Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 33353–33364.
DOI: 10.3934/math.20241591
Received: 14 August 2024
Revised: 17 October 2024
Accepted: 28 October 2024
Published: 25 November 2024

Research article

Application of q-starlike and q-convex functions under (u, v)-symmetrical constraints

Hanen Louati[1,2], Afrah Al-Rezami[3,4,*], Erhan Deniz[5], Abdulbasit Darem[6] and Robert Szasz[7]

[1] Department of Mathematics, College of Science, Northeren Border University, P.O. Box 1321, Arar, 73222, KSA

[2] Laboratory of PDEs and Applications (LR03ES04), Faculty of Science of Tunis, University of Tunis El Manar, Tunis, Tunisia

[3] Mathematics Department, Prince Sattam Bin Abdulaziz University, Al-Kharj 16278, Saudi Arabia

[4] Department of Statistics and Information, Sana'a University, Sana'a 1247, Yemen

[5] Department of Mathematics, Faculty of Science and Letters, Kafkas University, Campus, 36100, Kars-Türkiye

[6] Department of Computer Science at college of Science, Northern Border University, Arar, Saudi Arabia

[7] Department of Mathematics and Informatics, Sapientia Hungarian University of Transylvania, Târgu-Mures, Romania

* **Correspondence:** Email: a.alrezamee@psau.edu.sa.

Abstract: This research paper addressed a significant knowledge gap in the field of complex analysis by introducing a pioneering category of q-starlike and q-convex functions intricately interconnected with (u, v)-symmetrical functions. Recognizing the limited exploration of these relationships in existing literature, the authors delved into the new classes $\mathcal{S}_q(\alpha, u, v)$ and $\mathcal{T}_q(\alpha, u, v)$. The main contribution of this work was the establishment of a framework that amalgamates q-starlikeness and q-convexity with the symmetry conditions imposed by (u, v)-symmetrical functions. This comprehensive study include coefficient estimates, convolution conditions, and the properties underpinning the (ρ, q)-neighborhood, thereby enriching the understanding of these novel function classes.

Keywords: analytic functions; q-calculus; (u, v)-symmetrical functions (ρ, q)-neighborhood; coefficient inequality
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Geometric function theory is a fascinating branch of complex analysis that delves into the beautiful interplay between complex-valued functions and their geometric properties. It centers around understanding how these functions distort and transform shapes in the complex plane. Let $\widetilde{\mathcal{F}}(\Omega)$ denote the space of all analytic functions in the open unit disk $\Omega = \{\varpi \in \mathbb{C} : |\varpi| < 1\}$ and let $\widetilde{\mathcal{F}}$ denote the class of functions $\hbar \in \widetilde{\mathcal{F}}(\Omega)$ which has the form

$$\hbar(\varpi) = \varpi + \sum_{m=2}^{\infty} a_m \varpi^m. \tag{1.1}$$

Let $\widetilde{\mathcal{S}}$ denote the subclass of $\widetilde{\mathcal{F}}$ that includes all univalent functions within the domain Ω. The convolution, or Hadamard product, of two analytic functions $\hbar$ and g, both of which belong to $\widetilde{\mathcal{F}}$, is defined as follows: here, $\hbar$ is given by (1.1), while the function g takes the form $g(\varpi) = \varpi + \sum_{m=2}^{\infty} b_m \varpi^m$, as

$$(\hbar * g)(\varpi) = \varpi + \sum_{m=2}^{\infty} a_m b_m \varpi^m.$$

This research aims to define new starlike functions using the concepts of (u, v)-symmetrical functions and quantum calculus. Before delving into the discussion on (u, v)-symmetrical functions and quantum calculus (q-calculus), let us briefly review the essential concepts and symbols related to these theories.

The theory of (u, v)-symmetrical functions is a specific area within geometric function theory that explores functions exhibiting a unique kind of symmetry. Regular symmetric functions treat all variables alike, but (u, v)-symmetrical functions introduce a twist. Here, v denotes a fixed positive integer, and u can range from 0 to $v - 1$, (see [1]). A domain $\widetilde{\mathcal{D}}$ is said to be v-fold symmetric if a rotation of $\widetilde{\mathcal{D}}$ about the origin through an angle $\frac{2\pi}{v}$ carries $\widetilde{\mathcal{D}}$ onto itself. A function $\hbar$ is said to be v-fold symmetric in $\widetilde{\mathcal{D}}$ if for every ϖ in $\widetilde{\mathcal{D}}$ and $\hbar\left(e^{\frac{2\pi i}{v}} \varpi\right) = e^{\frac{2\pi i}{v}} \hbar(\varpi)$. A function $\hbar$ is considered (u, v)-symmetrical if for any element $\varpi \in \widetilde{\mathcal{D}}$ and a complex number ε with a special property ($\varepsilon = e^{\frac{2\pi i}{v}}$), the following holds:

$$\hbar(\varepsilon\varpi) = \varepsilon^u \hbar(\varpi),$$

where ε term introduces a rotation by a specific angle based on v, and the key concept is that applying this rotation to an element ϖ and then applying the function $\hbar$ has the same effect as applying h to ϖ first and then rotating the result by a power of ε that depends on u. In our work we need the following decomposition theorem

Lemma 1.1. *[1] For every mapping $\hbar : \Omega \to \mathbb{C}$ and a v-fold symmetric set Ω, there exists a unique sequence of (u, v)-symmetrical functions $\hbar_{u,v}$, such that*

$$\hbar(\varpi) = \sum_{u=0}^{v-1} \hbar_{u,v}(\varpi), \qquad \hbar_{u,v}(\varpi) = \frac{1}{v} \sum_{n=0}^{v-1} \varepsilon^{-nu} \hbar(\varepsilon^n \varpi), \ \varpi \in \Omega. \tag{1.2}$$

Remark 1.2. *In other words,* (1.2) *can also be formulated as*

$$\hbar_{u,v}(\varpi) = \sum_{m=1}^{\infty} \delta_{m,u} a_m \varpi^m, \quad a_1 = 1, \tag{1.3}$$

where

$$\delta_{m,u} = \frac{1}{v} \sum_{n=0}^{v-1} \varepsilon^{(m-u)n} = \begin{cases} 1, & m = lv + u; \\ 0, & m \neq lv + u; \end{cases}, \tag{1.4}$$

$$(l \in \mathbb{N}, \ v = 1, 2, \ldots, \ u = 0, 1, 2, \ldots, v-1).$$

The theory of (u, v)-symmetrical functions has many interesting applications; for instance, convolutions, fixed points and absolute value estimates. Overall, (u, v)-symmetrical functions are a specialized but powerful tool in geometric function theory. Their unique symmetry property allows researchers to delve deeper into the geometric behavior of functions and uncover fascinating connections. Denote be $\widetilde{\mathcal{F}}^{(u,v)}$ for the family of all (u, v)-symmetric functions. Let us observe that the classes $\widetilde{\mathcal{F}}^{(1,2)}$, $\widetilde{\mathcal{F}}^{(0,2)}$ and $\widetilde{\mathcal{F}}^{(1,v)}$ are well-known families of odd, even and of $v-$symmetrical functions, respectively.

The interplay between q-calculus and geometric function theory is a fascinating emerging area of mathematical research The literature recognizes the fundamental characteristics of q-analogs, which have various applications in the exploration of quantum groups, q-deformed super-algebras, fractals, multi-fractal measures, and chaotic dynamical systems. Certain integral transforms within classical analysis have their counterparts in the realm of q-calculus. Consequently, many researchers in q-theory have endeavored to extend key results from classical analysis to their q-analogs counterparts. To facilitate understanding, this paper presents essential definitions and concept explanations of q-calculus that are utilized. Throughout the discussion, it is assumed that the parameter q adheres to the condition $0 < q < 1$. Let's begin by reviewing the definitions of fractional q-calculus operators for a complex-valued function $\hbar$. In [2], Jackson introduced and explored the concept of the q-derivative operator $\partial_q \hbar(\varpi)$ as follows:

$$\partial_q \hbar(\varpi) = \begin{cases} \frac{\hbar(\varpi)-\hbar(q\varpi)}{\varpi(1-q)}, & \varpi \neq 0, \\ \hbar'(0), & \varpi = 0. \end{cases} \tag{1.5}$$

Equivalently (1.5), may be written as

$$\partial_q \hbar(\varpi) = 1 + \sum_{m=2}^{\infty} [m]_q a_m \varpi^{m-1} \quad \varpi \neq 0,$$

where

$$[m]_q = \frac{1-q^m}{1-q} = 1 + q + q^2 + ... + q^{m-1}. \tag{1.6}$$

Note that as $q \to 1^-$, $[m]_q \to m$. For a function $\hbar(\varpi) = \varpi^m$, we can note that

$$\partial_q \hbar(\varpi) = \partial_q(\varpi^m) = \frac{1-q^m}{1-q} \varpi^{m-1} = [m]_q \varpi^{m-1}.$$

Then

$$\lim_{q\to 1^-}\partial_q\hbar(\varpi)=\lim_{q\to 1^-}[m]_q\varpi^{m-1}=m\varpi^{m-1}=\hbar'(\varpi),$$

where $\hbar'(\varpi)$ represents the standard derivative.

The q-integral of a function $\hbar$ was introduced by Jackson [3] and serves as a right inverse, defined as follows:

$$\int_0^{\varpi}\hbar(\varpi)d_q\varpi=\varpi(1-q)\sum_{m=0}^{\infty}q^m\hbar(\varpi q^m),$$

provided that the series $\sum_{m=0}^{\infty}q^m\hbar(\varpi q^m)$ converges. Ismail et al. [4] was the first to establish a connection between quantum calculus and geometric function theory by introducing a q-analog of starlike (and convex) functions. They generalized a well-known class of starlike functions, creating the class of q-starlike functions, denoted by $\mathcal{S}_q^*$, which consists of functions $\hbar\in\widetilde{\mathcal{F}}$ that satisfy the inequality:

$$\left|\frac{\varpi\left(\partial_q\hbar(\varpi)\right)}{\hbar(\varpi)}-\frac{1}{1-q}\right|\leq\frac{1}{1-q},\varpi\in\Omega.$$

Numerous subclasses of analytic functions have been investigated using the quantum calculus approach in recent years by various authors, like how Naeem et al. [5], explored subclesses of q-convex functions. Srivastava et al. [6] investigated subclasses of q-starlike functions. Govindaraj and Sivasubramanian in [7], identified subclasses connected with q-conic domain. Alsarari et al. [8, 9]. examined the convolution conditions of q-Janowski symmetrical functions classes and studied (u, v)-symmetrical functions with q-calculus. Khan et al. [10] utilized the symmetric q-derivative operator. Srivastava [11] published a comprehensive review paper that serves as a valuable resource for researchers.

The (u, v)-symmetrical functions are crucial for the exploration of various subclasses of $\widetilde{\mathcal{F}}$. Recently, several authors have studied subclasses of analytic functions using the (u, v)-symmetrical functions approach, (see [12–15]). By incorporating the concept of the q-derivative into the framework of (u, v)-symmetrical functions, we will establish the following classes:

Definition 1.3. *Let q and α be arbitrary fixed numbers such that $0 < q < 1$ and $0 \leq \alpha < 1$. We define $\mathcal{S}_q(\alpha, u, v)$ as the family of functions $\hbar\in\widetilde{\mathcal{F}}$ that satisfy the following condition:*

$$\Re\left\{\frac{\varpi\partial_q\hbar(\varpi)}{\hbar_{u,v}(\varpi)}\right\}>\alpha,\quad \textit{for all }\ \varpi\in\Omega, \tag{1.7}$$

where $\hbar_{u,v}$ is defined in (1.2).

By selecting specific values for parameters, we can derive a variety of important subclasses that have been previously investigated by different researchers in their respective papers. Here, we enlist some of them:

- $\mathcal{S}_q(\alpha, 1, 1) = \mathcal{S}_q(\alpha)$ which was introduced and examined by Agrawal and Sahoo in [16].
- $\mathcal{S}_q(0, 1, 1) = \mathcal{S}_q$ which was initially introduced by Ismail et al. [4].
- $\mathcal{S}_1(\alpha, 1, 2) = \mathcal{S}(\alpha)$ the renowned class of starlike functions of order α established by Robertson [17].

- $\mathcal{S}_1(0,1,1) = \mathcal{S}^*$ the class introduced by Nevanlinna [18].
- $\mathcal{S}_1(0,1,v) = \mathcal{S}(0,k)$ the class introduced and studied by Sakaguchi [19].

We denote by $\mathcal{T}_q(\alpha,u,v)$ the subclass of $\widetilde{\mathcal{F}}$ that includes all functions $\hbar$ for which the following holds:

$$\varpi\partial_q\hbar(\varpi) \in \mathcal{S}_q(\alpha,u,v). \tag{1.8}$$

We must revisit the neighborhood concept initially introduced by Goodman [20] and further developed by Ruscheweyh [21].

Definition 1.4. *For any $\hbar \in \widetilde{\mathcal{F}}$, the ρ-neighborhood surrounding the function $\hbar$ can be described as:*

$$\mathcal{N}_\rho(\hbar) = \left\{ g \in \widetilde{\mathcal{F}} : g(\varpi) = \varpi + \sum_{m=2}^{\infty} b_m\varpi^m, \ \sum_{m=2}^{\infty} m|a_m - b_m| \le \rho \right\}. \tag{1.9}$$

For $e(\varpi) = \varpi$, we can see that

$$\mathcal{N}_\rho(e) = \left\{ g \in \widetilde{\mathcal{F}} : g(\varpi) = \varpi + \sum_{m=2}^{\infty} b_m\varpi^m, \ \sum_{m=2}^{\infty} m|b_m| \le \rho \right\}. \tag{1.10}$$

Ruschewegh [21] demonstrated, among other findings, that for all $\eta \in \mathbb{C}$, with $|\mu| < \rho$,

$$\frac{\hbar(\varpi) + \eta\varpi}{1+\eta} \in \mathcal{S}^* \Rightarrow \mathcal{N}_\rho(\hbar) \subset \mathcal{S}^*.$$

Our main results can be proven by utilizing the following lemma.

Lemma 1.5. *[20] Let $P(\varpi) = 1 + \sum_{m=1}^{\infty} p_m\varpi^m$, $\quad (\varpi \in \Omega)$, with the condition $\Re\{p(\varpi)\} > 0$, then*

$$|p_m| \le 2, \quad (m \ge 1).$$

In this paper, our main focus is on analyzing coefficient estimates and exploring the convolution property within the context of the class $\mathcal{S}_q(\alpha,u,v)$. Motivated by Definition 1.4, we introduce a new definition of neighborhood that is specific to this class. By investigating the related neighborhood result for $\mathcal{S}_q(\alpha,u,v)$, we seek to offer a thorough understanding of the properties and characteristics of this particular class.

2. Results

We will now examine the coefficient inequalities for the function $\hbar$ in $\mathcal{S}_q(\alpha,u,u)$ and $\mathcal{T}_q(\alpha,u,v)$.

Theorem 2.1. *If $\hbar \in \mathcal{S}_q(\alpha,u,v)$, then*

$$|a_m| \le \prod_{n=1}^{m-1} \frac{(1-2\alpha)\delta_{n,u} + [n]_q}{[n+1]_q - \delta_{n+1,u}}, \tag{2.1}$$

where $\delta_{n,u}$ is given by (1.4).

Proof. The function $p(\varpi)$ is defined by

$$p(\varpi) = \frac{1}{1-\alpha}\left(\frac{\varpi\partial_q\hbar(\varpi)}{\hbar_{u,v}(\varpi)} - \alpha\right) = 1 + \sum_{m=1}^{\infty} p_m\varpi^m,$$

where $p(\varpi)$ represents a Carathéodory function and $\hbar(\varpi)$ belongs to the class $\mathcal{S}_q(\alpha, u, v)$.

Since

$$\varpi\partial_q\hbar(\varpi) = (\hbar_{u,v}(\varpi))(\alpha + (1-\alpha)p(\varpi)),$$

we have

$$\sum_{m=2}^{\infty}([m]_q - \delta_{m,u})a_m\varpi^m = \left(\varpi + \sum_{m=2}^{\infty} a_m\delta_{m,u}\varpi^m\right)\left(1 + (1-\alpha)\sum_{m=1}^{\infty} p_m\varpi^m\right),$$

where $\delta_{m,u}$ is given by (1.4), $\delta_{1,u} = 1$.

By equating the coefficients of ϖ^m on both sides, we obtain

$$a_m = \frac{(1-\alpha)}{([m]_q - \delta_{m,u})}\sum_{i=1}^{m-1}\delta_{m-i,u}a_{m-i}p_i, \quad a_1 = 1.$$

By Lemma 1.5, we get

$$|a_m| \leq \frac{2(1-\alpha)}{|[m]_q - \delta_{m,u}|}.\sum_{i=1}^{m-1}\delta_{i,u}|a_i|, \quad a_1 = 1 = \delta_{1,u}. \tag{2.2}$$

It now suffices to prove that

$$\frac{2(1-\alpha)}{[m]_q - \delta_{m,u}}.\sum_{n=1}^{m-1}\delta_{m,u}|a_u| \leq \prod_{n=1}^{m-1}\frac{(1-2\alpha)\delta_{n,u} + [n]_q}{[n+1]_q - \delta_{n+1,u}}. \tag{2.3}$$

To accomplish this, we utilize the method of induction.

We can easily see that (2.3) is true for $m = 2$ and 3.

Let the hypotheses is be true for $m = i$.

From (2.2), we have

$$|a_i| \leq \frac{2(1-\alpha)}{[i]_q - \delta_{i,u}}\sum_{n=1}^{i-1}\delta_{n,u}|a_n|, \quad a_1 = 1 = \delta_{1,u}.$$

From (2.1), we have

$$|a_i| \leq \prod_{n=1}^{i-1}\frac{\delta_{n,u}(1-2\alpha) + [n]_q}{[1+n]_q - \delta_{1+n,u}}.$$

By the induction hypothesis , we have

$$\frac{(1-\alpha)2}{[i]_q - \ \delta_{i,u}}\sum_{n=1}^{i-1}\delta_{n,u}|a_n| \leq \prod_{n=1}^{i-1}\frac{(1-2\alpha)\delta_{n,u} + [n]_q}{[1+n]_q - \delta_{1+n,u}}.$$

Multiplying both sides by

$$\frac{(1-2\alpha)\delta_{i,u}+[i]_q}{[1+i]_q-\ \delta_{1+i,u}},$$

we have

$$\prod_{n=1}^{i}\frac{(1-2\alpha)\delta_{n,u}+[n]_q}{[n+1]_q-\delta_{n+1,u}}\geq\frac{(1-2\alpha)\delta_{i,u}+[i]_q}{[i+1]_q-\delta_{i+1,u}}\left[\frac{2(1-\alpha)}{[i]_q-\delta_{i,u}}\sum_{n=1}^{i-1}\delta_{n,u}|a_n|\right]$$

$$=\left\{\frac{2(1-\alpha)\delta_{i,u}}{[i]_q-\delta_{i,u}}\sum_{n=1}^{i-1}\delta_{n,u}|a_n|+\sum_{n=1}^{i-1}\delta_{n,u}|a_n|\right\}.\frac{(1-\alpha)2}{[1+i]_q-\delta_{1+i,u}}$$

$$\geq\frac{2(1-\alpha)}{[i+1]_q-\delta_{i+1,u}}\left\{\delta_{i,u}|a_i|+\sum_{n=1}^{i-1}\delta_{n,u}|a_n|\right\}$$

$$\geq\frac{2(1-\alpha)}{[i+1]_q-\delta_{i+1,u}}\sum_{n=1}^{i}\delta_{n,u}|a_n|.$$

Hence

$$\frac{(1-\alpha)2}{[1+i]_q-\delta_{1+i,u}}\sum_{n=1}^{i}\delta_{n,u}|a_n|\leq\prod_{n=1}^{i}\frac{\delta_{n,u}(1-2\alpha)+[n]_q}{[1+n]_q-\delta_{1+n,u}},$$

This demonstrates that the inequality (2.3) holds for $m=i+1$, confirming the validity of the result. □

For $q\to 1^-, u=1$ and $v=1$, we obtain the following well-known result (see [22]).

Corollary 2.2. *If $\hbar\in\mathcal{S}^*(\alpha)$, then*

$$|a_k|\leq\prod_{s=1}^{k-1}\frac{(s-2\alpha)}{(k-1)!}.$$

Theorem 2.3. *If $\hbar\in\mathcal{T}_q(\alpha,u,v)$, then*

$$|a_m|\leq\frac{1}{[m]_q}\prod_{n=1}^{m-1}\frac{(1-2\alpha)\delta_{n,u}+[n]_q}{[1+n]_q-\delta_{1+n,u}},\quad for\ \ m=2,3,4,..., \tag{2.4}$$

where $\delta_{m,u}$ is given by (1.4).

Proof. By using Alexander's theorem

$$\hbar(\varpi)\in\mathcal{T}_q(\alpha,u,v)\Leftrightarrow\varpi\partial_q\hbar(\varpi)\in\mathcal{S}_q(\alpha,u,v). \tag{2.5}$$

The proof follows by using Theorem 2.1. □

Theorem 2.4. *A function $\hbar\in\mathcal{S}_q(\alpha,u,v)$ if and only if*

$$\frac{1}{\varpi}\left[\hbar*\{k(\varpi)(1-e^{i\phi})+f(\varpi)(1+(1-2\alpha)e^{i\phi})\}\right]\neq 0, \tag{2.6}$$

where $0<q<1,\ 0\leq\alpha<1, 0\leq\phi<2\pi$ and f,k are given by (2.10).

Proof. Suppose that $f \in \mathcal{S}_q(\alpha, u, v)$, then

$$\frac{1}{1-\alpha}\left(\frac{z\varpi\partial_q\hbar(\varpi)}{\hbar_{u,v}(\varpi)}-\alpha\right)=p(\varpi),$$

if and only if

$$\frac{\varpi\partial_q\hbar(\varpi)}{\hbar_{u,v}(\varpi)} \neq \frac{1+(1-2\alpha)e^{i\phi}}{1-e^{i\phi}}. \tag{2.7}$$

For all $\varpi \in \Omega$ and $0 \leq \phi < 2\pi$, it is straightforward to see that the condition (2.7) can be expressed as

$$\frac{1}{\varpi}[\varpi\partial_q\hbar(\varpi)(1-e^{i\phi})-\hbar_{u,v}(\varpi)(1+(1-2\alpha)e^{i\phi})] \neq 0. \tag{2.8}$$

On the other hand, it is well-known that

$$\hbar_{u,v}(\varpi)=\hbar(\varpi)*f(\varpi), \qquad \varpi\partial_q\hbar(\varpi)=\hbar(\varpi)*k(\varpi), \tag{2.9}$$

where

$$f(\varpi)=\frac{1}{v}\sum_{n=0}^{v-1}\varepsilon^{(1-u)n}\frac{\varpi}{1-\varepsilon^n\varpi}=\varpi+\sum_{m=2}^{\infty}\delta_{m,u}\varpi^m, \qquad k(\varpi)=\varpi+\sum_{m=2}^{\infty}[m]_q\varpi^m. \tag{2.10}$$

Substituting (2.9) into (2.8) we get (2.6). □

Remark 2.5. *From Theorem 2.4, it is straightforward to derive the equivalent condition for a function $\hbar$ to be a member of the class $\mathcal{S}_q(\alpha, u, v)$ if and only if*

$$\frac{(\hbar * T_\phi)(\varpi)}{\varpi} \neq 0, \quad \varpi \in \Omega, \tag{2.11}$$

where $T_\phi(\varpi)$ has the form

$$T_\phi(\varpi)=\varpi+\sum_{m=2}^{\infty}t_m\varpi^m,$$
$$t_m=\frac{[m]_q-\delta_{m,u}-(\delta_{m,u}(1-2\alpha)+[m]_q)e^{i\phi}}{(\alpha-1)e^{i\phi}}. \tag{2.12}$$

In order to obtain neighborhood results similar to those found by Ruschewegh [21] for the classes, we define the following concepts related to neighborhoods.

Definition 2.6. *For any $\hbar \in \widetilde{\mathcal{F}}$, the ρ-neighborhood associated with the function $\hbar$ is defined as:*

$$\mathcal{N}_{\beta,\rho}(\hbar)=\left\{ g \in \widetilde{\mathcal{F}} : g(\varpi)=\varpi+\sum_{m=2}^{\infty}b_m\varpi^m, \ \sum_{m=2}^{\infty}\beta_m|a_m-b_m| \leq \rho \right\}, \ (\rho \geq 0). \tag{2.13}$$

For $e(\varpi)=\varpi$, we can see that

$$\mathcal{N}_{\beta,\rho}(e)=\left\{ g \in \widetilde{\mathcal{F}} : g(\varpi)=\varpi+\sum_{m=2}^{\infty}b_m\varpi^m, \ \sum_{m=2}^{\infty}\beta_m|b_m| \leq \rho \right\}, \ (\rho \geq 0), \tag{2.14}$$

where $[m]_q$ is given by Eq (1.6).

Remark 2.7. • *For $\beta_m = m$, from Definition 2.6, we get Definition 1.4.*

- *For $\beta_m = [m]_q$, from Definition 2.6, we get the definition of neighborhood with q-derivative $\mathcal{N}_{q,\rho}(\hbar), \mathcal{N}_{q,\rho}(e)$.*
- *For $\beta_m = |t_m|$ given by* (2.12)*, from Definition 2.6, we get the definition of neighborhood for the class $\mathcal{S}_q(\alpha, u, v)$ with $\mathcal{N}_{q,\rho}(\alpha, u, v; \hbar)$.*

Theorem 2.8. *Let $\hbar \in \mathcal{N}_{q,1}(e)$, defined in the form* (1.1)*, then*

$$\left|\frac{\varpi\partial_q\hbar(\varpi)}{\hbar_{u,v}(\varpi)} - 1\right| < 1, \tag{2.15}$$

where $0 < q < 1,\ \varpi \in \Omega$.

Proof. Let $\hbar \in \widetilde{\mathcal{F}}$, and $\hbar(\varpi) = \varpi + \sum_{m=2}^{\infty} a_m\varpi^m$, $\quad \hbar_{u,v}(\varpi) = \varpi + \sum_{m=2}^{\infty} \delta_{m,u}a_m\varpi^m$, where $\delta_{m,u}$ is given by (1.4).

Consider

$$\begin{aligned}|\varpi\partial_q\hbar(\varpi) - \hbar_{u,v}(\varpi)| &= \left|\sum_{m=2}^{\infty}([m]_q - \delta_{m,u})a_m\varpi^{m-1}\right| \\ &< |\varpi|\sum_{m=2}^{\infty}[m]_q|a_m| - \sum_{m=2}^{\infty}\delta_{m,u}|a_m|.|\varpi|^{m-1} \\ &= |\varpi| - \sum_{m=2}^{\infty}\delta_{m,u}|a_m|.|\varpi|^{m-1} \\ &\le |\hbar_{u,v}(\varpi)|,\ \varpi \in \Omega.\end{aligned}$$

This provides us with the desired result. □

Theorem 2.9. *Let $\hbar \in \widetilde{\mathcal{F}}$, and for any complex number η where $|\mu| < \rho$, if*

$$\frac{\hbar(\varpi) + \eta\varpi}{1 + \eta} \in \mathcal{S}_q(\alpha, u, v), \tag{2.16}$$

then

$$\mathcal{N}_{q,\rho}(\alpha, u, v; \hbar) \subset \mathcal{S}_q(\alpha, u, v).$$

Proof. Assume that a function g is defined as $g(\varpi) = \varpi + \sum_{m=2}^{\infty} b_m\varpi^m$ and is a member of the class $\mathcal{N}_{q,\rho}(\alpha, u, v; \hbar)$. To prove the theorem, we need to demonstrate that $g \in \mathcal{S}_q(\alpha, u, v)$. This will be shown in the following three steps.

First, we observe that Theorem 2.4 and Remark 2.5 are equivalent to

$$\hbar \in \mathcal{S}_q(\alpha, u, v) \Leftrightarrow \frac{1}{\varpi}[(\hbar * T_\phi)(\varpi)] \neq 0, \quad \varpi \in \Omega, \tag{2.17}$$

where $T_\phi(\varpi) = \varpi + \sum_{m=2}^{\infty} t_m\varpi^m$ and t_m is given by (2.12).

Second, we find that (2.16) is equivalent to

$$\left|\frac{\hbar(\varpi) * T_\phi(\varpi)}{\varpi}\right| \ge \rho. \tag{2.18}$$

Since $\hbar(\varpi) = \varpi + \sum_{m=2}^{\infty} a_m \varpi^m \in \widetilde{\mathcal{F}}$ which satisfies (2.16), then (2.17) is equivalent to

$$T_\phi \in \mathcal{S}_q(\alpha, u, v) \Leftrightarrow \frac{1}{\varpi}\left[\frac{\hbar(\varpi) * T_\phi(\varpi)}{1+\eta}\right] \neq 0, \qquad |\eta| < \rho.$$

Third, letting $g(\varpi) = \varpi + \sum_{m=2}^{\infty} b_m \varpi^m$ we notice that

$$\begin{aligned}
\left|\frac{g(\varpi) * T_\phi(\varpi)}{\varpi}\right| &= \left|\frac{\hbar(\varpi) * T_\phi(\varpi)}{\varpi} + \frac{(g(\varpi) - \hbar(\varpi)) * T_\phi(\varpi)}{\varpi}\right| \\
&\geq \rho - \left|\frac{(g(\varpi) - \hbar(\varpi)) * T_\phi(\varpi)}{\varpi}\right| \quad \text{(by using (2.18))} \\
&= \rho - \left|\sum_{m=2}^{\infty} (b_m - a_m) t_m \varpi^m\right| \\
&\geq \rho - |\varpi| \sum_{m=2}^{\infty} \frac{[m]_q - \delta_{m,u} - |[m]_q + \delta_{m,u}(1-2\alpha)|}{1-\alpha} |b_m - a_m| \\
&\geq \rho - |\varpi|\rho > 0.
\end{aligned}$$

This prove that

$$\frac{g(\varpi) * T_\phi(\varpi)}{\varpi} \neq 0, \qquad \varpi \in \Omega.$$

Based on our observations in (2.17), it follows that $g \in \mathcal{S}_q(\alpha, u, v)$. This concludes the proof of the theorem. □

When $u = v = 1$, $q \to 1^-$, and $\alpha = 0$ in the above theorem, we obtain (1.10), which was proven by Ruscheweyh in [21].

Corollary 2.10. *Let $\mathcal{S}^*$ represent the class of starlike functions. Let $\hbar \in \widetilde{\mathcal{F}}$, and for all complex numbers η such that $|\mu| < \rho$, if*

$$\frac{\hbar(\varpi) + \eta\varpi}{1+\eta} \in \mathcal{S}^*, \tag{2.19}$$

then $\mathcal{N}_\sigma(\hbar) \subset \mathcal{S}^*$.

3. Conclusions

In conclusion, this research paper successfully introduces and explores a novel category of q-starlike and q-convex functions, specifically $\mathcal{S}_q(\alpha, u, v)$ and $\mathcal{T}_q(\alpha, u, v)$, that are fundamentally linked to (u, v)-symmetrical functions. The findings highlight the intricate interplay between q-starlikeness, q-convexity, and symmetry conditions, offering a rich framework for further investigation. Through detailed analysis, including coefficient estimates and convolution conditions, this work lays a solid foundation for future studies in this area. The established properties within the (ρ, q)-neighborhood not only deepen our understanding of these function classes but also open avenues for potential applications in complex analysis and geometric function theory. Overall, this pioneering research marks a significant advancement in the study of special functions, inviting further exploration and development in this dynamic field.

Author contributions

Hanen Louati conceptualized and led the development of the study's methodology, focusing on the formulation of new mathematical frameworks for q-starlike and q-convex functions. Afrah Al-Rezami contributed significantly to data validation and the theoretical exploration of (u, v)-symmetrical functions. Erhan Deniz conducted the primary analysis of coefficient inequalities and convolution properties, offering critical insights into the results. Abdulbasit Darem provided computational support and assisted in exploring the applications of the (ρ, q)-neighborhood framework. Robert Szasz contributed to the literature review, linking the study to prior research and assisting in the interpretation of findings. All authors participated in drafting the manuscript, revising it critically for important intellectual content. All authors have read and agreed to the published version of the manuscript.

Acknowledgment

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA, for funding this research work through the project number "NBU-FPEJ-2024-2920-02". This study was also supported via funding from Prince Sattam bin Abdulaziz University, project number (PSAU/2024/R/1445).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. P. Liczberski, J. Polubinski, On (j, k)-symmtrical functions, *Math. Bohem.*. **120** (1995), 13–28.
2. F. H. Jackson, On q-functions and a certain difference operator, *T. Roy. Soc. Edinb.*, **46** (1909), 253–281. https://doi.org/10.1017/S0080456800002751
3. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193–203.
4. M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Var. Elliptic Equ.*, **14** (1990), 77–84. https://doi.org/10.1080/17476939008814407
5. M. Naeem, S. Hussain, S. Khan, T. Mahmood, M. Darus, Z. Shareef, Janowski type q-convex and q-close-to-convex functions associated with q-conic domain, *Mathematics*, **8** (2020), 440. https://doi.org/10.3390/math8030440
6. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, *Symmetry*, **11** (2019), 292. https://doi.org/10.3390/sym11020292
7. M. Govindaraj, S. Sivasubramanian, On a class of analytic functions related to conic domains involving q-calculus, *Anal. Math.*, **43** (2017), 475–487. https://doi.org/10.1007/s10476-017-0206-5

8. F. Alsarari, M. I. Faisal, A. A. Alzulaibani, Geometric properties of certain classes of analytic functions with respect to (x,y)-symmetric points, *Mathematics*, **11** (2023), 4180. https://doi.org/10.3390/math11194180
9. F. Alsarari, S. Alzahrani, Convolution properties of q-Janowski-type functions associated with (x, y)-symmetrical functions. *Mathematics*, **14** (2022), 1406. https://doi.org/10.3390/sym14071406
10. B. Khan, Z. G. Liu, T. G. Shaba, S. Araci, N. Khan, M. G. Khan, Applications of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials, *J. Math.*, **2022** (2022), 8162182. https://doi.org/10.1155/2022/8162182
11. H. M. Srivastava, Operators of basic (or $q-$) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344. https://doi.org/10.1007/s40995-019-00815-0
12. F. AlSarari, S. Latha, T. Bulboacă, On Janowski functions associated with (m, n)-symmetric points, *J. Taibah Univ. Sci.*, **13** (2019), 972–978. https://doi.org/10.1080/16583655.2019.1665487
13. H. Shamsan, F. S. M. Alsarari, S. Latha, Few results of q-sakaguchi type functions, *Palestin. J. Math.*, **9** (2020), 194–199.
14. F. Al Sarari, S. Latha, B. A. Frasin, Convex and starlike functions with respect to (j, k)-symmetric points, *Appl. Math. E-Notes* **17** (2017), 10–18.
15. F. Al Sarari, S. Latha, A new class of (j, k)-symmetric harmonic functions, *Stud. Univ. Babs-Bolyai Math.*, **60** (2015), 553–559.
16. S. Agrawal, S. K. Sahoo, A generalization of starlike functions of order alpha, *Hokkaido Math. J.*, **46** (2017), 15–27.
17. M. I. S. Robertson, On the theory of univalent functions, *Ann. Math.*, **37** (1936), 374–408. https://doi.org/10.2307/1968451
18. R. H. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, 1921.
19. K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan*, **11** (1959), 72–75. https://doi.org/10.2969/jmsj/01110072
20. A. W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8** (1957), 598–601. https://doi.org/10.1090/S0002-9939-1957-0086879-9
21. S. Ruschewyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81** (1981), 521–527. https://doi.org/10.1090/S0002-9939-1981-0601721-6
22. A. W. Goodman, *Univalent functions*, Tampa, Fla, USA: Mariner Publishing, 1983.

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)