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Research article

On the compactness via primal topological spaces

Ohud Alghamdi*

Department of Mathematics, Faculty of Science, Al-Baha University, Al-Baha 65527, Saudi Arabia

* **Correspondence:** Email: ofalghamdi@bu.edu.sa.

Abstract: In this paper, we introduce new concepts, including $\mathfrak{P}$-compactness, strongly $\mathfrak{P}$-compactness, and super $\mathfrak{P}$-compactness, in view of a primal topological space structure. We provide some results regarding to these concepts. Additionally, some examples are presented to illustrate the relations between these concepts.

Keywords: primal topological space; $\mathfrak{P}$-compactness; S$\mathfrak{P}$-compactness; SU$\mathfrak{P}$-compactness
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1. Introduction

Numerous topologies of significant applications have been characterized through the incorporation of some mathematical structures. For instance, Choquet developed the concept of a grill structure with topological spaces in [1]. Moreover, several topological concepts were presented, such as the ideal [2, 3] and the filter [4]. The concept of primal topological space $\mathcal{PS}$ was introduced by S. Acharjee et al. in [5]. Then, several papers discussed the topological properties in $\mathcal{PS}$, such as [6], which presented definitions of $\mathfrak{P}$-regularity, $\mathfrak{P}$-Hausdorff, and $\mathfrak{P}$-normality. Additionally, Al-Omari and Alqahtani provided definitions of new closure operators using a primal structure in [7]. Then, Alghamdi et al. introduced novel operators by leveraging the primal structure in [8]. Additional primal operators were defined in [9]. Moreover, Al-Saadi and Al-Malki discussed various categories of open sets within the framework of generalized topological spaces, thereby utilizing the primal structure [10]. In this paper, we introduce some properties concerning compactness in $\mathcal{PS}$. These properties are named $\mathfrak{P}$-compactness, strongly $\mathfrak{P}$-compactness, and super $\mathfrak{P}$-compactness. We provide some results and examples which connect these concepts together. Throughout this paper, $(\mathcal{T}, \mu, \mathfrak{P})$ represents a primal topological space $\mathcal{PS}$ such that μ is a topology on $\mathcal{T}$. Moreover, we use the symbol $\mathcal{CL}(A)$ for the closure of a set $A \subset \mathcal{T}$ and $\mathfrak{H}$ for an index set. Furthermore, we use the symbol $2^{\mathcal{T}}$ for the power set of the set $\mathcal{T}$.

Definition 1.1. ([5]) *For a nonempty set $\mathcal{T}$, we define a primal collection $\mathfrak{P} \subseteq 2^{\mathcal{T}}$ on $\mathcal{T}$ as follows:*
(1) $\mathcal{T} \notin \mathfrak{P}$,
(2) if $R \in \mathfrak{P}$ and $T \subseteq R$, then $T \in \mathfrak{P}$,
(3) if $R \cap T \in \mathfrak{P}$, then either $R \in \mathfrak{P}$ or $T \in \mathfrak{P}$.

Corollary 1.1. ([5]) *If $\mathcal{T} \neq \emptyset$, then $\mathfrak{P} \subseteq 2^{\mathcal{T}}$ is a primal collection on $\mathcal{T}$ if and only if:*
(1) $\mathcal{T} \notin \mathfrak{P}$,
(2) if $T \notin \mathfrak{P}$ and $T \subseteq R$, then $R \notin \mathfrak{P}$,
(3) if $R \notin \mathfrak{P}$ and $T \notin \mathfrak{P}$, then $R \cap T \notin \mathfrak{P}$.

Definition 1.2. ([5]) *A topological space $(\mathcal{T}, \nu)$ with a primal collection $\mathfrak{P}$ on $\mathcal{T}$ is called a primal topological space $\mathcal{PS}$ and is denoted by $(\mathcal{T}, \nu, \mathfrak{P})$.*

2. $\mathfrak{P}$-compact spaces

Definition 2.1. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. We say that $(\mathcal{T}, \rho, \mathfrak{P})$ is a primal compact space ($\mathfrak{P}$-compact space) if for every open cover $\{V_\eta\}_{\eta\in\mathfrak{H}}$ of $\mathcal{T}$, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ with $\bigcup_{\eta\in\mathfrak{H}_0} V_\eta \notin \mathfrak{P}$. Let $N \subseteq \mathcal{T}$. Then, N is called a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ if for every open cover $\{W_\eta\}_{\eta\in\mathfrak{H}}$ of N, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus [N \setminus \bigcup_{\eta\in\mathfrak{H}_0} W_\eta] \notin \mathfrak{P}$.*

Theorem 2.1. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and $B \subseteq \mathcal{T}$. If B is a compact subspace of $\mathcal{T}$, then B is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{V_\eta\}_{\eta\in\mathfrak{H}}$ be an open cover of B. Then, since B is a compact subspace of $\mathcal{T}$, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $B \subseteq \bigcup_{\eta\in\mathfrak{H}_0} V_\eta$. Hence,

$$\mathcal{T} \setminus \left[B \setminus \bigcup_{\eta\in\mathfrak{H}_0} V_\eta\right] = \mathcal{T} \notin \mathfrak{P}.$$

Therefore, B is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

The converse of Theorem 2.1 is not necessarily true as considered in the following example.

Example 2.1. *Let $(\mathbb{R}, \tau_1, \mathfrak{P}_1)$ be defined as follows: $U \in \tau_1$ if and only if either $U = \emptyset$ or $1 \in U$, see Example 10 in [11]. Let $\mathfrak{P}_1$ be defined on $\mathbb{R}$ as follows: $U \in \mathfrak{P}_1$ if and only if $1 \notin U$. Then, $(\mathbb{R}, \tau_1, \mathfrak{P}_1)$ is a $\mathcal{PS}$. Let $\mathbb{N}$ be the set of natural numbers and let $\{V_\eta\}_{\eta\in\mathfrak{H}}$ be any open cover of $\mathbb{N}$ such that $V_\eta \neq \emptyset$ for every $\eta \in \mathfrak{H}$. Let $\mathfrak{H}_0 = \{V_i\}_{i=1}^n \subseteq \{V_\eta\}_{\eta\in\mathfrak{H}}$. Then, $1 \in \mathbb{R} \setminus [\mathbb{N} \setminus \bigcup_{i=1}^n V_i]$, which means that $\mathbb{R} \setminus [\mathbb{N} \setminus \bigcup_{i=1}^n V_i] \notin \mathfrak{P}_1$. Hence, $\mathbb{N}$ is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. Note that $\mathbb{N}$ is not compact. Indeed, $\{j, 1\}_{j\in\mathbb{N}}$ is an open cover of $\mathbb{N}$, which has no finite subcover.*

Example 2.2. *Let $(\mathbb{R}, \mathcal{D}, \mathfrak{P})$ be a $\mathcal{PS}$ defined as follows: $U \in \mathfrak{P}$ if and only if $\mathbb{R} \setminus U$ is an infinite subset of $\mathbb{R}$. Moreover, $V \in \mathcal{D}$ if and only if $V \subseteq \mathbb{R}$ (the discrete topological space on $\mathbb{R}$, see Example 3 in [11]). Then, $\Lambda = \{r\}_{r\in\mathbb{R}}$ is an open cover of $\mathbb{R}$. If $\{V_1, V_2, ..., V_n\}$ is an arbitrary finite subfamily of Λ, then $\bigcup_{i=1}^n V_i = \{r_1, ..., r_n\} \in \mathfrak{P}$. Thus, $\mathbb{R}$ is not a $\mathfrak{P}$-compact space.*

Theorem 2.2. *$\mathfrak{P}$-compactness is hereditarily defined with respect to closed subspaces.*

Proof. Assume that $(\mathcal{T}, \rho, \mathfrak{P})$ is a $\mathfrak{P}$-compact space and $M \subseteq \mathcal{T}$ is any closed subspace. Suppose that $Q = \{V_\eta\}_{\eta\in\mathfrak{H}}$ is an open cover of M. Then, $\{V_\eta\}_{\eta\in\mathfrak{H}} \bigcup(\mathcal{T} \setminus M)$ is an open cover of $\mathcal{T}$. Hence, there exists a finite set $\mathfrak{H}_0 = \{V_1, V_2, ..., V_n\} \subseteq \{\mathcal{T} \setminus M\} \bigcup\{V_\eta : \eta \in \mathfrak{H}\}$ such that $\bigcup_{i=1}^n V_i \notin \mathfrak{P}$. Thus, $\mathcal{T} \setminus [M \setminus \bigcup_{i=1}^n V_i] \notin \mathfrak{P}$, which implies that M is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

The following example shows that if the subspace of $\mathcal{T}$ is not closed, then it may not be a $\mathfrak{P}$-compact subspace.

Example 2.3. *Let $(\mathbb{R}, \mathcal{F}, \mathfrak{P})$ be a $\mathcal{PS}$ defined as follows:*

$U \in \mathcal{F}$ if and only if either $\sqrt{2} \in \mathbb{R} \setminus U$ or $\mathbb{R} \setminus U$ is a finite subset of $\mathbb{R}$, see Example 24 in [11].

Let $\mathfrak{P}$ be defined as in Example 2.2. Let $Q = \{O_\eta\}_{\eta\in\mathfrak{H}}$ be an open cover of $\mathbb{R}$. Then, there exists $\lambda \in \mathfrak{H}$ such that $\sqrt{2} \in O_\lambda$. Hence, $\mathbb{R} \setminus O_\lambda$ is a finite subset of $\mathbb{R}$. Let $Q_0 = \{O_\lambda\} \subseteq Q$. Then, since $O_\lambda \notin \mathfrak{P}$, $\mathbb{R}$ is a $\mathfrak{P}$-compact space. Now, consider the subspace $\mathbb{R} \setminus \{\sqrt{2}\}$. Claim that $\mathbb{R} \setminus \{\sqrt{2}\}$ is not a $\mathfrak{P}$-compact subspace. Indeed, if Q_0 is any finite subfamily of $Q = \{t\}_{t\in\mathbb{R}\setminus\{\sqrt{2}\}}$, then $\bigcup_{O\in Q_0} O \in \mathfrak{P}$. Observe that $\mathbb{R} \setminus \{\sqrt{2}\}$ is a discrete subspace of $\mathbb{R}$ that is not closed.

Theorem 2.3. *Let $(\mathcal{T}, \nu, \mathfrak{P})$ be a $\mathcal{PS}$. For a subset K of $\mathcal{T}$, the following properties are equivalent:*
(1) K is a $\mathfrak{P}$-compact subspace; and
(2) for every family $\{L_\delta\}_{\delta\in\mathfrak{H}}$ of closed sets such that $K \cap \left(\bigcap_{\delta\in\mathfrak{H}} L_\delta\right) = \emptyset$, there exists a finite subset $\mathfrak{H}_0$ of $\mathfrak{H}$ such that

$$(\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}_0} (\mathcal{T} \setminus L_\delta)\right] \notin \mathfrak{P}.$$

Proof. (1) ⇒ (2): Let $\{L_\delta\}_{\delta\in\mathfrak{H}}$ be a collection of closed sets in $\mathcal{T}$ such that $K \cap \left(\bigcap_{\delta\in\mathfrak{H}} L_\delta\right) = \emptyset$. Then, we have the following:

$$K \subseteq \left(\mathcal{T} \setminus \bigcap_{\delta\in\mathfrak{H}} L_\delta\right) = \bigcup_{\delta\in\mathfrak{H}} (\mathcal{T} \setminus L_\delta).$$

Since $\mathcal{T} \setminus L_\delta$ is open for each $\delta \in \mathfrak{H}$ and K is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite subset $\mathfrak{H}_0$ of $\mathfrak{H}$ such that

$$\mathcal{T} \setminus \left[K \setminus \left(\bigcup_{\delta\in\mathfrak{H}_0} (\mathcal{T} \setminus L_\delta)\right)\right] \notin \mathfrak{P}.$$

Now, we have the following:

$$\begin{aligned}(\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}_0} (\mathcal{T} \setminus L_\delta)\right] &= \mathcal{T} \bigcap \left[\mathcal{T} \setminus \left[K \setminus \left(\bigcup_{\delta\in\mathfrak{H}_0} (\mathcal{T} \setminus L_\delta)\right)\right]\right] \\ &= \mathcal{T} \setminus \left[K \setminus \left(\bigcup_{\delta\in\mathfrak{H}_0} (\mathcal{T} \setminus L_\delta)\right)\right] \notin \mathfrak{P}.\end{aligned}$$

(2) ⇒ (1): Let $\{V_\delta\}_{\delta\in\mathfrak{H}}$ be any cover of K which consists of open sets in $\mathcal{T}$. Then, $K \cap \left(\mathcal{T} \setminus \bigcup_{\delta\in\mathfrak{H}} V_\delta\right) = K \cap \left[\bigcap_{\delta\in\mathfrak{H}} (\mathcal{T} \setminus V_\delta)\right] = \emptyset$.

Since $\mathcal{T} \setminus V_\delta$ is closed for each $\delta \in \mathfrak{H}$, then by (2), there exists a finite subset $\mathfrak{H}_0$ of $\mathfrak{H}$ such that

$$(\mathcal{T} \setminus K) \bigcup \left(\bigcup_{\delta \in \mathfrak{H}_0} V_\delta \right) \notin \mathfrak{P}.$$

Therefore, we have the following:

$$(\mathcal{T} \setminus K) \bigcup \left(\bigcup_{\delta \in \mathfrak{H}_0} V_\delta \right) = \mathcal{T} \setminus \left[K \setminus \left(\bigcup_{\delta \in \mathfrak{H}_0} V_\delta \right) \right] \notin \mathfrak{P}.$$

Hence, K is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Corollary 2.1. *If $(\mathcal{T}, \nu, \mathfrak{P})$ is a $\mathcal{PS}$ and $\{L_\delta\}_{\delta \in \mathfrak{H}}$ is a family of closed sets in $\mathcal{T}$ such that $\bigcap_{\delta \in \mathfrak{H}} L_\delta = \emptyset$, then $(\mathcal{T}, \nu, \mathfrak{P})$ is a $\mathfrak{P}$-compact space if and only if there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\bigcup_{\delta \in \mathfrak{H}_0} (\mathcal{T} \setminus L_\delta) \notin \mathfrak{P}$.*

Theorem 2.4. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. If $R, T \subseteq \mathcal{T}$ are both $\mathfrak{P}$-compact subspaces of $\mathcal{T}$, then $R \cup T$ is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{O_\delta\}_{\delta \in \mathfrak{H}}$ be an open cover of $R \cup T$. Since both R and T are $\mathfrak{P}$-compact subspaces of $\mathcal{T}$, then there are two finite subsets of $\mathfrak{H}$, namely $\mathfrak{H}_0$ and $\mathfrak{H}_1$, such that $\mathcal{T} \setminus \left(R \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta\right) \notin \mathfrak{P}$ and $\mathcal{T} \setminus \left(T \setminus \bigcup_{\delta \in \mathfrak{H}_1} O_\delta\right) \notin \mathfrak{P}$. Hence, $\mathcal{T} \setminus \left[(R \cup T) \setminus \bigcup_{\delta \in \mathfrak{H}_0 \cup \mathfrak{H}_1} O_\delta\right] \notin \mathfrak{P}$. Thus, $R \cup T$ is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Theorem 2.5. *Let $(\mathcal{T}, \nu, \mathfrak{P})$ be a $\mathcal{PS}$ and let R, S be any subsets of $\mathcal{T}$. If R is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ and S is a closed set, then $R \cap S$ is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{O_\delta\}_{\delta \in \mathfrak{H}}$ be an open cover of $R \cap S$. Then, $Q = \{O_\delta\}_{\delta \in \mathfrak{H}} \cup (\mathcal{T} \setminus S)$ is an open cover of R. Hence, there exists a finite subset of Q, namely Q_0, such that $\mathcal{T} \setminus \left[R \setminus \left(\bigcup_{O \in Q_0} O\right)\right] \notin \mathfrak{P}$. Since $\mathcal{T} \setminus \left[R \setminus \left(\bigcup_{O \in Q_0} O\right)\right] \subseteq \mathcal{T} \setminus \left[(R \cap T) \setminus \left(\bigcup_{O \in Q_0} O\right)\right]$, then $\mathcal{T} \setminus \left[(R \cap T) \setminus \left(\bigcup_{O \in Q_0} O\right)\right] \notin \mathfrak{P}$, which implies that $R \cap T$ is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Lemma 2.1. *Let $f : (\mathcal{T}, \rho) \to (\mathcal{Y}, \nu)$ be a function. Then, the following properties hold:*

(1) If f is a bijective function and $\mathfrak{P}$ is a primal collection on $\mathcal{T}$, then $f(\mathfrak{P}) = \{f(V) : V \in \mathfrak{P}\}$ is a primal collection on $\mathcal{Y}$; and

(2) If f is a bijective function and $\mathcal{J}$ is a primal collection on $\mathcal{Y}$, then $f^{-1}(\mathcal{J}) = \{f^{-1}(B) : B \in \mathcal{J}\}$ is a primal collection on $\mathcal{T}$.

Proof. (1) Since f is surjective, then $f(\mathcal{T}) = \mathcal{Y} \notin f(\mathfrak{P})$. Let $W \in f(\mathfrak{P})$ and let $Q \subseteq W$. Since $W \in f(\mathfrak{P})$, then $\exists M \in \mathfrak{P}$ such that $W = f(M) \Rightarrow f^{-1}(W) = M$. Hence, $f^{-1}(Q) \subseteq f^{-1}(W)$; then, $f^{-1}(Q) \in \mathfrak{P}$, which implies that $Q \in f(\mathfrak{P})$. Now, let $W \cap Q \in f(\mathfrak{P})$. Then, there exists $R \in \mathfrak{P}$ such that $W \cap Q = f(R)$. Thus, $f^{-1}(W \cap Q) = f^{-1}(W) \cap f^{-1}(Q) = R$. Hence, either $f^{-1}(W) \in \mathfrak{P}$ or $f^{-1}(Q) \in \mathfrak{P}$. Then, either $W \in f(\mathfrak{P})$ or $Q \in f(\mathfrak{P})$. Therefore, $f(\mathfrak{P})$ is a primal collection on $\mathcal{Y}$.

(2) We know that $f^{-1}(\mathcal{Y}) = \mathcal{T}$; since $\mathcal{Y} \notin \mathcal{J}$, then $f^{-1}(\mathcal{Y}) = \mathcal{T} \notin f^{-1}(\mathcal{J})$. Let $A \in f^{-1}(\mathcal{J})$ and let $B \subseteq A$. Then, $\exists C \in \mathcal{J}$ such that $A = f^{-1}(C)$. Hence, $f(A) = f(f^{-1}(C)) = C$. As $f(B) \subseteq f(A) = C$, then $f(B) \in \mathcal{J}$, which implies that $B \in f^{-1}(\mathcal{J})$. Now, suppose that $A \cap C \in f^{-1}(\mathcal{J})$. Then, $\exists R \in \mathcal{J}$ such that $A \cap C = f^{-1}(R)$. Then, $f(A \cap C) = f(f^{-1}(R)) = R$. Thus, $f(A) \cap f(C) = R \in \mathcal{J}$ implies that either $f(A) \in \mathcal{J}$ or $f(C) \in \mathcal{J}$. Therefore, either $A \in f^{-1}(\mathcal{J})$ or $C \in f^{-1}(\mathcal{J})$. □

Lemma 2.2. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. If $f : (\mathcal{T}, \rho, \mathfrak{P}) \to (\mathcal{Y}, \nu)$ is a function and $\mathcal{J}_{\mathfrak{P}} = \{B \subset \mathcal{Y} : f^{-1}(B) \in \mathfrak{P}\}$, then the following hold:*
(1) $\mathcal{J}_{\mathfrak{P}}$ is a primal collection on $\mathcal{Y}$;
(2) if f is injective, then $\mathfrak{P} \subseteq f^{-1}(\mathcal{J}_{\mathfrak{P}})$;
(3) if f is surjective, then $\mathcal{J}_{\mathfrak{P}} \subseteq f(\mathfrak{P})$; and
(4) if f is bijective, then $\mathcal{J}_{\mathfrak{P}} = f(\mathfrak{P})$.

Proof. (1) We know that $f^{-1}(\mathcal{Y}) = \mathcal{T} \notin \mathfrak{P}$. Then, $\mathcal{Y} \notin \mathcal{J}_{\mathfrak{P}}$. Let $A \in \mathcal{J}_{\mathfrak{P}}$ and let $B \subseteq A$. Then, $A \subset \mathcal{Y}$ and $f^{-1}(A) \in \mathfrak{P}$. Since $f^{-1}(B) \subseteq f^{-1}(A)$, then $f^{-1}(B) \in \mathfrak{P}$; hence $B \in \mathcal{J}_{\mathfrak{P}}$. Now, suppose that $A \cap B \in \mathcal{J}_{\mathfrak{P}}$. Then, $f^{-1}(A \cap B) \in \mathfrak{P}$, which implies that $f^{-1}(A) \cap f^{-1}(B) \in \mathfrak{P}$. Hence, either $f^{-1}(A) \in \mathfrak{P}$ or $f^{-1}(B) \in \mathfrak{P}$. Therefore, either $A \in \mathcal{J}_{\mathfrak{P}}$ or $B \in \mathcal{J}_{\mathfrak{P}}$.
(2) Let $A \in \mathfrak{P}$ and suppose that f is an injective function. Then, $f(A) \subset Y$ and $f^{-1}(f(A)) = A \in \mathfrak{P}$. Hence, $f(A) \in \mathcal{J}_{\mathfrak{P}}$, which implies that $A \in f^{-1}(\mathcal{J}_{\mathfrak{P}})$. Then, $\mathfrak{P} \subseteq f^{-1}(\mathcal{J}_{\mathfrak{P}})$.
(3) Suppose that $A \in \mathcal{J}_{\mathfrak{P}}$. Then, $f^{-1}(A) \in \mathfrak{P}$; hence, $f(f^{-1}(A)) = A \in f(\mathfrak{P})$.
(4) From (2) and (3), we have $\mathcal{J}_{\mathfrak{P}} = f(\mathfrak{P})$. □

Theorem 2.6. *If $f : (\mathcal{T}, \Gamma, \mathfrak{P}) \to (L, \nu, f(\mathfrak{P}))$ is a surjective continuous function and W is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$, then $f(W)$ is a $\mathfrak{P}$-compact subspace of L.*

Proof. Let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be an open cover of $f(W)$. Since f is a continuous function, then $\{f^{-1}(O_\delta)\}_{\delta\in\mathfrak{H}}$ is an open cover of $f^{-1}(f(W))$. As $W \subseteq f^{-1}(f(W))$, then $\{f^{-1}(O_\delta)\}_{\delta\in\mathfrak{H}}$ is an open cover of W. Since W is a $\mathfrak{P}$-compact space, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus \left[W \setminus \bigcup_{\delta\in\mathfrak{H}_0} f^{-1}(O_\delta)\right] \notin \mathfrak{P}$. Then, $f(\mathcal{T}) \setminus \left[f(W) \setminus f(f^{-1}(\bigcup_{\delta\in\mathfrak{H}_0} O_\delta))\right] \notin f(\mathfrak{P})$. Hence, $L \setminus \left[f(W) \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin f(\mathfrak{P})$, since f is a surjective function. Then, $f(W)$ is a $\mathfrak{P}$-compact subspace of L. □

Corollary 2.2. *If $f : (\mathcal{T}, \Gamma, \mathfrak{P}) \to (L, \nu, f(\mathfrak{P}))$ is a surjective continuous function and $(\mathcal{T}, \rho, \mathfrak{P})$ is a $\mathfrak{P}$-compact space, then $(L, \nu, f(\mathfrak{P}))$ is a $\mathfrak{P}$-compact space.*

Definition 2.2. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. A subset A of $\mathcal{T}$ is said to be as follows:*
(1) $\mathfrak{P}$g-closed if $\mathcal{CL}(A) \subseteq U$ whenever $\mathcal{T} \setminus (A \setminus U) = (\mathcal{T} \setminus A) \cup U \notin \mathfrak{P}$ and U is open; and
(2) g-closed if $\mathcal{CL}(A) \subseteq U$ whenever $A \subset U$ and U is open.

From the definition above, we have the following remark.

Remark 2.1.
(1) Every closed set is a g-closed set, but the converse is not true in general.
(2) The concept of $\mathfrak{P}$g-closed depends on the definition of the primal space.

To illustrate Remark 2.1, we present the following examples.

Example 2.4. *Let $\mathcal{T} = \{r, d, b\}$ and let $\rho = \{\mathcal{T}, \emptyset, \{r\}\}$. Consider the set $H = \{d\}$. Then, $H \subseteq U \in \rho$ if and only if $U = \mathcal{T}$; hence, H is g-closed but it is not a closed set since $\mathcal{CL}(H) = \{d, b\} \neq H$.*

Example 2.5. *Let $(\mathcal{T}, \rho)$ and H be defined as in Example 2.4. If $\mathfrak{P} = \{\emptyset\}$, then H is not a $\mathfrak{P}$g-closed since $\mathcal{CL}(H) \nsubseteq \{r\}$, although $(\mathcal{T} \setminus H) \cup \{r\} = \{r, b\} \notin \mathfrak{P}$.*
Now, let $\mathfrak{P} = 2^{\mathcal{T}} \setminus \{\mathcal{T}\}$. Then, H is $\mathfrak{P}$g-closed since $(\mathcal{T} \setminus H) \cup U \notin \mathfrak{P}$ if and only if $U = \mathcal{T}$.

Theorem 2.7. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let A, B be subsets of $\mathcal{T}$ such that $A \subseteq B \subseteq \mathcal{CL}(A)$. Then, the following properties hold:*

(1) If A is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ and $\mathfrak{P}g$-closed, then B is a compact subspace of $\mathcal{T}$; and

(2) If B is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ and A is g-closed, then A is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$.

Proof. (1) Suppose that A is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ and $\mathfrak{P}g$-closed. Let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be any open cover of B. Then, $\{O_\delta\}_{\delta\in\mathfrak{H}}$ is an open cover of A. Since A is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus \left[A \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin \mathfrak{P}$. Since A is $\mathfrak{P}g$-closed, then $\mathcal{CL}(A) \subseteq \bigcup_{\delta\in\mathfrak{H}_0} O_\delta$. Then, $B \subseteq \bigcup_{\delta\in\mathfrak{H}_0} O_\delta$. Therefore, B is a compact subspace of $\mathcal{T}$.

(2) Suppose that B is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ and A is g-closed. Let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be any open cover of A. Now, since $B \subseteq \mathcal{CL}(A)$ and A is a g-closed, then $B \subseteq \mathcal{CL}(A) \subseteq \bigcup_{\delta\in\mathfrak{H}} O_\delta$. Hence, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus \left[B \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin \mathfrak{P}$ because B is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. Then, $\mathcal{T} \setminus \left[A \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin \mathfrak{P}$ since $A \subseteq B$. Therefore, A is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Corollary 2.3. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. If A is $\mathfrak{P}g$-closed and $A \subseteq B \subseteq \mathcal{CL}(A)$, then A is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$ $\Leftrightarrow$ B is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

3. Strongly $\mathfrak{P}$-compact spaces

Definition 3.1. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. We say that $\mathcal{T}$ is a strongly $\mathfrak{P}$-compact space ($S\mathfrak{P}$-compact space) if for every family of open sets $\{O_\delta\}_{\delta\in\mathfrak{H}}$ such that $\bigcup_{\delta\in\mathfrak{H}} O_\delta \notin \mathfrak{P}$, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\bigcup_{\delta\in\mathfrak{H}_0} O_\delta \notin \mathfrak{P}$. A subset K of $\mathcal{T}$ is said to be an $S\mathfrak{P}$-compact subspace of $\mathcal{T}$ if for every family $\{O_\delta\}_{\delta\in\mathfrak{H}}$ of open sets of $\mathcal{T}$ such that $\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] \notin \mathfrak{P}$, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin \mathfrak{P}$.*

Example 3.1. *Let $(\mathbb{R}, \tau_1, \mathfrak{P}_1)$ be a $\mathcal{PS}$ defined in Example 2.1. Let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be any family of open sets. Then,*

Case 1. *$O_\delta = \emptyset$ for every $\delta \in \mathfrak{H}$. Then, since $\mathbb{R} \setminus \left[\mathbb{N} \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] \in \mathfrak{P}_1$, there is nothing to prove.*

Case 2. *$\exists\ \lambda \in \mathfrak{H}$ such that $O_\lambda \neq \emptyset$. Then, $\mathbb{R} \setminus \left[\mathbb{N} \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] \notin \mathfrak{P}_1$. Pick a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\lambda \in \mathfrak{H}_0$. Hence, $\mathbb{R} \setminus \left[\mathbb{N} \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin \mathfrak{P}_1$. Thus, $\mathbb{N}$ is an $S\mathfrak{P}$-compact subspace of $\mathbb{R}$.*

From the definition, it is clear that every S$\mathfrak{P}$-compact is a $\mathfrak{P}$-compact subspace of $\mathcal{T}$. However, this relation is not reversible, which is proven in next example.

Example 3.2. *Let $(\mathbb{R}, \mathcal{F}, \mathfrak{P})$ be as defined in Example 2.3. Consider the family $\mathcal{M} = \{\{x\} : x \in \mathbb{R} \text{ and } x \neq \sqrt{2}\}$. Then, $\bigcup_{x\in\mathbb{R}\setminus\{\sqrt{2}\}}\{x\} = \mathbb{R} \setminus \{\sqrt{2}\} \notin \mathfrak{P}$. Now, let $\{M_i : i \in \{1, ..., n\}\}$ be an arbitrary finite subfamily of $\mathcal{M}$. Then, $\bigcup_{i=1}^{n} M_i \in \mathfrak{P}$. Hence, $\mathbb{R}$ is not an $S\mathfrak{P}$-compact space. Observe that $\mathbb{R}$ is a $\mathfrak{P}$-compact space.*

Example 3.3. *Let $H = \mathbb{R} \times (\mathbb{R}^+ \cup \{0\})$. For $(n, m) \in H$ and $r > 0$. Define the set $M_r(n, m)$ as follows:*

$$M_r(n,m) = \begin{cases} B_r(n,m) & \text{if } r \leq m; \\ \\ B_r(n,r) \cup \{(n,0)\} \cup B_r(0,r), & \text{if } m = 0. \end{cases}$$

Let $\mathfrak{B} = \{M_r(n, m)\}$ be a base for the topology μ on the set H. Then, $(H, \mu, \mathfrak{P})$, where $\mathfrak{P} = \{\emptyset\}$ is a $\mathcal{PS}$. Hence, *(1)* $(H, \mu, \mathfrak{P})$ *is not a compact subspace of* H. *To show that, consider the family* $Q = \{M_1(n, 0)\} \cup \{M_1(n, m) : m \geq 1\}$. *Then,* Q *is an open cover of* H. *Since* $(t, 0) \notin \{M_1(n, m) : m \geq 1\}$ *and* $(t, 0) \in \{M_1(n, 0)\}$ *if and only if* $n = t$, *then the above open cover has no finite subcover. Thus,* H *is not compact.*

(2) $(H, \mu, \mathfrak{P})$ *is an* $S\mathfrak{P}$*-compact subspace of* H *since* $\mathfrak{P} = \emptyset$.

Theorem 3.1. *Let* $(\mathcal{T}, \Gamma, \mathfrak{P})$ *be a* $\mathcal{PS}$ *and let* $K \subseteq \mathcal{T}$. *Consider the family of closed sets* $\{C_\delta\}_{\delta\in\mathfrak{H}}$ *such that* $(\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}}(\mathcal{T} \setminus C_\delta)\right] \notin \mathfrak{P}$. *Then,* K *is an* $S\mathfrak{P}$*-compact subspace of* $\mathcal{T}$ *if and only if there exists a finite set* $\mathfrak{H}_0 \subseteq \mathfrak{H}$ *such that* $(\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}_0}(\mathcal{T} \setminus C_\delta)\right] \notin \mathfrak{P}$.

Proof. Suppose that K is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and let $\{C_\delta\}_{\delta\in\mathfrak{H}}$ be a family of closed sets such that $(\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}}(\mathcal{T} \setminus C_\delta)\right] \notin \mathfrak{P}$. Then,

$$\begin{aligned}
\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}}(\mathcal{T} \setminus C_\delta)\right] &= \mathcal{T} \setminus \left[K \setminus \left(\mathcal{T} \setminus \bigcap_{\delta\in\mathfrak{H}} C_\delta\right)\right] \\
&= \mathcal{T} \setminus \left[K \bigcap \left(\bigcap_{\delta\in\mathfrak{H}} C_\delta\right)\right] \\
&= (\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}} (\mathcal{T} \setminus C_\delta)\right] \notin \mathfrak{P}.
\end{aligned}$$

Since $\mathcal{T} \setminus C_\delta$ is an open set for each $\delta \in \mathfrak{H}$ and K is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that

$$\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}_0}(\mathcal{T} \setminus C_\delta)\right] \notin \mathfrak{P}.$$

Then,

$$\begin{aligned}
\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}_0}(\mathcal{T} \setminus C_\delta)\right] &= \mathcal{T} \setminus \left[K \setminus \left(\mathcal{T} \setminus \bigcap_{\delta\in\mathfrak{H}_0} C_\delta\right)\right] \\
&= (\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\delta\in\mathfrak{H}_0} (\mathcal{T} \setminus C_\delta)\right] \notin \mathfrak{P}.
\end{aligned}$$

Now, suppose that the condition in the theorem holds and let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be a family of open sets such that $\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] \notin \mathfrak{P}$. Then, $\{(\mathcal{T} \setminus O_\delta)\}_{\delta\in\mathfrak{H}}$ is a family of closed sets. Now, we have the following:

$$\begin{aligned}
\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] &= \mathcal{T} \setminus \left[K \bigcap \left(\mathcal{T} \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right)\right] \\
&= \mathcal{T} \setminus \left[K \bigcap \left(\bigcap_{\delta\in\mathfrak{H}} (\mathcal{T} \setminus O_\delta)\right)\right] = (\mathcal{T} \setminus K) \bigcup \left(\bigcup_{\delta\in\mathfrak{H}} O_\delta\right) \notin \mathfrak{P}.
\end{aligned}$$

Thus, there is a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that

$$(\mathcal{T} \setminus K) \bigcup \left(\bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right) \notin \mathfrak{P}.$$

Therefore, we have the following:

$$\begin{aligned}\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right] &= \mathcal{T} \setminus \left[K \bigcap \left(\mathcal{T} \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right) \right] \\ &= \mathcal{T} \setminus \left[K \bigcap \left(\bigcap_{\delta \in \mathfrak{H}_0} (\mathcal{T} \setminus O_\delta) \right) \right] \\ &= (\mathcal{T} \setminus K) \bigcup \left(\bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right) \notin \mathfrak{P}.\end{aligned}$$

This shows that K is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Corollary 3.1. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let $\{H_\eta\}_{\eta \in \mathfrak{H}}$ be a collection of closed sets such that $\bigcup_{\eta \in \mathfrak{H}} (\mathcal{T} \setminus H_\eta) \notin \mathfrak{P}$. Then, $(\mathcal{T}, \Gamma, \mathfrak{P})$ is an S$\mathfrak{P}$-compact space if and only if there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\bigcup_{\eta \in \mathfrak{H}_0} (\mathcal{T} \setminus H_\eta) \notin \mathfrak{P}$.*

Theorem 3.2. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. If A is $\mathfrak{P}g$-closed and $A \subseteq B \subseteq \mathcal{CL}(A)$, then A is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$ if and only if B is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. (1) Let A be an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and let $\{O_\delta\}_{\delta \in \mathfrak{H}}$ be a family of open sets such that $\mathcal{T} \setminus \left[B \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}$. Then, since $A \subseteq B$, we have $\mathcal{T} \setminus \left[A \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}$; then, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus \left[A \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right] \notin \mathfrak{P}$ because A is an S$\mathfrak{P}$-compact subspace. Now, as A is $\mathfrak{P}g$-closed, we have $\mathcal{CL}(A) \subseteq \bigcup_{\delta \in \mathfrak{H}_0} O_\delta$. Then, $\mathcal{T} \setminus \left[B \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right] = \mathcal{T} \notin \mathfrak{P}$. Hence, B is an S$\mathfrak{P}$-compact subspace.

(2) Let B be an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and let $\{O_\delta\}_{\delta \in \mathfrak{H}}$ be a family of open sets such that $\mathcal{T} \setminus \left[A \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}$. Since A is $\mathfrak{P}g$-closed, then $\mathcal{CL}(A) \subseteq \bigcup_{\delta \in \mathfrak{H}} O_\delta$. As $A \subseteq B \subseteq \mathcal{CL}(A)$, then $B \subseteq \bigcup_{\delta \in \mathfrak{H}} O_\delta$, which implies that $\mathcal{T} \setminus \left[B \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}$. Since B is an S$\mathfrak{P}$-compact space, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus \left[B \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right] \notin \mathfrak{P}$. Therefore, $\mathcal{T} \setminus \left[A \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta \right] \notin \mathfrak{P}$, which implies that A is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Theorem 3.3. *Let $(\mathcal{T}, \Gamma, \mathfrak{P})$ be a $\mathcal{PS}$. If $R, K \subseteq \mathcal{T}$ are both S$\mathfrak{P}$-compact subspaces of $\mathcal{T}$, then $R \cup K$ is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{O_\delta\}_{\delta \in \mathfrak{H}}$ be a family of open sets such that

$$\mathcal{T} \setminus \left[(R \cup K) \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}.$$

Then, $\mathcal{T} \setminus \left[R \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}$ and $\mathcal{T} \setminus \left[K \setminus \bigcup_{\delta \in \mathfrak{H}} O_\delta \right] \notin \mathfrak{P}$. Since R and K are both S$\mathfrak{P}$-compact, then there exist two finite sets $\mathfrak{H}_0 \subseteq \mathfrak{H}$ and $\mathfrak{H}_1 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus [R \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta] \notin \mathfrak{P}$ and $\mathcal{T} \setminus [K \setminus \bigcup_{\delta \in \mathfrak{H}_1} O_\delta] \notin \mathfrak{P}$, respectively. Hence, $[\mathcal{T} \setminus (R \setminus \bigcup_{\delta \in \mathfrak{H}_0} O_\delta)] \bigcap [\mathcal{T} \setminus (K \setminus \bigcup_{\delta \in \mathfrak{H}_1} O_\delta)] \notin \mathfrak{P}$. Thus, $\mathcal{T} \setminus \left[(R \cup K) \setminus \bigcup_{\delta \in \mathfrak{H}_0 \cup \mathfrak{H}_1} O_\delta \right] \notin \mathfrak{P}$, which implies that $R \cup T$ is an S$\mathfrak{P}$-compact space. □

Theorem 3.4. *Let $(\mathcal{T}, \Gamma, \mathfrak{P})$ be a $\mathcal{PS}$ and R, K be subsets of $\mathcal{T}$. If R is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and K is a closed set, then $R \cap K$ is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be a family of open sets such that

$$\mathcal{T} \setminus \left[(R \cap K) \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] \notin \mathfrak{P}.$$

Then, $[\mathcal{T} \setminus (R \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta)] \bigcup [\mathcal{T} \setminus (K \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta)] \notin \mathfrak{P}$. Let $G = \mathcal{T} \setminus \left[K \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right]$. Then, G is an open set. Since $\mathcal{T} \setminus [R \setminus (\bigcup_{\delta\in\mathfrak{H}} O_\delta \cup G)] \notin \mathfrak{P}$ and R is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite set $\{O_i\}_{i=1}^n \subseteq \{G, O_\delta : \delta \in \mathfrak{H}\}$ such that $\mathcal{T} \setminus [R \setminus \bigcup_{i=1}^n O_i] \notin \mathfrak{P}$. Now, since $\mathcal{T} \setminus [R \setminus \bigcup_{i=1}^n O_i] \subseteq \mathcal{T} \setminus [(R \cap K) \setminus \bigcup_{i=1}^n O_i]$, then $\mathcal{T} \setminus [(R \cap K) \setminus \bigcup_{i=1}^n O_i] \notin \mathfrak{P}$, which implies that $R \cap K$ is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Corollary 3.2. *Let $(\mathcal{T}, \Gamma, \mathfrak{P})$ be an S$\mathfrak{P}$-compact space and B be a closed set. Then, B is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Theorem 3.5. *If $h : (\mathcal{T}, \Gamma, \mathfrak{P}) \to (L, \nu, h(\mathfrak{P}))$ is a bijective continuous function and Q is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then $h(Q)$ is an S$\mathfrak{P}$-compact subspace of L.*

Proof. Suppose that $\{W_\eta\}_{\eta\in\mathfrak{H}}$ is a family of open sets such that

$$L \setminus \left[h(Q) \setminus \bigcup_{\eta\in\mathfrak{H}} W_\eta\right] \notin h(\mathfrak{P}).$$

Then, $h^{-1}(L) \setminus [h^{-1}(h(Q)) \setminus \bigcup_{\eta\in\mathfrak{H}} h^{-1}(W_\eta)] \notin \mathfrak{P}$. Hence, $\mathcal{T} \setminus [Q \setminus \bigcup_{\eta\in\mathfrak{H}} h^{-1}(W_\eta)] \notin \mathfrak{P}$, and $\{h^{-1}(W_\eta)\}_{\eta\in\mathfrak{H}}$ is a family of open sets in $\mathcal{T}$ since h is a continuous function. Therefore, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus [Q \setminus \bigcup_{\eta\in\mathfrak{H}_0} h^{-1}(W_\eta)] \notin \mathfrak{P}$, which implies that $L \setminus [h(Q) \setminus \bigcup_{\eta\in\mathfrak{H}_0} W_\eta] \notin h(\mathfrak{P})$. Hence, $h(Q)$ is an S$\mathfrak{P}$-compact subspace of L. □

Corollary 3.3. *If $d : (\mathcal{T}, \Gamma, \mathfrak{P}) \to (L, \nu, d(\mathfrak{P}))$ is a bijective continuous function and $\mathcal{T}$ is an S$\mathfrak{P}$-compact space, then $(L, \nu, d(\mathfrak{P}))$ is an S$\mathfrak{P}$-compact space.*

Theorem 3.6. *If $\hbar : (\mathcal{T}, \Gamma, \mathfrak{P}) \to (L, \nu, \mathcal{J}_{\mathfrak{P}})$ is a continuous bijective function and Q is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then $\hbar(Q)$ is an S$\mathfrak{P}$-compact subspace of L.*

Proof. Let $\{O_\delta\}_{\delta\in\mathfrak{H}}$ be a family of open sets such that

$$L \setminus \left[\hbar(Q) \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right] \notin \mathcal{J}_{\mathfrak{P}}.$$

Then, $\hbar^{-1}\left(L \setminus \left[\hbar(Q) \setminus \bigcup_{\delta\in\mathfrak{H}} O_\delta\right]\right) \notin \mathfrak{P}$. Therefore, $\mathcal{T} \setminus [Q \setminus \bigcup_{\delta\in\mathfrak{H}} \hbar^{-1}(O_\delta)] \notin \mathfrak{P}$. Since Q is an S$\mathfrak{P}$-compact subspace, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \setminus [Q \setminus \bigcup_{\delta\in\mathfrak{H}_0} \hbar^{-1}(O_\delta)] \notin \mathfrak{P}$. Hence,

$$L \setminus \left[\hbar(Q) \setminus \bigcup_{\delta\in\mathfrak{H}_0} O_\delta\right] \notin \mathcal{J}_{\mathfrak{P}}.$$

□

Corollary 3.4. *If $\hbar : (\mathcal{T}, \Gamma, \mathfrak{P}) \to (\mathcal{R}, \nu, \mathcal{J}_{\mathfrak{P}})$ is a bijective continuous function and $\mathcal{T}$ is an S$\mathfrak{P}$-compact space, then $(\mathcal{R}, \nu, \mathcal{J}_{\mathfrak{P}})$ is an S$\mathfrak{P}$-compact space.*

4. Super $\mathfrak{P}$-compact spaces

Definition 4.1. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$. We say that $(\mathcal{T}, \rho, \mathfrak{P})$ is a super $\mathfrak{P}$-compact space (SU$\mathfrak{P}$-compact space) if for every family of open sets $\{V_\eta\}_{\eta\in\mathfrak{H}}$ such that $\bigcup_{\eta\in\mathfrak{H}} V_\eta \notin \mathfrak{P}$, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $\mathcal{T} \subseteq \bigcup_{\eta\in\mathfrak{H}_0} V_\eta$. Let $A \subseteq \mathcal{T}$. Then, A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$ if for every family of open sets $\{V_\eta\}_{\eta\in\mathfrak{H}}$ such that $\mathcal{T} \setminus \left[A \setminus \bigcup_{\eta\in\mathfrak{H}} V_\eta\right] \notin \mathfrak{P}$, then there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $A \subseteq \bigcup_{\eta\in\mathfrak{H}_0} V_\eta$.*

Example 4.1. *Let $(\mathbb{R}, \Gamma_\mathbb{P}, \mathfrak{P})$, where $\mathbb{P}$ is the set of irrational numbers, be defined as follows: $U \in \Gamma_\mathbb{P}$ if and only if either $U \cap \mathbb{P} = \emptyset$ or $U = \mathbb{R}$ and $U \in \mathfrak{P}$ if and only if $\sqrt{2} \notin U$. Let $\{W_\eta\}_{\eta\in\mathfrak{H}}$ be any family of open sets such that $\bigcup_{\eta\in\mathfrak{H}} W_\eta \notin \mathfrak{P}$. Then, $\sqrt{2} \in \bigcup_{\eta\in\mathfrak{H}} W_\eta$, which implies that $\exists\ \gamma \in \mathfrak{H}$ such that $W_\gamma = \mathbb{R}$. Therefore, $(\mathbb{R}, \Gamma_\mathbb{P}, \mathfrak{P})$ is an SU$\mathfrak{P}$-compact space.*

Remark 4.1. *From the Definition 4.1, it is obvious that every SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$ is a compact subspace. Indeed, let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let $A \subseteq \mathcal{T}$ be an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$. Assume that $\{W_\eta\}_{\eta\in\mathfrak{H}}$ is an open cover of $A \subseteq \mathcal{T}$. Then, $\mathcal{T} \setminus [A \setminus \bigcup_{\eta\in\mathfrak{H}} W_\eta] = \mathcal{T} \notin \mathfrak{P}$. Hence, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $A \subseteq \bigcup_{\eta\in\mathfrak{H}_0} W_\eta$.*

The following example shows that not every compact space is an SU$\mathfrak{P}$-compact space.

Example 4.2. *Let $(\mathbb{R}, \rho_0, \mathfrak{P})$ be defined as follows: $U \in \rho_0$ if and only if either $0 \notin U$ or $U = \mathbb{R}$, and let $\mathfrak{P}$ be defined as in Example 2.2. Then, $\mathcal{V} = \{\{x\} : x \in \mathbb{R}$ and $x \neq 0\}$ is a family of open sets such that $\bigcup_{x\in\mathbb{R}\setminus\{0\}}\{x\} = \mathbb{R} \setminus \{0\} \notin \mathfrak{P}$. However, if $\mathcal{V}_0$ is any finite subfamily of $\mathcal{V}$, then $\mathbb{R} \nsubseteq \bigcup_{V\in\mathcal{V}_0} V$. Hence, $(\mathbb{R}, \rho_0, \mathfrak{P})$ is an example of a compact space that is not an SU$\mathfrak{P}$-compact space.*

On the other hand, every SU$\mathfrak{P}$-compact space is an S$\mathfrak{P}$-compact space. However, not every S$\mathfrak{P}$-compact space is an SU$\mathfrak{P}$-compact space, as shown in the following example.

Example 4.3. *Consider $(\mathbb{R}, \tau_1, \mathfrak{P}_1)$ that is defined in Example 2.1. In Example 3.1, we proved that $(\mathbb{R}, \tau_1, \mathfrak{P}_1)$ is an S$\mathfrak{P}$-compact space. Consider the family of open sets $\mathcal{V} = \{V_t = \{1, t\} : t \in \mathbb{N}\}$. Let $\mathcal{V}_0$ be any finite subfamily of $\mathcal{V}$. Then, $\bigcup_{V\in\mathcal{V}_0} V = \{1, t_1, t_2, ..., t_k\}$ for some $k \in \mathbb{N}$ and $\mathbb{N} \nsubseteq \bigcup_{V\in\mathcal{V}_0} V$. Hence, $\mathbb{N}$ is not an SU$\mathfrak{P}$-compact space.*

Theorem 4.1. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let $K \subseteq \mathcal{T}$. Suppose that $\{E_\eta\}_{\eta\in\mathfrak{H}}$ is a collection of closed sets such that $(\mathcal{T} \setminus K) \bigcup \left[\bigcup_{\eta\in\mathfrak{H}}(\mathcal{T} \setminus E_\eta)\right] \notin \mathfrak{P}$. Then, K is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$ if and only if there exists a finite subset $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $K \cap \left[\bigcap_{\eta\in\mathfrak{H}_0} E_\eta\right] = \emptyset$.*

Proof. First: Suppose that K is an SU$\mathfrak{P}$-compact space. Let $\{E_\eta\}_{\eta\in\mathfrak{H}}$ be a collection of closed sets of $\mathcal{T}$ such that

$$[\mathcal{T} \setminus K] \bigcup \left[\bigcup_{\eta\in\mathfrak{H}} (\mathcal{T} \setminus E_\eta)\right] \notin \mathfrak{P}.$$

$$\mathcal{T}\setminus\left[K\setminus\bigcup_{\eta\in\mathfrak{H}}(\mathcal{T}\setminus E_\eta)\right]=\mathcal{T}\setminus\left[K\setminus\left(\mathcal{T}\setminus\bigcap_{\eta\in\mathfrak{H}}E_\eta\right)\right]$$
$$=\mathcal{T}\setminus\left[K\bigcap\left(\bigcap_{\eta\in\mathfrak{H}}E_\eta\right)\right]$$
$$=(\mathcal{T}\setminus K)\bigcup\left[\bigcup_{\eta\in\mathfrak{H}}\left(\mathcal{T}\setminus E_\eta\right)\right]\notin\mathfrak{P}.$$

Since K is an SU$\mathfrak{P}$-compact subspace and $\{\mathcal{T}\setminus E_\eta\}_{\eta\in\mathfrak{H}}$ is a family of open sets, then $K\subseteq\bigcup_{\eta\in\mathfrak{H}_0}\left(\mathcal{T}\setminus E_\eta\right)$. Hence, $K\cap\left(\bigcap_{\eta\in\mathfrak{H}_0}E_\eta\right)=\emptyset$.

Second: Suppose that the condition in the theorem holds and let $\{W_\eta\}_{\eta\in\mathfrak{H}}$ be a family of open sets such that $\mathcal{T}\setminus\left[K\setminus\bigcup_{\eta\in\mathfrak{H}}W_\eta\right]\notin\mathfrak{P}$. Then, $\{\mathcal{T}\setminus W_\eta\}_{\eta\in\mathfrak{H}}$ is a family of closed sets; hence,

$$\mathcal{T}\setminus\left[K\setminus\bigcup_{\eta\in\mathfrak{H}}W_\eta\right]=(\mathcal{T}\setminus K)\bigcup\left(\bigcup_{\eta\in\mathfrak{H}}W_\eta\right)\notin\mathfrak{P}.$$

Thus, there exists a finite set $\mathfrak{H}_0\subseteq\mathfrak{H}$ such that

$$K\cap\left(\bigcap_{\eta\in\mathfrak{H}_0}(\mathcal{T}\setminus W_\eta)\right)=\emptyset.$$

Hence, $K\subseteq\bigcup_{\eta\in\mathfrak{H}_0}W_\eta$. This shows that $(\mathcal{T},\rho,\mathfrak{P})$ is an SU$\mathfrak{P}$-compact space. □

Corollary 4.1. *Let $(\mathcal{T},\rho,\mathfrak{P})$ be a $\mathcal{PS}$ and $\{E_\eta\}_{\eta\in\mathfrak{H}}$ be a collection of closed sets such that $\bigcup_{\eta\in\mathfrak{H}}(\mathcal{T}\setminus E_\eta)\notin\mathfrak{P}$. Then, $(\mathcal{T},\rho,\mathfrak{P})$ is an SU$\mathfrak{P}$-compact space if and only if there exists a finite subset $\mathfrak{H}_0\subseteq\mathfrak{H}$ such that $\bigcap_{\eta\in\mathfrak{H}_0}E_\eta=\emptyset$.*

Theorem 4.2. *Let $(\mathcal{T},\rho,\mathfrak{P})$ be a $\mathcal{PS}$ and $A,B\subseteq\mathcal{T}$ such that $A\subseteq B\subseteq\mathcal{CL}(A)$. Then, the following properties hold:*

(1) If A is an SU$\mathfrak{P}$-compact subspace and g-closed, then B is an SU$\mathfrak{P}$-compact subspace.

(2) If A is an S$\mathfrak{P}$-compact subspace and $\mathfrak{P}$g-closed, then B is an SU$\mathfrak{P}$-compact subspace.

(3) If B is a compact subspace and A is $\mathfrak{P}$g-closed, then A is an SU$\mathfrak{P}$-compact subspace.

Proof. (1) Suppose that A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and g-closed. Let $\{V_\eta\}_{\eta\in\mathfrak{H}}$ be a family of open sets such that $\mathcal{T}\setminus\left[B\setminus\bigcup_{\eta\in\mathfrak{H}}V_\eta\right]\notin\mathfrak{P}$. Then, $\mathcal{T}\setminus\left[A\setminus\bigcup_{\eta\in\mathfrak{H}}V_\eta\right]\notin\mathfrak{P}$. Since A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite subset $\mathfrak{H}_0\subseteq\mathfrak{H}$ such that $A\subseteq\bigcup_{\eta\in\mathfrak{H}_0}V_\eta$. Since A is g-closed, then $\mathcal{CL}(A)\subseteq\bigcup_{\eta\in\mathfrak{H}_0}V_\eta$. Hence, $B\subseteq\bigcup_{\eta\in\mathfrak{H}_0}V_\eta$. Therefore, B is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$.

(2) Suppose that A is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and $\mathfrak{P}g$-closed. Let $\{V_\eta\}_{\eta\in\mathfrak{H}}$ be a family of open sets such that $\mathcal{T}\setminus\left[B\setminus\bigcup_{\eta\in\mathfrak{H}}V_\eta\right]\notin\mathfrak{P}$. Then, $\mathcal{T}\setminus\left[A\setminus\bigcup_{\eta\in\mathfrak{H}}V_\eta\right]\notin\mathfrak{P}$. Since A is an S$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite set $\mathfrak{H}_0\subseteq\mathfrak{H}$ such that $\mathcal{T}\setminus\left[A\setminus\bigcup_{\eta\in\mathfrak{H}_0}V_\eta\right]\notin\mathfrak{P}$. Therefore, $\mathcal{CL}(A)\subseteq\bigcup_{\eta\in\mathfrak{H}_0}V_\eta$ because A is $\mathfrak{P}g$-closed. Thus, $B\subseteq\bigcup_{\eta\in\mathfrak{H}_0}V_\eta$. Hence, B is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$.

(3) Suppose that B is a compact subspace of $\mathcal{T}$ and A is $\mathfrak{P}g$-closed. Let $\{V_\eta\}_{\eta\in\mathfrak{H}}$ be any family of open sets such that $\mathcal{T} \setminus \left[A \setminus \bigcup_{\eta\in\mathfrak{H}} V_\eta\right] \notin \mathfrak{P}$. Since A is $\mathfrak{P}g$-closed, then we have $B \subseteq \mathcal{CL}(A) \subseteq \bigcup_{\eta\in\mathfrak{H}} V_\eta$. Hence, there exists a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$ such that $B \subseteq \bigcup_{\eta\in\mathfrak{H}_0} V_\eta$. Then, $A \subseteq \bigcup_{\eta\in\mathfrak{H}_0} V_\eta$, which implies that A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Corollary 4.2. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let A be $\mathfrak{P}g$-closed such that $A \subseteq B \subseteq \mathcal{CL}(A)$. Then, A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$ if and only if B is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Theorem 4.3. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let $A, B \subseteq \mathcal{T}$ both be SU$\mathfrak{P}$-compact subspaces of $\mathcal{T}$. Then, $A \cup B$ is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{O_\eta\}_{\eta\in\mathfrak{H}}$ be any family of open sets such that

$$\mathcal{T} \setminus \left[(A \cup B) \setminus \bigcup_{\eta\in\mathfrak{H}} O_\eta\right] \notin \mathfrak{P}.$$

Then, $\mathcal{T} \setminus \left[A \setminus \bigcup_{\eta\in\mathfrak{H}} O_\eta\right] \notin \mathfrak{P}$ and $\mathcal{T} \setminus \left[B \setminus \bigcup_{\eta\in\mathfrak{H}} O_\eta\right] \notin \mathfrak{P}$. Since A and B are both SU$\mathfrak{P}$-compact subspaces of $\mathcal{T}$, then there exist finite subsets of $\mathfrak{H}$, namely $\mathfrak{H}_A$ and $\mathfrak{H}_B$, such that $A \subseteq \bigcup_{\eta\in\mathfrak{H}_A} O_\eta$ and $B \subseteq \bigcup_{\eta\in\mathfrak{H}_B} O_\eta$. Hence, $A \cup B \subseteq \bigcup_{\eta\in\mathfrak{H}_A\cup\mathfrak{H}_B} O_\eta$. This shows that $A \cup B$ is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Theorem 4.4. *Let $(\mathcal{T}, \rho, \mathfrak{P})$ be a $\mathcal{PS}$ and let $A, B \subseteq \mathcal{T}$. If A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$ and B is closed, then $A \cap B$ is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Proof. Let $\{W_\delta\}_{\delta\in\mathfrak{H}}$ be a family of open sets such that

$$\mathcal{T} \setminus \left[(A \cap B) \setminus \bigcup_{\delta\in\mathfrak{H}} W_\delta\right] \notin \mathfrak{P}.$$

Then, $\{W_\delta\}_{\delta\in\mathfrak{H}} \cup \{\mathcal{T} \setminus B\}$ is a family of open sets such that

$$\mathcal{T} \setminus \left[A \setminus \left[(\mathcal{T} \setminus B) \bigcup \left(\bigcup_{\delta\in\mathfrak{H}} W_\delta\right)\right]\right] \notin \mathfrak{P}.$$

Since A is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then there exists a finite subfamily $\mathcal{W} = \{W_i\}_{i=1}^n \subseteq \{W_\delta : \delta \in \mathfrak{H}\} \cup \{\mathcal{T} \setminus B\}$ such that $A \subseteq \bigcup_{i=1}^n W_i$. Then, $A \cap B \subseteq \bigcup_{i=1}^n W_i$. This shows that $A \cap B$ is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$. □

Corollary 4.3. *If $(\mathcal{T}, \rho, \mathfrak{P})$ is an SU$\mathfrak{P}$-compact space and $B \subseteq \mathcal{T}$ is closed, then B is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$.*

Theorem 4.5. *If $\hbar : (\mathcal{T}, \Lambda, \mathfrak{P}) \to (L, \Gamma, \hbar(\mathfrak{P}))$ is a bijective continuous function and Q is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then $\hbar(Q)$ is an SU$\mathfrak{P}$-compact subspace of L.*

Proof. Let $\{V_\lambda\}_{\lambda\in\mathfrak{H}}$ be a family of open sets such that

$$L \setminus \left[\hbar(Q) \setminus \bigcup_{\lambda\in\mathfrak{H}} V_\lambda\right] \notin \hbar(\mathfrak{P}).$$

Then, $\mathcal{T} \setminus [Q \setminus \bigcup_{\lambda\in\mathfrak{H}} \hbar^{-1}(V_\lambda)] \notin \mathfrak{P}$. Hence, $Q \subseteq \bigcup_{\lambda\in\mathfrak{H}_0} \hbar^{-1}(V_\lambda)$ for a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$. Thus, $\hbar(Q) \subseteq \bigcup_{\lambda\in\mathfrak{H}_0} V_\lambda$, which implies that $\hbar(Q)$ is an SU$\mathfrak{P}$-compact subspace of L. □

Corollary 4.4. *If $\hbar : (\mathcal{T}, \Lambda, \mathfrak{P}) \to (L, \Gamma, \hbar(\mathfrak{P}))$ is a bijective continuous function and $(\mathcal{T}, \Lambda, \mathfrak{P})$ is an SU$\mathfrak{P}$-compact space, then $(L, \Gamma, \hbar(\mathfrak{P}))$ is an SU$\mathfrak{P}$-compact space.*

Theorem 4.6. *If $\hbar : (\mathcal{T}, \Lambda, \mathfrak{P}) \to (L, \Gamma, \mathcal{J}_{\mathfrak{P}})$ is a surjective continuous function and Q is an SU$\mathfrak{P}$-compact subspace of $\mathcal{T}$, then $\hbar(Q)$ is an SU$\mathfrak{P}$-compact subspace of L.*

Proof. Suppose that $\{V_\delta\}_{\delta\in\mathfrak{H}}$ is a family of open sets such that

$$L \setminus \left[\hbar(Q) \setminus \bigcup_{\delta\in\mathfrak{H}} V_\delta\right] \notin \mathcal{J}_{\mathfrak{P}}.$$

Then, $\mathcal{T} \setminus [Q \setminus \bigcup_{\delta\in\mathfrak{H}} \hbar^{-1}(V_\delta)] \notin \mathfrak{P}$. Hence, $Q \subseteq \bigcup_{\delta\in\mathfrak{H}_0} \hbar^{-1}(V_\delta)$ for a finite set $\mathfrak{H}_0 \subseteq \mathfrak{H}$. Therefore, $\hbar(Q) \subseteq \bigcup_{\delta\in\mathfrak{H}_0} V_\delta$, which implies that $\hbar(Q)$ is an SU$\mathfrak{P}$-compact subspace. □

Corollary 4.5. *If $f : (\mathcal{T}, \rho, \mathfrak{P}) \to (L, \nu, \mathcal{J}_{\mathfrak{P}})$ is a surjective continuous function and $(\mathcal{T}, \rho, \mathfrak{P})$ is an SU$\mathfrak{P}$-compact space, then $(L, \nu, \mathcal{J}_{\mathfrak{P}})$ is an SU$\mathfrak{P}$-compact space.*

Example 4.4. *Let $(\mathbb{R}, \mathcal{U}, \mathfrak{P})$ be defined as follows:*

$$T \in \mathfrak{P} \text{ if and only if } 0 \notin T,$$

$$W \in \mathcal{U} \text{ if and only if } W = \emptyset \text{ or } \forall\, r \in W\ \exists\, (a, b) \text{ such that } r \in (a, b) \subseteq W,$$

see Example 28 [11]. If $\{V_\delta\}_{\delta\in\mathfrak{H}}$ is a family of open sets, then we have the following two cases:
Case 1. *$0 \notin V_\delta$ for every $\delta \in \mathfrak{H}$. Then, there is nothing to prove since $\bigcup_{\delta\in\mathfrak{H}} V_\delta \in \mathfrak{P}$.*
Case 2. *There exists $\lambda \in \mathfrak{H}$ such that $0 \in V_\lambda$. Then, $V_\lambda \notin \mathfrak{P}$. Hence, $(\mathbb{R}, \mathcal{U}, \mathfrak{P})$ is an S$\mathfrak{P}$-compact space, which implies that $(\mathbb{R}, \mathcal{U}, \mathfrak{P})$ is a $\mathfrak{P}$-compact space.*

Consider the family $\mathcal{V} = \{V_n = (-n, n) : n \in \mathbb{N}\}$. Then, $\bigcup_{n\in\mathbb{N}} V_n = \mathbb{R} \notin \mathfrak{P}$. Let $\mathcal{V}_0 = \{V_k = (-k, k) : k \leq m, k \in \mathbb{N}\} \subseteq \mathcal{V}$ for some $m \in \mathbb{N}$. Then, since $\mathbb{R} \nsubseteq \bigcup_{k\leq m} V_k$, $(\mathbb{R}, \mathcal{U}, \mathfrak{P})$ is not an SU$\mathfrak{P}$-compact space.

Remark 4.2. We have the following relationships:

$$\begin{array}{ccc} \text{SU}\mathfrak{P}\text{-compact space} & \Rightarrow & \text{S}\mathfrak{P}\text{-compact space} \\ \Downarrow & & \Downarrow \\ \text{compact space} & \Rightarrow & \mathfrak{P}\text{-compact space} \end{array}$$

5. Conclusions

In this work, we introduced new notions using a primal structure. We started by providing a definition of $\mathfrak{P}$-compactness. Then, we proposed a definition of another concept called strongly $\mathfrak{P}$-compactness (S$\mathfrak{P}$-compactness) and observed that every S$\mathfrak{P}$-compact space is a $\mathfrak{P}$-compact space. A counterexample was discussed to show the converse of that relation is not necessary true. Furthermore, we defined super $\mathfrak{P}$-compact spaces (SU$\mathfrak{P}$-compact spaces). Additionally, more counterexamples and results were presented to illustrate the relations between SU$\mathfrak{P}$-compactness, S$\mathfrak{P}$-compactness, $\mathfrak{P}$-compactness, and compactness. It is worth noting that the primal structure was considered in both fuzzy and soft theories, as discussed in [12, 13]. In future work, we aim to define the concepts of $\mathfrak{P}$-compactness, S$\mathfrak{P}$-compactness, and SU$\mathfrak{P}$-compactness within the framework of a fuzzy primal structure.

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Conflict of interest

The author declares that they have no conflict of interest to report regarding the publication of this article.

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