub-case 1.2.1.2. $F_0 \neq \emptyset$. Let $s_0t_0 \in F_0$. Note that $\delta(Q_n^0 - (F_0 \setminus \{s_0t_0\})) \geq 3$.

Since $|M_0 \cup (F_0 \setminus \{s_0t_0\})| = 2n-6$, by induction hypothesis and Lemma 4.2, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - (F_0 \setminus \{s_0t_0\})$. Next, we distinguish four cases to consider.

If $s_0t_0 \notin E(P_{xy}^0)$, then choose an edge $u_0v_0 \in E(P_{xy}^0) \setminus M_0$ such that $u_1v_1 \notin M_1$ and $\{u_0u_1, v_0v_1\} \cap F_c = \emptyset$. Since $|E(P_{xy}^0) \setminus M_0| - |M_1| - 2|F_c| \geq 2^n - 1 - (2n-5) - 2 = 2^n - 2n + 2 > 1$ for $n \geq 4$, such an edge u_0v_0 exists. If $s_0t_0 \in E(P_{xy}^0)$, $s_1t_1 \notin M_1$, and $\{s_0s_1, t_0t_1\} \cap F_c = \emptyset$, then let $u_0v_0 = s_0t_0$. In the above two cases, since $|M_1 \cup F_1| \leq 1 < n-2$, by Corollary 2.4 there exists a Hamiltonian path $P_{u_1v_1}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xy}^0 + P_{u_1v_1}^1 + \{u_0u_1, v_0v_1\} - u_0v_0$ is the desired Hamiltonian path.

If $s_0t_0 \in E(P_{xy}^0)$ and $s_1t_1 \in M_1$, then now $|F_c \cup F_1| = 0$. Choose an edge $u_0v_0 \in E(P_{xy}^0) \setminus M_0$ such that $\{u_0, v_0\} \cap \{s_0, t_0\} = \emptyset$. Since $|E(P_{xy}^0) \setminus M_0| - 3 \geq 2^n - 1 - (2n-6) - 3 = 2^n - 2n + 2 > 1$ for $n \geq 4$, such an edge u_0v_0 exists. In Q_n^1, by Theorem 3.1 there exists a spanning 2-path $P_{u_1v_1}^1 + s_1t_1$. Hence, $P_{xy} = P_{xy}^0 + P_{u_1v_1}^1 + s_1t_1 + \{s_0s_1, t_0t_1, u_0u_1, v_0v_1\} - \{s_0t_0, u_0v_0\}$ is the desired Hamiltonian path.

It remains to consider the case $s_0t_0 \in E(P_{xy}^0)$ and $\{s_0s_1, t_0t_1\} \cap F_c \neq \emptyset$. Now, $|M_1 \cup F_1| = 0$. Without loss of generality, assume that $F_c = \{s_0s_1\}$ and s_0 is closer to x than t_0 on P_{xy}^0. Let $N(s_0) = \{v \in V(Q_n^0) \mid d_{Q_n^0}(s_0, v) = 1, d_{P_{xy}^0}(s_0, v) > 1$, and $vs_0 \notin F_0\}$. Note that $t_0 \notin N(s_0)$. Since $d_{Q_n^0 - F_0}(s_0) \geq 3$, we have $|N(s_0)| \geq 2$.

If there exists a vertex $r_0 \in N(s_0)$ such that $r_0 \in V(P_{xy}^0[x, s_0])$ and $r_0 \neq x$, then choose a neighbor w_0 of r_0 on P_{xy}^0 such that $r_0w_0 \notin M_0$. When $w_0 \in V(P_{xy}^0[r_0, s_0])$, by Theorem 1.1 there exists a Hamiltonian path $P^1_{t_1w_1}$ in Q^1_n. Hence, $P_{xy} = P^0_{xy} + P^1_{t_1w_1} + \{s_0r_0, t_0t_1, w_0w_1\} - \{s_0t_0, r_0w_0\}$ is the desired Hamiltonian path. When $w_0 \in V(P^0_{xy}[x, r_0])$, choose an edge $u_0v_0 \in E(P^0_{xy}[r_0, s_0]) \setminus M_0$ satisfying $s_0 \notin \{u_0, v_0\}$. Since $d_{P^0_{xy}}(r_0, s_0) \geq 3$, such an edge u_0v_0 exists. In Q^1_n, by Theorems 3.1 and 3.2 there exists a spanning 2-path $P^1_{w_1u_1} + P^1_{t_1v_1}$. Hence, $P_{xy} = P^0_{xy} + P^1_{w_1u_1} + P^1_{t_1v_1} + \{s_0r_0, w_0w_1, u_0u_1, v_0v_1, t_0t_1\} - \{w_0r_0, u_0v_0, s_0t_0\}$ is the desired Hamiltonian path.

Otherwise, there exists at least one vertex in $N(s_0)$ which lies on $P^0_{xy}[t_0, y]$. Denote the one closest to t_0 by z_0. Note that $z_0 \neq y$, otherwise, the vertex r_0 in the above case exists. A contraction occurs. Since $p(x) \neq p(y)$ and $p(s_0) \neq p(z_0)$, the number of odd vertices equals to the number of even vertices on $P^0_{xy}[x, s_0] + P^0_{xy}[z_0, y]$. Since s_0 has at least $|N(s_0)| + 1$ neighbors which lie on $P^0_{xy}[x, s_0] + P^0_{xy}[z_0, y]$, we have $|V(P^0_{xy}[x, s_0] + P^0_{xy}[z_0, y])| \geq 6$. Thus, $|E(P^0_{xy}[x, s_0] + P^0_{xy}[z_0, y])| \geq 4$.

Choose a neighbor w_0 of z_0 on P^0_{xy} such that $z_0w_0 \notin M_0$. If $w_0 \in V(P^0_{xy}[z_0, y])$, then by Theorem 1.1 there exists a Hamiltonian path $P^1_{t_1w_1}$ in Q^1_n. Hence, $P_{xy} = P^0_{xy} + P^1_{t_1w_1} + \{s_0z_0, t_0t_1, w_0w_1\} - \{s_0t_0, z_0w_0\}$ is the desired Hamiltonian path. If $w_0 \in V(P^0_{xy}[s_0, z_0])$, then choose an edge $u_0v_0 \in E(P^0_{xy}[x, s_0] + P^0_{xy}[z_0, y]) \setminus M_0$ satisfying $s_0 \notin \{u_0, v_0\}$. Since $|E(P^0_{xy}[x, s_0] + P^0_{xy}[z_0, y])| \geq 4$, such an edge u_0v_0 exists. In Q^1_n, by Theorems 3.1 and 3.2 there exists a spanning 2-path $P^1_{w_1u_1} + P^1_{t_1v_1}$. Hence, $P_{xy} = P^0_{xy} + P^1_{w_1u_1} + P^1_{t_1v_1} + \{s_0z_0, t_0t_1, w_0w_1, u_0u_1, v_0v_1\} - \{s_0t_0, z_0w_0, u_0v_0\}$ is the desired Hamiltonian path.

Sub-case 1.2.2. $x \in V(Q^0_n)$, $y \in V(Q^1_n)$ (or $x \in V(Q^1_n)$, $y \in V(Q^0_n)$).

Since $|M_0 \cup F_0| = 2n - 5 < 2n - 4$, by Theorem 1.6 when $M_0 \neq \emptyset$ and Corollary 1.5 when $M_0 = \emptyset$, there exists a Hamiltonian cycle C_0 containing M_0 in $Q^0_n - F_0$. Choose a neighbor s_0 of x on C_0 such that $xs_0 \notin M_0$.

Sub-case 1.2.2.1. $s_0s_1 \notin F_c$.

When $s_1y \notin M_1$, since $|M_1 \cup F_1| \leq 1 < n - 2$, by Corollary 2.4 there exists a Hamiltonian path $P^1_{s_1y}$ passing through M_1 in $Q^1_n - F_1$. Hence, $P_{xy} = C_0 + P^1_{s_1y} + s_0s_1 - xs_0$ is the desired Hamiltonian path. When $s_1y \in M_1$, now $|F_c \cup F_1| = 0$ and $|M_0| \leq 2n - 6$. Choose an edge $u_0v_0 \in E(C_0) \setminus M_0$ such that $s_0 \notin \{u_0, v_0\}$ and $y \notin \{u_1, v_1\}$. Since $|E(C_0) \setminus M_0| - 2 - 2 \geq 2^n - (2n - 6) - 4 = 2^n - 2n + 2 > 1$ for $n \geq 4$, such an edge u_0v_0 exists. In Q^1_n, by Theorem 3.1 there exists a spanning 2-path $P^1_{u_1v_1} + s_1y$ passing through M_1. Hence, $P_{xy} = C_0 + P^1_{u_1v_1} + s_1y + \{u_0u_1, v_0v_1, s_0s_1\} - \{xs_0, u_0v_0\}$ is the desired Hamiltonian path.

Sub-case 1.2.2.2. $s_0s_1 \in F_c$. Now $|M_1 \cup F_1| = 0$.

Since $d_{Q^0_n - F_0}(s_0) \geq 3$, we can choose a neighbor t_0 of s_0 in Q^0_n such that $d_{C_0}(t_0, s_0) \neq 1$ and $s_0t_0 \notin F_0$. Choose a neighbor r_0 of t_0 on C_0 such that $t_0r_0 \notin M_0$. If r_0 lies on one path joining t_0 and s_0 on C_0 and x lies on the other, then by Theorem 1.1 there exists a Hamiltonian path $P^1_{r_1y}$ in Q^1_n. Hence, $P_{xy} = C_0 + P^1_{r_1y} + \{s_0t_0, r_0r_1\} - \{xs_0, t_0r_0\}$ is the desired Hamiltonian path. If r_0 and x lie on the same path joining t_0 and s_0 on C_0, then let u_0 be the other neighbor of t_0 on C_0. Now, $t_0u_0 \in M_0$ and $u_0 \neq s_0$. Let v_0 be the other neighbor of u_0 on C_0. So, $u_0v_0 \notin M_0$. Since $p(r_1) = p(u_1) \neq p(v_1) = p(y)$ and $r_1 \neq u_1$, by Corollary 3.4 there exists a spanning 2-path $P^1_{r_1u_1} + P^1_{v_1y}$ in Q^1_n. Hence, $P_{xy} = C_0 + P^1_{r_1u_1} + P^1_{v_1y} + \{s_0t_0, u_0u_1, v_0v_1, r_0r_1\} - \{xs_0, u_0v_0, t_0r_0\}$ is the desired Hamiltonian path; see Figure 4.

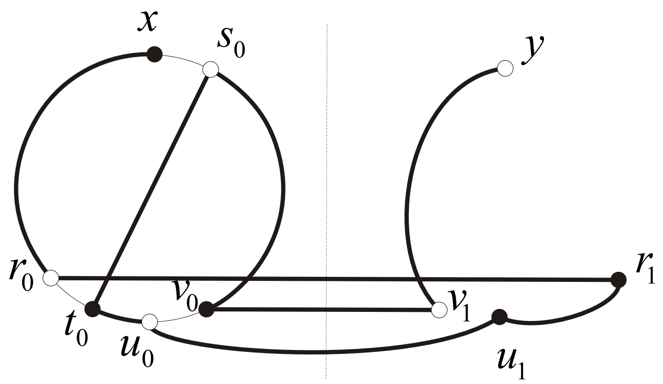

Figure 4. A sketch map for the construction of the Hamiltonian path.

Sub-case 1.2.3. $x, y \in V(Q_n^1)$.

Since $|M_0 \cup F_0| = 2n-5 < 2n-4$, by Theorem 1.6 when $M_0 \neq \emptyset$ and Corollary 1.5 when $M_0 = \emptyset$, there exists a Hamiltonian cycle C_0 containing M_0 in $Q_n^0 - F_0$. Choose an edge $s_0t_0 \in E(C_0) \setminus M_0$ such that $\{s_0s_1, t_0t_1\} \cap F_c = \emptyset$, $s_1t_1 \neq xy$, and $\{s_1, t_1\} \cap V(M_1) = \emptyset$. Since $|E(C_0) \setminus M_0| - 2|F_c| - 1 - 4|M_1| = |E(C_0)| - |M| - (2|F_c| + 3|M_1|) - 1 \geq 2^n - (2n-5) - 3 - 1 = 2^n - 2n + 1 > 1$ for $n \geq 4$, such an edge s_0t_0 exists. Since $|(M_1 \cup \{s_1t_1\}) \cup (F_1 \setminus \{s_1t_1\})| \leq 2 \leq n-2$ for $n \geq 4$ and $M_1 \cup \{s_1t_1\}$ is a matching, by Corollary 2.4 there exists a Hamiltonian path P_{xy}^1 passing through $M_1 \cup \{s_1t_1\}$ in $Q_n^1 - (F_1 \setminus \{s_1t_1\})$. Hence, $P_{xy} = C_0 + P_{xy}^1 + \{s_0s_1, t_0t_1\} - \{s_0t_0, s_1t_1\}$ is the desired Hamiltonian path.

Sub-case 1.3. $|M_0 \cup F_0| = 2n-4$. Now, $|F_c| + |M_1 \cup F_1| = 0$ and $|M_0| \geq 1$.

Since $|M_0 \cup F_0| = 2n-4$ and $|M_0| \geq 1$, by Theorem 1.6 there exists a Hamiltonian cycle C_0 containing M_0 in $Q_n^0 - F_0$.

Sub-case 1.3.1. $x, y \in V(Q_n^0)$.

Choose neighbors s_0, t_0 of x, y on C_0 such that $xs_0 \notin M_0$ and $t_0y \notin M_0$. If s_0 lies on one path joining x and y on C_0 and r_0 lies on the other, then by Theorem 1.1 there exists a Hamiltonian path $P_{s_1t_1}^1$ in Q_n^1. Hence, $P_{xy} = C_0 + P_{s_1t_1}^1 + \{s_0s_1, t_0t_1\} - \{xs_0, t_0y\}$ is the desired Hamiltonian path. If s_0 and t_0 lie on the same path joining x and y on C_0, then choose an edge $r_0w_0 \in E(C_0) \setminus M_0$ on the other path. We may assume that r_0 is closer to x than w_0 on one path joining x and y on C_0. By Theorems 3.1 and 3.2, there exists a spanning 2-path $P_{r_1s_1}^1 + P_{w_1t_1}^1$ in Q_n^1. Hence, $P_{xy} = C_0 + P_{r_1s_1}^1 + P_{w_1t_1}^1 + \{s_0s_1, t_0t_1, r_0r_1, w_0w_1\} - \{xs_0, t_0y, r_0w_0\}$ is the desired Hamiltonian path.

Sub-case 1.3.2. $x \in V(Q_n^0), y \in V(Q_n^1)$ (or $x \in V(Q_n^1), y \in V(Q_n^0)$).

Choose a neighbor s_0 of x on C_0 such that $xs_0 \notin M_0$. In Q_n^1, by Theorem 1.1 there exists a Hamiltonian path $P_{s_1y}^1$. Hence, $P_{xy} = C_0 + P_{s_1y}^1 + s_0s_1 - xs_0$ is the desired Hamiltonian path.

Sub-case 1.3.3. $x, y \in V(Q_n^1)$.

Choose an edge $s_0t_0 \in E(C_0) \setminus M_0$ satisfying $s_0t_0 \neq xy$. Since $|E(C_0) \setminus M_0| - 1 \geq 2^n - (2n-5) - 1 = 2^n - 2n + 4 > 1$ for $n \geq 4$, such an edge s_0t_0 exists. In Q_n^1, by Lemma 2.2 there exists a Hamiltonian path P_{xy}^1 passing through s_1t_1. Hence, $P_{xy} = C_0 + P_{xy}^1 + \{s_0s_1, t_0t_1\} - \{s_0t_0, s_1t_1\}$ is the desired Hamiltonian path.

Case 2. $|M \cap E_j| = 1$. Now, $M_c = \{w_0w_1\}$, $\{w_0, w_1\} \cap \{x, y\} = \emptyset$, $|F_c| = 0$, and $|M_1 \cup F_1| \leq \lfloor \frac{2n-4-1}{2} \rfloor = n-3 < n-2$.

Sub-case 2.1. $|M_0 \cup F_0| \leq 2n-6$.

Sub-case 2.1.1. $x, y \in V(Q_n^0)$ (or $x, y \in V(Q_n^1)$).

By induction hypothesis and Lemma 4.2, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - F_0$. Choose a neighbor r_0 of w_0 on P_{xy}^0. By Corollary 2.4, there exists a Hamiltonian path $P_{r_1w_1}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xy}^0 + P_{r_1w_1}^1 + \{r_0r_1, w_0w_1\} - r_0w_0$ is the desired Hamiltonian path.

Sub-case 2.1.2. $x \in V(Q_n^0), y \in V(Q_n^1)$ (or $x \in V(Q_n^1), y \in V(Q_n^0)$).

When $p(x) \neq p(w_0)$, by induction hypothesis and Lemma 4.2, there exists a Hamiltonian path $P_{xw_0}^0$ passing through M_0 in $Q_n^0 - F_0$. By Corollary 2.4, there exists a Hamiltonian path $P_{w_1y}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xw_0}^0 + P_{w_1y}^1 + w_0w_1$ is the desired Hamiltonian path.

When $p(x) = p(w_0)$, choose a neighbor s_1 of y in Q_n^1 such that $s_1 \notin V(M_1)$ and $xs_0 \notin M_0$. Since $n > (n-3) + 1$, such a vertex s_1 exists. By induction hypothesis and Lemma 4.2, there exists a Hamiltonian path $P_{xs_0}^0$ passing through M_0 in $Q_n^0 - F_0$.

If there exists a neighbor r_0 of w_0 on $P_{xs_0}^0$ such that $r_0 \neq s_0$ and $r_1y \notin M_1$, then since $|M_1 \cup F_1| \leq n-3$, by Lemma 3.8 there exists a spanning 2-path $P_{s_1y}^1 + P_{r_1w_1}^1$ passing through M_1 in $Q_n^1 - F_1$. Hence, $P_{xy} = P_{xs_0}^0 + P_{s_1y}^1 + P_{r_1w_1}^1 + \{s_0s_1, w_0w_1, r_0r_1\} - r_0w_0$ is the desired Hamiltonian path; see Figure 5(1).

Otherwise, $d_{P_{xs_0}^0}(s_0, w_0) = 1$, and the other neighbor r_0 of w_0 on $P_{xs_0}^0$ satisfies $r_1y \in M_1$. Choose an edge $u_0v_0 \in E(P_{xs_0}^0) \setminus M_0$ such that $u_1v_1 \neq s_1w_1$ and $\{u_1, v_1\} \cap V(M_1) = \emptyset$. Since $|E(P_{xs_0}^0) \setminus M_0| - 1 - 4|M_1| = |E(P_{xs_0}^0)| - (|M_0| + |M_1|) - 1 - 3|M_1| \geq (2^n - 1) - (2n - 6) - 1 - 3(n-3) = 2^n - 5n + 13 > 1$ for $n \geq 4$, such an edge u_0v_0 exists. Since $|(M_1 \cup \{u_1v_1\} \setminus \{r_1y\}) \cup F_1| \leq n - 3$ and $s_1w_1r_1ys_1$ is a cycle of length four, by Lemma 3.12 there exists a spanning 2-path $P_{s_1w_1}^1 + r_1y$ passing through $M_1 \cup \{u_1v_1\}$ in $Q_n^1 - (F_1 \setminus \{u_1v_1\})$. Hence, $P_{xy} = P_{xs_0}^0 + P_{s_1w_1}^1 + r_1y + \{s_0s_1, w_0w_1, r_0r_1, u_0u_1, v_0v_1\} - \{w_0r_0, u_0v_0, u_1v_1\}$ is the desired Hamiltonian path; see Figure 5(2).

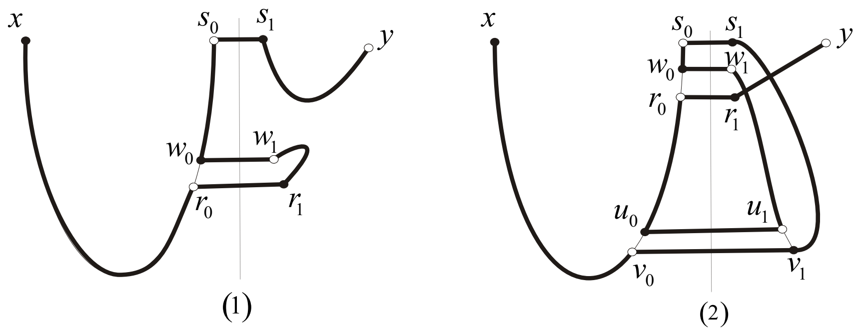

Figure 5. A sketch map for the construction of the Hamiltonian path in $Q_{n+1} - F$.

Sub-case 2.2. $|M_0 \cup F_0| = 2n - 5$. Now, $|M_1 \cup F_1| = 0$ and $|F_0| \geq 1$.

Sub-case 2.2.1. $x, y \in V(Q_n^0)$.

Let $u_0v_0 \in F_0$. By induction hypothesis and Lemma 4.2, there exists a Hamiltonian path P_{xy}^0 passing through M_0 in $Q_n^0 - (F_0 \setminus \{u_0v_0\})$. If $u_0v_0 \notin E(P_{xy}^0)$, then choose a neighbor r_0 of w_0 on P_{xy}^0. If $u_0v_0 \in E(P_{xy}^0)$ and $d_{P_{xy}^0}(w_0, u_0v_0) = 0$, then let $r_0w_0 = u_0v_0$. In Q_n^1, by Theorem 1.1 there exists a Hamiltonian path $P_{r_1w_1}^1$. Hence, $P_{xy} = P_{xy}^0 + P_{r_1w_1}^1 + \{r_0r_1, w_0w_1\} - r_0w_0$ is the desired Hamiltonian path. If $u_0v_0 \in E(P_{xy}^0)$ and $d_{P_{xy}^0}(w_0, u_0v_0) \geq 1$, then since $w_0 \notin \{x, y\}$, we can choose a neighbor r_0 of w_0 on

P^0_{xy} such that $r_0 \notin \{u_0, v_0\}$. In Q^1_n, by Theorem 3.1 there exists a spanning 2-path $P^1_{u_1v_1} + P^1_{r_1w_1}$. Hence, $P_{xy} = P^0_{xy} + P^1_{u_1v_1} + P^1_{r_1w_1} + \{r_0r_1, w_0w_1, u_0u_1, v_0v_1\} - \{r_0w_0, u_0v_0\}$ is the desired Hamiltonian path.

Sub-case 2.2.2. $x \in V(Q^0_n)$, $y \in V(Q^1_n)$ (or $x \in V(Q^1_n)$, $y \in V(Q^0_n)$).

Since $|M_0 \cup F_0| = 2n - 5 < 2n - 4$, by Theorem 1.6 when $M_0 \neq \emptyset$ and Corollary 1.5 when $M_0 = \emptyset$, there exists a Hamiltonian cycle C_0 containing M_0 in $Q^0_n - F_0$. If $d_{C_0}(w_0, x) = 1$, then by Theorem 1.1 there exists a Hamiltonian path $P^1_{w_1y}$ in Q^1_n. Hence, $P_{xy} = C_0 + P^1_{w_1y} + w_0w_1 - xw_0$ is the desired Hamiltonian path. If $d_{C_0}(w_0, x) = 2$, then choose neighbors s_0, r_0 of x, w_0 on C_0, respectively, such that $xs_0 \notin M_0$ and $r_0 \neq s_0$. Now, $p(s_1) = p(r_1) \neq p(w_1) = p(y)$ and s_1, y, r_1, w_1 are distinct vertices. If $d_{C_0}(w_0, x) \geq 3$, then choose neighbors s_0, r_0 of x, w_0 on C_0, respectively, such that $r_1 \neq y$ and $xs_0 \notin M_0$. Now, $p(s_1) \neq p(y)$, $p(r_1) \neq p(w_1)$, and s_1, y, r_1, w_1 are distinct vertices. In the above two cases, by Theorem 3.1 there exists a spanning 2-path $P^1_{s_1y} + P^1_{r_1w_1}$ in Q^1_n. Hence, $P_{xy} = C_0 + P^1_{s_1y} + P^1_{r_1w_1} + \{s_0s_1, r_0r_1, w_0w_1\} - \{xs_0, r_0w_0\}$ is the desired Hamiltonian path.

Sub-case 2.2.3. $x, y \in V(Q^1_n)$.

Since $|M_0 \cup F_0| = 2n - 5 < 2n - 4$, by Theorem 1.6 when $M_0 \neq \emptyset$ and Corollary 1.5 when $M_0 = \emptyset$, there exists a Hamiltonian cycle C_0 containing M_0 in $Q^0_n - F_0$. Choose a neighbor r_0 of w_0 on C_0. Since $r_1w_1 \neq xy$, by Lemma 2.2 there exists a Hamiltonian path P^1_{xy} passing through r_1w_1 in Q^1_n. Hence, $P_{xy} = P^1_{xy} + C_0 + \{r_0r_1, w_0w_1\} - \{r_0w_0, r_1w_1\}$ is the desired Hamiltonian path. □

5. Conclusions

We investigate the existence of a Hamiltonian path in Q_n passing through a matching and avoiding faulty edges. From Lemmas 4.1, 4.2, and 4.3, we can obtain the following conclusion.

Theorem 5.1. *For $n \geq 4$, let M be a matching of Q_n, and let F be a set of edges in $Q_n - M$ with $|M \cup F| \leq 2n - 6$. Let $x, y \in V(Q_n)$ be such that $p(x) \neq p(y)$ and $xy \notin M$. If the degree of every vertex in $Q_n - F$ is at least 2, and there is neither x/y–DS nor (x, y)–DS in $Q_n - F$, then there exists a Hamiltonian path joining x and y passing through M in $Q_n - F$.*

Author contributions

Shenyang Zhao: Methodology and Writing-original & draft; Fan Wang: Conceptualization and Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest in this work.

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