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Research article

Smoothing properties of the fractional Gauss-Weierstrass semi-group in Morrey smoothness spaces

Franka Baaske[1,*], **Romaric Kana Nguedia**[2] **and Hans-Jürgen Schmeißer**[2]

[1] University of Applied Sciences Mittweida, Faculty Applied Computer Sciences & Biosciences, Technikumplatz 17, 09648 Mittweida, Germany

[2] Friedrich Schiller University Jena, Faculty of Mathematics and Computer Sciences, Ernst-Abbe Platz 2, 07743 Jena, Germany

* **Correspondence:** Email: baaske@hs-mittweida.de; Tel: +493727581326.

Abstract: In this paper we derive caloric smoothing estimates in Morrey smoothness spaces using decomposition techniques by means of wavelets and molecules. Our new estimate extends results for Gauss-Weierstrass, Cauchy-Poisson and fractional Gauss-Weierstrass semigroups.

Keywords: Morrey smoothness spaces of Besov and Triebel-Lizorkin type; caloric smoothing; wavelets; molecules; fractional Gauss-Weierstrass semi-group
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1. Introduction

The aim of this paper is to derive a caloric smoothing estimate of the fractional Gauss-Weierstrass semigroup in Morrey smoothness spaces of Besov and Triebel-Lizorkin type. More precisely, we are interested in the inequality

$$\|W_t^\alpha \omega \,|\rho\text{-}A^{s+d}_{p,q}(\mathbb{R}^n)\| \le C t^{-\frac{d}{2\alpha}} \|\, \omega \,|\rho\text{-}A^{s}_{p,q}(\mathbb{R}^n)\|, \tag{1.1}$$

where $0 < t \le 1$, $d \ge 0$. Here, $\rho\text{-}A^s_{p,q}(\mathbb{R}^n)$ denotes the so-called ρ-clan of Morrey smoothness spaces where $A = B$ stands for spaces of Besov type and $A = F$ for spaces of Triebel-Lizorkin type. These spaces were introduced in [10] and provide a unified approach to several types of Morrey(-Campanato) spaces, global and hybrid spaces; see Subsection 2.2 below. Estimates of type (1.1) play a significant role in the analysis of evolution equations such as (nonlinear) heat, Burgers or Navier-Stokes equations. Due to these applications, we restrict our consideration to Banach spaces, hence, to $s \in \mathbb{R}$, $1 \le p < \infty$, and $1 \le q \le \infty$. Moreover, we focus on $-n < \rho < 0$; see explanations in Remark 3. Further, W_t^α

denotes the fractional Gauss-Weierstrass semigroup, formally defined as

$$W_t^\alpha \omega(x) = (e^{-t|\xi|^{2\alpha}}\widehat{\omega})^\vee(x), \quad \omega \in \rho\text{-}A^s_{p,q}(\mathbb{R}^n), \quad \alpha > 0, \tag{1.2}$$

where $\wedge$ and $\vee$ stand for the Fourier transform and its inverse, respectively. Estimate (1.1) extends known results with respect to considered function spaces and gives an alternative proof concerning α compared with former results. Concerning the global spaces $A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$, we refer to [1] for $\alpha \in \mathbb{N}$ based on the ideas for $\alpha = 1$ developed in [25]. The case $\alpha > 0$ has been considered in [11, 13]. An alternative proof was presented in [4] using characterizations of the spaces in terms of fractional Gauss-Weierstrass semigroups. For local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, we refer again to [25] and for hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ to [26] (both consider $\alpha = 1$). As mentioned above, the definition of W_t^α in terms of a convolution is rather formal due to the fact that the fractional heat kernel is not smooth in $\xi = 0$. More precisely, $e^{-|\xi|^{2\alpha}}$ does not belong to the Schwartz space of rapidly decreasing functions but still has a sufficiently fast polynomial decay; see [17] Lemmas 1 and 2. Our proof of (1.1) combines the results presented in [1, 17] using characterizations by means of wavelets and molecules adapted to Morrey smoothness spaces. The paper is structured as follows. Section 2 provides necessary notation, definitions of global function spaces, a Fourier analytic approach to Morrey smoothness spaces and their characterization in terms of wavelets and molecules. Moreover, we recall useful embeddings and coincidences of related spaces. Our main result is contained in Theorem 1 in Section 3. We start by introducing the fractional Gauss-Weierstrass semigroup and provide estimates of their fractional derivatives which turn out to be our main tool besides decomposition techniques of $\rho\text{-}A^s_{p,q}(\mathbb{R}^n)$-spaces. Finally, Section 4 sketches some application.

2. Preliminaries

2.1. Notation and basic definitions

Let $\mathbb{R}^n$ be the Euclidean n - space with $n \in \mathbb{N}$ where $\mathbb{N}$ indicates the collection of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. $S(\mathbb{R}^n)$ denotes the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ and $S'(\mathbb{R}^n)$ its dual, the space of all tempered distributions. Furthermore, let $L_p(\mathbb{R}^n)$ with $0 < p < \infty$ be the standard complex quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p}$$

with the natural modification if $p = \infty$. Similarly, we define $L_p^{\mathrm{loc}}(\mathbb{R}^n)$ which consists of all f whose restriction to bounded Lebesgue measurable sets $M \subset \mathbb{R}^n$ belongs to $L_p(M)$. As usual, $\mathbb{Z}$ is the collection of all integers and $\mathbb{Z}^n$ defines the set of all lattice points $m = (m_1, \ldots, m_n)$, $m_i \in \mathbb{Z}$. If $\phi \in S(\mathbb{R}^n)$, then

$$\widehat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\phi(x)\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{2.1}$$

denotes the Fourier transform of ϕ. $\mathcal{F}^{-1}\phi$ and $\phi^\vee$ stand for the inverse Fourier transform, given by the righthand side of (2.1) with i in place of $-i$, and $x\xi$ stands for the scalar product in $\mathbb{R}^n$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended in the usual way to $S'(\mathbb{R}^n)$. Let $\phi_0 \in S(\mathbb{R}^n)$ with

$$\phi_0(x) = 1 \text{ if } |x| \leq 1 \text{ and } \phi_0(x) = 0 \text{ if } |x| \geq 3/2$$

and let

$$\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}.$$

Then

$$\sum_{j\in\mathbb{N}_0} \phi_j(x) = 1, \ x \in \mathbb{R}^n,$$

i.e., $\{\phi_j\}$ forms a dyadic resolution of unity. Recall that $(\phi_j \widehat{f})^\vee$ is an entire analytic function and makes sense point-wise for any $f \in S'(\mathbb{R}^n)$. All function spaces, which we consider here, are subspaces of $S'(\mathbb{R}^n)$. Let

$$Q_{J,M} = 2^{-J}M + 2^{-J}(0,1)^n, \quad J \in \mathbb{Z}, \ M \in \mathbb{Z}^n, \tag{2.2}$$

be the dyadic cube in $\mathbb{R}^n$ (where $(0,1)^n$ denotes the open cube with side length 1) with side length 2^{-J} parallel to the coordinate axes and $2^{-J}M$ as the lower left corner. For a cube Q, we denote by dQ with $d > 0$ the cube concentric with Q and its side length multiplied by d. $|\Omega|$ denotes the Lebesgue measure of the Lebesgue measurable set $\Omega \subset \mathbb{R}^n$. We write $a_+ := \max(0, a)$ for $a \in \mathbb{R}$.

Finally, we write

$$a \sim b \text{ (equivalence)}$$

if there exist two positive constants $c_1, c_2 > 0$ such that $c_1\, a \leq b \leq c_2\, a$. Sometimes we use the symbol "$\lesssim$" instead of "$\leq$". The meaning of $a \lesssim b$ is given by: There exists a positive constant C such that $a \leq Cb$.

Definition 1. Let $\phi = \{\phi_j\}_{j=0}^\infty$ be the above dyadic resolution of unity.

(i) Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. We define the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ as the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|_\phi = \Big(\sum_{j=0}^\infty 2^{jsq}\|(\phi_j\widehat{f})^\vee|L_p(\mathbb{R}^n)\|^q\Big)^{1/q}$$

is finite (with the usual modification if $q = \infty$).

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. We define the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ as the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|F^s_{p,q}(\mathbb{R}^n)\|_\phi = \Big\|\Big(\sum_{j=0}^\infty 2^{jsq}|(\phi_j\widehat{f})^\vee(\cdot)|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^n)\Big\|$$

is finite (with the usual modification if $q = \infty$).

(iii) Let $0 < q < \infty$ and $s \in \mathbb{R}$. We define the spaces $F^s_{\infty,q}(\mathbb{R}^n)$ as the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|F^s_{\infty,q}(\mathbb{R}^n)\|_\phi = \sup_{J\in\mathbb{Z},\, M\in\mathbb{Z}^n} 2^{Jn/q}\Big(\int_{Q_{J,M}} \sum_{j\geq J_+} 2^{jsq}\big|(\phi_j\widehat{f})^\vee(x)\big|^q\,\mathrm{d}x\Big)^{1/q}$$

is finite.

Remark 1. These are the classical (global) spaces $A^s_{p,q}(\mathbb{R}^n)$, where A stands either for B or F. In what follows, we will write $A^s_{p,q}(\mathbb{R}^n)$ if an assertion applies both to B- and F- spaces. The above definition coincides with [28, Definition 1.1]. A detailed study of these spaces, including their history and properties, can be found in [21–23]. They are independent (in the sense of equivalent quasi-norms) of the chosen resolution of unity. Therefore, we will omit the subscript ϕ in the sequel.

2.2. *Morrey smoothness spaces*

Next, we introduce Morrey smoothness spaces following closely the new approach presented in [10]. We present two different possibilities to characterize these types of spaces. We start with the Fourier analytic approach followed by characterizations of the spaces by means of wavelets and molecules. This is one of the main tools for proving estimate (1.1). We recall the necessary definitions and assertions as far as we need them for our considerations. Standard references with respect to wavelets are, e.g., [6, 14, 30]. For molecules, we refer, e.g., to [20], [8, Section 12], [9, Section 5].

2.2.1. Fourier analytic approach

Recall that $L^{\mathrm{loc}}_p(\mathbb{R}^n)$ consists of all f whose restriction to bounded Lebesgue measurable sets $M \subset \mathbb{R}^n$ belongs to $L_p(M)$, $1 \leq p \leq \infty$.

Definition 2. Let $n \in \mathbb{N}$, $0 < p < \infty$ and $-n \leq \rho \leq 0$. Then, $\Lambda^\rho_p(\mathbb{R}^n)$ collects all $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ such that

$$\|f|\Lambda^\rho_p(\mathbb{R}^n)\| = \sup_{J\in\mathbb{Z},\, M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)}\|f|L_p(Q_{J,M})\|$$

is finite.

Remark 2. Note that we have the coincidence

$$\Lambda^{-n}_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$$

in the sense of equivalent (quasi-)norms.
Let $0 < p \leq u < \infty$. The Morrey space $\mathcal{M}^u_p(\mathbb{R}^n)$ collects all $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ such that

$$\|f|\mathcal{M}^u_p(\mathbb{R}^n)\| = \sup_{J\in\mathbb{Z},\, M\in\mathbb{Z}^n} 2^{Jn(\frac{1}{p}-\frac{1}{u})}\|f|L_p(Q_{J,M})\| \tag{2.3}$$

is finite. Compared with Definition 2, one has

$$\Lambda^\rho_p(\mathbb{R}^n) = \mathcal{M}^u_p(\mathbb{R}^n) \qquad \text{with } 0 < p \leq u < \infty, \quad u\rho + np = 0\,. \tag{2.4}$$

Definition 3. Let $n \in \mathbb{N}$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$.

(i) Let $-n \le \rho \le 0$. Then, $\Lambda^\rho B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|\Lambda^\rho B^s_{p,q}(\mathbb{R}^n)\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \Big(\sum_{j\ge J^+} 2^{jsq} \|(\phi_j \hat{f})^\vee | L_p(Q_{J,M})\|^q \Big)^{1/q} \tag{2.5}$$

is finite and $\Lambda^\rho F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|\Lambda^\rho F^s_{p,q}(\mathbb{R}^n)\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \Big\| \Big(\sum_{j\ge J^+} 2^{jsq} \big|(\phi_j \hat{f})^\vee(\cdot)\big|^q \Big)^{1/q} \Big| L_p(Q_{J,M})\Big\| \tag{2.6}$$

is finite (usual modification if $q = \infty$).

(ii) Let $-n \le \rho < 0$. Then, $\Lambda_\rho B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|\Lambda_\rho B^s_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\phi_j \hat{f})^\vee | \Lambda^\rho_p(\mathbb{R}^n)\|^q \Big)^{1/q} \tag{2.7}$$

is finite (usual modification if $q = \infty$). The space $\Lambda_\rho F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|\Lambda_\rho F^s_{p,q}(\mathbb{R}^n)\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |(\phi_j \hat{f})^\vee(\cdot)|^q \Big)^{1/q} \Big| \Lambda^\rho_p(\mathbb{R}^n)\Big\| \tag{2.8}$$

is finite (usual modification if $q = \infty$).

Remark 3. Let $s \in \mathbb{R}$ and $A \in \{B, F\}$.

(i) The spaces introduced in Definition 3 coincide with the well-known global spaces $A^s_{p,q}(\mathbb{R}^n)$ when $\rho = -n$. Thus,

$$\Lambda^{-n} A^s_{p,q}(\mathbb{R}^n) = \Lambda_{-n} A^s_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n), \tag{2.9}$$

see [10, Remarks 2.7 and 2.9].
Moreover, if $\rho = 0$, we have the coincidence

$$\Lambda^0 F^s_{p,q}(\mathbb{R}^n) = F^s_{\infty,q}(\mathbb{R}^n)$$

(see [10, Proposition 2.12 (iii)]).

(ii) The Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p \le u < \infty$ were introduced by Kozono and Yamazaki in [12]. The Triebel-Lizorkin Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ go back to Tang and Xu in [29]. They coincide with the above scales $\Lambda_\rho A^s_{p,q}(\mathbb{R}^n)$ as follows:

$$\Lambda_\rho B^s_{p,q}(\mathbb{R}^n) = \mathcal{N}^s_{u,p,q}(\mathbb{R}^n), \quad u\rho + np = 0, \ -n \le \rho < 0 \tag{2.10}$$

$$\Lambda_\rho F^s_{p,q}(\mathbb{R}^n) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^n), \quad u\rho + np = 0, \ -n \le \rho < 0. \tag{2.11}$$

For more details; see [10, Remark 2.7] and the references given there.

(iii) The spaces in Definition 3, part (i), are reformulations of corresponding Morrey smoothness spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ (see [10, Remark 2.9]). It holds

$$\Lambda^\rho A^s_{p,q}(\mathbb{R}^n) = A^{s,\tau}_{p,q}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \le \infty, \tag{2.12}$$

where

$$\tau = \frac{1}{p}\left(1 + \frac{\rho}{n}\right), \quad -n \le \rho \le 0.$$

(iv) Note that the spaces defined in Definition 3, part (i), coincide for all admitted parameters with the hybrid spaces as introduced by Triebel in [26]. We have

$$L^r A^s_{p,q}(\mathbb{R}^n) = \Lambda^\rho A^s_{p,q}(\mathbb{R}^n), \quad -n \le \rho \le 0,\ 0 < p < \infty,\ r = \frac{\rho}{p}, \tag{2.13}$$

and $0 < q \le \infty$ (see [10, Remark 2.9]).

(v) Moreover, it holds

$$\Lambda^\rho F^s_{p,q}(\mathbb{R}^n) = \Lambda_\rho F^s_{p,q}(\mathbb{R}^n),\ s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \le \infty,\ -n \le \rho < 0. \tag{2.14}$$

We refer to [10, formula (2.45), p. 1309]) and the references given there to [19, Theorem 1.1 (ii)], and [18, Theorem 6.35, p. 794]. The situation is different if $A = B$. We have the strict embedding

$$\Lambda_\rho B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^\rho B^s_{p,q}(\mathbb{R}^n) \tag{2.15}$$

if $0 < q < \infty$ and

$$\Lambda_\rho B^s_{p,\infty}((\mathbb{R}^n)) \ = \ \Lambda^\rho B^s_{p,\infty}((\mathbb{R}^n)) \tag{2.16}$$

(see [10, Theorem 2.21,(iv)]).

In other words, one has only one $\Lambda^\rho F = \Lambda_\rho F$ scale but two $\Lambda^\rho B$ and $\Lambda_\rho B$ scales.

(vi) Furthermore, spaces $\Lambda_\rho B^s_{p,q}(\mathbb{R}^n)$ can be obtained by means of real interpolation. More precisely, let $n < \rho < 0,\ 0 < p < \infty$ and $0 < q_1, q_2 \le \infty$. Then it holds

$$\Lambda_\rho B^s_{p,q}(\mathbb{R}^n) = \Big(\Lambda^\rho F^{s_1}_{p,q_1}(\mathbb{R}^n),\ \Lambda^\rho F^{s_2}_{p,q_2}(\mathbb{R}^n)\Big)_{\theta,q} \tag{2.17}$$

if

$$\infty < s_1 < s_2 < \infty,\ s = (1-\theta)s_1 + \theta s_2,\ \text{and } 0 < \theta < 1$$

(see [10, Proposition 3.3 and Remark 3.4]).

Following [10, Definition 2.15], for $-n \le \rho < 0$ the so-called ρ-clan ρ-$A^s_{p,q}(\mathbb{R}^n)$ stands for the three families

$$\Lambda^\rho B^s_{p,q}(\mathbb{R}^n),\ \Lambda_\rho B^s_{p,q}(\mathbb{R}^n) \text{ and } \Lambda^\rho F^s_{p,q}(\mathbb{R}^n) = \Lambda_\rho F^s_{p,q}(\mathbb{R}^n) \tag{2.18}$$

with

$$s \in \mathbb{R},\ 0 < p < \infty \text{ and } 0 < q \le \infty.$$

2.2.2. Lifts and embeddings

We summarize shortly some properties of ρ-$A^s_{p,q}(\mathbb{R}^n)$ - spaces. Here, we adapt parameters $p,\ q$, and ρ to our later needs.

Proposition 1. *(see [10, Theorem 5.3])*
Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q \le \infty$ and $-n < \rho < 0$. Then

$$\Lambda^\rho A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \text{ if, and only if, } s > \frac{|\rho|}{p} \tag{2.19}$$

$A \in \{B, F\}$, and

$$\Lambda_\rho B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \text{ if, and only if, } s > \frac{|\rho|}{p}. \tag{2.20}$$

Let σ be a real number. The operator I_σ given by

$$I_\sigma f = \mathcal{F}^{-1}(1 + |x|^2)^{-\sigma/2}\mathcal{F} f \tag{2.21}$$

is a one-to-one map onto itself both in $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$. Furthermore, I_σ is a lift for the spaces $A^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < p, q \le \infty$. Thus, we have

$$I_\sigma A^s_{p,q}(\mathbb{R}^n) = A^{s+\sigma}_{p,q}(\mathbb{R}^n) \tag{2.22}$$

in the sense of equivalent quasi-norms. We recall the corresponding results for Morrey smoothness spaces.

Proposition 2. *(see [10, Theorem 3.8])*
Let $s, \sigma \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q \le \infty$ and $-n < \rho < 0$. Then I_σ maps $\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$ isomorphically onto $\Lambda^\rho A^{s+\sigma}_{p,q}(\mathbb{R}^n)$ and $\Lambda_\rho A^s_{p,q}(\mathbb{R}^n)$ isomorphically onto $\Lambda_\rho A^{s+\sigma}_{p,q}(\mathbb{R}^n)$, where $A \in \{B, F\}$. Further,

$$I_\sigma \Lambda^\rho A^s_{p,q}(\mathbb{R}^n) = \Lambda^\rho A^{s+\sigma}_{p,q}(\mathbb{R}^n), \quad A \in \{B, F\}, \tag{2.23}$$
$$I_\sigma \Lambda_\rho A^s_{p,q}(\mathbb{R}^n) = \Lambda_\rho A^{s+\sigma}_{p,q}(\mathbb{R}^n), \quad A \in \{B, F\}. \tag{2.24}$$

Remark 4. We have the following properties.

(i) It holds

$$I_\sigma^{-1} = I_{-\sigma} \qquad \text{for} \quad \sigma \in \mathbb{R}.$$

(ii) Let σ_1 and σ_2 be real numbers. Then

$$I_{\sigma_1} \cdot I_{\sigma_2} = I_{\sigma_1+\sigma_2}.$$

2.2.3. Representation by means of wavelets

We recall the necessary definitions and assertions as far as we need them for our considerations. Standard references with respect to wavelets are, e.g., [6, 14, 30]. Throughout the following sections, we restrict parameters p, q and ρ as indicated above to $1 \le p < \infty$, $1 \le q \le \infty$ and $-n < \rho < 0$. Let $C^u(\mathbb{R})$, $u \in \mathbb{N}$ denote the space of all complex-valued u- times continuously differentiable functions with bounded derivatives in $\mathbb{R}$. Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \tag{2.25}$$

be real compactly supported Daubechies wavelets with $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ and

$$\int_{\mathbb{R}} x^v \psi_M(x) \mathrm{d}x = 0 \quad \text{for all } v \in \{0, \ldots, u-1\}. \tag{2.26}$$

ψ_F is called the scaling function (father wavelet) and ψ_M is the so-called associated wavelet (mother wavelet). We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual multi-resolution procedure. Let either

$$G = (G_1, \ldots, G_n) \in G^0 = \{F, M\}^n, \tag{2.27}$$

which means that the components G_r of G where $r \in \{1, \ldots, n\}$ are either F or M or let

$$G = (G_1, \ldots, G_n) \in G^j = \{F, M\}^{n*}, \quad j \in \mathbb{N}. \tag{2.28}$$

Here, $*$ indicates that at least one of the components of G must be an M. In the sequel, we denote such a set G^j with G^*. Let

$$\Psi^j_{G,m}(x) = \prod_{r=1}^{n} \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j, \ m \in \mathbb{Z}^n, \tag{2.29}$$

(where m_r denote the components of m), $x \in \mathbb{R}^n$, now with $j \in \mathbb{N}_0$. We always assume that ψ_F and ψ_M have an L_2-norm 1. Then, for any $u \in \mathbb{N}$,

$$\Psi = \{2^{jn/2} \Psi^j_{G,m} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\} \tag{2.30}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m} =: \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn} \int_{\mathbb{R}^n} f(x) \Psi^j_{G,m}(x) \mathrm{d}x = 2^{jn} \left\langle f, \Psi^j_{G,m} \right\rangle$$

is the corresponding expansion, where $2^{-jn/2} \Psi^j_{G,m}$ are uniformly bounded functions with respect to j and m. For more detailed explanations, cf. [22, 23] and [24, Subsection 1.2.1].

Let $\chi_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ be the characteristic function of the usual dyadic cubes $Q_{j,m}$ as defined in (2.2).

Definition 4. Let $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ and $-n < \rho < 0$. Let
$$\lambda := \{\lambda_m^{j,G} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}. \tag{2.31}$$

(i) Then
$$\Lambda^\rho b^s_{p,q} = \{\lambda : \|\lambda|\Lambda^\rho b^s_{p,q}\| < \infty\}$$
with
$$\|\lambda|\Lambda^\rho b^s_{p,q}\| = \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)}\Bigg(\sum_{j\ge J^+}^{\infty} 2^{j(s-\frac{n}{p})q}\Bigg(\sum_{\substack{m:Q_{j,m}\subset Q_{J,M}\\ G\in G_j}} |\lambda_m^{j,G}|^p\Bigg)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}}.$$

(ii) Let $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ and $-n < \rho < 0$. Let
$$\lambda := \{\lambda_m^{j,G} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}. \tag{2.32}$$
Then
$$\Lambda^\rho f^s_{p,q} = \{\lambda : \|\lambda|\Lambda^\rho f^s_{p,q}\| < \infty\} = \Lambda_\rho f^s_{p,q},$$
where
$$\|\lambda|\Lambda^\rho f^s_{p,q}\| = \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)}\Bigg\|\Big(\sum_{j\ge J^+}^{\infty} 2^{jsq}\sum_{\substack{m:Q_{j,m}\subset Q_{J,M}\\ G\in G_j}} |\lambda_m^{j,G}\chi_{j,m}(\cdot)|^q\Big)^{\frac{1}{q}}|L_p(\mathbb{R}^n)\Bigg\|,$$
(usual modification if $q = \infty$).

Remark 5. Part (i) is covered by [28, Definition 1.13] with a reference to [26, 3.26]. Concerning part (ii), we refer to [26, Definition 3.24] which is already adapted to our notation.

We shall use the notation $a^s_{p,q}$ with $a = b$ or $a = f$. Based on [26, Theorem 3.26] and Remark 3 part (iv), we have the following wavelet characterization of $\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$.

Proposition 3. *Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le, q \le \infty$, and $-n < \rho < 0$. Let Ψ be the wavelet system* (2.30) *based on* (2.25)–(2.29)*, where $u > \max(s, -s)$.*

Let $f \in S'(\mathbb{R}^n)$. Then $f \in \Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as
$$f = \sum_{\substack{j\in\mathbb{N}_0,G\in G^j\\ m\in\mathbb{Z}^n}} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in \Lambda^\rho a^s_{p,q}, \tag{2.33}$$
where $a = b$ if $A = B$ and $a = f$ if $A = F$. The series converges unconditionally in $S'(\mathbb{R}^n)$. The representation (2.33) *is unique,*
$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\left\langle f, \Psi^j_{G,m}\right\rangle \tag{2.34}$$
and
$$I : f \mapsto \{\lambda_m^{j,G}(f)\} \tag{2.35}$$
is an isomorphic map of ρ-$A^s_{p,q}(\mathbb{R}^n)$ onto $\Lambda^\rho a^s_{p,q}$. Hence,
$$\|f|\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda(f)|\Lambda^\rho a^s_{p,q}\|. \tag{2.36}$$

Remark 6. For a detailed discussion of how to understand the dual pairing $\left\langle f, \Psi^j_{G,m}\right\rangle$ in (2.34) we refer to [26, Thm 3.26] and the references given there.

2.2.4. Representation by means of molecules

Next, we introduce molecular decompositions of the spaces $\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$. We recall first the definition of molecules related to $Q_{j,m}$ according to [25, Section 2.4.2]. Molecular decompositions have been considered, for instance, in [8, 9, 20]. We refer also to [32, Chapter 3] where one finds, in particular, corresponding representations for the spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ briefly mentioned in Remark 3 part (iii).

Definition 5. Let $K \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $L > N + n - 1$. Let j be a natural number and $m \in \mathbb{Z}^n$. The $L_\infty-$ functions $b_{j,m} : \mathbb{R}^n \mapsto \mathbb{C}$ are called $(K, N, L)-$ molecules, related to $Q_{j,m}$, if

$$\left|D^\zeta b_{j,m}(x,t)\right| \le c 2^{j|\zeta|} \left(1 + 2^j |x - 2^{-j} m|\right)^{-L}, \quad |\zeta| \le K, \tag{2.37}$$

and

$$\int_{\mathbb{R}^n} x^\beta b_{j,m}(x) dx = 0, \quad |\beta| < N. \tag{2.38}$$

We introduce corresponding sequence spaces.

Definition 6. Let $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ and $-n < \rho < 0$. Let

$$\mu = \{\mu^j_m : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}. \tag{2.39}$$

Then

$$\Lambda^\rho \overline{b}^s_{p,q} = \{\mu : \|\mu|\Lambda^\rho \overline{b}^s_{p,q}\| < \infty\}$$

with

$$\|\mu|\Lambda^\rho \overline{b}^s_{p,q}\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left(\sum_{j\ge J^+}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m: Q_{j,m}\subset Q_{J,M}} |\mu^j_m|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{2.40}$$

and

$$\Lambda^\rho \overline{f}^s_{p,q} = \{\mu : \|\mu|\Lambda^\rho \overline{f}^s_{p,q}\| < \infty\}$$

with

$$\|\mu|\Lambda^\rho \overline{f}^s_{p,q}\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \Big\| \Big(\sum_{j\ge J^+} 2^{jsq} \sum_{m:\, Q_{j,m}\subset Q_{J,M}} |\mu^j_m \chi_{j,m}(\cdot)|^q \Big)^{1/q} |L_p \Big\|, \tag{2.41}$$

(usual modification if $q = \infty$).

The following molecular characterization of $\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$ is already adapted to our needs based on [25, Proposition 2.35].

Proposition 4. *$1 \le p < \infty$, $1 \le q \le \infty$, $s \ge 0$ and $-n < \rho < 0$. Let $K \in \mathbb{N}_0$, $N = 1$ and $L \in \mathbb{R}$ with*

$$K > s \text{ and } L > N. \tag{2.42}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in \Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \mu^j_m b_{j,m}, \quad \mu \in \Lambda^\rho \overline{a}^s_{p,q}, \tag{2.43}$$

where $b_{j,m}$ are (K, N, L)-molecules, unconditional convergence being in $S'(\mathbb{R}^n)$. Additionally,

$$\|f|\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)\| \sim \inf \|\mu|\Lambda^\rho \overline{a}^s_{p,q}\|, \tag{2.44}$$

where the infimum is taken over all admissible representations (2.43).

3. Caloric smoothing

In this section we prove the smoothing property

$$\|W^\alpha_t \omega|\rho\text{-}A^{s+d}_{p,q}(\mathbb{R}^n)\| \le Ct^{-\frac{d}{2\alpha}} \|\omega|\rho\text{-}A^s_{p,q}(\mathbb{R}^n)\| \quad 0 < t \le 1,\ d \ge 0. \tag{3.1}$$

where

$$A \in \{B, F\}, -n < \rho < 0,\ s \in \mathbb{R},\ 1 \le p < \infty,\ \text{and } 1 \le q \le \infty.$$

Note that in view of Remark 3, part (i), the case $\rho = -n$ is already covered by the papers [4, 13]. At first, we prove (3.1) for the $\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$-spaces. The corresponding result for $\Lambda_\rho A^s_{p,q}(\mathbb{R}^n)$ follows then from Remark 3 parts (v) and (vi).

3.1. Fractional Gauss-Weierstrass semigroup

We start with some observations concerning fractional heat kernels. Consider the function

$$\varphi(\xi) := e^{-|\xi|^{2\alpha}}, \quad \xi \in \mathbb{R},\ \alpha > 0. \tag{3.2}$$

Clearly, the function φ is not smooth in $\xi = 0$ if $\alpha \notin \mathbb{N}$. To define the Gauss-Weierstrass semigroup in a proper way, consider first

$$G^\alpha(x) := \left(e^{-|\xi|^{2\alpha}}\right)^\vee (x),\ x \in \mathbb{R}^n,\ \alpha > 0. \tag{3.3}$$

Moreover, we need the fractional Laplacian, formally given by

$$(-\Delta)^\sigma \omega = (|\xi|^{2\sigma}\ \widehat{\omega})^\vee, \quad \sigma > 0, \tag{3.4}$$

for ω in an appropriate function space. We define

$$G^{\alpha,\sigma}(x) := (-\Delta)^{\sigma/2}\ G^\alpha = \left(|\xi|^\sigma e^{-|\xi|^{2\alpha}}\right)^\vee (x),\ x \in \mathbb{R}^n,\ \alpha > 0,\ \sigma > 0. \tag{3.5}$$

The following two estimates can be found in [17, Lemmas 1 and 2].

Lemma 1. *The kernel function G^α satisfies the point-wise estimate*

$$\left|G^\alpha(x)\right| \le c\,(1 + |x|)^{-n-2\alpha}, \quad x \in \mathbb{R}^n, \tag{3.6}$$

for $\alpha > 0$. Consequently, one has

$$G^\alpha \in L_p(\mathbb{R}^n) \quad \textit{for all } 1 \le p \le \infty. \tag{3.7}$$

Lemma 2. *The kernel function $G^{\alpha,\sigma}$ has the point-wise estimate*

$$\left|G^{\alpha,\sigma}(x)\right| \leq c\,(1+|x|)^{-n-\sigma}, \quad x \in \mathbb{R}^n \tag{3.8}$$

for α, $\sigma > 0$. Consequently, one has

$$G^{\alpha,\sigma} \in L_p(\mathbb{R}^n) \quad \text{for all } 1 \leq p \leq \infty. \tag{3.9}$$

These lemmas show that G^α and $G^{\alpha,\sigma}$ provide a sufficiently fast polynomial decay which will be of great use later in the proof of Proposition 5. Now, we consider the fractional heat kernel given by

$$G_t^\alpha(x) = (2\pi)^{-n/2}\left(e^{-t|\xi|^{2\alpha}}\right)^\vee(x),\ x \in \mathbb{R}^n,\ t > 0,\ \alpha > 0. \tag{3.10}$$

Obviously, it holds

$$G_t^\alpha(x) = (2\pi)^{-n/2} t^{-n/2\alpha}\, G^\alpha(t^{-1/2\alpha}x). \tag{3.11}$$

Note that $G_t^{\alpha,\sigma}$ has the same scaling properties as G_t^α, namely,

$$G_t^{\alpha,\sigma}(x) = (2\pi)^{-n/2}\, t^{-\sigma/2\alpha}\, t^{-n/2\alpha}\, G^\alpha(t^{-1/2\alpha}x), \tag{3.12}$$

see [17, p. 6].

Based on Propositions 1 and 2 and Lemma 1, we define the fractional Gauss-Weierstrass semigroup W_t^α as follows.

Definition 7. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $-n < \rho < 0$. Let $\omega \in \rho\text{-}A^s_{p,q}(\mathbb{R}^n)$ and G_t^α be as above.

(1) We define

$$W_t^\alpha \omega(x) := (G_t^\alpha * \omega)(x) \quad \text{if } s > \tfrac{|\rho|}{p}. \tag{3.13}$$

(2) Let $\sigma \in \mathbb{R}$ such that $s + \sigma > \frac{|\rho|}{p}$. Then we define

$$W_t^\alpha \omega := I_{-\sigma}[W_t^\alpha(I_\sigma \omega)] \quad \text{if } s \leq \tfrac{|\rho|}{p}. \tag{3.14}$$

Note that the definition in part (ii) is independent of σ. According to Proposition 2, $I_\sigma\omega$ is smooth enough to justify the application of W_t^α in the sense of part (i) .

3.2. Preparations

Let $\omega \in \Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$. Under the conditions of Proposition 3, we can represent

$$\omega = \sum_{j,G,m} \lambda_m^{j,G} 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in\ \Lambda^\rho a^s_{p,q},\ a \in \{b, f\}, \tag{3.15}$$

with coefficients

$$\lambda_m^{j,G} = \lambda_m^{j,G}(\omega) = 2^{jn/2}\left\langle \omega, \Psi^j_{G,m}\right\rangle \tag{3.16}$$

in the interpretation of (2.34). We are interested in a similar decomposition of $W_t^\alpha \omega$ in terms of molecules.

Let

$$\begin{aligned} b^j_{G,m}(x,t) &:= 2^{-jn/2} W_t^\alpha \Psi^j_{G,m}(x) = \int_{\mathbb{R}^n} G_t^\alpha(x-y) 2^{-jn/2} \Psi^j_{G,m}(y) \mathrm{d}y \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee (x-y) \prod_{l=1}^n \psi_{G_l}(2^j y_l - m_l) \mathrm{d}y, \end{aligned} \tag{3.17}$$

based on (2.29), where $\psi_F,\ \psi_M \in C^u(\mathbb{R})$ are the Daubechies wavelets as in (2.25) and (2.26). According to the case $\alpha = 1$, cf. [25, Subsection 2.4.2], the functions $b^j_{G,m}(x,t)$ are called α-caloric wavelets. As already mentioned, we show that after a slight modification, they are molecules in the sense of Definition 5 for appropriately chosen parameters N, K, L. We put

$$b^j_{G,m}(x,t)_d := c\, 2^{jd} t^{d/2\alpha} b^j_{G,m}(x,t), \quad j \in \mathbb{N}_0,\ G \in G^*,\ m \in \mathbb{Z}. \tag{3.18}$$

Proposition 5. *Let* $\alpha > 0,\ d \geq 0$ *and* $(b^j_{G,m})_d$ *have the meaning of* (3.18). *Let* $u \in \mathbb{N}$ *such that*

$$u > d \cdot \frac{n+2\alpha}{2\alpha}. \tag{3.19}$$

Then, there exists $L > n$ *such that* $(b^j_{G,m})_d$ *are* $(u,1,L)$*-molecules according to Definition 5 for some* $c > 0$ *and any fixed* t *with* $2^j t^{1/2\alpha} \geq 1$.

Proof. Step 1: We prove the vanishing moment conditions for $b^j_{G,m}(x,t)_d$ first. Let β be a lattice point on $\mathbb{N}_0^n$ such that $|\beta| < u$. We have

$$\begin{aligned} \int_{\mathbb{R}^n} x^\beta b^j_{G,m}(x,t)_d\, dx &\sim \int_{\mathbb{R}^n} x^\beta b^j_{G,m}(x,t) dx \\ &= \int_{\mathbb{R}^n} x^\beta \int_{\mathbb{R}^n} G_t^\alpha(x-y) \Psi^j_{G,m}(y) dy\, dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} x^\beta G_t^\alpha(x-y) dx \right) \Psi^j_{G,m}(y) dy \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (y+z)^\beta G_t^\alpha(z) dz \right) \Psi^j_{G,m}(y) dy \\ &= \int_{\mathbb{R}^n} G_t^\alpha(z) \left(\prod_{l=1}^n \int_{\mathbb{R}} (y_l+z_l)^{\beta_l} \psi_{G_l}(2^j y_l - m_l) dy_l \right) dz \\ &= 0. \end{aligned} \tag{3.20}$$

In (3.20), we used the fact that because of $G \in G^*$, there exists $l \in \{1, \cdots, n\}$ such that $G_l = M$. Hence, using the moment condition (2.26) and $0 \leq \beta_l < u$, at least one of the factors is equal to zero, which can be seen by elementary calculations.

Step 2: Now, we prove that there exists an $L > 0$ satisfying (2.42) such that

$$\left|D^{\zeta} b^{j}_{G,m}(x,t)_d\right| \le C\, 2^{j|\zeta|} \left(1 + 2^j|x - 2^{-j}m|\right)^{-L} \quad |\zeta| \le u. \tag{3.21}$$

Due to (3.17) we may assume $m = 0$. Let $|\zeta| = 0$ and consider

$$b^{j}_{G,0}(x,t) = \int_{\mathbb{R}^n} G^{\alpha}_t(x-y) \prod_{r=1}^{n} \psi_{G_r}(2^j y_r) dy,$$

$j \in \mathbb{N}_0$, $G \in G^*$ and $2^j t^{1/2\alpha} \ge 1$. We rewrite

$$b^{j}_{G,0}(x,t) = \int_{\mathbb{R}^n} t^{-n/2\alpha} G^{\alpha}\left(\frac{x-y}{t^{1/2\alpha}}\right) \prod_{r=1}^{n} \psi_{G_r}(2^j y_r) dy,$$

where G^{α} is defined as in (3.3). Apparently, it holds

$$b^{j}_{G,0}(t^{1/2\alpha}x,t) \;=\; \int_{\mathbb{R}^n} G^{\alpha}(x-y) \prod_{r=1}^{n} \psi_{G_r}(2^j t^{1/2\alpha} y_r) dy. \tag{3.22}$$

We expand G^{α} in a Taylor polynomial about the origin with a remainder term of order u (which is possible according to Lemma 2) and substitute it into (3.22). Because of the moment conditions, terms of order less than u vanish such that we have the estimate

$$\begin{aligned} b^{j}_{G,0}(t^{1/2\alpha}x,t) &\lesssim \int_{\mathbb{R}^n} \frac{1}{\beta!} \sum_{|\beta|=u} |(D^{\beta}G^{\alpha})(x-\xi)\, y^{\beta} \prod_{r=1}^{n} \psi_{G_r}(2^j t^{1/2\alpha} y_r)| dy \\ &\lesssim \int_{\mathbb{R}^n} \sum_{|\beta|=u} |y^{\beta}| \left|\prod_{r=1}^{n} \psi_{G_r}(2^j t^{1/2\alpha} y_r)\right| dy \\ &\lesssim \int_{\mathbb{R}^n} |y|^u \left|\prod_{r=1}^{n} \psi_{G_r}(2^j t^{1/2\alpha} y_r)\right| dy, \end{aligned} \tag{3.23}$$

where we used the boundedness of the derivatives of G^{α} in (3.23). The integrand in (3.23) is zero outside a ball of radius $c\, 2^{-j} t^{-1/2\alpha}$ centered at the origin. Hence, we obtain

$$\left|b^{j}_{G,0}(t^{1/2\alpha}x,t)\right| \le C \int_{|y| \le c\, 2^{-j} t^{-1/2\alpha}} |y|^u \mathrm{d}u \le C \left(2^{-j} t^{-1/2\alpha}\right)^{u+n} \tag{3.24}$$

for all $x \in \mathbb{R}^n$. On the other hand, it follows from (3.22) and Lemma 1 that

$$\begin{aligned} \left|b^{j}_{G,0}(t^{1/2\alpha}x,t)\right| &\le C_1 \int_{|y| \le c 2^{-j} t^{-1/2\alpha}} \left|G^{\alpha}(x-y) \prod_{r=1}^{n} \psi_{G_r}(2^j t^{1/2\alpha} y_r)\right| dy \\ &\le C_2 \int_{|y| \le c 2^{-j} t^{-1/2\alpha}} \frac{1}{1 + (|x-y|)^{n+2\alpha}} dy \end{aligned} \tag{3.25}$$

$$\leq \quad C_2 \frac{1}{(1+|x|)^{n+2\alpha}} \int_{|y|\leq c2^{-j}t^{-1/2\alpha}} (1+|y|)^{n+2\alpha}\, dy.$$

Since $2^j t^{1/2\alpha}$ and $|y| \leq c2^{-j}t^{-1/2\alpha}$, we have $1+|y| \leq c$.
Thus,

$$\begin{aligned} \left|b^j_{G,0}(t^{1/2\alpha}x,t)\right| &\leq C_3 \frac{1}{(1+|x|)^{n+2\alpha}} \int_{|y|\leq c2^{-j}t^{-1/2\alpha}} dy \\ &\lesssim \frac{1}{(1+|x|)^{n+2\alpha}} \left(2^{-j}t^{-1/2\alpha}\right)^n. \end{aligned} \tag{3.26}$$

Let $0<\varepsilon<1$. Combining (3.24) and (3.26), we have

$$\begin{aligned} \left|b^j_{G,0}(t^{1/2\alpha}x,t)\right| &= \left|b^j_{G,0}(t^{1/2\alpha}x,t)\right|^{\varepsilon} \left|b^j_{G,0}(t^{1/2\alpha}x,t)\right|^{1-\varepsilon} \\ &\leq \frac{c_{\varepsilon,\alpha}}{(1+|x|)^{(n+2\alpha)\varepsilon}} \left(2^{-j}t^{-1/2\alpha}\right)^{\varepsilon n} \left(2^{-j}t^{-1/2\alpha}\right)^{(n+u)(1-\varepsilon)} \\ &= C' \frac{1}{(1+|x|)^{(n+2\alpha)\varepsilon}} \left(2^{-j}t^{-1/2\alpha}\right)^{n+(1-\varepsilon)u}. \end{aligned} \tag{3.27}$$

Since $2^j t^{1/2\alpha} \geq 1$, it holds

$$\frac{1}{(1+|x|)^{(n+2\alpha)\varepsilon}} \leq \frac{\left(2^j t^{1/2\alpha}\right)^{(n+2\alpha)\varepsilon}}{(1+2^j t^{1/2\alpha}|x|)^{(n+2\alpha)\varepsilon}}.$$

Hence,

$$\begin{aligned} \left|b^j_{G,0}(t^{1/2\alpha}x,t)\right| &\lesssim \frac{\left(2^j t^{1/2\alpha}\right)^{(n+2\alpha)\varepsilon}}{(1+2^j t^{1/2\alpha}|x|)^{(n+2\alpha)\varepsilon}} \left(2^{-j}t^{-1/2\alpha}\right)^{(1-\varepsilon)u+n} \\ &\sim \frac{\left(2^{-j} t^{-1/2\alpha}\right)^{-(n+2\alpha)\varepsilon}}{(1+2^j t^{1/2\alpha}|x|)^{(n+2\alpha)\varepsilon}} \left(2^{-j}t^{-1/2\alpha}\right)^{(1-\varepsilon)u+n} \\ &\lesssim \frac{1}{(1+2^j t^{1/2\alpha}|x|)^{(n+2\alpha)\varepsilon}} \left(2^{-j}t^{-1/2\alpha}\right)^{(1-\varepsilon)(u+n)-2\alpha\varepsilon}. \end{aligned} \tag{3.28}$$

Replacing $t^{1/2\alpha}x$ by x in (3.28) yields the estimate

$$\left|b^j_{G,0}(x,t)\right| \leq C' \frac{1}{(1+2^j|x|)^{(n+2\alpha)\varepsilon}} \left(2^{-j}t^{-1/2\alpha}\right)^{(1-\varepsilon)(u+n)-2\alpha\varepsilon}. \tag{3.29}$$

We define $g(\varepsilon) := (1-\varepsilon)(u+n) - 2\alpha\varepsilon = u+n-\varepsilon(u+n+2\alpha)$ with $0<\varepsilon<1$. Obviously, the graph of g is a strictly decreasing straight line. Because of $u>0$, it holds that

$$0 < \frac{n}{n+2\alpha} < \frac{u+n}{u+n+2\alpha}. \tag{3.30}$$

Hence,

$$0 = g\left(\frac{u+n}{u+n+2\alpha}\right) < g\left(\frac{n}{n+2\alpha}\right) = \left(\frac{2\alpha}{n+2\alpha}\right)u. \tag{3.31}$$

According to (3.19), we have

$$g\left(\frac{u+n}{u+n+2\alpha}\right) \;=\; 0 \;\le\; d \;<\; g\left(\frac{n}{n+2\alpha}\right).$$

Hence, there exists a uniquely determined $\varepsilon \in (0,1)$, more precisely,

$$\frac{n}{n+2\alpha} < \varepsilon \le \frac{u+n}{u+n+2\alpha} < 1,$$

such that $g(\varepsilon) = d$. For this choice of ε, we put $L := (n+2\alpha)\varepsilon > n$. Inserting this in (3.29) leads finally to the estimate

$$\left|b^j_{G,0}(x,t)\right| \le C' \left(2^{-j} t^{-1/2\alpha}\right)^d \frac{1}{(1+2^j|x|)^L}. \tag{3.32}$$

Step 3: Let now $1 \le |\zeta| \le u$. Recall that

$$b^j_{G,0}(t^{1/2\alpha}x, t) = 2^{-j\frac{n}{2}} (G^\alpha_t * \Psi^j_{G,0})(t^{1/2\alpha}x).$$

The derivatives D^ζ_x can be shifted to $\Psi^j_{G,0}$ and we get

$$\begin{aligned} D^\zeta_x \left(b^j_{G,0}(t^{1/2\alpha}x, t)\right) &= t^{|\zeta|/2\alpha} D^\zeta_x b^j_{G,0}(t^{1/2\alpha}x, t) \\ &\le t^{|\zeta|/2\alpha} \int_{\mathbb{R}^n} G^\alpha_t(t^{1/2\alpha}x - y) D^\zeta_y \left[\prod_{r=1}^n \psi_{G_r}(2^j y_r)\right] dy \\ &\sim t^{|\zeta|/2\alpha} 2^{j|\zeta|} \int_{\mathbb{R}^n} G^\alpha_t(t^{1/2\alpha}x - y) \left(D^\zeta \prod_{r=1}^n \psi_{G_r}\right)(2^j y_r) dy. \end{aligned}$$

Note that $\left(D^\zeta \prod_{r=1}^n \psi_{G_r}\right)(2^j y_r)$ fulfills the vanishing moment condition for $|\zeta| \le u$, which can be seen by iterative integration by parts. Thus, we obtain, using the same arguments as in the case $\zeta = 0$,

$$\left|D^\zeta_x b^j_{G,0}(x,t)\right| \le C' \left(2^{-j} t^{-1/2\alpha}\right)^d \frac{\left(2^j t^{1/2\alpha}\right)^{|\zeta|}}{(1+2^j|x|)^L}, \qquad |\zeta| \le u. \tag{3.33}$$

To be in correlation with the Definition 5, the choice of $N = 1$ leads directly to $N \le u$. The condition on L as discussed in Step 2 remains unchanged. We conclude from (3.32) and (3.33) that $b^j_{G,m}(\cdot;t)_d$ are (u,N,L)-molecules according to Definition 5. □

Remark 7. Note that throughout the previous proof, we assumed $m = 0$. This is due to the fact that the case $m \neq 0$ can be transformed to $m = 0$ by applying a change of variable.

3.3. Main result

In this section, we prove the estimate (1.1) which is the key estimate to derive existence and uniqueness results for several nonlinear heat and Navier-Stokes equations.

Theorem 1. *Let* $1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R},\ \alpha > 0,\ d \geq 0$ *and* $-n < \rho < 0$. *Then there exists a constant* $C > 0$ *such that*

$$\|W_t^\alpha \omega|\rho\text{-}A^{s+d}_{p,q}(\mathbb{R}^n)\| \leq C\, t^{-\frac{d}{2\alpha}} \|\omega|\rho\text{-}A^s_{p,q}(\mathbb{R}^n)\| \tag{3.34}$$

for all $0 < t \leq 1$ *and for all* $\omega \in \rho\text{-}A^s_{p,q}(\mathbb{R}^n)$.

Proof. Step 1: We assume first $s > \frac{|\rho|}{p}$. Let $\omega \in \Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$. Then by Proposition 3 we have the wavelet representation

$$\omega = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m} \tag{3.35}$$

with $(\lambda_m^{j,G}) \in \Lambda^\rho a^s_{p,q}$, where we choose sufficiently smooth wavelets $\Psi^j_{G,m}$. More precisely, we assume for $u \in \mathbb{N}$, according to (2.25) and (3.19), respectively, that

$$u > \max\left(s+d, d\cdot \frac{n+2\alpha}{2\alpha}\right). \tag{3.36}$$

Let $k \in \mathbb{N}$. We split (3.35) as follows:

$$\begin{aligned} \omega &= \underbrace{\sum_{j\leq k} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}}_{\omega_k^0} + \underbrace{\sum_{j> k} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}}_{\omega_k} \\ &= \omega_k^0 + \omega_k. \end{aligned}$$

Applying the Gauss-Weierstrass semigroup on the previous representation yields

$$W_t^\alpha \omega = W_t^\alpha \omega_k^0 + W_t^\alpha \omega_k. \tag{3.37}$$

We consider the second summand on the righthand side of (3.37) and assume $2^{-2\alpha k} < t \leq 2^{-2\alpha(k-1)}$. Since $j > k$, it follows $2^j t^{1/2\alpha} \geq 2^{j-k} \geq 1$. Applying Proposition 5, we want to derive a molecular representation of $W_t^\alpha \omega_k$. We have

$$\begin{aligned} W_t^\alpha \omega_k &= \sum_{j>k} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2} W_t^\alpha \Psi^j_{G,m} \\ &= \sum_{j>k} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \left(c^{-1}\, 2^{-jd} t^{-d/2\alpha} \lambda_m^{j,G}\right) \cdot \left(c\, 2^{jd}\, t^{d/2\alpha}\, 2^{-jn/2} W_t^\alpha \Psi^j_{G,m}\right) \\ &= \sum_{j>k} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} c^{-1}\, 2^{-jd}\, t^{-d/2\alpha} \lambda_m^{j,G}\, b^j_{G,m}(\cdot,t)_d \\ &= \sum_{j>k} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \mu_m^{j,G} b^j_{G,m}(\cdot,t)_d, \end{aligned}$$

where $\mu_m^{j,G} = c^{-1} 2^{-jd} t^{-d/2\alpha} \lambda_m^{j,G}$ and $b^j_{G,m}(\cdot,t)_d$ has the meaning of (3.18). We set

$$b_{j,m} = \begin{cases} \sum\limits_{G\in G^j} \frac{\mu_m^{j,G}}{\mu_m^j} b^j_{G,m}(\cdot,t)_d, & \text{if } j > k, \\ 0, & \text{otherwise}, \end{cases} \tag{3.38}$$

and $\mu_m^j = 0$, if $j = 0, \ldots, k$. For $j > k$, the choice of $\{\mu_m^j\}_{j,m} \in \Lambda^\rho \overline{a}_{p,q}^{s+d}$ depends on $\overline{a} = \overline{b}$ or $\overline{a} = \overline{f}$ (according to Definition 6). It follows from Proposition 5 that $\{b_{j,m}\}_{j,m}$ with $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$ are $(u, 1, L)$-molecules in the sense of Definition 5, where $L > n$ has the meaning as in Step 2, Proposition 5. In order to show that $W_t^\alpha \omega_k \in \Lambda^\rho A_{p,q}^{s+d}(\mathbb{R}^n)$ and that

$$\|W_t^\alpha \omega_k | \Lambda^\rho A_{p,q}^{s+d}(\mathbb{R}^n)\| \leq C\, t^{-\frac{d}{2\alpha}} \|\omega | \Lambda^\rho A_{p,q}^{s}(\mathbb{R}^n)\|, \tag{3.39}$$

we use Proposition 4 with $N = 1$, $K = u$, L as above and $s + d$ in place of s. It remains to be shown the estimate

$$\|\mu^* \| \Lambda^\rho \overline{a}_{p,q}^{s+d}\| \leq C\, t^{-\frac{d}{2\alpha}} \|\lambda | \Lambda^\rho a_{p,q}^{s}\|, \tag{3.40}$$

where $\mu^* = \{\mu_m^j : \ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$ and

$$\mu_m^j = \begin{cases} 0, & \text{if } j = 0, \cdots, k \text{ and } m \in \mathbb{Z}^n, \\ \left(\sum_{G \in G^j} |\mu_m^{j,G}|^p\right)^{\frac{1}{p}}, & \text{if } j > k, m \in \mathbb{Z}^n \text{ and } a = b, \\ \left(\sum_{G \in G^j} |\mu_m^{j,G}|^q\right)^{\frac{1}{q}}, & \text{if } j > k, m \in \mathbb{Z}^n \text{ and } a = f. \end{cases} \tag{3.41}$$

If $a = b$, we have

$$\begin{aligned} &\|\mu^* | \Lambda^\rho \overline{b}_{p,q}^{s+d}\| \\ &= \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left(\sum_{\substack{j \geq J^+ \\ j > k}}^{\infty} 2^{j(s+d-\frac{n}{p})q} \left(\sum_{m: Q_{j,m} \subset Q_{J,M}} |\mu_m^j|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &= \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left(\sum_{\substack{j \geq J^+ \\ j > k}}^{\infty} 2^{j(s+d-\frac{n}{p})q} \left(\sum_{m: Q_{j,m} \subset Q_{J,M}} \sum_{G \in G^j} |\mu_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &\lesssim t^{-d/2\alpha} \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left(\sum_{\substack{j \geq J^+ \\ j > k}}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \left(\sum_{m: Q_{j,m} \subset Q_{J,M}} |\lambda_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}. \end{aligned}$$

Thus,

$$\|\mu^* | \Lambda^\rho \overline{b}_{p,q}^{s+d}\| \lesssim t^{-d/2\alpha} \|\lambda | \Lambda^\rho b_{p,q}^{s}\|. \tag{3.42}$$

If a=f, we have

$$\begin{aligned} &\|\mu^* | \Lambda^\rho \overline{f}_{p,q}^{s+d}\| \\ &= \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left\| \left(\sum_{\substack{j \geq J^+,\ j > k \\ m: Q_{j,m} \subset Q_{J,M}}} 2^{j(s+d)q} |\mu_m^j \chi_{j,m}(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \end{aligned}$$

$$= \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left\| \Bigg(\sum_{\substack{j\geq J^+,\ j>k \\ m:Q_{j,m}\subset Q_{J,M}}} 2^{j(s+d)q} \left| \left(\sum_{G\in G^j} |\mu_m^{j,G}|^q \right)^{\frac{1}{q}} \chi_{j,m}(\cdot) \right|^q \Bigg)^{1/q} |L_p(\mathbb{R}^n) \right\|$$

$$= \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left\| \Bigg(\sum_{\substack{j\geq \max(J^+,\ k) \\ m:Q_{j,m}\subset Q_{J,M}}} \sum_{G\in G^j} 2^{j(s+d)q} |\mu_m^{j,G} \chi_{j,m}(\cdot)|^q \Bigg)^{1/q} |L_p(\mathbb{R}^n) \right\|$$

$$= c^{-1} t^{-d/2\alpha} \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left\| \Bigg(\sum_{\substack{j\geq \max(J^+,\ k) \\ G\in G^j \\ m:Q_{j,m}\subset Q_{J,M}}} 2^{jsq} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^q \Bigg)^{1/q} |L_p(\mathbb{R}^n) \right\|.$$

Thus,

$$\|\mu^* | \Lambda^\rho \overline{f}^{s+d}_{p,q}\| \lesssim t^{-d/2\alpha} \|\lambda | \Lambda^\rho f^s_{p,q}\|. \tag{3.43}$$

Estimates (3.42) and (3.43) imply (3.40) and thus, (3.39).
Now, we consider the first term on the righthand side of (3.37), i.e., $j \leq k$ and assume $A = B$. Then,

$$\begin{aligned} &\|\omega_k^0 | \Lambda^\rho B^{s+d}_{p,q}(\mathbb{R}^n)\| \\ &\lesssim \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left(\sum_{J^+\leq j\leq k} 2^{jdq} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \left(\sum_{m:Q_{j,m}\subset Q_{J,M}} |\lambda_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}. \end{aligned}$$

Since $j \leq k$, we have $2^{jd} \leq 2^{kd}$. This leads to

$$\begin{aligned} &\|\omega_k^0 | \Lambda^\rho B^{s+d}_{p,q}(\mathbb{R}^n)\| \\ &\lesssim 2^{kd} \sup_{J\in\mathbb{Z},M\in\mathbb{Z}^n} 2^{\frac{J}{p}(n+\rho)} \left(\sum_{J^+\leq j\leq k} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \left(\sum_{m:Q_{j,m}\subset Q_{J,M}} |\lambda_m^{j,G}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &\lesssim 2^{kd} \|\lambda | \Lambda^\rho b^s_{p,q}\|. \end{aligned} \tag{3.44}$$

Reasoning in the same way but now with the F-spaces and the $\Lambda^\rho f^s_{p,q}$ sequences, we also obtain

$$\|\omega_k^0 | \Lambda^\rho F^{s+d}_{p,q}(\mathbb{R}^n)\| \leq 2^{kd} \|\lambda | \Lambda^\rho f^s_{p,q}\|. \tag{3.45}$$

Hence, it follows from (3.44) and (3.45) that

$$\|\omega_k^0 | \Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\| \leq 2^{kd} \|\lambda | \Lambda^\rho a^s_{p,q}\| \sim 2^{kd} \|\omega_k^0 | \Lambda^\rho A^s_{p,q}(\mathbb{R}^n)\|. \tag{3.46}$$

We assumed at the very beginning that $s > \frac{|\rho|}{p}$. Hence, $s + d > \frac{|\rho|}{p}$ and

$$(W_t^\alpha \omega_k^0)(x) = (G_t^\alpha * \omega_k^0)(x) = \int_{\mathbb{R}^n} G_t^\alpha(y) \omega_k^0(x-y) dy$$

is well-defined.
Applying Minkowski's inequality and the translation invariance of the spaces, we can estimate

$$\|W_t^\alpha \omega_k^0 | \Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\| \quad \leq \quad \int_{\mathbb{R}^n} |G_t^\alpha(y)| dy \|\omega_k^0(\cdot - y) | \Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\|$$

$$\leq \quad C\|\omega_k^0|\Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\|,$$

where the constant is independent of t. Together with (3.44) and (3.45), we achieve

$$\|W_t^\alpha \omega_k^0|\rho\text{-}A^{s+d}_{p,q}(\mathbb{R}^n)\| \leq C2^{kd}\|\lambda|\Lambda^\rho a^s_{p,q}\|.$$

Since $2^{-k2\alpha} < t \leq 2^{-2\alpha(k-1)}$, we have the equivalence $t^{-d/2\alpha} \sim 2^{kd}$. Together with the wavelet characterization of ω, this leads to the estimate

$$\|W_t^\alpha \omega_k^0|\Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\| \quad \leq \quad C\, t^{-d/2\alpha}\|\omega|\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)\|, \tag{3.47}$$

where $0 < t \leq 1$ and $s > \frac{|\rho|}{p}$, $d \geq 0$.

Step 2: Let $s \leq \frac{|\rho|}{p}$.

We choose $\sigma > 0$ such that $s + \sigma > \frac{|\rho|}{p}$. Let $\omega \in \Lambda^\rho A^s_{p,q}(\mathbb{R}^n)$. Recall from part (ii) of Definition 7 that

$$W_t^\alpha \omega := I_{-\sigma}[W_t^\alpha(I_\sigma \omega)],$$

where $I_\sigma \omega \in \Lambda^\rho A^{s+\sigma}_{p,q}$. Let $d \geq 0$.
Then,

$$\begin{aligned}
\|W_t^\alpha \omega|\Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\| &= \|I_{-\sigma}[W_t^\alpha(I_\sigma\omega)]|\Lambda^\rho A^{s+d}_{p,q}(\mathbb{R}^n)\| \\
&\sim \|W_t^\alpha(I_\sigma\omega)|\Lambda^\rho A^{s+\sigma+d}_{p,q}(\mathbb{R}^n)\| \\
&\lesssim t^{-d/2\alpha}\|I_\sigma\omega|\Lambda^\rho A^{s+\sigma}_{p,q}(\mathbb{R}^n)\| \\
&\sim t^{-d/2\alpha}\|\omega|\Lambda^\rho A^s_{p,q}(\mathbb{R}^n)\|.
\end{aligned}$$

Finally, the corresponding result for the spaces $\Lambda_\rho A^s_{p,q}(\mathbb{R}^n)$ follows by real interpolation according to Remark 3, part (vi). This completes the proof of (3.34). □

3.4. Final remarks

Let us give a short outlook in view of possible applications. As mentioned in the introduction, estimates of type (1.1) play a significant role in the analysis of (fractional) evolution equations e.g., heat, Navier-Stokes, quasi-geostrophic, Keller-Segel or Burger's equations. We refer to the approach developed and elaborated in the monographs [25–27] which is related to the classical Gauss-Weierstrass semigroup (i.e., $\alpha = 1$ in (1.1)). Refined mathematical models, for example, in physics and chemotaxis, suggest and require us to replace the Laplacian by the fractional Laplacian $(-\Delta)^\alpha$ in related (nonlinear) evolution equations. As far as the study of corresponding Cauchy problems is concerned, let us mention, for example, the papers [5, 7, 15–17, 31]. For example, let us consider the Cauchy problem

$$\partial_t u(x,t) + (-\Delta)^\alpha u(x,t) = f(u(x,t)), \qquad x \in \mathbb{R}^n,\ 0 < t < T, \tag{3.48}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{3.49}$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha > 0$ and

$$f(u(x,t)) := Du^2(x,t) = \sum_{i=1}^n \frac{\partial}{\partial x_i} u^2(x,t)$$

stands for the nonlinear term. It serves as a scalar model case for fractional Navier-Stokes equations. For further types of nonlinear terms, we refer to the abovementioned papers. The standard approach to prove the existence and uniqueness of mild solutions is to consider the related fixed point problem $T_{u_0}u = u$, where the operator T_{u_0} is defined as

$$T_{u_0}u(x,t) := W_t^{\alpha}u_0(x) + \int_0^t W_{t-\tau}^{\alpha} f(u(x,\tau))\,\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T, \tag{3.50}$$

in appropriate function spaces. We are interested in vector-valued weighted Lebesgue spaces

$$L_v((0,T),\, b,\, X) := \left\{ u : (0,T) \to X,\ \int_0^T t^{bv}\, \|u(\cdot,t)|X\|^v \,\mathrm{d}t < \infty \right\},$$

as solution spaces. Here, $1 \leq v < \infty$ (usual modification if $v = \infty$), $b \in \mathbb{R}$, $0 < T \leq \infty$, and X is an appropriately chosen Banach space according to given initial data. For initial data belonging to Besov or Triebel-Lizorkin spaces $A^{s_0}_{p,q}(\mathbb{R}^n)$, this has been investigated in the hyper-dissipative case $\alpha \in \mathbb{N}$ in [1–3] as well as in [4] in the case of fractional α, where $X = A^{s}_{p,q}(\mathbb{R}^n)$. Here, parameters $s,\ b$, and v depend on $\alpha,\ p,\ s_0$ and the dimension n.

4. Conclusions

The smoothing property (1.1) paves the way to deal with Cauchy problems of the above type for initial data belonging to Morrey smoothness spaces $\rho\text{-}A^{s_0}_{p,q}(\mathbb{R}^n)$. As far as the case $\alpha = 1$ is concerned, partial results can be found in [26, Chapters 4 and 5] for hybrid spaces $\Lambda^{\rho}A^{s_0}_{p,q}(\mathbb{R}^n)$ (see also Remark 3, part (iv)). We intend to consider the general case of fractional α and certain classes of nonlinear terms f in (3.48) in forthcoming papers.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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