AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 33386–33400.
DOI: 10.3934/math.20241593
Received: 03 September 2024
Revised: 27 October 2024
Accepted: 18 November 2024
Published: 25 November 2024

Research article

Novel analytical superposed nonlinear wave structures for the eighth-order (3+1)-dimensional Kac-Wakimoto equation using improved modified extended tanh function method

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Abstract: Higher-order nonlinear partial differential equations, such as the eighth-order Kac-Wakimoto model, are useful for studying wave turbulence in fluids, where energy transfers across a range of wave numbers. This phenomenon is observed in oceanographic research involving sea surface and internal waves, where intricate multi-dimensional interactions play a crucial role. In this work, we use the improved modified extended tanh function method for the first time to extract the exact solutions of the eighth-order (3+1)-dimensional Kac-Wakimoto equation, which describes the dynamics of fields and the structure of solutions in various physical and mathematical contexts. The proposed method is simple and quick to execute, and it offers more innovative solutions than other methods. As a consequence, through the donation of suitable assumptions for the parameters, some new solutions for dark and singular soliton, as well as Jacobi elliptic, exponential, hyperbolic, and singular periodic forms, are developed. Furthermore, to enhance understanding, graphical representations of certain solutions are included.

Keywords: Kac-Wakimoto equation; soliton solutions; hyperbolic solutions; wave profile
Mathematics Subject Classification: 35C07, 35C08, 35C09, 35G20

1. Introduction

Nonlinear partial differential equations (NLPDEs) are used to model a variety of physical phenomena in a broad range of scientific sectors, including solid state physics, quantum mechanics, optical fibers, and chemical physics, such as Kadomtsev-Petviashvili equation [1, 2], Manakov model [3], Non-linear Schrödinger equation [4], Korteweg-de Vries-Zakharov-Kuznetsov equation [5], Biswas-Milovic model [6], Sasa-Satsuma equation [7, 8], Lakshmanan-Porsezian-Daniel equation [9], concatenation model [10], and Schrödinger-Hirota equation [11]. The exact solutions of NLPDEs provide both symbolic and physical insights into the mechanisms underlying many nonlinear complex phenomena. Various techniques have been developed to investigate the exact wave structures of NLPDEs, for example, extended F-expansion method [12], variational method [13] and improved modified extended tanh-function method [14].

The study of nonlinear waves and transitions in higher-dimensional models with variable coefficients is an essential area of research in understanding complex nonlinear phenomena across different physical systems. These systems encompass a diverse range of applications such as magnetic media, water waves, optics, plasmas, and Bose-Einstein condensates; for example, Yang et al. [15] investigated the soliton structures of a variable-coefficient Gross-Pitaevskii equation with partially nonlocal nonlinearity under a harmonic potential. Zhu and Xu [16] discussed the dynamics of solitons for a variable-coefficient partially nonlocal coupled Gross-Pitaevskii equation in a harmonic potential. Yan et al. [17] obtained the soliton solutions and lump-type solutions to the (2+1)-dimensional Kadomtsev-Petviashvili equation with variable coefficient. Wu and Sun [18] studied the dynamic mechanism of nonlinear waves for the (3+1)-dimensional generalized variable-coefficient shallow water wave equation. Adeyemo and Khalique [19] discussed the dynamical soliton wave structures for a higher-dimensional soliton equation with various applications in ocean physics and mechatronics engineering. Hamed et al. [20] obtained soliton solutions and modulation instability for a generalized (3+1)-dimensional coupled variable-coefficient nonlinear Schrödinger equations in nonlinear optics. Kumar and Mohan [21] studied multi-soliton solutions, breast her, lumps, and their interactions for the Kadomtsev-Petviashvili equation. The Kac-Wakimoto equation is a mathematical framework that encompasses various intricate concepts in algebraic structures and their interplay with soliton equations, vertex operators, and singularity theory. The Kac-Wakimoto equation is deeply rooted in the realms of representation theory, affine Kac-Moody algebras, and the study of hierarchies of soliton equations associated with pseudo unitary modular tensor categories [22, 23]. The (3+1)-dimensional variable-coefficient eight-order nonintegrable Kac-Wakimoto equation is a significant model that captures complex nonlinear phenomena in various physical systems.

In this work, we aim to delve into the intertwined spatial-temporal modulations leading to multiple non-linear mechanisms, by exploring the eight-order (3+1)-dimensional Kac-Wakimoto equation. The suggested model is represented by the subsequent equation [24]:

$$\begin{aligned}\zeta(y)(&420\,\mathcal{U}_x\,\mathcal{U}_{xx}\mathcal{U}_{xxx}+420\,\mathcal{U}_x^3\,\mathcal{U}_{xx}+210\mathcal{U}_x^2\,\mathcal{U}_{xxxx}+28\mathcal{U}_x\mathcal{U}_{xxxxxx}+28\mathcal{U}_{xx}\mathcal{U}_{xxxxx}\\&+70\mathcal{U}_{xxx}\mathcal{U}_{xxxx}+\mathcal{U}_{xxxxxxxx})-\hbar(y)\left(3\mathcal{U}_x\mathcal{U}_{xy}+3\mathcal{U}_{xx}\mathcal{U}_y+\mathcal{U}_{xxxy}\right)\\&-\sigma(y)\mathcal{U}_{xt}+\mu(y)\mathcal{U}_{zz}=0,\end{aligned}\tag{1.1}$$

where $\mathcal{U}=\mathcal{U}(x,y,z,t)$ is the nonlinear wave evolution field, while $\zeta(y),\hbar(y)$, and $\sigma(y)$ are dispersive nonlinearity coefficients that change based on the spatial dimension y, and the subscripts indicate partial derivatives concerning spatial dimensions (x, y, z) and time t.

In this study, we utilize the improved modified extended tanh function (IMETF) method to explore the traveling wave solutions of the equation represented by Eq (1.1). Through this method, various types of precise wave solutions are attainable, including dark and singular solitons, as well as Jacobi elliptic, exponential, hyperbolic, and singular periodic solutions. These unique solutions showcase the method's effectiveness and robustness, not reported elsewhere. Additionally, contour, 2D, and 3D plots illustrate the model's physical behaviors.

The layout of the paper is organized in the following manner: Section 2 acts as the fundamental framework of this paper by outlining a recap of the suggested approach. Moving on to Section 3, the suggested approach was put into practice on the designated model, allowing us to derive a fresh resolution for the model. Within Section 4, a range of numerical solutions is offered through visual depictions of a selection of the inferred solutions, and lastly. Section 5 delivers the summary and ultimate remarks of this paper.

2. Proposed strategy outline

This segment outlines the foundational structure of this paper, providing a succinct overview of the proposed method (see [25, 26]).

In order to explain the IMETF method, we assume that the nonlinear (NL) model of the PDE under study can be represented as follows:

$$R\Big(\mathcal{U},\ \mathcal{U}_t,\ \mathcal{U}_x,\ \mathcal{U}_y,\ \mathcal{U}_z,\ \mathcal{U}_{xx},\ \mathcal{U}_{yy},\ \mathcal{U}_{zz},\ \mathcal{U}_{xt},...\Big) = 0, \tag{2.1}$$

which is a function of the dependent variable $\mathcal{U}$ and its various partial derivatives with respect to time t and spatial coordinates $x,\ y$, and z.

Equation (2.1) can be simplified into a nonlinear ordinary differential equation (NLODE) through the transformation provided below:

$$\mathcal{U}\ (x,\ y,\ z,\ t) = \mathcal{H}(\hbar), \qquad \hbar = xW_1 + yW_2 + zW_3 + tW_4. \tag{2.2}$$

This transformation captures the interactions between the different dimensions in a single variable. Therefore, Eq (2.1) could be reformulated as

$$V(\mathcal{H}, \mathcal{H}', \mathcal{H}'', \ldots) = 0. \tag{2.3}$$

Following the proposed method, we can represent the general solution of Eq (2.3) as

$$\mathcal{H}(\hbar) = \sum_{i=0}^{N} b_i\, \psi^i(\hbar) + \sum_{j=1}^{N} \mathcal{A}_j\, \Psi^{-j}(\hbar), \tag{2.4}$$

where $\Psi(\hbar)$ satisfies the following auxiliary equation:

$$\Psi'(\hbar) = \epsilon \sqrt{d_0 + d_1\, \Psi(\hbar) + d_2\, \Psi^2(\hbar) + d_3\, \Psi^3(\hbar) + d_4\, \Psi^4(\hbar)}, \tag{2.5}$$

where $\epsilon = \pm 1$.

The highest power term and the highest derivative term of Eq (2.3) will be balanced to get the positive integer N.

Equation (2.5) has the following cases for its solutions:

Case 1: When $d_0 = d_1 = d_3 = 0$, the following solutions are brought up:

$$\Psi(\hbar) = \sqrt{-\frac{d_2}{d_4}}\operatorname{sech}\left[\hbar\sqrt{d_2}\right], \quad d_2 > 0,\ d_4 < 0,$$

$$\Psi(\hbar) = \sqrt{-\frac{d_2}{d_4}}\sec\left[\hbar\sqrt{-d_2}\right], \quad d_2 < 0,\ d_4 > 0,$$

$$\Psi(\hbar) = -\frac{1}{\hbar\sqrt{d_4}}, \qquad d_2 = 0,\ d_4 > 0.$$

Case 2: When $d_1 = d_3 = 0$, the following solutions are brought up:

$$\Psi(\hbar) = \sqrt{\frac{-d_2}{2d_4}}\tanh\left[\hbar\sqrt{\frac{-d_2}{2}}\right], \quad d_2 < 0,\ d_4 > 0,\ d_0 = \frac{d_2^2}{4d_4},$$

$$\Psi(\hbar) = \sqrt{\frac{d_2}{2d_4}}\tan\left[\hbar\sqrt{\frac{d_2}{2}}\right], \quad d_2 > 0,\ d_4 > 0,\ d_0 = \frac{d_2^2}{4d_4},$$

$$\Psi(\hbar) = \sqrt{-\frac{m^2 d_2}{(2m^2-1)\,d_4}}\operatorname{cn}\left[\hbar\sqrt{\frac{d_2}{2m^2-1}}\right], \qquad d_2 > 0,\ d_4 < 0,\ d_0 = \frac{m^2\left(1-m^2\right)d_2^2}{4\left(2m^2-1\right)^2 d_4},$$

$$\Psi(\hbar) = \sqrt{-\frac{m^2}{(2-m^2)\,d_4}}\operatorname{dn}\left[\hbar\sqrt{\frac{d_2}{2-m^2}}\right], \qquad d_2 > 0,\ d_4 < 0,\ d_0 = \frac{\left(1-m^2\right)d_2^2}{\left(2-m^2\right)^2 d_4},$$

$$\Psi(\hbar) = \sqrt{-\frac{m^2 d_2}{(m^2+1)\,d_4}}\operatorname{sn}\left[\hbar\sqrt{\frac{-d_2}{m^2+1}}\right], \qquad d_2 < 0,\ d_4 > 0,\ d_0 = \frac{m^2 d_2^2}{\left(m^2+1\right)^2 d_4}.$$

Case 3: When $d_3 = d_4 = 0$, the following solution is brought up:

$$\Psi(\hbar) = -\frac{d_1}{2d_2} + e^{\hbar\sqrt{d_2}}, \qquad d_2 > 0,\ d_0 = \frac{d_1^2}{4d_2}.$$

Case 4: When $d_1 = d_3 = d_4 = 0$, the following solutions are brought up:

$$\Psi(\hbar) = \sqrt{-\frac{d_0}{d_2}}\sin\left[\hbar\sqrt{-d_2}\right], \quad d_0 > 0,\ d_2 < 0,$$

$$\Psi(\hbar) = \sqrt{\frac{d_0}{d_2}}\sinh\left[\hbar\sqrt{d_2}\right], \qquad d_0 > 0,\ d_2 > 0.$$

Case 5: When $d_0 = d_1 = 0$, the following solutions are brought up:

$$\Psi(\hbar) = \frac{d_2\sec^2\left[\frac{1}{2}\hbar\sqrt{-d_2}\right]}{2\sqrt{-d_2 d_4}\tan\left[\frac{1}{2}\hbar\sqrt{-d_2}\right] + d_3}, \qquad d_2 < 0,\ d_4 > 0,\ d_3 \neq \pm 2\sqrt{d_2 d_4},$$

$$\Psi(\hbar) = \frac{d_2 \operatorname{sech}^2\left[\frac{1}{2}\hbar\sqrt{d_2}\right]}{2\sqrt{d_2 d_4}\tanh\left[\frac{1}{2}\hbar\sqrt{d_2}\right] - d_3}, \qquad d_2 > 0,\ d_4 > 0,\ d_3 \neq \pm 2\sqrt{d_2 d_4},$$

$$\Psi(\hbar) = \frac{1}{2}\sqrt{\frac{d_2}{d_4}}\left(\tanh\left[\frac{1}{2}\hbar\sqrt{d_2}\right] + 1\right), \qquad d_2 > 0,\ d_4 > 0,\ d_3 = \pm 2\sqrt{d_2 d_4}.$$

The highest power term and the highest derivative term of Eq (2.3) will be balanced to get the positive integer N.

By putting Eq (2.4) through Eq (2.5) into Eq (2.3), the outcome is a polynomial in $\Psi(\hbar)$. Next, we set the sum of terms with matching powers equal to zero. To solve the resulting system, we will use Mathematical software.

3. Deriving exact solutions for the Kac-Wakimoto equation

In this section, the method proposed in Section 2 is applied to Eq (1.1). When we use the wave transform shown in Eq (2.2) on Eq (1.1), under constraints $\zeta(y) = a_1$, $\hbar(y) = a_2$, $\mu(y) = a_3$ and $\sigma(y) = a_4$, where $a_l (l = 1, 2, 3, 4)$ are constants, then, we can formulate Eq (1.1) as follows:

$$\begin{aligned} &a_1 W_1^8 \mathcal{H}^{(8)} + 28 a_1 W_1^7 \mathcal{H}^{(6)} \mathcal{H}' + 28 a_1 W_1^7 \mathcal{H}^{(5)} \mathcal{H}'' + 70 a_1 W_1^7 \mathcal{H}^{(3)} \mathcal{H}^{(4)} \\ &+ 210 a_1 W_1^6 \mathcal{H}^{(4)} (\mathcal{H}')^2 - a_2 W_2 W_1^3 \mathcal{H}^{(4)} + 420 a_1 W_1^5 (\mathcal{H}')^3 \mathcal{H}'' - 6 a_2 W_2 W_1^2 \mathcal{H}' \mathcal{H}'' \\ &+ 420 a_1 W_1^6 \mathcal{H}^{(3)} \mathcal{H}' \mathcal{H}'' + \left(a_3 W_3^2 - a_4 W_1 W_4\right) \mathcal{H}'' = 0. \end{aligned} \tag{3.1}$$

Integrating Eq (3.1) with respect to $\hbar$ and setting the constant of integration to zero yields the following expression:

$$\begin{aligned} &a_1 W_1^8 \mathcal{H}^{(7)} + 28 a_1 W_1^7 \mathcal{H}' \mathcal{H}^{(5)} + 35 a_1 W_1^7 (\mathcal{H}''')^2 + 210 a_1 W_1^6 \mathcal{H}'^2 \mathcal{H}''' + 105 a_1 W_1^5 \mathcal{H}'^4 \\ &- a_2 W_1^3 W_2 \mathcal{H}''' - 3 a_2 W_2 W_1^2 \mathcal{H}'^2 + \left(a_3 W_3^2 - a_4 W_1 W_4\right) \mathcal{H}' = 0. \end{aligned} \tag{3.2}$$

To keep it straightforward, we assume that

$$\mathcal{H}'(\hbar) = \mathcal{G}(\hbar). \tag{3.3}$$

Thus, Eq (3.2) can be expressed as

$$\begin{aligned} &a_1 W_1^8 \mathcal{G}^{(6)} + 28 a_1 W_1^7 \mathcal{G} \mathcal{G}^{(4)} + 35 a_1 W_1^7 (\mathcal{G}'')^2 + 210 a_1 W_1^6 \mathcal{G}^2 \mathcal{G}'' + 105 a_1 W_1^5 \mathcal{G}^4 \\ &- a_2 W_1^3 W_2 \mathcal{G}'' - 3 a_2 W_2 W_1^2 \mathcal{G}^2 + \left(a_3 W_3^2 - a_4 W_1 W_4\right) \mathcal{G} = 0. \end{aligned} \tag{3.4}$$

We perform the application of the principle of balance to the equation presented (3.4), for $\mathcal{G}^{(6)}$ and $\mathcal{G}^4$, resulting in $N = 2$.

Therefore, the solution to Eq (3.4) can be written as

$$\mathcal{G}(\hbar) = b_0 + b_1 \psi(\hbar) + b_2 \psi(\hbar)^2 + \frac{\mathcal{A}_1}{\psi(\hbar)} + \frac{\mathcal{A}_2}{\psi(\hbar)^2}, \tag{3.5}$$

the constants b_i $(i = 0, 1, 2)$ and $\mathcal{A}_j$ $(j = 1, 2)$ require further analysis to compute, while it is important to note that $b_2^2 + \mathcal{A}_2^2 \neq 0$.

Substituting Eq (3.5) and Eq (2.5) into Eq (3.4), we set the coefficients of common powers to zero. This system is then solved using Mathematical to derive the solutions of Eq (1.1) under the condition $b_0 = 0$, resulting in the following cases:
Case 1: $d_0 = d_1 = d_3 = 0$.

$$\mathcal{A}_1 = \mathcal{A}_2 = b_1 = 0, \quad b_2 = -2\, d_4\, W_1, \quad W_2 = \frac{64\, a_1\, d_2^3\, W_1^8 - a_4\, W_4\, W_1 + a_3\, W_3^2}{4 a_2\, d_2\, W_1^3}.$$

The outcomes of these cases will classify the solution to Eq(1.1) as follows:
Sub-case 1.1: Dark soliton, when $d_2 > 0$ and $d_4 < 0$:

$$\mathcal{U}_{1.1} = 2\, W_1\, \sqrt{d_2}\, \tanh\left[(xW_1 + yW_2 + zW_3 + tW_4)\, \sqrt{d_2}\right]. \tag{3.6}$$

Sub-case 1.2: Singular periodic solution, when $d_2 < 0$ and $d_4 > 0$:

$$\mathcal{U}_{1.2} = -2\, W_1\, \sqrt{-d_2}\, \tan\left[(xW_1 + yW_2 + zW_3 + tW_4)\, \sqrt{-d_2}\right]. \tag{3.7}$$

Sub-case 1.3: When $d_2 = 0$ and $d_4 > 0$, a rational-type solution is available, as shown:

$$\mathcal{U}_{1.3} = \frac{2\, W_1}{xW_1 + yW_2 + zW_3 + tW_4}. \tag{3.8}$$

Case 2: $d_1 = d_3 = 0$.
Sub-case 2.1:

$$\mathcal{A}_1 = b_1 = b_2 = 0,\ \mathcal{A}_2 = -2d_0\, W_1,\ W_2 = \frac{4a_1 W_1^5 \left(4d_2^2 - 17 d_0\, d_4\right)}{a_2},\ a_4 = \frac{432 a_1 d_0 d_2 d_4 W_1^8 + a_3 W_3^2}{W_1\, W_4}.$$

Sub-case 2.2:

$$\mathcal{A}_1 = b_1 = \mathcal{A}_2 = 0,\ b_2 = -2\, d_4\, W_1,\ W_2 = \frac{4a_1 W_1^5 \left(4d_2^2 - 17\, d_0\, d_4\right)}{a_2},\ a_4 = \frac{432\, a_1\, d_0\, d_2\, d_4\, W_1^8 + a_3\, W_3^2}{W_1\, W_4}.$$

Sub-case 2.3:

$$\mathcal{A}_1 = b_1 = 0,\ \mathcal{A}_2 = -2d_0 W_1,\ b_2 = -2d_4 W_1,\ W_2 = \frac{16 a_1 W_1^5 \left(31 d_2^2 + 17 d_0 d_4\right)}{a_2},$$

$$a_4 = \frac{-1920 a_1 d_2^3 W_1^8 - 2496 a_1 d_0 d_2 d_4 W_1^8 + a_3 W_3^2}{W_1 W_4}.$$

The outcomes of ***Sub-case 2.1*** will classify the solution to Eq (1.1) as follows:
(2.1, 1) Singular soliton, when $d_2 < 0$, $d_4 > 0$ and $d_0 = \frac{d_2^2}{4\, d_4}$:

$$\mathcal{U}_{2.1.1} = -\, W_1\, \sqrt{-2\, d_2} \Bigg((xW_1 + yW_2 + zW_3 + tW_4)\, \sqrt{-\frac{d_2}{2}} - \coth\left[(xW_1 + yW_2 + zW_3 + tW_4)\, \sqrt{-\frac{d_2}{2}}\right]\Bigg). \tag{3.9}$$

(2.1, 2) When $d_2 > 0$, $d_4 > 0$ and $d_0 = \frac{d_2^2}{4\ d_4}$, a singular periodic-type solution is available, as shown:

$$\mathcal{U}_{2.1.2} = W_1 \sqrt{2\ d_2}\Bigg((xW_1 + yW_2 + zW_3 + tW_4) \sqrt{\frac{d_2}{2}} + \cot\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{\frac{d_2}{2}} \right] \Bigg). \tag{3.10}$$

(2.1, 3) When $d_2 < 0$, $d_4 > 0$, $d_0 = \frac{d_2^2\ m^2}{d_4\ (1+m^2)^2}$ and $0 \leq m \leq 1$, a Jacobi elliptic solution (JES) is available, as shown:

$$\mathcal{U}_{2.1.3} = \left(\hbar \sqrt{-\frac{d_2}{m^2+1}} - \mathrm{cs}\left[\hbar \sqrt{-\frac{d_2}{m^2+1}} \right] \mathrm{dn}\left[\hbar \sqrt{-\frac{d_2}{m^2+1}} \right] - \mathrm{JacobiEpsilon}\left[\hbar \sqrt{-\frac{d_2}{m^2+1}} \right] \right) \times \left[-2W_1 \sqrt{-\frac{d_2}{m^2+1}} \right], \tag{3.11}$$

where $\hbar = xW_1 + yW_2 + zW_3 + tW_4$.

Setting $m = 1$ or $m = 0$ in Eq (3.11) leads to the singular soliton-type solution or singular periodic-type solution, as shown:

$$\mathcal{U}_{2.1.4} = W_1(\sqrt{-2d_2} \coth\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{d_2} \right] + d_2\left[xW_1 + yW_2 + zW_3 + tW_4 \right]), \tag{3.12}$$

or

$$\mathcal{U}_{2.1.5} = 2W_1 \sqrt{-d_2}\ \cot\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-d_2} \right]. \tag{3.13}$$

The outcomes of ***Sub-case 2.2*** will classify the solution to Eq (1.1) as follows:

(2.2, 1) When $d_2 > 0$, $d_4 < 0$ and $d_0 = \frac{d_2^2 m^2 (1-m^2)}{d_4 (2m^2-1)^2}$, a dark soliton-type solution is available, as shown:

$$\mathcal{U}_{2.2.1} = -W_1 \sqrt{-2d_2}\Bigg[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{-\frac{d_2}{2}} - \tanh\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{-\frac{d_2}{2}} \right] \Bigg]. \tag{3.14}$$

(2.2, 2) Singular periodic solution, when $d_2 > 0$, $d_4 < 0$ and $d_0 = \frac{d_2^2\ (1-m^2)}{d_4\ (2-m^2)^2}$:

$$\mathcal{U}_{2.2.2} = W_1 \sqrt{2d_2}[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{\frac{d_2}{2}} - \tan\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{\frac{d_2}{2}} \right]]. \tag{3.15}$$

(2.2, 3) When $d_2 > 0$, $d_4 < 0$ and $d_0 = \frac{m^2 d_2^2 (1-m^2)}{d_4 (2m^2-1)^2}$ a Jacobi elliptic solution is available, as shown:

$$\mathcal{U}_{2.2.3} = 2m\ W_1 \sqrt{\frac{d_2}{2\ m^2-1}} \left[(m-1)\ \hbar \sqrt{\frac{d_2}{2\ m^2-1}} - \mathrm{JacobiEpsilon}\left(\hbar \sqrt{\frac{d_2}{2\ m^2-1}} \right) \right], \tag{3.16}$$

where $\hbar = xW_1 + yW_2 + zW_3 + tW_4$.

Setting $m = 1$ in Eq (3.16) leads to a dark soliton-type solution that is available, as shown:

$$\mathcal{U}_{2.2.4} = 2W_1 \sqrt{d_2} \tanh\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{d_2}\right]. \quad (3.17)$$

(2.2, 4) When $d_2 > 0$, $d_4 < 0$ and $d_0 = \frac{m^2 d_2^2 (1-m^2)}{d_4 (2-m^2)^2}$ a Jacobi elliptic solution is available, as shown:

$$\mathcal{U}_{2.2.5} = \frac{2\, m^2\, W_1}{\sqrt{d_2\ (2 - m^2)}}\ \text{JacobiEpsilon}\left[xW_1 + yW_2 + zW_3 + tW_4) \sqrt{\frac{d_2}{2 - m^2}}\right]. \quad (3.18)$$

Setting $m = 1$ in Eq (3.18) leads to a dark soliton-type solution that is available, as shown:

$$\mathcal{U}_{2.2.6} = \frac{2\, W_1}{\sqrt{d_2}} \tanh\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{d_2}\right]. \quad (3.19)$$

The outcomes of ***Sub-case 2.3*** will classify the solution to Eq (1.1) as follows:

(2.3, 1) When $d_2 < 0$, $d_4 > 0$ and $d_0 = \frac{d_2^2}{4\, d_4}$, a singular soliton-type solution is available, as shown:

$$\begin{aligned}\mathcal{U}_{2.3.1} = 2\ W_1 \Big(& (xW_1 + yW_2 + zW_3 + tW_4)\ d_2 \\ & - \sqrt{-2d_2}\ \coth\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-2d_2}\right]\Big).\end{aligned} \quad (3.20)$$

(2.3, 2) When $d_2 > 0$, $d_4 > 0$ and $d_0 = \frac{d_2^2}{4\, d_4}$, a singular periodic-type solution is available, as shown:

$$\begin{aligned}\mathcal{U}_{2.3.2} = W_1\ (2\ d_2\ & (xW_1 + yW_2 + zW_3 + tW_4) \\ & + \sqrt{\frac{d_2}{2}} \cot\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{2\ d_2}\right]).\end{aligned} \quad (3.21)$$

(2.3, 3) Jacobi elliptic solution (JES), when $d_2 > 0$, $d_4 < 0$, $d_0 = \frac{d_2^2\ m^2\ (1-m^2)}{d_4\ (2\ m^2-1)^2}$ and $0 \le m \le 1$:

$$\begin{aligned}\mathcal{U}_{2.3.3} = & -2W_1 \sqrt{\frac{d_2}{2\ m^2 - 1}} \Big((xW_1 + yW_2 + zW_3 + tW_4)(m - 1) \\ & + \text{JacobiEpsilon}\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2\ m^2 - 1}}\right] \\ & - \frac{(m + 1)\, \text{dn}[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2\ m^2-1}}]\, \text{sn}[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2\ m^2-1}}]}{\text{cn}[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2\ m^2-1}}]}\Big).\end{aligned} \quad (3.22)$$

Setting $m = 1$ or $m = 0$ in Eq (3.22) leads to the dark soliton-type solution or singular periodic solution are available, as shown:

$$\mathcal{U}_{2.3.4} = 2\ W_1 \sqrt{d_2}\ \tanh\left[(xW_1 + yW_2 + zW_3 + tW_4) \sqrt{d_2}\right], \quad (3.23)$$

or

$$\mathcal{U}_{2.3.5} = 2\ W_1 \sqrt{-d_2}\ \tan\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-d_2}\right]. \tag{3.24}$$

(2.3, 4) When $d_2 > 0$, $d_4 < 0$, $d_0 = \frac{d_2^2\ (1-m^2)}{d_4\ (2-m^2)^2}$ and $0 < m \leq 1$, a JES is available, as shown:

$$\mathcal{U}_{2.3.6} = \frac{2W_1}{m^2\ \sqrt{d_2\ (2-\ m^2)}}\ [(m^4 + d_2^2(m+1))\ \text{JacobiEpsilon}\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-\frac{d_2}{m^2-2}}\right]$$

$$-\frac{m(m+1)\ \text{cn}[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2-\ m^2}}]\ \text{sn}[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2-\ m^2}}]}{\text{dn}[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{\frac{d_2}{2-\ m^2}}]}]. \tag{3.25}$$

Setting $m = 1$ in Eq (3.25) leads to a dark soliton-type solution that's available, as shown:

$$\mathcal{U}_{2.3.7} = \frac{2\ W_1}{\sqrt{d_2}}\ \tanh\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{d_2}\right]. \tag{3.26}$$

Case 3: $d_3 =\ d_4 = 0$.
Sub-case 3.1:

$$\mathcal{A}_1 = -d_1\ W_1,\ \ b_1 = b_2 = 0,\ \ \mathcal{A}_2 = -2\ d_0\ W_1,\ \ W_2 = \frac{21a_1\ d_2^2\ W_1^5}{a_2},\ \ a_4 = \frac{a_3\ W_3^2 - 20\ a_1\ d_2^3\ W_1^8}{W_1\ W_4}.$$

Sub-case 3.2:

$$\mathcal{A}_1 = -2d_1W_1,\ b_1 = b_2 = 0,\ \mathcal{A}_2 = -4d_0W_1,\ d_2 = \frac{d_1^2}{4d_0},\ W_2 = \frac{21a_1d_1^4W_1^5}{16a_2d_0^2},\ a_4 = \frac{16a_3d_0^3W_3^2 - 5a_1d_1^6W_1^8}{16\ d_0^3W_1W_4}.$$

The outcomes of ***Sub-case 3.1*** will classify the solution to Eq (1.1) as follows:
(3.1, 1) When $d_2 > 0$, $d_0 = \frac{d_1^2}{4d_2}$, $d_1 \neq 0$ and $d_1 \neq 2\ d_2\ e^{(W_1\ x+W_2\ y+W_3\ z+W_4\ t+\Phi)\sqrt{d_2}}$, the exponential solution is available, as shown:

$$\mathcal{U}_{3.1,1} = -\frac{2\ d_1\ W_1\ \sqrt{d_2}}{d_1 - 2\ d_2\ e^{(xW_1+yW_2+zW_3+tW_4)\sqrt{d_2}}}. \tag{3.27}$$

(3.1, 2) When $d_2 < 0$ and $d_0 = 0$, the singular periodic solution is available, as shown:

$$\mathcal{U}_{3.1,2} = \frac{4\ W_1\ \sqrt{-d_2}}{1 - \cot\left[\frac{(xW_1+yW_2+zW_3+tW_4)\ \sqrt{-d_2}}{2}\right]}. \tag{3.28}$$

(3.1, 3) When $d_2 > 0$ and $d_0 = 0$, the hyperbolic solution is available, as shown:

$$\mathcal{U}_{3.1,3} = 2\ W_1\ \sqrt{2\ d_2}\ \tanh^{-1}\left[\frac{1 + \tanh\left[\frac{1}{2}(W_1\ x + W_2\ y + W_3\ z + W_4\ t)\ \sqrt{d_2}\right]}{\sqrt{2}}\right]. \tag{3.29}$$

(3.1, 4) Singular periodic solution, when $d_2 < 0$, $d_1 = 0$ and $d_0 > 0$:

$$\mathcal{U}_{3.1,4} = 2\ W_1\ \sqrt{-d_2}\ \cot\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-d_2}\right]. \tag{3.30}$$

(3.1, 5) Singular soliton, when $d_2 > 0$, $d_1 = 0$ and $d_0 > 0$:

$$\mathcal{U}_{3.1,4} = -2\ W_1\ \sqrt{-d_2}\ \coth\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-d_2}\right]. \tag{3.31}$$

The outcomes of ***Sub-case 3.2*** will classify the solution to Eq (1.1) as follows:
(3.2) When $d_2 > 0$, $d_0 = \frac{d_1^2}{4d_2}$, $d_1 \neq 0$ and $d_1 \neq 2\ d_2\ e^{(W_1\ x+W_2\ y+W_3\ z+W_4\ t+\Phi)\sqrt{d_2}}$, the exponential solution is available, as shown:

$$\mathcal{U}_{3.2} = -\frac{2\ d_1\ W_1\ \sqrt{d_2}}{d_1 - 2\ d_2\ e^{(xW_1+yW_2+zW_3+tW_4)\sqrt{d_2}}}. \tag{3.32}$$

Case 4: $d_0 = d_1 = 0$.
Sub-case 4.1:

$$\mathcal{A}_1 = b_1 = 0,\ \ b_2 = -2d_4W_1,\ \ \mathcal{A}_2 = d_3 = 0,\ \ a_4 = \frac{64\ a_1\ d_2^3\ W_1^8 - 4\ a_2\ d_2\ W_2\ W_1^3 + a_3\ W_3^2}{W_1\ W_4}.$$

Sub-case 4.2:

$$\mathcal{A}_1 = 0,\ b_1 = \pm 2W_1\sqrt{d_2d_4},\ b_2 = -2d_4W_1,\ \mathcal{A}_2 = 0,\ d_3 = \mp 2\sqrt{d_2d_4},\ a_4 = \frac{a_1d_2^3W_1^8 - a_2d_2W_2W_1^3 + a_3W_3^2}{W_1W_4}.$$

The outcomes of ***Sub-case 4.1*** will classify the solution to Eq (1.1) as follows:
(4.1, 1) Singular periodic solution, when $d_2 < 0$:

$$\mathcal{U}_{4.1,1} = 2\ W_1\ \sqrt{-d_2}\ \cot\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{-d_2}\right]. \tag{3.33}$$

(4.1, 2) Singular soliton, when $d_2 > 0$, and $d_3^4 \neq d_2\ d_4$:

$$\mathcal{U}_{4.1,2} = 2\ W_1\ \sqrt{d_2}\ \coth\left[(xW_1 + yW_2 + zW_3 + tW_4)\ \sqrt{d_2}\right]. \tag{3.34}$$

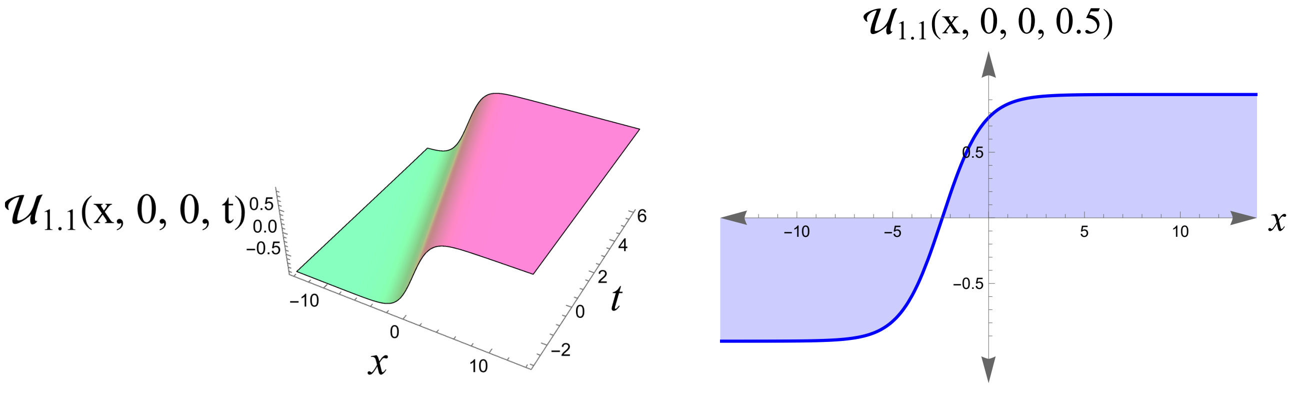
The outcomes of ***Sub-case 4.2*** will classify the solution to Eq (1.1) as follows:
(4.2) When $d_2 > 0$, and $d_3^4 = 4\ d_2\ d_4$, a dark soliton-type solution is available, as shown:

$$\mathcal{U}_{4.2,1} = W_1\ \sqrt{d_2}\ \left[\tanh\left(\frac{\hbar\sqrt{d_2}}{2}\right) + 4\log\left[1 - \tanh\left(\frac{\hbar\sqrt{d_2}}{2}\right)\right]\right], \tag{3.35}$$

where $\hbar = (xW_1 + yW_2 + zW_3 + tW_4)$.

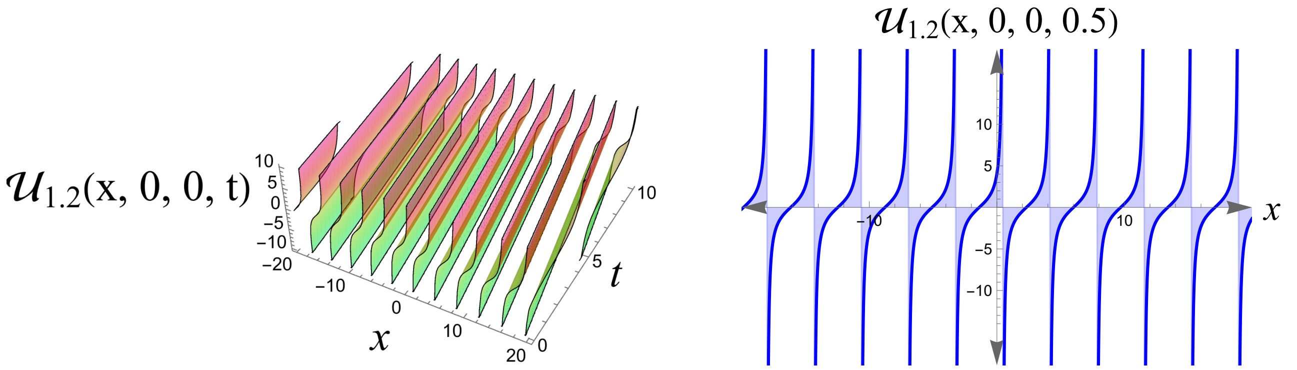
4. Graphical representation

This section presents graphical representations of some derived solutions in two and three dimensions. Figure 1 depicts a dark soliton from Eq (3.6) with $W_1 = 0.47$, $d_2 = 1$ and $W_4 = 0.47$. Dark solitons are localized zones of reduced amplitude within a surrounding medium, often linked to lower fluid density or pressure in fluid dynamics. Figure 2 depicts a singular periodic solution from Eq (3.7) with $W_1 = 0.85$, $d_2 = -1$ and $W_4 = 0.57$. A system displaying these periodic solutions alongside abrupt shifts or severe events is characterized by unique periodic behavior in physical phenomena. Such singularities or discontinuities may arise from non-linearities, boundary conditions, or external stimuli.



(a) 3-Dimension **(b)** 2-Dimension

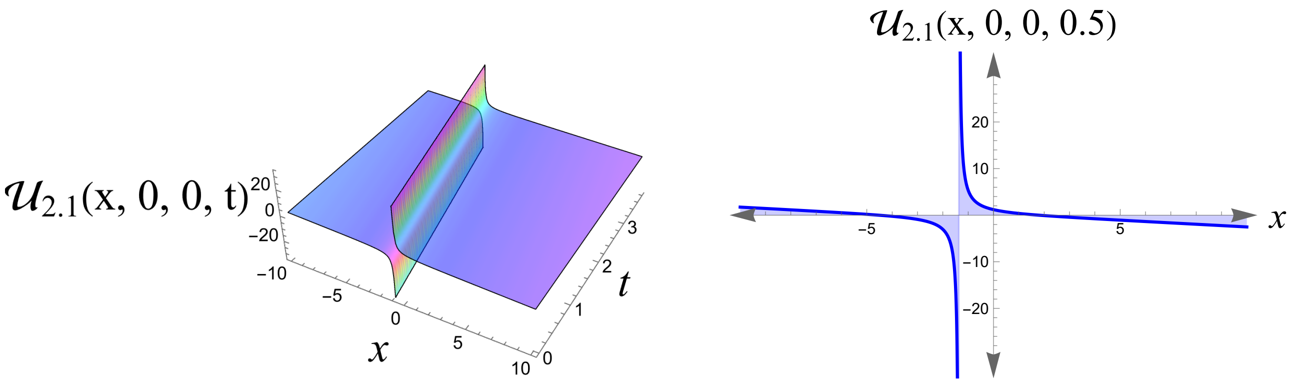
Figure 1. Dark soliton.



(a) 3-Dimension **(b)** 2-Dimension

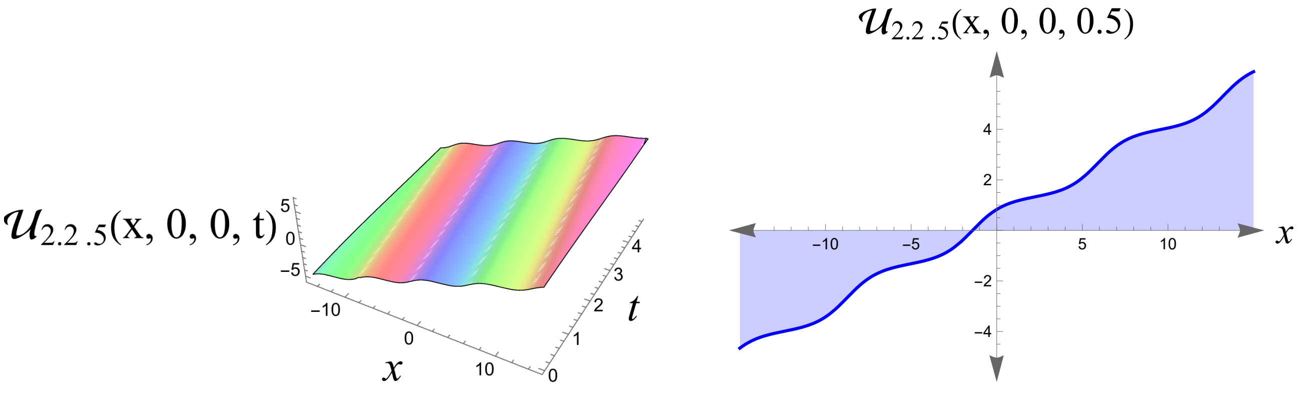
Figure 2. Singular periodic solution.

Figure 3 depicts a singular soliton from Eq (3.9) with $W_1 = 0.76$, $d_2 = -0.5$ and $W_4 = 0.7$. These solutions are primarily characterized by a narrow region approaching a peak at infinity. Figure 4 depicts a Jacobi elliptic solution from Eq (3.18) with $\rho_1 = 0.86$, $d_2 = 0.7$, $W_4 = 0.85$ and $m = 0.8$.



(a) 3-Dimension **(b)** 2-Dimension

Figure 3. Singular soliton.



(a) 3-Dimension **(b)** 2-Dimension

Figure 4. Jacobi elliptic solution.

5. Conclusions

The study significantly opens up numerous avenues for further comprehensive investigation into various nonlinear phenomena, delving into intricate soliton interactions. This exploration paves the way for considerable advancements in our understanding of complex wave dynamics, which is crucial for various scientific applications. Additionally, the research highlights the vital importance of obtaining exact solutions to nonlinear equations. This is essential for achieving a deeper and more refined understanding of nonlinear systems, especially within the realms of applied sciences and the broader field of nonlinear dynamics. The findings not only enhance theoretical knowledge but also provide practical insights that can lead to innovative applications and methodologies in related disciplines. The IMETF method has been used to study the eight-order (3+1)-dimensional Kac-Wakimoto equation, successfully yielding new solutions not previously reported in [24]. We were successful in achieving the primary purpose of this work, which was to suggest novel and different solutions for our proposed model that had not before been presented in the literature, with the anticipation that this research will have a significant impact on future studies.

This work introduced the improved modified extended tanh function approach for the first time to derive exact solutions of the proposed model, facilitating the analysis of fluid wave behavior. The reason behind using this method over the other methods is that it offers more types of complex solutions, making it suitable for several types of NLPDEs. We presented various solutions, including dark and singular solitons, as well as Jacobi elliptic, exponential, hyperbolic, and singular periodic solutions. The obtained solutions are novel ones that haven't been documented in previous works. Additionally, these solutions–which come from this model–showcase the efficiency and resilience of the employed approach in comparison to alternative approaches. We believe this work will significantly influence future research in the field of solitons. To enhance the physical representation of the model's dynamic features, we provided several important solutions displayed through contour, 2D, and 3D plots.

Author contributions

Wafaa B. Rabie: Formal analysis, Software; Hamdy M. Ahmed: Validation, Methodology; Taher A. Nofal: Resources, Writing–review & editing; E. M. Mohamed: Software, Investigation. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The authors extend their appreciation to Taif University, Saudi Arabia, for supporting this work through project number (TU-DSPP-2024-46).

Funding

This research was funded by Taif University, Saudi Arabia, Project No. (TU-DSPP-2024-46).

Conflict of interest

The authors declare that they have no conflicts of interest.

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