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Research article

Square-free numbers in the intersection of Lehmer set and Piatetski-Shapiro sequence

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Abstract: Let q be a sufficiently large odd integer, and let $c \in \left(1, \frac{4}{3}\right)$. We denote $R(c; q)$ as the count of square-free numbers in the intersection of the Lehmer set and the Piatetski-Shapiro sequence. By employing additive character properties to transform congruence equations and applying Kloosterman sums and methods of exponential sums, we derive a sharp asymptotic formula as q approaches infinity, which is significant for understanding the distribution properties of the Lehmer problem.

Keywords: Lehmer set; Piatetski-Shapiro sequence; square-free numbers; estimate methods of exponential sum; asymptotic properties
Mathematics Subject Classification: 11B83, 11L05, 11N69

1. Introduction

Let q be an integer. For each integer a with

$$1 \leqslant a < q, \ \ (a, q) = 1,$$

we know that [1] there exists one and only one $\bar{a}$ with

$$1 \leqslant \bar{a} < q$$

such that

$$a\bar{a} \equiv 1(q).$$

Define

$$R(q) := \{a : 1 \leqslant a \leqslant q, (a, q) = 1, 2 \nmid a + \bar{a}\},$$

$$r(q) := \#R(q).$$

The work [2] posed the problem of investigating a nontrivial estimation for $r(q)$ when q is an odd prime. Zhang [3, 4] gave several asymptotic formulas for $r(q)$, one of which is:

$$r(q) = \frac{1}{2}\phi(q) + O\left(q^{\frac{1}{2}} d^2(q) \log^2 q\right),$$

where $\phi(q)$ is the Euler function and $d(q)$ is the divisor function. Lu and Yi [5] studied a generalization of the Lehmer problem over short intervals. Let $n \geqslant 2$ be a fixed positive integer, $q \geqslant 3$ and c be integers with

$$(nc, q) = 1.$$

They defined

$$r_n(\theta_1, \theta_2, c; q) = \#\left\{(a, b) \in [1, \theta_1 q] \times [1, \theta_2 q] \mid ab \equiv c(\bmod q), n \nmid a + b\right\},$$

where $0 < \theta_1, \theta_2 \leqslant 1$, and obtained

$$r_n(\theta_1, \theta_2, c; q) = \left(1 - \frac{1}{n}\right)\theta_1\theta_2\varphi(q) + O\left(q^{1/2}\tau^6(q) \log^2 q\right),$$

where the O constant depends only on n. In addition, Xi and Yi [6] considered generalized Lehmer problem over short intervals. Han and Liu [7] gave an upper bound estimation for another generalization of the Lehmer problem over incomplete interval.

Guo and Yi [8] also found the Lehmer problem has good distribution properties on Beatty sequences. For fixed real numbers α and β, defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}.$$

Beatty sequences are linear sequences. Based on the results obtained, we conjecture the Lehmer problem also has good distribution properties in some non-linear sequences.

The Piatetski-Shapiro sequence is a non-linear sequence, defined by

$$\mathbb{N}^c = \{\lfloor n^c \rfloor : n \in \mathbb{N}\},$$

where $c \in \mathbb{R}$ is non-integer with $c > 1$ and $z \in \mathbb{R}$. This sequence was first introduced by Piatetski-Shapiro [9] to study prime numbers in sequences of the form $\lfloor f(n) \rfloor$, where $f(n)$ is a polynomial. A positive integer is called square-free if it is a product of distinct primes. The distribution of square-free numbers in the Piatetski-Shapiro sequence has been studied extensively. Stux [10] found that, as x tends to infinity,

$$\sum_{\substack{n \le x \\ \lfloor n^c \rfloor \text{ is square-free}}} 1 = \left(\frac{6}{\pi^2} + o(1)\right)x, \quad \text{for } 1 < c < \frac{4}{3}. \tag{1.1}$$

In 1978, Rieger [11] improved the range to $1 < c < 3/2$ and obtained

$$\frac{6x}{\pi^2} + O\left(x^{(2c+1)/4+\varepsilon}\right), \quad \text{for } 1 < c < \frac{3}{2}.$$

Considering the results obtained, we develop this problem by investigating

$$R(c;q) := \sum_{\substack{n\in\mathbb{N}^c\cap R(q)\\ n \text{ is square-free}}} 1$$

and range of c when q tends to infinity. By methods of exponential sum and Kloosterman sums and fairly detailed calculations, we get the following result, which is significant for understanding the distribution properties of the Lehmer problem.

Theorem 1.1. *Let q be an odd integer and large enough,*

$$\gamma := 1/c \quad \textit{and} \quad c \in (1, \frac{4}{3}),$$

we obtain

$$\begin{aligned} R(c;q) =& \frac{3}{\pi^2}\prod_{p|q}(1+p^{-1})^{-1}q^{\gamma} \\ &+ O\left(\sum_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\gamma-\frac{1}{2}}\right) + O\left(q^{\frac{7}{13}\gamma+\frac{4}{13}}\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}\log q\right) \\ &+ O\left(q^{\frac{3}{4}}d^3(q)\log q\right) + O\left(q^{\gamma-\frac{1}{6}}d^2(q)\log^3 q\right), \end{aligned}$$

where the O constant only depends on c.

This paper consists of three main sections. Introduction covers the origins and developments of the Lehmer problem, along with several interesting results. It also presents relevant findings related to the Piatetski-Shapiro sequences. The second section includes some definitions and lemmas throughout the paper. The third section outlines the calculation process, where we use additive characteristics to convert the congruence equations into exponential sum problems. We then employ the Kloosterman sums and exponential sums methods to derive an interesting asymptotic formula.

2. Preliminary lemmas

To complete the proof of the theorem, we need the following several definitions and lemmas.

In this paper, we denote by $\lfloor t\rfloor$ and $\{t\}$ the integral part and the fractional part of t, respectively. As is customary, we put

$$\mathbf{e}(t) := e^{2\pi i t} \quad \text{and} \quad \{t\} := t - \lfloor t\rfloor.$$

The notation $\|t\|$ is used to denote the distance from the real number t to the nearest integer; that is,

$$\|t\| := \min_{n\in\mathbb{Z}} |t-n|.$$

And $\sum'$ indicates that the variable summed over takes values coprime to the number q. Throughout the paper, ε always denotes an arbitrarily small positive constant, which may not be the same at different occurrences; the implied constants in symbols $O, \ll$, and $\gg$ may depend (where obvious) on the parameters c and ε, but are absolute otherwise. For given functions F and G, the notations

$$F \ll G, \quad G \gg F \quad \text{and} \quad F = O(G)$$

are all equivalent to the statement that the inequality

$$|F| \leqslant C|G|$$

holds with some constant $C > 0$.

Lemma 2.1. *Let* $\mathbf{1}_c(m)$ *denote the characteristic function of numbers in a Piatetski-Shapiro sequence, then*

$$\mathbf{1}_c(m) = \gamma m^{\gamma-1} + O(m^{\gamma-2}) + \psi\left(-(m+1)^{\gamma}\right) - \psi(-m^{\gamma}),$$

where

$$\psi(t) = t - \lfloor t \rfloor - \frac{1}{2} \quad \textit{and} \quad \gamma = 1/c.$$

Proof. Note that an integer m has the form

$$m = \lfloor n^c \rfloor$$

for some integer n if and only if

$$m \leqslant n^c < m + 1, \quad -(m+1)^{\gamma} < -n \leqslant -m^{\gamma}.$$

So

$$\begin{aligned}
\mathbf{1}_c(m) &= \lfloor -m^{\gamma} \rfloor - \lfloor -(m+1)^{\gamma} \rfloor \\
&= -m^{\gamma} - \psi(-m^{\gamma}) + (m+1)^{\gamma} + \psi(-(m+1)^{\gamma}) \\
&= \gamma m^{\gamma-1} + O(m^{\gamma-2}) + \psi\left(-(m+1)^{\gamma}\right) - \psi(-m^{\gamma}).
\end{aligned}$$

Thie completes the proof. □

Lemma 2.2. *Let* $H \geqslant 1$ *be an integer,* a_h, b_h *be real numbers, we have*

$$\left|\psi(t) - \sum_{0<|h|\leqslant H} a_h \mathbf{e}(th)\right| \leqslant \sum_{|h|\leqslant H} b_h \mathbf{e}(th), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.$$

Proof. In 1985, Vaaler showed how Beurling's function could be used to construct a trigonometric polynomial approximation to $\psi(x)$. For each positive integer N, Vaaler's construction yields a trigonometric polynomial ψ^* of degree N which satisfies

$$|\psi^*(x) - \psi(x)| \leq \frac{1}{2N+2} \sum_{|n|\leq N} \left(1 - \frac{|n|}{N+1}\right) \mathbf{e}(nx),$$

where

$$\begin{aligned}
\psi^*(x) &= -\sum_{1\leq|n|\leq N} (2\pi i n)^{-1} \hat{J}_{N+1}(n) \mathbf{e}(nx), \\
H(z) &= \frac{\sin^2 \pi z}{\pi^2} \left\{ \sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} + \frac{2}{z} \right\}, \quad J(z) = \frac{1}{2} H'(z),
\end{aligned}$$

$$H_N(z) = \frac{\sin^2 \pi z}{\pi^2} \left\{ \sum_{|n| \le N} \frac{\text{sgn}\,(n)}{(z-n)^2} + \frac{2}{z} \right\}, \quad J_N(z) = \frac{1}{2} H_N'(z),$$

and sgn (n) is the sign of n. The Fourier transform $\hat{J}(t)$ satisfies

$$\hat{J}(t) = \begin{cases} 1, & t = 0; \\ \pi t(1-|t|)\cot \pi t + |t|, & 0 < |t| < 1; \\ 0, & t \ge 1. \end{cases}$$

To be short, we denote

$$a_h = -(2\pi i h)^{-1} \hat{J}_{H+1}(h) \ll \frac{1}{|h|},$$
$$b_h = \frac{1}{2H+2}\left(1 - \frac{|h|}{H+1}\right) \ll \frac{1}{H}.$$

There are more details in Appendix Theorem A.6. of [12]. □

Lemma 2.3. *Denote*

$$\mathbf{Kl}(m,n;q) = \sum_{\substack{a=1 \\ ab\equiv 1(\bmod q)}}^{q} \sum_{b=1}^{q} \mathbf{e}\left(\frac{ma+nb}{q}\right),$$

then

$$\mathbf{Kl}(m,n;q) \ll (m,n,q)^{\frac{1}{2}} q^{\frac{1}{2}} d(q),$$

where (m,n,q) is the greatest common divisor of m,n and q and $d(q)$ is the number of positive divisors of q.

Proof. The proof is given in [13]. □

Lemma 2.4. *(Korobov [14]) Let α be a real number, Q be an integer, and P be a positive integer, then*

$$\left| \sum_{x=Q+1}^{Q+P} \mathbf{e}(\alpha x) \right| \leqslant \min\left(P, \frac{1}{2\|\alpha\|}\right).$$

Lemma 2.5. *(Karatsuba [15]) For any number b, $U < 0$, $K \geqslant 1$, let*

$$a = \frac{s}{r} + \frac{\theta}{r^2}, \quad (r,s) = 1, \quad r \geqslant 1, \quad |\theta| \leqslant 1,$$

then

$$\sum_{k \leqslant K} \min\left(U, \frac{1}{\|ak+b\|}\right) \ll \left(\frac{K}{r} + 1\right)(U + r \log r).$$

Lemma 2.6. *Suppose f is continuously differentiable, $f'(n)$ is monotonic, and*

$$\|f'(n)\| \geqslant \lambda_1 > 0$$

on I, then

$$\sum_{n \in I} \mathbf{e}\,(f(n)) \ll \lambda_1^{-1}.$$

Proof. See [12, Theorem 2.1]. □

Lemma 2.7. *Let k be a positive integer, $k \geqslant 2$. Suppose that $f(n)$ is a real-valued function with k continuous derivatives on $[N, 2N]$, Further suppose that*

$$0 < F \leqslant f^{(k)}(n) \leqslant hF.$$

Then

$$\left| \sum_{N<x\leqslant 2N} \mathbf{e}(f(n)) \right| \ll F^{\kappa} N^{\lambda} + F^{-1},$$

where the implied constant is absolute.

Proof. See [12, Chapter 3]. □

3. Proof of theorem

By the definition of Mobius function

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \forall p|n, p^2 \nmid n, \\ 0, & \exists p^2|n, \end{cases}$$

it is clear that n is square-free if and only if

$$\mu^2(n) = 1,$$

where $\omega(n)$ is the number of prime divisor of n. So

$$\begin{aligned} R(c;q) &= \frac{1}{2} \sideset{}{'}\sum_{n=1}^{q} \left(1 - (-1)^{n+\bar{n}}\right) \mu^2(n) \mathbf{1}_c(n) \\ &= \frac{1}{2}(R_1 - R_2), \end{aligned} \tag{3.1}$$

where

$$R_1 = \sideset{}{'}\sum_{n=1}^{q} \mu^2(n) \mathbf{1}_c(n)$$

and

$$R_2 = \sideset{}{'}\sum_{n=1}^{q} (-1)^{n+\bar{n}} \mu^2(n) \mathbf{1}_c(n).$$

3.1. Estimation of R_1

From Lemma 2.1, we have

$$\begin{aligned} R_1 &= \sum_{n=1}^{q}{}' \mu^2(n)\mathbf{1}_c(n) \\ &= \sum_{n=1}^{q}{}' \mu^2(n)\left(\gamma n^{\gamma-1} + O(n^{\gamma-2}) + \psi\left(-(n+1)^{\gamma}\right) - \psi(-n^{\gamma})\right) \\ &= R_{11} + R_{12}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} R_{11} &:= \sum_{n=1}^{q}{}' \mu^2(n)\left(\gamma n^{\gamma-1} + O(n^{\gamma-2})\right) \\ &= \sum_{n=1}^{q}{}' \mu^2(n)\gamma n^{\gamma-1} + O\left(\sum_{n=1}^{q}{}' \mu^2(n) n^{\gamma-2}\right). \end{aligned}$$

Let

$$\mathcal{D} = \{d : p|d \Rightarrow p|q\}$$

and $\lambda(n)$ is Liouville function. When $n \in R(q)$,

$$\mu^2(n) = \begin{cases} \sum\limits_{dm=n, d\in\mathcal{D}} \lambda(d)\mu^2(m), & (n,q) = 1, \\ 0, & (n,q) > 1. \end{cases} \tag{3.3}$$

We just consider the first term of R_{11}. Applying Euler summuation [1],

$$\begin{aligned} \sum_{n=1}^{q}{}' \mu^2(n)\gamma n^{\gamma-1} &= \sum_{n=1}^{q} \sum_{\substack{dm=n \\ d\in\mathcal{D}}} \lambda(d)\mu^2(m)\gamma(dm)^{\gamma-1} \\ &= \sum_{d\in\mathcal{D}} \lambda(d) d^{\gamma-1} \sum_{m\leqslant \frac{q}{d}} \mu^2(m)\gamma m^{\gamma-1} \\ &= \sum_{d\in\mathcal{D}} \lambda(d) d^{\gamma-1} \sum_{m\leqslant \frac{q}{d}} \left(\sum_{l^2|m} \mu(l)\right) \gamma m^{\gamma-1} \\ &= \sum_{d\in\mathcal{D}} \lambda(d) d^{\gamma-1} \sum_{l\leqslant (\frac{q}{d})^{\frac{1}{2}}} \mu(l) l^{2\gamma-2} \sum_{m\leqslant \frac{q}{dl^2}} \gamma m^{\gamma-1} \\ &= \sum_{d\in\mathcal{D}} \lambda(d) d^{\gamma-1} \sum_{l\leqslant (\frac{q}{d})^{\frac{1}{2}}} \mu(l) l^{2\gamma-2} \left((\frac{q}{dl^2})^{\gamma} + O\left((\frac{q}{dl^2})^{\gamma-1} \right) \right) \\ &= q^{\gamma} \sum_{d\in\mathcal{D}} \lambda(d) d^{-1} \sum_{l\leqslant (\frac{q}{d})^{\frac{1}{2}}} \mu(l) l^{-2} + O\left(\sum_{d\in\mathcal{D}} \sum_{l\leqslant (\frac{q}{d})^{\frac{1}{2}}} q^{\gamma-1} \right) \end{aligned}$$

$$= q^{\gamma}\sum_{d\in\mathcal{D}}\lambda(d)d^{-1}\left(\sum_{l}\mu(l)l^{-2}+O\left((\frac{q}{d})^{-\frac{1}{2}}\right)\right)+O\left(\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\gamma-\frac{1}{2}}\right)$$
$$= \frac{6}{\pi^2}\prod_{p|q}(1+p^{-1})^{-1}q^{\gamma}+O\left(\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\gamma-\frac{1}{2}}\right),$$

thus

$$R_{11}=\frac{6}{\pi^2}\prod_{p|q}(1+p^{-1})^{-1}q^{\gamma}+O\left(\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\gamma-\frac{1}{2}}\right). \tag{3.4}$$

For R_{12}, by Lemma 2.2, we have

$$\begin{aligned} R_{12} &:= \sum_{n=1}^{q}{}' \mu^2(n)\left(\psi\left(-(n+1)^{\gamma}\right)-\psi(-n^{\gamma})\right) \\ &= R_{121}+O(R_{122}), \end{aligned} \tag{3.5}$$

where

$$R_{121}:=\sum_{n=1}^{q}{}'\mu^2(n)\left(\sum_{0<|h|\leqslant H}a_h\left(\mathbf{e}\left(-(n+1)^{\gamma}h\right)-\mathbf{e}\left(-n^{\gamma}h\right)\right)\right)$$

and

$$R_{122}:=\sum_{n=1}^{q}{}'\mu^2(n)\left(\sum_{|h|\leqslant H}b_h\left(\mathbf{e}\left(-(n+1)^{\gamma}h\right)+\mathbf{e}\left(-n^{\gamma}h\right)\right)\right).$$

Define

$$f(t)=\mathbf{e}\left(\left((dt)^{\gamma}-(dt+1)^{\gamma}\right)h\right)-1,$$

then

$$\begin{aligned} f(t) &\ll |h|(dt)^{\gamma-1}, \\ \frac{\partial f(t)}{\partial t} &\ll |h|d^{\gamma-1}t^{\gamma-2}. \end{aligned}$$

By Lemma 2.2 and Eq (3.3),

$$\begin{aligned} R_{121} &= \sum_{0<|h|\leqslant H}a_h\sum_{d\in\mathcal{D}}\lambda(d)\left(\sum_{1<m\leqslant\frac{q}{d}}\mu^2(m)\mathbf{e}\left(-(dm)^{\gamma}h\right)f(m)\right) \\ &\ll \sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\in\mathcal{D}}\left|\int_0^{\frac{q}{d}}f(t)\mathrm{d}\left(\sum_{1<m\leqslant t}\mu^2(m)\mathbf{e}\left(-(dm)^{\gamma}h\right)\right)\right| \\ &\ll \sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\in\mathcal{D}}\left|f(\frac{q}{d})\left(\sum_{1<m\leqslant\frac{q}{d}}\mu^2(m)\mathbf{e}\left(-(dm)^{\gamma}h\right)\right)\right| \end{aligned}$$

$$+\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\in\mathcal{D}}\left|\int_0^{\frac{q}{d}}\frac{\partial f(t)}{\partial t}\sum_{1<m\leqslant t}\mu^2(m)\mathbf{e}\left(-(dm)^\gamma h\right)\mathrm{d}t\right|.$$

Let

$$(\kappa,\lambda)=(\frac{1}{6},\frac{2}{3})$$

be an exponential pair. Applying Lemma 2.7, it's easy to see

$$\begin{aligned}\sum_{1<m\leqslant t}\mu^2(m)\mathbf{e}\left(-(dm)^\gamma h\right)&=\sum_{m\leqslant t}\left(\sum_{l^2|m}\mu(l)\right)\mathbf{e}\left(-(dm)^\gamma h\right)\\&\ll\sum_{l\leqslant t^{\frac{1}{2}}}\left|\sum_{m\leqslant\frac{t}{l^2}}\mathbf{e}\left(-(dl^2m)^\gamma h\right)\right|\\&\ll\sum_{l\leqslant t^{\frac{1}{2}}}\log q\left(\left((dl^2)^\gamma|h|(\frac{t}{l^2})^{\gamma-1}\right)^{\frac{1}{6}}(\frac{t}{l^2})^{\frac{2}{3}}+\left((dl^2)^\gamma|h|(\frac{t}{l^2})^{\gamma-1}\right)^{-1}\right)\\&\ll\log q\sum_{l\leqslant t^{\frac{1}{2}}}\left((d^\gamma|h|)^{\frac{1}{6}}t^{\frac{1}{6}\gamma+\frac{1}{2}}l^{-1}+(d^\gamma|h|)^{-1}t^{1-\gamma}l^{-2}\right)\\&\ll(d^\gamma|h|)^{\frac{1}{6}}t^{\frac{1}{6}\gamma+\frac{1}{2}}\log^2q+(d^\gamma|h|)^{-1}t^{1-\gamma}\log q\\&\ll(d^\gamma|h|)^{\frac{1}{6}}t^{\frac{1}{6}\gamma+\frac{1}{2}}\log^2q,\end{aligned}$$

thus

$$\begin{aligned}R_{121}\ll&\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\in\mathcal{D}}|h|q^{\gamma-1}(d^\gamma|h|)^{\frac{1}{6}}(\frac{q}{d})^{\frac{1}{6}\gamma+\frac{1}{2}}\log^2q\\&+\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\in\mathcal{D}}\int_0^{\frac{q}{d}}|h|d^{\gamma-1}t^{\gamma-2}(d^\gamma|h|)^{\frac{1}{6}}t^{\frac{1}{6}\gamma+\frac{1}{2}}\log^2q\mathrm{d}t\\\ll&\sum_{0<|h|\leqslant H}|h|^{\frac{1}{6}}\sum_{d\in\mathcal{D}}d^{-\frac{1}{2}}q^{\frac{7}{6}\gamma-\frac{1}{2}}\log^2q\\&+\sum_{0<|h|\leqslant H}|h|^{\frac{1}{6}}\sum_{d\in\mathcal{D}}d^{\frac{7}{6}\gamma-1}\log^2q\int_0^{\frac{q}{d}}t^{\frac{7}{6}\gamma-\frac{3}{2}}\mathrm{d}t\\\ll&H^{\frac{7}{6}}\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\frac{7}{6}\gamma-\frac{1}{2}}\log^2q.\end{aligned}\tag{3.6}$$

For R_{122}, the contribution from $h\neq0$ can be bounded by similar methods of Eq (3.6). Taking

$$H=q^{\frac{9}{13}-\frac{7}{13}\gamma}\geqslant1,$$

we obtain

$$R_{122}=b_0\sum_{n=1}^{q}{}'\mu^2(n)+\sum_{0<|h|\leqslant H}b_h\sum_{n=1}^{q}{}'\mu^2(n)\left(e\left(-(n+1)^\gamma h\right)+e\left(-n^\gamma h\right)\right)$$

$$
\begin{aligned}
&\ll H^{-1}q + H^{\frac{7}{6}}\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\frac{7}{6}\gamma-\frac{1}{2}}\log q\\
&\ll q^{\frac{7}{13}\gamma+\frac{4}{13}}\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}\log^2 q. \qquad (3.7)
\end{aligned}
$$

It follows from Eqs (3.5)–(3.7),

$$
R_{12} \ll q^{\frac{7}{13}\gamma+\frac{4}{13}}\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}\log^2 q.
$$

Hence

$$
R_1 = \frac{6}{\pi^2}\prod_{p|q}(1+p^{-1})^{-1}q^{\gamma} + O\left(\sum_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\gamma-\frac{1}{2}}\right) + O\left(q^{\frac{7}{13}\gamma+\frac{4}{13}}\prod_{p|q}(1-p^{-\frac{1}{2}})^{-1}\log^2 q\right). \qquad (3.8)
$$

3.2. Estimation of R_2

Similarly,

$$
\begin{aligned}
R_2 &= \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\mathbf{1}_c(n)\\
&=RP_{21}+RP_{22}, \qquad (3.9)
\end{aligned}
$$

where

$$
R_{21} = \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\left(\gamma n^{\gamma-1} + O(n^{\gamma-2})\right)
$$

and

$$
R_{22} = \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\left(\psi\left(-(n+1)^{\gamma}\right)-\psi(-n^{\gamma})\right).
$$

We also just consider the first term of R_{21}.

$$
\begin{aligned}
\sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\gamma n^{\gamma-1} &= \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\left(\sum_{d^2|n}\mu(d)\right)\gamma n^{\gamma-1}\\
&= \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\sum_{\substack{d^2|n\\ d\leqslant q^{\frac{1}{4}}}}\mu(d)\gamma n^{\gamma-1} + \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\sum_{\substack{d^2|n\\ q^{\frac{1}{4}}<d\leqslant q^{\frac{1}{2}}}}\mu(d)\gamma n^{\gamma-1}. \qquad (3.10)
\end{aligned}
$$

It is easy to see

$$
\sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\sum_{\substack{d^2|n\\ q^{\frac{1}{4}}<d\leqslant q^{\frac{1}{2}}}}\mu(d)\gamma n^{\gamma-1} \ll \sum_{n=1}^{q}{}'\sum_{\substack{d^2|n\\ q^{\frac{1}{4}}<d\leqslant q^{\frac{1}{2}}}}\gamma n^{\gamma-1} \ll q^{\gamma-\frac{1}{4}}. \qquad (3.11)
$$

Since for integers m and a, one has

$$\frac{1}{q}\sum_{s=1}^{q}\mathbf{e}(\frac{s(m-a)}{q}) = \begin{cases}1, & m \equiv a(\bmod q);\\ 0, & m \not\equiv a(\bmod q).\end{cases}$$

This gives

$$\begin{aligned}
\sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\sum_{\substack{d^2|n\\ d\leqslant q^{\frac14}}}\mu(d)\gamma n^{\gamma-1} &= \sum_{\substack{n=1\\ nm\equiv 1(\bmod q)}}^{q}\sum_{m=1}^{q}(-1)^{n+m}\sum_{\substack{d^2|n\\ d\leqslant q^{\frac14}}}\mu(d)\gamma n^{\gamma-1}\sum_{\substack{a=1\\ a=m}}^{q-1}\sum_{\substack{b=1\\ b=n}}^{q-1}1\\
&= \sum_{\substack{n=1\\ nm\equiv 1(\bmod q)}}^{q}\sum_{m=1}^{q}\sum_{\substack{a=1\\ a=m}}^{q-1}\sum_{\substack{b=1\\ b=n}}^{q-1}(-1)^{a+b}\sum_{\substack{d^2|b\\ d\leqslant q^{\frac14}}}\mu(d)\gamma b^{\gamma-1}\\
&= \sum_{\substack{n=1\\ nm\equiv 1(\bmod q)}}^{q}\sum_{m=1}^{q}\sum_{\substack{a=1\\ a\equiv m(\bmod q)}}^{q-1}\sum_{\substack{b=1\\ b\equiv n(\bmod q)}}^{q-1}(-1)^{a+b}\sum_{\substack{d^2|b\\ d\leqslant q^{\frac14}}}\mu(d)\gamma b^{\gamma-1}\\
&= \sum_{\substack{n=1\\ nm\equiv 1(\bmod q)}}^{q}\sum_{m=1}^{q}\sum_{a=1}^{q-1}\sum_{b=1}^{q-1}(-1)^{a+b}\sum_{\substack{d^2|b\\ d\leqslant q^{\frac14}}}\mu(d)\gamma b^{\gamma-1}\\
&\quad\times\left(\frac{1}{q}\sum_{s=1}^{q}\mathbf{e}(\frac{s(m-a)}{q})\right)\left(\frac{1}{q}\sum_{t=1}^{q}\mathbf{e}(\frac{t(n-b)}{q})\right)\\
&=\frac{1}{q^2}\sum_{s=1}^{q}\sum_{t=1}^{q}\left(\sum_{nm\equiv 1(\bmod q)}\mathbf{e}(\frac{sm+tn}{q})\right)\\
&\quad\times\left(\sum_{a=1}^{q-1}(-1)^{a}\mathbf{e}(-\frac{sa}{q})\right)\left(\sum_{b=1}^{q-1}(-1)^{b}\mathbf{e}(-\frac{tb}{q})\sum_{\substack{d^2|b\\ d\leqslant q^{\frac14}}}\mu(d)\gamma b^{\gamma-1}\right).
\end{aligned} \tag{3.12}$$

From Lemma 2.3,

$$\sum_{nm\equiv 1(\bmod q)}\mathbf{e}(\frac{sm+tn}{q}) = \mathbf{Kl}(s,t;q) \ll (s,t,q)^{\frac12}q^{\frac12}d(q). \tag{3.13}$$

Note the estimate

$$\left|\sum_{a=1}^{q-1}(-1)^{a}\mathbf{e}(-\frac{sa}{q})\right| \ll \frac{1}{\left|e(\frac12-\frac{s}{q})-1\right|} \ll \frac{1}{|\cos\frac{s}{q}\pi|} \tag{3.14}$$

holds. By Abel summation and Lemma 2.4, we have

$$\begin{aligned}
\sum_{b=1}^{q-1}(-1)^b\mathbf{e}(-\frac{tb}{q})\sum_{\substack{d^2|b\\ d\leqslant q^{\frac14}}}\mu(d)\gamma b^{\gamma-1} &= \sum_{d\leqslant q^{\frac14}}\mu(d)\sum_{b=1}^{\lfloor\frac{q-1}{d^2}\rfloor}(-1)^{d^2b}\mathbf{e}(-\frac{td^2b}{q})\gamma(d^2b)^{\gamma-1}\\
&\ll \sum_{d\leqslant q^{\frac14}} d^{2(\gamma-1)}\left|\sum_{b=1}^{\lfloor\frac{q-1}{d^2}\rfloor}(-1)^{d^2b}\mathbf{e}(-\frac{td^2b}{q})\gamma b^{\gamma-1}\right|\\
&\ll \sum_{d\leqslant q^{\frac14}} d^{2(\gamma-1)}\gamma(\frac{q}{d^2})^{\gamma-1}\max_{1\leqslant\beta\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\left|\sum_{b=1}^{\beta}(-1)^{d^2b}\mathbf{e}(-\frac{td^2b}{q})\right|\\
&\ll \sum_{\substack{d\leqslant q^{\frac14}\\ 2|d}}\gamma q^{\gamma-1}\max_{1\leqslant\beta\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\left|\sum_{b=1}^{\beta}\mathbf{e}(-\frac{td^2b}{q})\right|\\
&\quad+\sum_{\substack{d\leqslant q^{\frac14}\\ 2\nmid d}}\gamma q^{\gamma-1}\max_{1\leqslant\beta\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\left|\sum_{b=1}^{\beta}(-1)^{b}\mathbf{e}(-\frac{td^2b}{q})\right|\\
&\ll \sum_{\substack{d\leqslant q^{\frac14}\\ 2|d}} q^{\gamma-1}\min\left(\lfloor\frac{q-1}{d^2}\rfloor,\frac{1}{2\|\frac{d^2}{q}t\|}\right)+\sum_{\substack{d\leqslant q^{\frac14}\\ 2\nmid d}} q^{\gamma-1}\min\left(\lfloor\frac{q-1}{d^2}\rfloor,\frac{1}{2\|\frac12-\frac{d^2}{q}t\|}\right).
\end{aligned}\tag{3.15}$$

To be short, combining Eqs (3.13)–(3.15), we denote

$$\begin{aligned}
R_{211} :&= q^{-2}\sum_{s=1}^{q}\sum_{t=1}^{q}(s,t,q)^{\frac12}d(q)q^{\frac12}\frac{1}{|\cos\frac{s}{q}\pi|}\sum_{\substack{d\leqslant q^{\frac14}\\ 2|d}} q^{\gamma-1}\min\left(\lfloor\frac{q-1}{d^2}\rfloor,\frac{1}{2\|\frac{d^2}{q}t\|}\right)\\
&\ll q^{\gamma-3}\sum_{s=1}^{q}\sum_{t=1}^{q}(s,t,q)^{\frac12}d(q)q^{\frac12}\frac{1}{|\cos\frac{s}{q}\pi|}\sum_{d\leqslant q^{\frac14}}\min\left(\frac{q-1}{d^2},\frac{1}{2\|\frac{d^2}{q}t\|}\right)\\
&\ll q^{\gamma-3}\sum_{u|q}u^{\frac12}d(q)q^{\frac12}\sum_{s=1}^{\frac{q}{u}}\frac{1}{|\cos\frac{su}{q}\pi|}\sum_{d\leqslant q^{\frac14}}\sum_{t=1}^{\frac{q}{u}}\min\left(\frac{q-1}{d^2},\frac{1}{2\|\frac{d^2}{\frac{q}{u}}t\|}\right)
\end{aligned}$$

and

$$R_{212} := q^{-2}\sum_{s=1}^{q}\sum_{t=1}^{q}(s,t,q)^{\frac12}d(q)q^{\frac12}\frac{1}{|\cos\frac{s}{q}\pi|}\sum_{\substack{d\leqslant q^{\frac14}\\ 2\nmid d}} q^{\gamma-1}\min\left(\lfloor\frac{q-1}{d^2}\rfloor,\frac{1}{2\|\frac12-\frac{d^2}{q}t\|}\right).$$

Let

$$(d^2,\frac{q}{u}) = r,\quad (\frac{d^2}{r},\frac{q}{ur}) = 1,$$

making use of Lemma 2.5, we have

$$\sum_{t=1}^{\frac{q}{u}} \min\left(\frac{q-1}{d^2}, \frac{1}{2\|\frac{d^2}{\frac{q}{u}}t\|}\right) \ll (\frac{\frac{q}{u}}{\frac{q}{ur}}+1)(\frac{q-1}{d^2}+\frac{q}{ur}\log q) \ll \frac{qr}{d^2}+\frac{q}{u}\log q.$$

Insert it to R_{211}, then

$$\begin{aligned} R_{211} \ll & q^{\gamma-3}\sum_{u|q} u^{\frac{1}{2}}d(q)q^{\frac{1}{2}}\sum_{s=1}^{\frac{q}{u}}\frac{1}{|1-2\frac{su}{q}|}\sum_{d\leqslant q^{\frac{1}{4}}}\sum_{(d^2,\frac{q}{u})=r}(\frac{qr}{d^2}+\frac{q}{u}\log q) \\ \ll & q^{\gamma-3}\sum_{u|q} u^{\frac{1}{2}}d(q)q^{\frac{1}{2}}\frac{q}{u}\log q\sum_{d\leqslant q^{\frac{1}{4}}}\sum_{\substack{r|d^2 \\ r|\frac{q}{u}}}(\frac{qr}{d^2}+\frac{q}{u}\log q) \\ \ll & q^{\gamma-3}\sum_{u|q} u^{\frac{1}{2}}d(q)q^{\frac{1}{2}}\frac{q}{u}\log q\sum_{r|\frac{q}{u}}\sum_{d\leqslant\frac{q^{\frac{1}{4}}}{r^{\frac{1}{2}}}}(\frac{q}{d^2}+\frac{q}{u}\log q) \\ \ll & q^{\gamma-3}\sum_{u|q} u^{\frac{1}{2}}d(q)q^{\frac{1}{2}}\frac{q}{u}\log q\left(qd(q)+\frac{q^{\frac{5}{4}}}{u}d(q)\log q\right) \\ \ll & q^{\gamma-\frac{1}{4}}d^3(q)\log^3 q. \end{aligned}$$

By the same method of R_{211},

$$R_{212} \ll q^{\gamma-\frac{1}{4}}d^3(q)\log^3 q.$$

Following from Eqs (3.10) and (3.11), estimations of R_{211} and R_{212},

$$R_{21} \ll q^{\gamma-\frac{1}{4}} + RP_{211} + RP_{212} \ll q^{\gamma-\frac{1}{4}}d^3(q)\log^3 q. \tag{3.16}$$

By the similar method of R_{12} and R_{21},

$$R_{22} = R_{221} + O(R_{222}), \tag{3.17}$$

where

$$R_{221} := \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\left(\sum_{0<|h|\leqslant H} a_h\left(\mathbf{e}\left(-(n+1)^\gamma h\right)-\mathbf{e}(-n^\gamma h)\right)\right)$$

and

$$R_{222} := \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\left(\sum_{|h|\leqslant H} b_h\left(\mathbf{e}\left(-(n+1)^\gamma h\right)+\mathbf{e}(-n^\gamma h)\right)\right).$$

It is obvious that

$$R_{221} = \sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\left(\sum_{d^2|n}\mu(d)\right)\left(\sum_{0<|h|\leqslant H} a_h\left(\mathbf{e}\left(-(n+1)^\gamma h\right)-\mathbf{e}(-n^\gamma h)\right)\right)$$

$$= \sum_{n=1}^{q}{}' (-1)^{n+\bar{n}} \sum_{\substack{d^2|n \\ d \leqslant q^{\frac{1}{6}}}} \mu(d) \left(\sum_{0<|h|\leqslant H} a_h \left(\mathbf{e}\left(-(n+1)^{\gamma} h\right) - \mathbf{e}(-n^{\gamma} h) \right) \right)$$

$$+ \sum_{n=1}^{q}{}' (-1)^{n+\bar{n}} \sum_{\substack{d^2|n \\ q^{\frac{1}{6}} < d \leqslant q^{\frac{1}{2}}}} \mu(d) \left(\sum_{0<|h|\leqslant H} a_h \left(\mathbf{e}\left(-(n+1)^{\gamma} h\right) - \mathbf{e}(-n^{\gamma} h) \right) \right). \tag{3.18}$$

From the estimate

$$\mathbf{e}\left(n^{\gamma} h - (n+1)^{\gamma} h\right) - 1 \ll (n^{\gamma} - (n+1)^{\gamma}) h \ll \gamma n^{\gamma-1} h,$$

by partial summation,

$$\sum_{n=1}^{q}{}' (-1)^{n+\bar{n}} \sum_{\substack{d^2|n \\ q^{\frac{1}{6}} < d \leqslant q^{\frac{1}{2}}}} \mu(d) \left(\sum_{0<|h|\leqslant H} a_h \left(\mathbf{e}\left(-(n+1)^{\gamma} h\right) - \mathbf{e}(-n^{\gamma} h) \right) \right)$$

$$\ll \sum_{n=1}^{q}{}' \sum_{\substack{d^2|n \\ q^{\frac{1}{6}} < d \leqslant q^{\frac{1}{2}}}} \left| \sum_{0<|h|\leqslant H} a_h \mathbf{e}(-n^{\gamma} h) \left(\mathbf{e}\left(n^{\gamma} h - (n+1)^{\gamma} h\right) - 1 \right) \right|$$

$$\ll \sum_{n=1}^{q}{}' \sum_{\substack{d^2|n \\ q^{\frac{1}{6}} < d \leqslant q^{\frac{1}{2}}}} \gamma n^{\gamma-1} H \log H$$

$$\ll q^{\gamma - \frac{1}{6}} H \log H. \tag{3.19}$$

For another term of R_{221},

$$\sum_{n=1}^{q}{}' (-1)^{n+\bar{n}} \sum_{\substack{d^2|n \\ d \leqslant q^{\frac{1}{6}}}} \mu(d) \left(\sum_{0<|h|\leqslant H} a_h \left(\mathbf{e}\left(-(n+1)^{\gamma} h\right) - \mathbf{e}(-n^{\gamma} h) \right) \right)$$

$$= \sum_{n=1}^{q}{}' (-1)^{n+\bar{n}} \sum_{\substack{d^2|n \\ d \leqslant q^{\frac{1}{6}}}} \mu(d) \left(\sum_{0<|h|\leqslant H} a_h \left(\mathbf{e}\left(-(n+1)^{\gamma} h\right) - \mathbf{e}(-n^{\gamma} h) \right) \right)$$

$$\times \left(\frac{1}{q} \sum_{a=1}^{q} \sum_{s=1}^{q} \mathbf{e}\Big(\frac{s(m-a)}{q}\Big) \right) \left(\frac{1}{q} \sum_{b=1}^{q} \sum_{t=1}^{q} \mathbf{e}\Big(\frac{t(n-b)}{q}\Big) \right)$$

$$= \frac{1}{q^2} \sum_{s=1}^{q} \sum_{t=1}^{q} \left(\sum_{nm \equiv 1 (\bmod q)} \mathbf{e}\Big(\frac{sm+tn}{q}\Big) \right) \left(\sum_{a=1}^{q-1} (-1)^a \mathbf{e}\Big(-\frac{sa}{q}\Big) \right)$$

$$\times \left(\sum_{b=1}^{q-1} (-1)^b \mathbf{e}\Big(-\frac{tb}{q}\Big) \sum_{\substack{d^2|b \\ d \leqslant q^{\frac{1}{6}}}} \mu(d) \sum_{0<|h|\leqslant H} a_h \left(\mathbf{e}\left(-(b+1)^{\gamma} h\right) - \mathbf{e}(-b^{\gamma} h) \right) \right). \tag{3.20}$$

We just need to give an estimation of the last part in (3.20). Similarly, let

$$g(x) = \mathbf{e}\left(\left((d^2x)^\gamma - (d^2x+1)^\gamma\right)h\right) - 1,$$

then

$$g(x) \ll |h|(d^2x)^{\gamma-1},$$
$$\frac{\partial g(x)}{\partial x} \ll |h|d^{2\gamma-2}x^{\gamma-2}.$$

By partial summation,

$$\begin{aligned}
&\sum_{b=1}^{q-1}(-1)^b\mathbf{e}(-\frac{tb}{q})\sum_{\substack{d^2|b\\ d\leqslant q^{\frac{1}{6}}}}\mu(d)\sum_{0<|h|\leqslant H}a_h\left(\mathbf{e}\left(-(b+1)^\gamma h\right)-\mathbf{e}(-b^\gamma h)\right)\\
&=\sum_{0<|h|\leqslant H}a_h\sum_{d\leqslant q^{\frac{1}{6}}}\mu(d)\sum_{1\leqslant b\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right)g(b),\\
&\ll\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\leqslant q^{\frac{1}{6}}}\left|\int_1^{\lfloor\frac{q-1}{d^2}\rfloor}g(x)\mathrm{d}\left(\sum_{1<b\leqslant x}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right)\right)\right|\\
&\ll\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\leqslant q^{\frac{1}{6}}}\left|g(\lfloor\frac{q-1}{d^2}\rfloor)\sum_{1\leqslant b\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right)\right|\\
&\quad+\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{d\leqslant q^{\frac{1}{6}}}\left|\int_1^{\lfloor\frac{q-1}{d^2}\rfloor}\frac{\partial g(x)}{\partial x}\sum_{1\leqslant b\leqslant x}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right)\mathrm{d}x\right|,
\end{aligned}$$

where

$$g(\lfloor\frac{q-1}{d^2}\rfloor) \ll |h|q^{\gamma-1}$$

and

$$\begin{aligned}
&\sum_{1\leqslant b\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right)\\
&=\sum_{1\leqslant b\leqslant q^{\frac{1}{6}}}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right)+\sum_{q^{\frac{1}{6}}<b\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right).
\end{aligned}$$

It is obvious that

$$\sum_{1\leqslant b\leqslant q^{\frac{1}{6}}}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right) \ll q^{\frac{1}{6}}.$$

Suppose q be large enough and for $b > q^{\frac{1}{6}}$, when $2 \nmid d$ or $q \nmid td^2$,

$$\left\|(\frac{d^2}{2}-\frac{td^2}{q})-\gamma d^{2\gamma}b^{\gamma-1}h\right\|^{-1} \geqslant \frac{1}{2}\left\|(\frac{1}{2}-\frac{t}{q})d^2\right\|^{-1} > 0,$$

and applying Lemma 2.6, we have

$$\begin{aligned}\sum_{q^{\frac{1}{6}}<b\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right) &\ll \left\|(\frac{d^2}{2}-\frac{td^2}{q})-\gamma d^{2\gamma}b^{\gamma-1}h\right\|^{-1}\\ &\ll \left\|(\frac{1}{2}-\frac{t}{q})d^2\right\|^{-1}.\end{aligned}$$

So

$$\sum_{1\leqslant b\leqslant\lfloor\frac{q-1}{d^2}\rfloor}\mathbf{e}\left((\frac{d^2}{2}-\frac{td^2}{q})b-(d^2b)^\gamma h\right) \ll \begin{cases} q^{\frac{1}{6}}+\left\|(\frac{1}{2}-\frac{t}{q})d^2\right\|^{-1}, & 2\nmid d \text{ or } q\nmid td^2;\\ \frac{q}{d^2}, & 2\mid d \text{ and } q\mid td^2;\end{cases}$$

which means

$$\begin{aligned}&\sum_{b=1}^{q-1}(-1)^b\mathbf{e}(-\frac{tb}{q})\sum_{\substack{d^2|b\\ d\leqslant q^{\frac{1}{6}}}}\mu(d)\sum_{0<|h|\leqslant H}a_h\left(\mathbf{e}\left(-(b+1)^\gamma h\right)-\mathbf{e}(-b^\gamma h)\right)\\ &\ll \sum_{0<|h|\leqslant H}|h|^{-1}\sum_{\substack{d\leqslant q^{\frac{1}{6}}\\ 2\nmid d \text{ or } q\nmid td^2}}|h|q^{\gamma-1}\left(q^{\frac{1}{6}}+\left\|(\frac{1}{2}-\frac{t}{q})d^2\right\|^{-1}\right)+\sum_{0<|h|\leqslant H}|h|^{-1}\sum_{\substack{d\leqslant q^{\frac{1}{6}}\\ 2|d \text{ and } q|td^2}}|h|q^\gamma d^{-2}\\ &\ll Hq^{\gamma-1}\left(q^{\frac{1}{3}}+\sum_{\substack{d\leqslant q^{\frac{1}{6}}\\ 2\nmid d \text{ or } q\nmid td^2}}\left\|(\frac{1}{2}-\frac{t}{q})d^2\right\|^{-1}\right)+Hq^\gamma\sum_{\substack{d\leqslant q^{\frac{1}{6}}\\ 2|d \text{ and } q|td^2}}d^{-2}.\end{aligned}$$

We denote

$$\begin{aligned}T(c) := &\sum_{d\leqslant q^{\frac{1}{6}}}\#\left\{(\frac{q}{u}-2t)d^2\equiv c(\bmod 2\frac{q}{u}), t\leqslant\frac{q}{u}\right\}\\ &\ll \sum_{d\leqslant q^{\frac{1}{6}}}(\frac{q}{u},d^2)\\ &\ll q^{\frac{1}{3}}d(q),\end{aligned}$$

thus,

$$\sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\sum_{\substack{d^2|n\\ d\leqslant q^{\frac{1}{6}}}}\mu(d)\left(\sum_{0<|h|\leqslant H}a_h\left(\mathbf{e}\left(-(n+1)^\gamma h\right)-\mathbf{e}(-n^\gamma h)\right)\right)$$

$$\ll q^{-2}\sum_{s=1}^{q}\sum_{t=1}^{q}(s,t,q)^{\frac{1}{2}}d(q)q^{\frac{1}{2}}\frac{Hq^{\gamma-1}}{|\cos\frac{s}{q}\pi|}\left(q^{\frac{1}{3}}+\sum_{\substack{d\leqslant q^{\frac{1}{6}}\\ 2\nmid d \text{ or } q\nmid td^2}}\left\|(\frac{1}{2}-\frac{t}{q})d^2\right\|^{-1}\right)$$
$$+\,q^{-2}\sum_{s=1}^{q}\sum_{t=1}^{q}(s,t,q)^{\frac{1}{2}}d(q)q^{\frac{1}{2}}\frac{Hq^{\gamma}}{|\cos\frac{s}{q}\pi|}\sum_{\substack{d\leqslant q^{\frac{1}{6}}\\ 2|d \text{ and } q|td^2}}d^{-2}$$
$$\ll Hq^{\gamma-\frac{5}{2}}\sum_{u|q}u^{\frac{1}{2}}d(q)\sum_{s=1}^{\frac{q}{u}}\frac{1}{|1-2\frac{su}{q}|}\sum_{t=1}^{\frac{q}{u}}\left(q^{\frac{1}{3}}+\sum_{d\leqslant q^{\frac{1}{6}}}\left\|(\frac{1}{2}-\frac{ut}{q})d^2\right\|^{-1}\right)$$
$$+\,Hq^{\gamma-\frac{3}{2}}\sum_{u|q}u^{\frac{1}{2}}d(q)\sum_{s=1}^{\frac{q}{u}}\frac{1}{|1-2\frac{su}{q}|}\sum_{d\leqslant q^{\frac{1}{6}}}\sum_{\substack{t=1\\ q|td^2}}^{\frac{q}{u}}d^{-2}$$
$$\ll Hq^{\gamma-\frac{1}{6}}d^2(q)\log q+Hq^{\gamma-\frac{1}{3}}d^2(q)\log q$$
$$+\,Hq^{\gamma-\frac{3}{2}}\sum_{u|q}u^{-\frac{1}{2}}d(q)\log^2 q\max_{C}\sum_{C<c\leqslant 2C}\|\frac{C}{2\frac{q}{u}}\|^{-1}T(c)$$
$$\ll Hq^{\gamma-\frac{1}{6}}d^2(q)\log^2 q. \tag{3.21}$$

With Eqs (3.18) and (3.19), we have

$$R_{221}\ll Hq^{\gamma-\frac{1}{6}}d^2(q)\log^2 q+Hq^{\gamma-\frac{1}{6}}\log H. \tag{3.22}$$

For R_{222}, the contribution from $h=0$ can be bounded by similar methods of R_{21}, and the contribution from $h\neq 0$ can be bounded by similar methods of R_{221}. Taking

$$H=\log q,$$

we obtain

$$R_{222}=b_0\sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)+\sum_{n=1}^{q}{}'(-1)^{n+\bar{n}}\mu^2(n)\left(\sum_{0<|h|\leqslant H}b_h\left(e\left(-(n+1)^{\gamma}h\right)+e\left(-n^{\gamma}h\right)\right)\right)$$
$$\ll H^{-1}q^{\frac{3}{4}}d^3(q)\log^2 q+Hq^{\gamma-\frac{1}{6}}d^2(q)\log^2 q$$
$$\ll q^{\frac{3}{4}}d^3(q)\log q+q^{\gamma-\frac{1}{6}}d^2(q)\log^3 q. \tag{3.23}$$

Following from Eqs (3.16), (3.22), and (3.23),

$$R_2\ll q^{\gamma-\frac{1}{6}}d^2(q)\log^3 q+q^{\frac{3}{4}}d^3(q)\log q. \tag{3.24}$$

Hence, from Eqs (3.1), (3.8), and (3.24), we derive that

$$R(c;q)=\frac{3}{\pi^2}\prod_{p|q}(1+p^{-1})^{-1}q^{\gamma}+O\left(\sum_{p|q}(1-p^{-\frac{1}{2}})^{-1}q^{\gamma-\frac{1}{2}}\right)$$

$$+ O\left(q^{\frac{7}{13}\gamma+\frac{4}{13}} \prod_{p|q}(1 - p^{-\frac{1}{2}})^{-1} \log q\right) + O\left(q^{\frac{3}{4}} d^3(q) \log q\right) + O(q^{\gamma-\frac{1}{6}} d^2(q) \log^3 q).$$

We need the error terms to be smaller than the main term, so

$$\begin{cases} \frac{7}{13}\gamma + \frac{4}{13} < \gamma, \\ \frac{3}{4} < \gamma, \end{cases}$$

which means the range of c is $(1, \frac{4}{3})$. The reason why the range of c is changed is that $R(c; q)$ requires q large enough.

4. Conclusions

In this paper, we generalize the Lehmer problem by considering the count of square-free numbers in the intersection of the Lehmer set and Piatetski-Shapiro sequence when q is an odd integer and large enough. By methods of exponential sum and Kloosterman sum, we study its asymptotic properties and give a sharp asymptotic formula as q tends to infinity.

Based on this result, we will consider some distribution problems similar to the Lehmer problem with more special sequences, which is significant for understanding the distribution properties of those problems.

Author contributions

Xiaoqing Zhao: calculations, writing and editing; Yuan Yi: methodology and reviewing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

In preparing this manuscript, we employed the language model ChatGPT-4 for the purpose of grammatical corrections. It did not influence the calculations and conclusion in this paper.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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