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Research article

Nonlinear differential equations with neutral term: Asymptotic behavior of solutions

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Abstract: The aim of this work is to study some oscillation behavior of solutions of a class of third-order neutral differential equations with multi delays. We present new oscillation criteria that complete and simplify some previous results. We also provide an example to clarify the significance of our results.

Keywords: oscillation; third-order; neutral; differential equation
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1. Introduction

Differential equations (DEs) are mathematical models used to study phenomena that occur in nature, where each dependent variable represents a quantity in the modeled phenomenon. Differential equations made it possible to understand many complex phenomena in our daily lives and play a pivotal role in many applications in engineering [1–3]. They have become important tools in applied sciences and technology, used for studying telephone signals, media, conversations, and the statistics of online purchasing. More traditionally, they were used in astronomy to describe the orbits of planets and the motion of stars [4–6]. They also have many applications in biology and the medical sciences. By describing those phenomena with variables that symbolize time and place, differential equations can provide insights about the phenomena on future.

Due to the huge advantage of neutral differential equations in describing several neutral phenomena, there is great scientific and academic values theoretically and practically for studying neutral differential equations [7–9]. Hence, a large amount of research attention has been focused on the oscillation problem of third-order linear and nonlinear neutral differential equations in recent years; see, for example [10–12].

Recent years have seen a surge in research on the oscillation and non-oscillation of solutions to third/fourth-order differential equations [13–15]. For further exploration, readers can refer to the references provided [16–18].

The authors in [19–21] discussed several oscillatory properties of higher-order equations in canonical form, and used different methods to find those properties, such as Riccati transformations. Moreover, they applied the comparison method to inequalities of different orders that are oscillatory [22, 23].

The purpose of this work is to investigate the oscillatory and asymptotic behavior of the third-order neutral delay differential equations

$$\left(r_2\left(r_1 w'\right)'\right)'(t)+\sum_{i=1}^{j} a_i(t)x(\varsigma_i(t))=0 \quad t\geq t_0>0, \tag{E}$$

where $w(t)=x(t)+b(t)x(g(t))$. We also assume that the following conditions are satisfied:

(H_1) $\varsigma_i, g \in C'([t_0,\infty),\mathbb{R}), \varsigma_i(t)<t, g(t)<t, g'(t)\geq 0$ and $\lim_{t\to\infty} g(t)=\lim_{t\to\infty}\varsigma_i(t)=\infty, i=1,2,..,j$;
(H_2) $b, a_i \in C([t_0,\infty),\mathbb{R}^+), 0\leq b(t)\leq b_0<\infty$ and a_i does not vanish identically;
(H_3) $r_1, r_2 \in C([t_0,\infty),(0,\infty))$ satisfy

$$\int_{t_0}^{\infty}\frac{1}{r_1(t)}dt<\infty \text{ and } \int_{t_0}^{\infty}\frac{1}{r_2(t)}dt<\infty, \tag{1.1}$$

that is, (E) is in noncanonical form;
(H_4) g and ς_i commute.

By a solution of (E), we mean a function $x\in C([T_x,\infty),\mathbb{R})$ with $T_x>t_0$, which has the property $L_i\, w\in C^1([T_x,\infty),\mathbb{R}), i=0,1,2$, and satisfies $(E) on\, [T_x,\infty)$. We only consider those solutions of (E) which exist on some half-line $[T_x,\infty)$ and satisfy the condition $\sup\{|x(t)|: T\leq t<\infty\}>0$ for any $T\geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is said to be *oscillatory* if it is neither eventually negative nor eventually positive, and it is called *nonoscillatory* otherwise. The equation itself is referred to as oscillatory if all of the solutions are oscillatory.

The main motivation for studying this paper is to contribute to the development of the oscillation theory for third-order equations by finding sufficient conditions that guarantee that the solutions of this type of equations are oscillatory.

Chatzarakis et al. [24] established new oscillation criteria for the differential equation

$$(r_2(r_1 w')')'(t)+a(t)x(\varsigma(t))=0, \tag{1.2}$$

in the canonical form. Recently, techniques have been developed to study the oscillatory behavior of solutions to third-order equations.

Candan [25] established some sufficient conditions for oscillation of the following class of third-order neutral differential equations

$$(r_2(r_1 w')')'(t)+a(t)f(x(\varsigma(t)))=0, \tag{1.3}$$

under conditions

$$\int_{t_0}^{\infty}\frac{1}{r_1(t)}dt=\int_{t_0}^{\infty}\frac{1}{r_2(t)}dt=\infty, \tag{1.4}$$

Li et al. [26] also studied the cases

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)}dt < \infty \text{ and } \int_{t_0}^{\infty} \frac{1}{r_1(t)}dt = \infty, \tag{1.5}$$

also under conditions

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)}dt = \infty \quad \text{and} \int_{t_0}^{\infty} \frac{1}{r_2(t)}dt < \infty. \tag{1.6}$$

So during these years, it was found that sufficient criteria were found to ensure that the solutions of (E) were oscillatory. The first of these findings for (E) was reported in [3], in canonical type under the conditions $0 \leqslant b(t) \leqslant b_0 < \infty$ and $\varsigma_i o g = g o \varsigma_i$. Very recently in [4, 5], the authors provided enough parameters for (E) to oscillate.in the noncanonical or semi-canonical case with an unbounded neutral coefficient, that is, $b(t) \geq b_0 > 1$ since in this case one can easily find the relation between $x(t)$ and $w(t)$. This is generally essential to obtain oscillation criteria for neutral-type differential equations.

Our literature review indicates a scarcity of research on the oscillatory behavior of solutions to Eq (E) when it takes the semi-canonical form. This paper tackles Eq (E) in its less-studied semi-canonical form. We begin by transforming it into the more common canonical form. This transformation allows us to then establish new criteria for determining when solutions to Eq (E) oscillate.

In the sequel, we use the following notations for a compact presentation of our results:

$$L_0 w = w, L_1 w = r_1 w', L_2 w = r_2 (L_1 w)', L_3 w = (r_2 L_2 w)'.$$

We remark that in the study of the asymptotic behaviour of the positive solutions of (E), there are four cases:

$$\begin{aligned} S_1 &: w > 0, L_1 w < 0, L_2 w < 0, L_3 w \leq 0, \\ S_2 &: w > 0, L_1 w < 0, L_2 w > 0, L_3 w \leq 0, \\ S_3 &: w > 0, L_1 w > 0, L_2 w > 0, L_3 w \leq 0, \\ S_4 &: w > 0, L_1 w > 0, L_2 w < 0, L_3 w \leq 0. \end{aligned}$$

2. Main results

In view of (H_3), one can use the following notations:

$$\varpi_j(t) = \int_t^{\infty} \frac{ds}{r_j(s)}, \quad j = 1, 2, \quad \beta_1(t) = r_1(t)\varpi_1^2(t),$$

$$\beta_2(t) = \frac{r_2(t)}{\varpi_1(t)}, \quad F(t) = \min\{\sum_{i=1}^{j} a_i(t), \sum_{i=1}^{j} a_i(g(t))\},$$

$$M(t) = F(t)\varpi_1(\varsigma_i(t)), \quad A(t,u) = \int_u^t \frac{1}{\beta_1(s)} \int_s^t \frac{ds_1}{\beta_2(s_1)} ds,$$

for all $t > u \geq t_1 \geq t_0$.

From the form of (E), it is enough to consider positive solutions for nonoscillatory solutions of (E). The following is a standard one and can be found in [1].

Hence, if we want to derive oscillation conditions for (E), we have to eliminate the above mentioned four cases. However, if we transform (E) into semi-canonical type, then the number of cases is reduced to three without making any additional assumptions. Thus, this greatly streamlines the analysis of (E) oscillation.

Theorem 1. *The noncanonical operator L_3w has the semi-canonical representation*

$$L(t) = \left(\frac{r_2}{\varpi_1}\left(r_1\varpi_1^2\left(\frac{w}{\varpi_1}\right)'\right)'\right)'(t). \tag{2.1}$$

Proof. Direct calculation shows that

$$\begin{aligned}\tfrac{r_2(t)}{\varpi_1(t)}\left(r_1(t)\varpi_1^2(t)\left(\tfrac{w(t)}{\varpi_1(t)}\right)'\right)' &= \tfrac{r_2(t)}{\varpi_1(t)}\left(\varpi_1(t)r_1(t)w'(t) + w(t)\right)', \\ &= \tfrac{r_2(t)}{\varpi_1(t)}(\varpi_1(t)\left(r_1(t)w'(t)'\right),\end{aligned}$$

that is,

$$\left(\frac{r_2(t)}{\varpi_1(t)}\left(r_1(t)\varpi_1^2(t)\left(\frac{w(t)}{\varpi_1(t)}\right)'\right)'\right)' = \left(r_2(t)\left(r_1(t)w'(t)\right)'\right)'.$$

Further note that

$$\int_{t_0}^{\infty}\frac{dt}{r_1(t)\varpi_1^2(t)} = \int_{t_0}^{\infty} d\left(\frac{1}{\varpi_1(t)}\right) = \lim_{t\to\infty}\frac{1}{\varpi_1(t)} - \frac{1}{\varpi_1(t_0)} = \infty,$$

and

$$\int_{t_0}^{\infty}\frac{\varpi_1(t)}{r_2(t)}dt = \varpi_1(t_0)\int_{t_0}^{\infty}\frac{1}{r_2(t)}dt < \infty.$$

Hence L_3w transformed into semi-canonical form. This ends the proof.

Now it follows from Theorem 2.2 that (E) can be written in the equivalent semi-canonical form

$$\left(\beta_2(t)\left(\beta_1(t)\left(\frac{w(t)}{\varpi_1(t)}\right)'\right)'\right)' + a_i(t)x(\varsigma_i(t)) = 0.$$

By letting $\gamma(t) = \frac{w(t)}{\varpi_1(t)}$, the following result is at once.

Theorem 2. *Noncanonical equation (E) has a solution $x(t)$ if and only if the semi-canonical Eq (E_s)*

$$\left(\beta_2(v)\left(\beta_1(t)\gamma'(t)'\right)' + a_i(t)x(\varsigma_i(t)) = 0,$$

has the solution $x(t)$.

Corollary 1. *The function x is identified as the ultimate positive solution to (E) if and only if the semi-canonical Eq (E_s) has the same solution x .*

Now set

$$B_0\gamma = \gamma, B_1\gamma = \beta_1\gamma', B_2\gamma = \beta_2\left(\beta_1\gamma'\right)', B_3\gamma = \left(\beta_2\left(\beta_1\gamma'\right)'\right)'.$$

Corollary 2.4 clearly simplifies the investigation of (E) since for (E_s) we deal with only three cases of positive solutions; see, for example [3, Theorem 2.2], namely

$$\begin{aligned}
&O_1 : \gamma(t) > 0, B_1\gamma(t) < 0, B_2\gamma(t) > 0, B_3\gamma(t) \leq 0,\\
&O_2 : \gamma(t) > 0, B_1\gamma(t) > 0, B_2\gamma(t) > 0, B_3\gamma(t) \leq 0,\\
&O_3 : \gamma(t) > 0, B_1\gamma(t) > 0, B_2\gamma(t) < 0, B_3\gamma(t) \leq 0,
\end{aligned}$$

eventually.

Lemma 1. *Let x be an eventually positive solution of (E) then the corresponding function satisfies the inequality*

$$B_3\gamma(t) + \frac{b_0}{g_0} B_3\gamma(g(t)) + M(t)\gamma(\varsigma_i(t)) \leqslant 0, \tag{2.2}$$

for all $t \geq t_1 \geq t_0$.

Proof. The function θ is identified as the ultimate positive solution to (E) Let θ be an eventually positive solution of (E). Then we have that $\theta(\zeta) > 0, \theta(\tau(\zeta)) > 0$ and $\theta(\delta(\zeta)) > 0$ for all $t \geq \zeta_1$. From Corollary 2.4, the function $\alpha(\zeta)$ is a positive solution of (E_s) for all $\zeta \geq \zeta_1$. Now, from (E_s), (H_1) and (H_4), we see that

$$\begin{aligned}
0 &= \frac{g_0}{\tau'(\zeta)}(D_2\alpha(\tau(\zeta))' + g_0 f(\tau(\zeta))\theta(\delta(\tau(\zeta))),\\
&\geq \frac{g_0}{\tau_0}\,(D_2\alpha(\tau(\zeta)))' + g_0 f(\tau(\zeta))\theta(\delta(\tau(\zeta))),\\
&= \frac{g_0}{\tau_0}\,(D_2\alpha(\tau(\zeta)))' + g_0 f(\tau(\zeta))\theta(\tau(\delta(\zeta))).
\end{aligned} \tag{2.3}$$

Combining (E_s) along with the last inequality, we obtain

$$\begin{aligned}
0 &\geq D_3\alpha(\zeta) + \tfrac{g_0}{\tau_0} D_3\alpha(\tau(\zeta)) + f(\zeta)\theta(\delta(\zeta)) + g_0 f(\tau(\zeta))\theta(\tau(\delta(\zeta))),\\
&\geq D_3\alpha(\zeta) + \tfrac{g_0}{\tau_0} D_3\alpha(\tau(\zeta)) + F(\zeta)\,(\theta(\delta(\zeta)) + g_0\theta(\tau(\delta(\zeta))).
\end{aligned}$$

Using (H_2) in the definition of $\phi(\zeta)$, we obtain

$$\begin{aligned}
&\Omega_1(\delta(\zeta))\alpha(\delta(\zeta)) = \phi(\delta(\zeta)) = \theta(\delta(\zeta)) + g(\delta(\zeta))\theta(\tau(\delta(\zeta)))\\
&\leq \theta(\delta(\zeta)) + g_0\theta(\tau(\delta(\zeta))).
\end{aligned}$$

In view of the latter, inequality (2.3) becomes

$$D_3\alpha(\zeta) + \frac{g_0}{\tau_0} D_3\alpha(\tau(\zeta)) + N(\zeta)\alpha(\delta(\zeta)) \leq 0,$$

or

$$(D_2\alpha(\zeta) + \frac{g_0}{\tau_0} D_2\alpha(\tau(\zeta)))' + N(\zeta)\alpha(\delta(\zeta)) \leq 0, \tag{2.4}$$

which proves(2.2). □

Before we state and prove our main results, let us define

$$G_1(t) = \int_{t_0}^{t} \frac{1}{\beta_1(s)} ds, \quad G(t) = \int_{t}^{\infty} \frac{1}{\beta_2(s)} ds,$$

for all $t \geq t_0$.

Theorem 3. *Given that γ constitutes the final positive solution of (E_s). If*

$$\int_{t_0}^{\infty} M(t)G_1(t)dt = \infty, \tag{2.5}$$

then class O_2 is empty.

Proof. Assume to the contrary that class O_2 is not empty. Then there exists a $\zeta_1 \geq \zeta_0$ such that $\alpha(\zeta) > 0$, $\alpha(\tau(\zeta)) > 0$, $\alpha(\delta(\zeta)) > 0$ for all $\zeta \geq \zeta_1$, such that the function $\alpha(\zeta)$ in class O_2 for all $\zeta \geq \zeta_1$. Since $\beta_1(\zeta)\alpha'(\zeta) > 0$ is increasing, we have

$$\beta_1(\zeta)\alpha'(\zeta) \geq \beta_1(\zeta_1)\alpha'(\zeta_1) = M \text{ on } [\zeta_1, \infty).$$

Dividing this inequality by $\beta_1(\zeta)$, then integrating the resulting inequality, we obtain

$$\alpha(\delta(\zeta)) \geq MB_1(\delta(\zeta)), \zeta \geq \zeta_2 > \zeta_1 . \tag{2.6}$$

Integrating (E_s) from ζ_2 to ζ and using (2.6) in the resulting inequality, we obtain

$$\begin{aligned} D_2\alpha(\zeta) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta)) &= D_2\alpha(\zeta_2) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta_2)) - \int_{\zeta_2}^{\zeta} N(s)\alpha(\delta(s))ds \\ &\leq D_2\alpha(t_2) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta_0)) - M\int_{\zeta_2}^{t} N(s)B_1(\delta(s))ds, \end{aligned}$$

which tends $\zeta_0 \to \infty$ as $\zeta \to \infty$. This contradiction ends the proof. □

Lemma 2. *Let γ be an eventually positive increasing solution of (E_s). If*

$$\int_{t_0}^{\infty} \frac{1}{\beta_2(t)} \left(\int_{t_0}^{t} M(s)G_1(\varsigma_i(s))ds \right) dt = \infty, \tag{2.7}$$

then γ satisfies the class O_3 for $t \geq t_1$ for some $t_1 \geq t_0$ and further

$$\gamma(t) \geq G_1(t)\beta_1(t)\gamma'(t) \text{ for } t \geq t_1. \tag{2.8}$$

Proof. Since α is a positive increasing solution, so class O_1 is empty, and hence, by Theorem 2.6, $\alpha \in O_2 \cup O_3$ for $\zeta \geq \zeta_1$, where $\zeta_1 \geq \zeta_0$ is such that $\alpha(\delta(\zeta)) > 0$ and $\alpha(\tau(\zeta)) > 0$ for $\zeta \geq \zeta_1$. In view of (H_3), we see that (2.7) implies (2.5), and hence α satisfies class O_3 for $\zeta \geq \zeta_1$. Since $D_1\alpha$ is positive and decreasing, we see that

$$\alpha(\zeta) = \alpha(\zeta_1) + \int_{\zeta_1}^{\zeta} \frac{\beta_1(s)\alpha'(s)}{\beta_1(s)}ds \geq B_1(\zeta)\beta_1(\zeta)\alpha'(\zeta).$$

This ends the proof. □

Theorem 4. *Let γ be an eventually positive solution of (E_s). If*

$$\liminf_{t\to\infty} \int_{\varsigma_i(t)}^{t} \frac{1}{\beta_2(s)} \left(\int_{t_0}^{s} M(s_1)G_1(\varsigma_i(s_1))ds_1 \right) ds > \frac{g_0 + b_0}{eg_0}, \tag{2.9}$$

then the classes O_2 and O_3 are empty.

Proof. Assume that (2.9) holds but α belongs to classes O_2 and O_3. Pick $\zeta_1 \geq \zeta_o$ such that $\alpha(\tau(\zeta)) > 0$ and $\alpha(\delta(\zeta)) > 0$ for $\zeta \geq \zeta_1$. Clearly, it is necessary for the validity of (2.9) that (2.7) holds. Hence, by Theorem 2.6 and Lemma 2.7, one can see that α satisfies class O_3. Proceeding as in the proof of Lemma 2.7, we see that (2.8) holds, and so we obtain

$$\alpha(\delta(\zeta)) \geq B_1(\delta(\zeta))\beta_1(\delta(\zeta))\alpha'(\delta(\zeta)),$$

for $\zeta \geq \zeta_2$ for some $\zeta_2 \geq \zeta_1$. From the latter inequality and Eq (E_s), we observe that

$$-\left(D_3\alpha(\zeta) + \frac{g_0}{\tau_0}D_3\alpha(\tau(\zeta))\right) = N(\zeta)\alpha(\delta(v)) \geq N(\zeta)B_1\left(\delta(\zeta)\right)\beta_1\left(\delta(\zeta)\right)\alpha'(\delta(\zeta))\,.$$

Integrating from ζ_2 to ζ, we obtain

$$\begin{aligned} -\left(D_2\alpha(\zeta) + \tfrac{g_0}{\tau_0}D_2\alpha(\tau(\zeta))\right) &\geq \int_{\zeta_2}^{\zeta} N(s)B_1\left(\delta(s)\right)\beta_1(\delta(s))\alpha'(\delta(s))ds \\ &\geq \beta_1(\delta(\zeta))\alpha'(\delta(\zeta))\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s))ds. \end{aligned} \tag{2.10}$$

Since $D_2\alpha(\zeta)$ is decreasing and $\tau(\zeta) < \zeta$, we have $D_2\alpha(\zeta) \leq D_2\alpha(\tau(\zeta))$, and using this in (2.10), we obtain

$$\begin{aligned} -\left(1 + \tfrac{g_0}{\tau_0}\right)D_2\alpha(\zeta) &\geq \beta_1(\delta(\zeta))\alpha'(\delta(\zeta))\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s))ds, \\ -\left(\beta_1(\zeta)\alpha'(\zeta)\right)' &\geq \left(\tfrac{\tau_0}{\tau_0+g_0}\right)\tfrac{\beta_1(\delta(\zeta))}{\beta_2(\zeta)}\alpha'(\delta(\zeta))\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s))ds. \end{aligned} \tag{2.11}$$

Let $\omega(\zeta) = \beta_1(\zeta)\alpha'(\zeta) > 0$ is a positive solution of the first-order delay differential inequality

$$\omega'(\zeta) + \left(\frac{\tau_0}{\tau_0 + g_0}\right)\left(\frac{1}{\beta_2(\zeta)}\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s))ds\right)\omega(\delta(\zeta)) \leq 0. \tag{2.12}$$

However, by [13, Theorem 2.11], the inequality (2.12) does not have a positive solution. This contradicts our initial assumption, and the proof is complete. □

Theorem 5. *Assume that (2.5) holds. If*

$$\limsup_{t\to\infty} G(t)\int_{t_0}^{t} M(s)G_1(\varsigma_i(s))ds > \frac{b_0 + g_0}{g_0}, \tag{2.13}$$

then the classes O_2 *and* O_3 *are empty.*

Proof. Assume to the contrary that α satisfies class O_2 or O_3 for $\zeta \geq \zeta_1$. First note that $\lim_{\zeta\to\infty} B(\zeta) = 0$ holds, which together with (2.13) implies (2.5). So by Lemma 2.7, we conclude that α satisfies O_3 and the asymptotic property (2.8) for all $\zeta \geq \zeta_1 \geq \zeta_0$. Proceeding as in the proof of Theorem 2.8, we arrive at (2.11). Now from the monotonicity of $D_2\alpha(\zeta)$, we obtain

$$\begin{aligned} \beta_1(\zeta)\alpha'(\zeta) &\geq -\int_{\zeta}^{\infty} \frac{1}{\beta_2(s)}\beta_2(s)\left(\beta_1(s)\alpha'(s)\right)' ds, \\ &\geq -B(\zeta)\beta_2(\zeta)\left(\beta_1(\zeta)\alpha'(\zeta)\right)', \end{aligned}$$

and using this in (2.11), we obtain

$$-\left(\beta_1(\zeta)\alpha'(\zeta)\right)' \geq -\left(\frac{\tau_0}{g_0 + \tau_0}\right)B(\zeta)\left(\beta_1(\zeta)\alpha'(\zeta)\right)'\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s)ds, \tag{2.14}$$

where we have used $\beta_1\left(\delta(\zeta)\alpha'(\delta(\zeta)\right) \geq \beta_1(\zeta)\alpha'(\zeta)$. From (2.14) we obtain

$$\frac{g_0+\tau_0}{\tau_0} \geq B(\zeta)\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s))ds.$$

But the last inequality contradicts (2.13), and the proof is complete.

□

Theorem 6. *Let γ be an eventually positive solution of (E_s). If $\varsigma_i(t) < g(g(t))$ and*

$$\limsup_{t\to\infty}\int_{g(t)}^{t} M(s)A(\varsigma_i(t),\varsigma_i(s))ds > \frac{g_0+b_0}{g_0}, \tag{2.15}$$

then the class O_1 *is empty.*

Proof. Assume the contrary that (2.15) holds, but α belongs to class O_1. Choose $\zeta_1 \geq \zeta_0$ such that $\delta(\zeta) \geq \zeta_1$ for $\zeta \geq \zeta_1$. From the monotonicity of $D_2\alpha(\zeta)$ that for $v \geq u$

$$-\beta_1(u)\alpha'(u) \geq \int_u^v \frac{\beta_2(s)}{\beta_2(s)}\left(\beta_1(s)\alpha'(s)\right)' ds \geq \beta_2(v)\left(\beta_1(v)\alpha'(v)\right)' \int_u^v \frac{ds}{\beta_2(s)}.$$

Dividing by $\beta_1(u)$ and then integrating from u to $v \geq u$ in u for the resulting inequality, we find

$$\alpha(u) \geq \beta_2(v)\left(\beta_1(v)\alpha'(v)\right)' \int_u^v \frac{1}{\beta_1(x)}\int_x^v \frac{ds}{\beta_2(s)}dx = \beta_2(v)\left(\beta_1(v)\alpha'(v)\right)' A(v,u). \tag{2.16}$$

Integrating (2.2) from $\tau(\zeta)$ to ζ and using (2.16) with $u = \delta(s)$ and $v = \delta(\zeta)$, we obtain

$$D_2\alpha(\tau(\zeta)) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\tau(\zeta))) \geq \int_{\tau(\zeta)}^{\zeta} N(s)\alpha(\delta(s))ds,$$

$$\geq D_2\alpha(\delta(\zeta))\int_{\tau(\zeta)}^{\zeta} N(s)A(\delta(\zeta),\delta(s))ds. \tag{2.17}$$

From $\delta(\zeta) < \tau(\tau(\zeta))$ and $\tau(\tau(\zeta)) < \tau(\zeta)$, we find

$$D_2\alpha(\delta(\zeta)) \geq D_2\alpha(\tau(\tau(\zeta))) \text{ and } \mathrm{D}_2\,\alpha(\tau(\tau(\zeta))) \geq D_2\alpha(\tau(\zeta)),$$

and using these in (2.17), we obtain

$$\left(1+\frac{g_0}{\tau_0}\right) \geq \int_{\tau(\zeta)}^{\zeta} N(s)A(\delta(\zeta),\delta(s))ds,$$

which contradicts (2.15) and the proof is complete. □

Theorem 7. *Given that γ constitutes the final positive solution of (E_s). If the function $\sigma(t) \in C([t_0,\infty),(0,\infty))$ satisfying $\varsigma_i(t) < \sigma(t) < g(t)$ such that*

$$\liminf_{t\to\infty}\int_{g^{-1}(\sigma(t))}^{t} M(s)A(\sigma(s),\varsigma_i(s))ds > \frac{g_0+b_0}{g_0 e}, \tag{2.18}$$

then the class O_1 *is empty.*

Proof. Let (2.18) holds, but α belongs to class O_1. Proceeding as in the prof of Theorem 2.10 we arrive at (2.10). Setting $u = \delta(\zeta)$ and $v = \xi(\zeta), \zeta \geq x \geq \zeta_1$, in (2.10), we obtain

$$\alpha(\delta(\zeta)) \geq D_2\alpha(\xi(\zeta))A(\xi(\zeta), \delta(\zeta)). \tag{2.19}$$

On the other hand, using (2.19) in (2.4) yields

$$(D_2\alpha(\zeta) + \frac{g_0}{\tau_0} D_2\alpha(\tau(\zeta)))' + N(\zeta)A(\xi(\zeta), \delta(\zeta))D_2\alpha(\xi(\zeta)) \leqslant 0. \tag{2.20}$$

Now, let

$$\omega(\zeta) = D_2\alpha(\zeta) + \frac{g_0}{\tau_0} D_2\alpha(\tau(\zeta)) > 0.$$

Using the fact that $\tau(\zeta) < \zeta$ and $D_2\alpha(\zeta)$ is nonincreasing, we have

$$\omega(\zeta) \leq \left(1 + \frac{g_0}{\tau_0}\right) D_2\alpha(\tau(\zeta)),$$

or equivalently

$$D_2\alpha(\xi(\zeta)) \geq \frac{\tau_0}{g_0 + \tau_0} \omega\tau^{-1}(\xi(\zeta)). \tag{2.21}$$

From (2.21) and (2.20), we see that $w(\zeta)$ is a positive solution of the first-order delay differential inequality

$$\omega'(\zeta) + \frac{\tau_0}{\tau_0 + g_0} N(\zeta)A(\xi(\zeta), \delta(\zeta))\omega\left(\tau^{-1}\xi(\zeta)\right) \leq 0. \tag{2.22}$$

If we apply [13, Theorem 2.11], we obtain that $w(t)$ is not a positive solution to (2.22), and thus the proof is complete. □

The primary outcome of the study is as follows: oscillation condition for (E).

Theorem 8. *Assume that* (H_1)–(H_4) *hold. If (2.9) (or (2.13)) and (2.15) (or (2.18)) satisfied, then Eq (E) is oscillatory.*

Proof. Let θ be a nonoscillatory solution of (E), and without loss of generality, assume that there exists a $\zeta_1 \geq \zeta_0$ such that $\theta(\zeta) > 0, \theta(\tau(\zeta)) > 0$ and $\theta(\delta(\zeta)) > 0$ for all $\zeta \geq \zeta_1$. Then, by Corollary 2.4, the function $\theta(\zeta)$ is also a positive solution of (E_s) as well as the related function $\alpha(\zeta)$, which satisfies one of the three classes O_1 or O_2 or O_3 for $\zeta \geq \zeta_1$.

In view of Theorem 2.8 (or Theorem 2.9), the classes O_2 and O_3 are empty. On the other hand from Theorem 2.10 (or Theorem 2.11), the class O_1 is empty. This contradiction implies that the Eq (E) is oscillatory. This concludes the proof. □

We provide an example at the end of this section to highlight the significance of our primary findings.

Example 1. *Examine the third-order Euler type neutral differential equation*

$$\left(t^2\left(t^2\left(x(t) + b_0 x\left(\beta_1 \eta\right)\right)'\right)'\right)' + a_0 t x\left(\beta_2 t\right) = 0, \quad t \geq 1, \tag{2.23}$$

where $a_0 > 0, b_0 > 0, \beta_1 \in (0,1)$ *and* $\beta_2 \in (0,1)$. *A simple calculation shows that*

$$\varpi_1(t) = \varpi_2(t) = \frac{1}{t}, \beta_1(t) = 1, \quad \beta_2(t) = \eta^3 \quad \text{and } g_0 = \beta_1.$$

We apply this data to obtain the transformed equation in semi-canonical form

$$\left(t^3\gamma''(t)\right)' + a_0 t x(\beta_2 t) = 0,$$

so, we find

$$M(t) = \frac{a_0\beta_1}{\beta_2}, G_1(t) \approx t \text{ and } G(t) = \frac{1}{2t^2}.$$

The condition (2.9) becomes.

$$\liminf_{t\to\infty} \int_{\beta_2\eta}^{t} \left(\frac{1}{s^3}\int_1^s \frac{a_0\beta_1}{\beta_2}\beta_2 s_1 ds_1\right) ds = \frac{a_0\beta_1}{2}\ln\frac{1}{\beta_2} > \frac{\beta_1 + b_0}{\beta_1 e},$$

that is, condition (2.9) satisfied if

$$a_0 > \frac{2(\beta_1 + b_0)}{\beta_1^2 e \ln\frac{1}{\beta_2}}.$$

Choose β_3 *such that* $\beta_2 < \beta_3 < \beta_1$ *then the condition (2.18) becomes*

$$\begin{aligned}\lim_{t\to\infty} inf \int_{\frac{\beta_3 t}{\beta_1}}^{t} \frac{a_0\beta_1}{\beta_2}\left(\frac{1}{2\beta_2} - \frac{1}{\beta_3} + \frac{\beta_2}{2\beta_3^2}\right)\frac{1}{s}ds \\ = \frac{a_0\beta_1}{\beta_2}\left(\frac{1}{2\beta_2} - \frac{1}{\beta_3} + \frac{\beta_2}{2\beta_3^2}\right)\ln\frac{\beta_1}{\beta_3} > \frac{\beta_1 + b_0}{\beta_1 e},\end{aligned}$$

that is, condition (2.18) is satisfied if

$$a_0\left(\frac{1}{2\beta_2} - \frac{1}{\beta_3} + \frac{\beta_2}{2\beta_3^2}\right) > \frac{\beta_2(\beta_1 + b_0)}{\beta_1^2 e \ln\frac{\beta_1}{\beta_3}}.$$

Therefore Eq (2.23) is oscillatory if

$$a_0 > \frac{2(\beta_1 + b_0)}{\beta_1^2 e \ln\frac{1}{\beta_2}},$$

and

$$a_0\left(\frac{1}{2\beta_2} - \frac{1}{\beta_3} + \frac{\beta_2}{2\beta_3^2}\right) > \frac{\beta_2(\beta_1 + b_0)}{\beta_1^2 e \ln\left(\frac{\beta_1}{\beta_3}\right)}.$$

In particular if we assume $\beta_1 = 1/2 \quad \beta_2 = 1/4 \quad \beta_3 = 1/3 \quad b_0 = 1/2$ *then we get* $a_0 > 29.033674$. *So in this case, the Eq (2.23) is oscillatory if* $a_0 > 29.033674$.

Take note that none of the outcomes listed in [2–4] can yield this conclusion since $b_0 < 1$ *and the equation is noncanonical.*

3. Conclusions

The aim of this paper is to investigate the oscillatory characteristics inherent in third-order differential equations featuring a noncanonical term. This investigation is conducted through the application of integral averaging and comparison techniques, ultimately leading to the derivation of oscillation criteria. The study culminates in the establishment of a central theorem pertaining to the oscillation behavior of equations. Additionally, three examples of the effectiveness of these criteria were discussed. In future work, we will study fractional order delay differential equations in their non-canonical form to find oscillatory properties that will contribute to enriching oscillation theory.

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Competing interests

There are no competing interests.

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