Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 31947–31961.
DOI:10.3934/math.20241535
Received: 16 September 2024
Revised: 28 October 2024
Accepted: 01 November 2024
Published: 11 November 2024

Research article

Note on p-ideals set of orthomodular lattices

Ziteng Zhao[1], **Jing Wang** [2] **and Yali Wu**[1,*]

[1] School of Mathematics and Science, Hebei GEO University, Shijiazhuang 050031, China
[2] School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China

* **Correspondence:** Email: wuyl_hb@163.com.

Abstract: This paper mainly discusses the problems raised in Kalmbach's book: When are the p-ideals of an irreducible orthomodular lattice well ordered under set inclusion? We give three classes of orthomodular lattices whose p-ideals set is a chain under set inclusion. Furthermore, this article also provides a sufficient and necessary condition for the p-ideals set of orthomodular lattices to be a chain under set inclusion from the perspective of L-algebras, which gives a new point to solve the questions. Moveover, we also give some characterizations about central elements of orthomodular lattice.

Keywords: orthomodular lattices; OM-L-algebras; p-ideals; L-ideals; well ordered
Mathematics Subject Classification: 03G10, 03G12, 06B10

1. Introduction

The concept of orthomodular lattices (referred to as OML for convenience) originated from von Neumann algebraic theory. In 1936, Birkhoff and von Neumann proposed using lattices of closed subspaces in Hilbert spaces as the fundamental mathematical model for studying quantum mechanics. However, the projection lattice of a Hilbert space is modular only if the Hilbert space is finite dimensional. However, the projection lattice of type II von Neumann algebras related to continuous geometries—especially those favoured by John von Neumann—is modular. Therefore, if the projection lattice is required to be a modular lattice, the case of infinite-dimensional Hilbert spaces is excluded. As a remedy, Husimi proposed to study the following orthomodular law instead:

$$p \leq q \Rightarrow q = p \vee (p' \wedge q),$$

this law, indeed, holds in arbitrary Hilbert spaces.

Finch [1] shows that a congruence relation in an OML L is completely determined by its kernel and that an ideal I of L is called a p-ideal or orthomodular ideal. Let $I_P(L)$ denote the set of all p-ideals of an orthomoular lattice. Then, $I_p(L)$ is a complete distributive lattice. In the literature [2], it is raised: when are p-ideals of an irreducible orthomodular lattice well ordered under set inclusion? In the paper, we exhibit three big classes of orthomodular lattices where the answer is positive.

During the past decade, L-algebras occurred in various contexts, including algebraic logic, combinatorial group theory, operator algebras, and number theory (see, e.g., [3–6]). Wu and Yang provide a succinct description of orthomodular lattices [7], which is called OM-L-algebra. The ideals of an OM-L-algebra L are equivalent to p-filters of orthomodular L. The paper provides a sufficient and necessary condition for the p-ideals set of orthomodular lattices to be a chain under set inclusion from the perspective of L-algebra.

The paper is organized as follows: In Section 2, we recall some basic definitions and facts related to OML L and we exhibit three big classes of OMLs where its p-ideals are set to be a chain under the set inclusion relation. In Section 3, we focus on the OM-L-algebras. A sufficient and necessary condition for the L-ideals set of OM-L-algebra to be a chain is given.

2. P-Ideals set of orthomodular lattices

At the beginning of this section, we introduce our basic terminology and present some results.

Definition 2.1. [2] An *ortholattice* is a structure $(L, \leq, ', 0, 1)$, where $(L, \leq, ', 0, 1)$ is a lattice with maximum 1 and minimum 0, and $'$ is a unary operation on L such that the following conditions are satisfied:

$$\begin{aligned} &(Involutive\ law) && a = a''; \\ &(Antitony) && if\ a \leq b,\ then\ b' \leq a'; \\ &(Exclude\ middle\ law) && a \vee a' = 1; \\ &(Non-contradiction\ law) && a \wedge a' = 0. \end{aligned}$$

An orthomodular lattice (OML) is an ortholattice $(L, \leq, ', 0, 1)$, that satisfies the following condition for $\forall\ a, b \in L$:

$$(Orthomodular\ identity) \qquad if\ a \leq b,\ then\ b = a \vee (a' \wedge b). \tag{2.1}$$

In an ortholattice L, we write aCb (in words, a commutes with b) if $a = (a \wedge b) \vee (a \wedge b'), a, b \in L$. Let L be an OML. It follows that aCb, aCb', $a'Cb$ and $a'Cb'$ are equivalent. In particular, aCb is characterized by $a = (a \vee b) \wedge (a \vee b')$. Let L be a lattice. For $x, y, z \in L$, we call $\{x, y, z\}$ is a distributive triple, if the elements of a triple $\{x, y, z\}$ can generate a distributive sublattice. In any orthomodular lattice L, let aCb and aCc. Then $\{a, b, c\}$ generate a distributive sublattice of L [8]. If M is a subset of an orthomodular lattice L, the set $C(M) = \{x \in L \mid xCb, \forall b \in M\}$ is called the *commutant* of M in L. The elements of $C(L)$ are called *central* and $C(L)$ *the centre of* L. For every subset M of L, the set $C(M)$ is a subalgebra of L containing arbitrary joins and meets, provided they exist in L. In an OML L, $x \in L$, then x has a unique complement if and only if $x \in C(L)$ [2].

Definition 2.2. *Let a, b be elements of an orthomodular lattice L [2].*

(i). $a, b \in L$, a is said to be *perspective* to b, $a \sim b$, if a and b have a common complement, i.e., if there exists c in L such that $a \vee c = 1 = b \vee c, a \wedge c = 0 = b \wedge c$ hold; elements a, b are said to be *strongly* *perspective*, $a \sim_s b$, if a and b have a common complement c in the sublattice $[0, a \vee b]$, the element c then is called a *relative complement of a and b*; it is easy to see that strong perspectivity implies perspectivity. Note, by [2] (the parallelogram law for $\sim$), we have $\forall a, b \in L$, $a \wedge (a' \vee b) \sim b \wedge (b' \vee a)$.

(ii). *x is projective to y*, denoted $x \approx y$, if there exists a sequence of elements $x_1, ..., x_n$ such that $x_1 = x$, $x_n = y$, and $x_i \sim x_{i+1}$ hold for $1 \leq i \leq n-1$.

(iii). A *p-ideal* in L is an ideal I closed under perspectivity, i.e., $a \in I$ and $a \sim b$ imply $b \in I$. Dually, a *p-filter* in L is a filter F closed under perspectivity, i.e., $a \in F$ and $a \sim b$ imply $b \in F$.

Let L be an OML; $a, b, c \in L$. [2] presents: If $c \leq a$ and $a \sim b$, then there exists $d \in L$ such that $d \leq b$ and $c \sim d$ holds. Dual, we have the following result:

Lemma 2.1. *Let L be an orthomodular lattice. Assume that $a, b, c \in L$ satisfies $c \sim a$ and $a \leq b$. Then, there exists d such that $c \leq d$ and $d \sim b$ holds.*

Proof. Let y be the common complement of c and a, satisfy $y \vee c = y \vee a = 1, y \wedge c = y \wedge a = 0$. Define $z = y \wedge (b' \vee y') \leq y$, $d = c \vee (c' \wedge z') \geq c$. Then, z is the common complement of d and b, i.e., $d \vee z = b \vee z = 1, d \wedge z = b \wedge z = 0$. In fact, $d \vee z = (c \vee (c' \wedge z')) \vee z = (c \vee z) \vee (c' \wedge z') = (c \vee z) \vee (c \vee z)' = 1$. Since $b', y' \leq b' \vee y'$, $\{b, y, b' \vee y'\}$ is a distributive triple. We have $b \vee z = b \vee (y \wedge (b' \vee y')) = (b \vee y) \wedge (b' \vee y' \vee b) = b \vee y \geq a \vee y = 1$, so $b \vee z = 1$. Since $c' \wedge z' \leq c', z'$, $\{c, z, c' \wedge z'\}$ is a distributive triple. We have $d \wedge z = (c \vee (c' \wedge z')) \wedge z = (c \wedge z) \vee (c' \wedge z' \wedge z) = (c \wedge z) \vee 0 = c \wedge z \leq c \wedge y = 0$, $b \wedge z = b \wedge (y \wedge (b' \vee y')) = (b \wedge y) \wedge (b' \vee y') = (b \wedge y) \wedge (b \wedge y)' = 0$. □

Corollary 2.1. Let L be an orthomodular lattice. For $a, b, c \in L$ with $a \approx b \leq c$. Then, there exists d such that $a \leq d \approx c$ holds.

Proof. Assume first that $a \approx b \leq c$. There exist $z_1, z_2, \cdots, z_n$ such that $a = z_1 \sim \cdots \sim z_{n-2} \sim z_{n-1} \sim z_n = b \leq c$. For $z_{n-1} \sim z_n = b \leq c$. By Lemma 2.1, $\exists\, d_n \in L$, we have $a = z_1 \sim \cdots \sim z_{n-2} \sim z_{n-1} \leq d_n \sim c$. Repeat to use the lemma 2.1, $\exists\, d_{n-1}, d_{n-2}, \cdots, d_2$, such that $a = z_1 \leq d_2 \cdots \sim d_{n-2} \sim d_{n-1} \sim d_n \sim c$, i.e., $\exists\, d = d_2$, $a \leq d \approx c$. □

Let L be an orthomodular lattice. In [1], Finch shows that $Con(L)$ is isomorphic to $I_p(L)$, where $Con(L)$ represents the set of all congruence relations on L, $I_p(L)$ represents the set of all p-ideals on L, i.e., congruence relations in an OML L is completely determined by its kernel, and an ideal I of L is the kernel of a congruence relation if and only if it is p-ideal (or orthomodular ideal). There are a lot of other conditions that can be used to define p-ideal. More precisely, we have:

Lemma 2.2. [2] *If I is an ideal of orthomodular latticeL, then, the following statements are equivalent:*

(i) I is a p-ideal;
(ii) $a \in I$ implies $x \wedge (x' \vee a) \in I$ for all $x \in L$.

Dual to Lemma 2.2, we have the following result:

Lemma 2.3. [9] *If F is a filter of orthomodular lattice L, then, the following statement are equivalent:*

(i) F is a p-filter;
(ii) $a \in F$ implies $x \vee (x' \wedge a) \in F$ for all $x \in L$.

Remark 2.1. *Let L be an orthomodular lattice. If I be an ideal of L, $I' = \{x' \in L \mid x \in I\}$, by Lemmas 2.2 and 2.3, we can obtain I is a p-ideal of L if and only if I' is a p-filter of L, and vice versa.*

Lemma 2.4. *Let L be an orthomodular lattice, $y \in L$. Then, $[0, y]$ is a p-ideal if and only if $y \in C(L)$.*

Proof. Let $y \in C(L)$. Clearly, $[0, y]$ is the principle ideal of L. Set $a \in [0, y]$, i.e., $0 \leq a \leq y$. Since $y \in C(L)$, then $\forall x \in L$, xCy. Then, $0 \leq x \wedge (x' \vee a) \leq x \wedge (x' \vee y) \leq (x \vee y) \wedge (x' \vee y) = y$. Consequently, $x \wedge (x' \vee a) \in [0, y]$. By Lemma 2.2, we have $[0, y]$ is a p-ideal.

Conversely, if $[0, y]$ is a p-ideal, $\forall x \in L$, since $x', y \leq x' \vee y$, then $x'C(x' \vee y), yC(x' \vee y)$. Then, we have $(x' \vee y)Cx, (x' \vee y)Cy$, so $\{x, y, x' \vee y\}$ is a distributive triple; hence, $(x' \vee y) \wedge (x \vee y) = ((x' \vee y) \wedge x) \vee ((x' \vee y) \wedge y) = ((x' \vee y) \wedge x) \vee y$. Since $[0, y]$ is a p-ideal and $y \in [0, y]$, by Lemma 2.2, we have $x \wedge (x' \vee y) \in [0, y]$, i.e., $x \wedge (x' \vee y) \leq y$. So, $(x' \vee y) \wedge (x \vee y) = ((x' \vee y) \wedge x) \vee y = y$, i.e., $y \in C(L)$. □

The properties of commute listed above, together with Lemma 2.4, imply.

Corollary 2.2. *Let L be an orthomodular lattice, $[y, 1]$ is a p-filter if and only if $y \in C(L)$.*

Proposition 2.1. *Let L be an orthomodular lattice. Then, $c \in C(L)$ if and only if $\forall a \in L, c \sim a$ implies $a = c$.*

Proof. Let $c \in C(L)$, $c \sim a$. Then, a and c have a common complement $c' \in L$. Since $aCc \Leftrightarrow a = (c \vee a) \wedge (c' \vee a) = (c \vee a) \wedge 1 = c \vee a$, then $c \leq a$. Similarly, $cCa \Leftrightarrow c = (c \vee a) \wedge (c \vee a') = (c \vee a) \wedge (c' \wedge a)' = (c \vee a) \wedge 0' = (c \vee a) \wedge 1 = c \vee a$, which implies $a \leq c$. Thus, $a = c$.

Conversely, assuming $c \notin C(L)$, we have $c' \notin C(L)$. Then, c' has at least two different complements, a, c. Obviously, if $c \sim a$, which is contrary to $a = c$. Hence, $c \in C(L)$. □

Corollary 2.3. *Let L be an orthomodular lattice. Then, $c \in C(L)$ if and only if $\forall a \in L, c \approx a$ implies $a = c$.*

Proof. Let $c \in C(L)$ and $c \approx a$. Then, $\exists\, d_1, d_2, ..., d_n \in L$, $c = d_1 \sim d_2 \sim ... \sim d_n = a$. Since $c \sim d_2$, By proposition 2.1, we have $c = d_2$, and repeat to use proposition 2.1, we have $c = d_1 = d_2 = ... = d_n = a$.

Conversely, Let $\forall a \in L, c \approx a$ implies $a = c$ and $c \notin C(L)$. Proposition 2.1 yields $\exists\, a \in L, c \sim a$ and $a \neq c$. Clearly, $c \sim a$ implies $c \approx a$, it can be inferred that $a = c$, which is a contradiction. Hence, $c \in C(L)$. □

An *order ideal* in a lattice L is a subset I, such that $x \in L, y \in I$, and $x \leq y$ imply $x \in I$.

Lemma 2.5. [2] Let L be an orthomodular lattice and M be an order ideal of L that is closed under perspectivity:

(i) The smallest p-ideal containing M is the set $B = \{c \in L\ |$, there exists a finite subset C of M with c= $\bigvee C\}$,

(ii) The set $N_a = \{d \in L \mid d \approx e, e \leq a\}$ is an order ideal and is closed under perspectivity. P_a denotes the smallest p-ideal containing N_a.

Proposition 2.2. *Let L be an orthomodular lattice. Then, $c \in C(L)$ if and only if $N_c = [0, c]$.*

Proof. Let $c \in C(L)$. Lemma 2.5 gives $[0, c] \subseteq N_c$. Let $\forall d \in N_c$. Then, $\exists\, e \leq c, d \approx e$. By Corollary 2.1, we have $\exists\, d_1 \in L, d \leq d_1$ and $d_1 \approx c$. By Corollary 2.3, we have $d_1 = c$. Hence, $d \leq d_1 = c$, and then $d \in [0, c]$, hence $N_c \subseteq [0, c]$, so $N_c = [0, c]$.

If $c \notin C(L)$, we have $c' \notin C(L)$. Then, c' has at least two different complements a, c, whence $a \in N_c = [0, c]$. Thus, $a \leq c$, whence aCc. Then, $a = (a \vee c) \wedge (a \vee c') = (a \vee c) \wedge 1 = c \vee a = c$, which is a contradiction. Hence, $c \in C(L)$. □

Proposition 2.3. *Let L be an orthomodular lattice. Then, $N_c \cap N_{c'} = \{0\}$ if and only if $c \in C(L)$.*

Proof. Let $c \sim a$. Then, $c' \sim a'$. Since $a \sim c \in N_c, a' \sim c' \in N_{c'}$. By Lemma 2.5, we have $a \in N_c, a' \in N_{c'}$. Then, $c \wedge (c' \vee a') \sim a' \wedge (a \vee c)$, since $a' \wedge (a \vee c) \leq a' \in N_{c'}$, then $c \wedge (c' \vee a') \in N_{c'}$. $c \wedge (c' \vee a') \leq c$ yields $c \wedge (c' \vee a') \in N_c$. Thus, $c \wedge (c' \vee a') \in N_c \cap N_{c'}$. Similarly, $a \wedge (a' \vee c') \in N_c \cap N_{c'} = \{0\}$. Then, $c \wedge (c' \vee a') = c \wedge (c \wedge a)' = 0$, since $N_c \cap N_{c'} = \{0\}$. Then, $c \leq a$. Similarly, $a \leq c$, i.e., $a = c$. Thus, proposition 2.1 implies that $c \in C(L)$.

Conversely, let $c \in C(L)$. By Proposition 2.2, we get $N_c = [0, c], N_{c'} = [0, c']$. Assume that $d \in N_c \cap N_{c'}$, then $d \leq c, c' \Leftrightarrow d \leq c \wedge c' = 0$, i.e., $d = 0$. Hence, $N_c \cap N_{c'} = \{0\}$. □

Proposition 2.4. *Let L be an orthomodular lattice. M and N are order ideals and closed under perspectivity; P_M and P_N are the smallest p-ideals containing M and N, respectively. Then, $M \cap N = \{0\} \Leftrightarrow P_M \cap P_N = \{0\}$.*

Proof. Let $d \in P_M \cap P_N$; we have $d \in P_M, P_N$. Then, Lemma 2.5 yields that there exists a finite subset C of M, with $d = \vee C$. Let $e_i \in C, \forall i \in I$, then $e_i \leq \vee C = d \in P_M \cap P_N$. Since $P_M \cap P_N$ is an ideal, then $e_i \in P_M \cap P_N \subseteq P_N$. By Lemma 2.5, there exists a finite subset D_i of N, with $e_i = \vee D_i$. Let $f_{ij} \in D_i, \forall j \in J$, we have $f_{ij} \in D_i \subseteq N$ and $f_{ij} \leq \vee D_i = e_i \in C \subseteq M$. Then, $f_{ij} \in M \cap N = \{0\}$, i.e., $f_{ij} = 0$. Hence, $d = \vee C = \underset{i \in I}{\vee} e_i = \underset{i \in I}{\vee} \underset{j \in J}{\vee} f_{ij} = 0$. Therefore, $P_M \cap P_N = \{0\}$.

Conversely, since $M \subseteq P_M$, $N \subseteq P_N$, $M \cap N \subseteq P_M \cap P_N = \{0\}$. So, $M \cap N = \{0\}$. □

Recall that a poset P satisfies the *ascending condition* (*ACC*) and is called *Noetherian* when every nonvoid subset of P has a maximal element. It satisfies the descending chain condition (DCC) when its dual satisfies the ACC. Nonvoid chains that satisfy the *DCC* are called *well-ordered sets*. The *length* $l(P)$ of a poset P is defined as the least upper bound of the lengths of the chains in P. When $l(P)$ is finite, P is said to be *finite length*. A poset P is *chain-finite* if all chains in L are finite. It is easy to show that a finite length lattice must be chain-finite; a chain-finite lattice must be Noetherian. The *dual* of a poset P is that poset $\widetilde{P}$ defined by the converse partial ordering relation on the same elements.

Lemma 2.6. [8] The following conditions about a poset P are equivalent:
(i) P is *Noetherian*,
(ii) every chain in the dual $\widetilde{P}$ of P is well-ordered,
(iii) every well-ordered subset of P is finite.

Lemma 2.7. [10] *In every infinite boolean algebra, there is an infinite pairwise disjoint family, a strictly increasing infinite sequence is and a strictly decreasing infinite sequence.*

Next, we give a characterization of Noetherian orthomodular lattice and boolean lattice.

Proposition 2.5. *Let L be a boolean lattice. Then, L is Noetherian if and only if it is finite.*

Proof. Let L be a Noetherian boolean lattice. Assume that L is infinite. By Lemma 2.7, L has a strictly increasing sequence A. Since L is a Noetherian lattice, then $A' = \{x \in L | x' \in A\}$ is also a Noetherian lattice. Let B be any subset of A, then B' is a subset of A', so B' has a maximal element b', which is equivalent to B having a minimal element, i.e., A satisfies the descending chain condition, thus A is well-ordered. By Lemma 2.6, A is finite, which is a contradiction. Thus, L is finite. Conversely, any well ordered subset of a finite lattice is finite. By Lemma 2.6, a finite lattice is a Noetherian lattice. □

Proposition 2.6. *Let L be an orthomodular lattice. Then, L is Noetherian if and only if it is chain-finite.*

Proof. Assume that L has an infinite chain, say A. Then A generates a boolean subalgebra B of L; B is also Noetherian. By Proposition 2.5, B is finite; it follows that A is finite, which is a contradiction. Then L is chain-finite. The converse is trivial. □

Corollary 2.4. *Let L be an orthomodular lattice. Then, L is Noetherian ⇔ L satisfies DCC.*

Proof. L is Noetherian ⇔ L is chain-finite ⇔ $\widetilde{L}$ is chain-finite ⇔ $\widetilde{L}$ is Noetherian ⇔ L satisfies DCC. □

Let $(P; \leq)$ be a partially ordered set, $a, b \in P$ and $a \leq b$. If $a < b$ and $[a, b] = \{a, b\}$, then b *covers* a (or a is covered by b). We use the symbol $b \succ a$ (or $a \prec b$)to indicate that b covers a (or a is covered by b). Let P have a minimum element of 0. We call $a \in L$ an atom of P if $0 \prec a$. Let L be a lattice. if for all $x \in L \setminus \{0\}$, there exists an atom a of L such that $a \leq x$. Then, L is called an atomic lattice. Dually, let L be a lattice with a maximum element 1. And we call $b \in L$ is a coatom of L if $b \prec 1$. For all $y \in L \setminus \{1\}$, there exists a coatom b of L such that $y \leq b$. Then, L is called a coatomic lattice. Let L be an atomic orthomodular lattice. For $\forall x \in L - \{1\}$, we have $x = \bigvee\{a \in L \mid a \leq x, a \textit{ is an atom}\}$. Dually, let L be a coatomic orthomodular lattice. For $\forall y \in L - \{0\}$, we have $x = \bigwedge\{b \in L \mid b \geq x, b \textit{ is a coatom}\}$. An element b of an atomic orthomodular lattice is finite if it is the join of finitely many atoms.

Corollary 2.5. *Let L be a Noetherian orthomodular lattice. Then, L is atomic.*

Proof. L is Noetherian. Corollary 2.4 implies that L satisfies DCC (Descending Chain Condition), i.e., every nonvoid subset of P has a minimal element. Thus, for $\forall x \in L$ and $0 < x$, $(0, x]$ has a minimal element, which is an atom. This proves that L is atomic. □

We are now ready to have a look at the p-ideals set of irreducible OML. First, however, we shall need some additional machinery.

Definition 2.3. [2] *Every orthomodular lattice L is* irreducible*, i.e., L is isomorphism to a product* $M \times N$ *of orthomodular lattices implies that* $|M| = 1$ *or* $|N| = 1$*, where* $|M|$ *denotes cardinality of M.*

Remark 2.2. It is easy to see that $C(L) = \{0, 1\}$ is equivalent to the irreducibility of L.

Lemma 2.8. [2] *Let L be an orthomodular lattice. If L is chain-finite. Then,* $Con(L)$ *is isomorphic to* $C(L)$*. In particular,* $Con(L)$ *is a boolean algebra.*

Theorem 2.1. *Let L be an irreducible orthomodular lattice and* Noetherian*. Then, the p-ideals are well ordered under set inclusion.*

Proof. Let L be *Noetherian*. Proposition 2.6 implies that L is chain-finite. Since L is irreducible, Remark 2.2 implies that $C(L) = \{0, 1\}$. By Lemma 2.8 and the fact that $Con(L)$ and $I_p(L)$ are isomorphic, we have $I_p(L) = \{\{0\}, L\}$, Hence, the p-ideals set is well ordered under set inclusion. □

Next, we introduce our basic terminology and notations, which are borrowed from [2]:
Let L be an OML. The binary relation $\nabla, S^0, S^1, S^\infty$ are defined in an orthomodular lattice L by:

$$a \nabla b \quad \text{if } (a \vee x) \wedge b = x \wedge b \text{ for all } x \in L.$$
$$aS^0 b \quad \text{if } c \leq a, d \leq b \text{ and } c \sim_s d \text{ imply } c = 0 = d.$$

$$aS\,b \quad \text{if } a \wedge b = 0 \text{ and } a \in C([0, a \vee b]).$$
$$aS^1b \quad \text{if } c \leq a, d \leq b \text{ and } c \sim d \text{ imply } c = 0 = d.$$
$$aS^\infty b \quad \text{if } c \leq a, d \leq b \text{ and } c \approx d \text{ imply } c = 0 = d.$$

Immediately, we can obtain $aS^\infty b \Rightarrow aS^1b \Rightarrow aS^0b$. Furthermore, we have the following results:

Lemma 2.9. [2] *Let L be an orthomodular lattice. For $\forall a, b \in L$, $aS^0b \Leftrightarrow aS\,b$.*

Lemma 2.10. [2] *For elements a, b of an orthomodular lattice L, the following statements are equivalent:*
(i) $a\nabla b$,
(ii) aS^1b,
(iii) $a \vee x = 1$ *implies* $b \leq x$,
(iv) $(a \vee x) \wedge (b \vee x) = x, \forall x \in L$.

Proposition 2.7. *Let L be an orthomodular lattice, $a, b \in L$. Then, $N_a \cap N_b = \{0\}$ if and only if $aS^\infty b$.*

Proof. For $c, d \in L$, let $c \leq a, d \leq b, c \approx d$, then $c, d \in N_a \cap N_b = \{0\}$, whence $c = 0 = d$. Hence, $aS^\infty b$.
Conversely, let $d \in N_a \cap N_b$. There exists c, e, such that $d \approx c \leq a, d \approx e \leq b$. Then, $c \approx e$. Since $aS^\infty b$, we have $c = 0 = e$. Thus, $d = 0$, which proves that $N_a \cap N_b = \{0\}$. □

Corollary 2.6. *Let L be an orthomodular lattice, $a \in L$. Then, $a\nabla a' \Leftrightarrow aS^1a' \Leftrightarrow aS^0a' \Leftrightarrow aS\,a' \Leftrightarrow aS^\infty a' \Leftrightarrow a \in C(L)$.*

Proof. By Lemma 2.9, we have $aS^0b \Leftrightarrow aS\,b$. By Lemma 2.10, this gives $a\nabla a' \Leftrightarrow aS^1a' \Leftrightarrow a \in C(L)$ and $aS^\infty b \Rightarrow aS^1b \Rightarrow aS^0b$. It suffices to prove $a \in C(L) \Rightarrow aS^\infty a'$ and $aS\,a' \Rightarrow a \in C(L)$. For $a \in C(L) \Rightarrow aS^\infty a'$, by Propositions 2.3 and 2.7. We have $N_a \cap N_{a'} = \{0\}$ and $aS^\infty a'$, then $a \in C(L) \Rightarrow aS^\infty a'$. Let $aS\,a'$, we obtain $a \in C([0, a \vee a']) = C([0, 1]) = C(L)$. So, $aS^\infty a' \Rightarrow aS^0a' \Rightarrow aS\,a' \Rightarrow a \in C(L)$. □

Let L be an orthomodular lattice, $a \in L$, $M = \{z \in C(L) \mid a \leq z\}$. If the infimum of M exists, we denote $v(a) = \bigwedge M$ and say $v(a)$ is the *center cover* of a. If for $\forall a \in L$, $C([0, a]) = \{a \wedge c \mid c \in C(L)\}$, we say that the *relative centre property* holds.

Lemma 2.11. [2] *Assume that the orthomodular lattice L has the relative centre property. For elements a, b of L, the following statements are equivalent:*

(i) There exists $c \in C(L)$ with $a \leq c$ and $b \leq c'$,
(ii) aS^1b,
(iii) aS^0b.

Lemma 2.12. [2] *Let L be a complete orthomodular lattice. For $\forall a, b \in L$, there exists $c, d, e, f \in L$ such that the following conditions hold:*

(i) $a = c \vee d, b = e \vee f$,
(ii) $c \leq d', e \leq f'$,
(iii) $c \sim_s e$,
(iv) dS^0f.

Theorem 2.2. *Let L be a complete irreducible orthomodular lattice with the relative centre property. Then, the p-ideals set $I_p(L)$ is a chain under set inclusion.*

Proof. If L is irreducible, then by Remark 2.2, we get $C(L) = \{0, 1\}$. If L has the relative centre property, by Lemma 2.11, we have $aS^0b \Leftrightarrow a = 0$ or $b = 0$. Next, we prove $\forall a, b \neq 0, 1$, $P_a \subseteq P_b$ or $P_b \subseteq P_a$. In fact, Lemma 2.12 implies that there exists $c = a, d = 0, b = e \vee f, e \leq f', c \sim_s e$, or $b = e, f = 0, a = c \vee d, c \leq d', c \sim_s e$. In the first case, $a = c \sim e \leq b$ implies $a \in N_b$ and thus $N_a \subseteq N_b$. Similarly, $N_b \subseteq N_a$, which yields $P_a \subseteq P_b$ or $P_b \subseteq P_a$. Let $I, J \in I_p(L), I \nsubseteq J, J \nsubseteq I$. Then, $\exists\, x \in I - J, y \in J - I$. If $P_x \subseteq P_y$, we have $x \in P_y \subseteq J$, thus $x \in J$, which is a contradiction; if $P_y \subseteq P_x$, we obtain $y \in P_x \subseteq I$, which is impossible. Then, $I_p(L)$ is a chain. □

Now, we consider the OML with exchange axiom, which states the following:

We say that the atomic orthomodular lattice L satisfies the *exchange axiom* if one of the equivalent conditions of the following conditions holds for L:

(i) For atoms p, q of L, the conditions $p \leq q \vee a$ and $p \wedge a = 0$ imply $q \leq p \vee a$;

(ii) If for an atom p of L holds $p \neq a$, then $p \vee a$ covers a;

(iii) If a covers $a \wedge b$, then $a \vee b$ covers b.

Lemma 2.13. [2] *Let L be an atomic complete orthomodular lattice satisfying the exchange axiom. Then, L is irreducible if and only if every two atoms of L are perspective.*

Theorem 2.3. *Let L be an atomic irreducible complete orthomodular lattice satisfying the exchange axiom. Then* 1 *is finite* $\Leftrightarrow P_r = L \Leftrightarrow \forall x \in L, x$ *is finite, where* $\forall r \in A(L)$, $A(L)$ *is the set of all atoms of L. In particular, in the case,* $I_p(L)$ *is well ordered under set inclusion relations.*

Proof. First, we show that p is an atom, $p \sim q$, then q is an atom. In fact, let a be a common complement of p, q. Then, $p \wedge a = 0 \prec p \Rightarrow a \prec p \vee a = 1 = q \vee a \Rightarrow 0 = q \wedge a \prec q$, i.e., q is an atom. Then, $\forall r \in A(L)$, $N_r = \{x | x \approx d \leq r\} = \{0\} \cup A(L)$. By Lemma 2.5, we have $P_r = \{x = q_1 \vee q_2 \vee \cdots \vee q_n | q_i \in A(L),\ i = 1, 2, \cdots, n\}$. Then, 1 is finite $\Rightarrow 1 \in P_r \Rightarrow P_r = L \Rightarrow \forall x \in L, x \in P_r \Rightarrow x$ is finite $\Rightarrow$ 1 is finite. For $\forall 0 \neq a \in L, \forall r \in A(L)$, we have $r \in N_a$. In fact, since $0 < a$, then $\exists\, s \in A(L)$, $s \leq a$. Thus, $r \sim s \leq a$ implies $r \in N_a$. It is a routine to show $P_r \subseteq P_a, \forall 0 \neq a \in L, \forall r \in A(L)$. So, $I_p(L) = \{\{0\}, L\}$ is well ordered, as desired. □

3. L-Ideals set of OM-L-algebras

Definition 3.1. [7] *An* L-algebra *is an algebra* $(L, \to, 1)$ *of type* $(2, 0)$ *satisfying:*

$$x \to x = x \to 1 = 1, 1 \to x = x. \tag{3.1}$$

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \tag{3.2}$$

$$x \to y = y \to x = 1 \Rightarrow x = y. \tag{3.3}$$

For $\forall x, y, z \in L$, condition (3.1) states that 1 is a logical unit. Note that a logical unit is always unique. There is a partial ordering by [7, P. 2332, Prop.2].

$$x \leq y \Leftrightarrow x \to y = 1. \tag{3.4}$$

Let L be an L-algebra. By [3], $x \leq y$ implies that $z \to x \leq z \to y$ for $\forall x, y, z \in L$. An L-algebra L is a *KL-algebra*. If it satisfies:

$$x \leq y \to x. \tag{3.5}$$

Definition 3.2. [11] *Let L be an L-algebra. A subset Y of L is said to be an* L-subalgebra *if* $\forall x, y \in Y$ *implies* $x \to y \in Y$. *If* $x \to y \in Y$ *holds for* $\forall x \in L, y \in Y$, *we call Y an* invariant L-subalgebra.

Definition 3.3. [3] *Let* $(L, \to, 1)$ *be an L-algebra. We call* $I \subseteq L$ *an* ideal *if the following holds for all* $x, y \in L$:

$$1 \in I, \tag{I0}$$

$$x, x \to y \in I \Rightarrow y \in I, \tag{I1}$$

$$x \in I \Rightarrow (x \to y) \to y \in I, \tag{I2}$$

$$x \in I \Rightarrow y \to x \in I, y \to (x \to y) \in I. \tag{I3}$$

We call the ideal of L-algebra is an L-ideal. For an element $x \in L$, $\langle x \rangle$ represents the L-ideal generated by x. We will use $I(L)$ to denote the set of all L-ideals of an L-algebra L.

Regarding the relevant background of L-algebra, we refer to [3].

We call an L-algebra $(L, \to, 1)$ self-similar if, for $\forall x \in L$, the left multiplication $\varepsilon_x : y \mapsto x \to y$ induces a bijection $\downarrow x \to L$. The inverse of ε_x gives rise to a multiplication $xy := \varepsilon_y^{-1}(x)$ that happens to be associative. This allows to describe a self-similar L-algebra equationally as a monoid with an operation $\to$ satisfying:

$$a \to ba = b, \tag{3.6}$$

$$ab \to c = a \to (b \to c), \tag{3.7}$$

$$(a \to b)a = (b \to a)b. \tag{3.8}$$

Axiom (3.6) implies that as a monoid; a self-similar L-algebra is right cancellative. Let L be a self-similar L-algebra. For $a, b \in L$, we define

$$a \wedge b := (a \to b)a. \tag{3.9}$$

Then, (3.9) is an infimum, and for $a, b, c \in L$, we have

$$a \to (b \wedge c) = (a \to b) \wedge (a \to c). \tag{3.10}$$

Let L be a self-similar L-algebra. Then L satisfies

$$a \to bc = ((c \to a) \to b)(a \to c). \tag{3.11}$$

Every L-algebra L admits a self-similar closure, a self-similar L-algebra $S(L)$ with L as an L-subalgebra such that as a monoid, $S(L)$ is generated by L. Up to isomorphism of left hoops, $S(L)$ is the unique self-similar closure of L.

In this article, we regard $(L, \to, 1, 0)$ as OM-L-algebra; $S(L)$ is the self-similar closure of L; for $\forall a \in S(L)$, define $a' = a \to 0$. We call $C(L) = \{a \in L \mid ab = ba, \forall b \in L\}$ the centre of L.

Definition 3.4. [7] *We call* $(L, \to)$ *an* OM-L-algebra *if it is an L-algebra with* 0 *(i.e., L admits the smallest element* 0*) and satisfies the following condition, where* $x' := x \to 0$.

$$x' \leq y \Rightarrow y \to x = x. \tag{3.12}$$

Lemma 3.1. [7] *Let L be an OM-L-algebra. Then, L is an orthomodular lattice and satisfies the following conditions, where* $x' := x \to 0$.

$$x \wedge y = (x \to (x \to y)')'. \tag{3.13}$$

$$x \vee y = (x' \to y') \to x. \tag{3.14}$$

According to the proof process of [7], (3.12) is equivalent to $x' \leq y \Leftrightarrow x \to y = y \Leftrightarrow y \to x = x$. In particular, $\forall x, y \in L, x' \leq x \to y \Leftrightarrow x' \to (x \to y) = 1$, whence $x \to (x \to y) = x \to y$, and $(x \to y) \to x = x$. This shows that OM-L-algebra is sharp [12].

Lemma 3.2. [7] *Every orthomodular lattice L gives rise to an OM-L-algebra* $(L, \to, 1, 0)$*, where*

$$x \to y := x' \vee (x \wedge y). \tag{3.15}$$

Lemma 3.3. [7] *Let L be an OM-L-algebra. Then,* $0 \leq a \Leftrightarrow a \in L$.

Lemma 3.4. [7] *Let L be an OM-L-algebra,* $a, b \in L$*. The equivalence* $aCb \Leftrightarrow ab = ba$ *is valid.*

Lemma 3.5. [7] *Let L be an OM-L-algebra.* $a, b \in L$*. The equivalence* $ab = ba \Leftrightarrow a \leqslant b \to a$ *holds. In particular,* $a \leq b$ *implies* $ab = ba$.

Lemma 3.6. [7] *Let L be an OM-L-algebra. For all* $a, b \in L$*, the following are satisfied:*

$$a \leq b \to a \Leftrightarrow b \to a = a' \to b'. \tag{3.16}$$

$$a \leq b \to a \Leftrightarrow b \leq a' \to b. \tag{3.17}$$

$$a \leq b \to a \Rightarrow (b \to a) \to a = a \vee b = (a \to b) \to b. \tag{3.18}$$

Remark 3.1. *Let L be an OM-L-algebra. Then, for all* $a, x \in L, a \leq x \to a$ *or* $a \parallel x \to a$*, where* $x \parallel y$ *denotes* x, y *are incomparable, i.e.,* $x \nleq y$ *and* $y \nleq x$*. In fact, if not, then there exists* $b \in L$ *such that* $b \to a < a$*, i.e.,* $b' \vee (b \wedge a) < a$*, which implies* $b' < a$*. Thus,* $a > b' \vee (b \wedge a) = a$*, which is impossible.*

Proposition 3.1. *Let L be an OM-L-algebra. For* $x, y, z \in L$*, we have the following properties:*

(i) Let $x \leq y, z$ and $y \to x = z \to x$. Then, $y = z$.
(ii) Let $x \leq y$. Then, there exists a unique element $z \in L$ such that $x \leqslant z$ and $y = z \to x$.

Proof. (i) Set $x \leqslant y, z$ and $y \to x = z \to x$; we have $(y \to x) \to x = (z \to x) \to x$. Then, Lemma 3.5 and (3.18) imply that $y = z$.

(ii) Set $x \leq y$. By Lemma 3.5, we get $xy = yx \Leftrightarrow x \leq y \to x$. Then, $\exists\, z = y \to x$, such that $x \leq z$. (3.18) yields $y = (y \to x) \to x = z \to x$. By (i), the uniqueness holds. □

Definition 3.5. [13] *We define a semibrace A to be a commutative monoid* $(A, \wedge)$ *with an additional binary operation* $A \times A \dot{\to} A$ *such that*

$$a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c) \quad a \cdot 1 = 1; \tag{3.19}$$

$$(a \wedge b) \cdot c = (a \cdot b) \cdot (a \cdot c) \quad 1 \cdot a = a. \tag{3.20}$$

Let L be an L-algebra. We define a $\wedge$-*closure* of L to be an L-algebra $\theta(L)$, which is a semibrace such that L is an L-subalgebra of $\theta(L)$, and every $a \in \theta(L)$ is of the form $a = x_1 \wedge \cdots \wedge x_n$ with $x_i \in L$.

Definition 3.6. [14] *We call an L-algebra L* $\wedge$-closed *if* $L = \theta(L)$.

Proposition 3.2. *Every OM-L-algebra is a* $\wedge$*-closed L-algebra.*

Proof. It suffices to prove that the intersection in L is consistent with the intersection in $S(L)$. For $\forall x, y \in L$, by (3.11), we have $0 \to (x \to y)x = ((x \to 0) \to (x \to y))(0 \to x) = ((0 \to x) \to (0 \to y)) = 1$, then $0 \leq (x \to y)x$. By Lemma 3.3 and (3.9), we have $x \wedge y = (x \to y)x \in L$. Thus, every OM-L-algebra is a $\wedge$-closed L-algebra. □

Lemma 3.7. [15] *Let L be a* $\wedge$*-closed L-algebra.* $x, y \in L$. *Then, the following are satisfied.*

$$\langle x \rangle \vee \langle y \rangle = \langle x \wedge y \rangle. \tag{3.21}$$

$$\langle x \rangle \vee \langle x \to y \rangle = \langle x \wedge y \rangle. \tag{3.22}$$

Corollary 3.1. *Let L be a* $\wedge$*-closed L-algebra.* $a, b \in L$. *If* $a \wedge b = 0$ *and* $\langle a \rangle \subseteq \langle b \rangle$. *Then,* $\langle b \rangle = L$.

Proof. By (3.21), we have $\langle b \rangle = \langle a \rangle \vee \langle b \rangle = \langle a \wedge b \rangle = \langle 0 \rangle = L$. □

Proposition 3.3. *Let L be an L-algebra. For* $\forall I \in I(L)$. *Then,* $I = \bigcup_{i \in I_0} \{\langle x_i \rangle \mid \forall x_i \in I\}$, *where* I_0 *represents the index set.*

Proof. For $\forall t \in I$, we have $t \in \langle t \rangle \subseteq \bigcup_{i \in I_0} \{\langle x_i \rangle \mid \forall x_i \in I\}$. Then, $I \subseteq \bigcup_{i \in I_0} \{\langle x_i \rangle \mid \forall x_i \in I\}$. For $\forall t \in \bigcup_{i \in I_0} \{\langle x_i \rangle \mid \forall x_i \in I\}$, $\exists\, x_0 \in I$, such that $t \in \langle x_0 \rangle \subseteq I$, which yields $\bigcup_{i \in I_0} \{\langle x_i \rangle \mid \forall x_i \in I\} \subseteq I$. Thus, $I = \bigcup_{i \in I_0} \{\langle x_i \rangle \mid \forall x_i \subseteq I\}$. □

Definition 3.7. *Let L be an L-algebra. L is* irreducible *if L is isomorphic to a product* $L_1 \times L_2$ *of L-algebra implies that* $|L_1| = 1$ *or* $|L_2| = 1$. $|L_1|$, $|L_2|$, *respectively, the cardinality of L-algebras* L_1 *and* L_2.

Proposition 3.4. *Let L be an irreducible OM-L-algebra. Assume that* $a \in L$*; a is an atom. Then,* $\langle a \rangle = L$.

Proof. Let a be an atom, and L is irreducible. Lemma 3.5 and Remark 3.1 imply that there exists $x_0 \in L$ and $x_0 \neq 0, 1$, such that $ax_0 \neq x_0 a \Leftrightarrow a \nleq x_0 \to a \Leftrightarrow (x_0 \to a) \parallel a$. Since a is an atom, then $a \wedge (x_0 \to a) = 0$. Since $a, x_0 \to a \in \langle a \rangle$, then $0 = a \wedge (x_0 \to a) \in \langle a \rangle$, thus $\langle a \rangle = L$. □

Let L be an OM-L-algebra, a subset $I \subseteq L$. By [16] and [11], we obtain that I is an L-ideal if and only if I is an invariant L-subalgebra satisfying $(I1)$. Next, we will prove the p-filters in L (L as an orthomodular lattice) are consistent with the L-ideals (L as OM-L-algebra).

Proposition 3.5. *Let L be an OM-L-algebra. The p-filter of orthomodular lattice L is the same as the L-ideals of OM-L-algebra L.*

Proof. Let F be a p-filter of L. We need to prove that F satisfies $(I1)$ and is an invariant L-subalgebra of L. Let $x, x \to y \in F$. Since $x \to y = x' \vee (x \wedge y) \in F$ and F is a p-filter, we have $x \wedge (x \to y) \in F$. Then, $x \wedge (x \to y) = (x \to (x \to y))x = (x \to y)x = x \wedge y \leq y \in F$. So $y \in F$. For $\forall y \in L, x \in F$, we get $y \to x = y' \vee (y \wedge x) \in F$, whence F is an invariant L-subalgebra of L.

Conversely, let I be an L-ideal of an OM-L-algebra L. Set $a, b \in I$. We have $a \to b \in I$. Then, $(a \to b) \to (a \wedge b) = (a \to b) \to (a \to b)a = ((a \to (a \to b)) \to (a \to b))((a \to b) \to a) = ((a \to b) \to (a \to b))a = 1a = a \in I$. By $(I1)$, $a \wedge b \in I$. Let $a \in I, b \in L, a \leq b$. Then $a \to b = 1 \in I$. By $(I1)$, we have $b \in I$. Then, I is a filter. If $a \in I$, for $\forall x \in L$, by (3.15), we obtain $x \vee (x' \wedge a) = x' \to a \in I$. □

Proposition 3.6. *Let L be an OM-L-algebra. $a \in L$, $a \in C(L)$ if and only if $a^{\uparrow} = \{x \to a \mid \forall x \in X\}$.*

Proof. Let a be the central element; $\forall y \in a^{\uparrow}$, $a \leq y$. By Proposition 3.1, there exists a unique element $z \in L$ such that $a \leq z$ and $y = z \to a$. Then, $y \in \{x \to a \mid \forall x \in L\}$. Hence, $a^{\uparrow} \subseteq \{x \to a \mid \forall x \in L\}$. Set $\forall y \in \{x \to a \mid \forall x \in L\}$, there exists $x_0 \in L$, such that $y = x_0 \to a$. Since a is the central element, then Lemma 3.4 implies that $ax_0 = x_0 a$. Since $x_0 a \leq a \Leftrightarrow ax_0 \leq a \Leftrightarrow a \leq x_0 \to a = y$, we have $y \in a^{\uparrow}$, then $\{x \to a \mid \forall x \in L\} \subseteq a^{\uparrow}$. This shows that $a^{\uparrow} = \{x \to a \mid \forall x \in L\}$.

Conversely, if $a^{\uparrow} = \{x \to a \mid \forall x \in L\}$, for $\forall x \in L$, we have $a \leq x \to a$. Lemma 3.5 implies $\forall x \in L$, $a \leq x \to a \Leftrightarrow ax = xa$. Thus, $a \in C(L)$. □

Proposition 3.7. *Let L be an OM-L-algebra, $a \in L$. Then, $a \in C(L)$ if and only if $\langle a \rangle = a^{\uparrow}$.*

Proof. If $a \in C(L)$, by Corollary 2.2 and Proposition 3.5, $a^{\uparrow} = [a, 1]$ is an L-ideal. Since $a \in a^{\uparrow}$, we have $\langle a \rangle \subseteq a^{\uparrow}$. For $\forall y \in a^{\uparrow}$, we obtain $a \leq y$. Since $\langle a \rangle$ is an upper set, it follows that $y \in \langle a \rangle$. Hence, $a^{\uparrow} \subseteq \langle a \rangle$. Thus, we have $\langle a \rangle = a^{\uparrow}$.

Conversely, if $a \notin C(L)$, then $\exists\, x_0 \in L$, such that $ax_0 \neq x_0 a \Leftrightarrow a \nleq x_0 \to a$, which is impossible since $x_0 \to a \in \langle a \rangle = a^{\uparrow}$. Thus, $a \in C(L)$. □

Definition 3.8. [17] *An element p of an L-algebra L is said to be* prime. *If $p < 1$ and $x \leq p$ or $x \to p \leq p$ for all $x \in L$.*

Definition 3.9. [17] *We call an L-*algebra prime *if all of its elements $p < 1$ are prime.*

Remark 3.2. *Let Ω be a partially ordered set with the greatest element* 1. *By* [3],

$$x \to y := \begin{cases} 1, & if\ x \leq y, \\ y, & if\ x \nleq y, \end{cases} \tag{3.23}$$

makes Ω into an L-algebra which, satisfies the inequality (3.5). Thus, Ω is a prime KL-algebra. Note that for any KL-algebra, $x \to p \leq p$ implies that $x \to p = p$.

A prime KL-algebra L is completely determined by its underlying partial order. So, the above example shows that there is a unique prime KL-algebra for each partially ordered set with greatest element 1.

Definition 3.10. [11] *We call an element x of an L-algebra L* spatial *if $x = \bigwedge\{p \in P(L) \mid x \leq p\}$ holds in $\theta(L)$. If all $x \in L$ are spatial, we say that L has* enough primes.

Definition 3.11. [11] *We define a closure L-algebra to be an L-algebra L such that for each $x \in L$, the meet $\bar{x} := \bigwedge\{p \in P(L) \mid p \geq x\}$ exists in $\theta(L)$ and belongs to L. We call an L-algebra L complete if $L = \theta(L)$ and its underlying $\wedge$-semilattice is a* complete lattice.

Definition 3.12. [11] *We call an L-algebra L* distributive *if its lattice of ideals is distributed.*

Lemma 3.8. [17] *The lattice of ideals of an L-algebra L is distributive.*

Definition 3.13. [18] *A complete lattice is said to be a* locale *if it satisfies the infinite distributive law*

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i). \tag{3.24}$$

Lemma 3.9. [11] *For the distributive L-algebra L. The ideal lattice $I(L)$ is spatial locale, and the operation on $I(L)$ is*

$$I \to J = \{x \in L \mid \langle x \rangle \cap I \subseteq J\}, \tag{3.25}$$

where $\langle x \rangle$ represents the ideal generated by x. Then, $\forall I, J, K \in I(L)$. We have

$$I \cap J \subseteq K \Leftrightarrow \forall x \in I : \langle x \rangle \cap J \subseteq K \Leftrightarrow I \subseteq J \to K, \tag{3.26}$$

and

$$I \to (J \cap K) = (I \to J) \cap (I \to K). \tag{3.27}$$

Theorem 3.1. *Let L be an L-algebra. Then, $I(L)$ is a chain under the set inclusion if and only if $I(L)$ is a bounded prime KL-algebra.*

Proof. Let $I(L)$ be a chain, $I \in I(L)$, $J \in I(L)$. If $J \subseteq I$. Then, $J \to I = L$. If $J \nsubseteq I$, since $I(L)$ is a chain, then $I \subsetneq J$. Let $\forall x \in J \to I$, we have $\langle x \rangle \subseteq J \to I$, by (3.26), $\langle x \rangle \subseteq J \to I \Leftrightarrow \langle x \rangle \cap J \subseteq I$. Then, by Lemma 3.8, we can obtain $I = I \vee (\langle x \rangle \cap J) = (I \vee \langle x \rangle) \cap (I \vee J) = (I \vee \langle x \rangle) \cap J$. If $J \subseteq J \to I$, by (3.26), we get $J = J \cap J \subseteq I$, which is a contradiction. Then, $x \in J \to I \subseteq J$. Obviously, $x \in \langle x \rangle \subseteq I \vee \langle x \rangle$, we have $x \in (I \vee \langle x \rangle) \cap J = I$, i.e., $J \to I \subseteq I$. Since $I(L)$ is a KL-algebra, we have $I \subseteq J \to I$, then $I = J \to I$. Remark 3.2 gives $I(L)$ is a prime KL-algebra. Clearly, $I(L)$ is bounded. Thus, $I(L)$ is a bounded prime KL-algebra.

Conversely, assume that $I, J \in I(L)$ is incomparable. Since $I \parallel J$, we obtain $I \neq I \cap J$, i.e., $I \nsubseteq I \cap J$. Since $I(L)$ is a bounded prime KL-algebra, by Remark 3.2, we have $I \to J = J$ and $I \to (I \cap J) = I \cap J$. By (3.27), we have $J = L \to J = (I \to I) \to (I \to J) = I \to (I \cap J) = I \cap J \subseteq I$, which is a contradiction. Thus, $I(L)$ is a chain under the set inclusion. □

It immediately, we have:

Corollary 3.2. *Let L be an OML. Then, $I_p(L)$ is a chain under the set inclusion relationship if and only if $I_p(L)$ is a bounded prime KL-algebra.*

Proof. By Lemma 3.2, Proposition 3.5, Remark 2.1, and Theorem 3.1, it clearly. □

4. Conclusions

This paper aims to give an answer to the question raised in Kalmbach's book: When are the p-ideals of an irreducible orthomodular lattice well ordered under set inclusion? For irreducible orthomodular lattice L, we prove that if L is Noetherian or completely with the relative central properties or complete atomic satisfying the exchange axiom, then the p-ideal sets of L are a chain (or well ordered) under set inclusion. In Section 3, from the perspective of L-algebra, we obtain that L-ideals lattice is a chain under the set inclusion if and only if L-ideals lattice is a bounded prime KL-algebra. Remarkably, this paper only partially answers the issues mentioned above. Therefore, we look forward to a more in-depth study about it.

Author contributions

Z. T. Zhao and J. Wang are mainly responsible for writing the initial draft; Y. L. Wu is responsible for reviewing, commenting, and revising before and after publication. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

This research is supported by the following grant: Cultivation Fund of Qilu University of Technology (Shandong Academy of Sciences) (2023PX068).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. P. D. Finch, Congruence relations in orthomodular lattices, *J. Aust. Math. Soc.*, **6** (1966), 46–54. https://doi.org/10.1017/s1446788700003992
2. G. Kalmbach, *Orthomodular lattices*, Cambridge: Cambridge University Press, 1983.
3. W. Rump, L-Algebras, self-similarity, and l-groups, *J. Algebra*, **320** (2008), 2328–2348. https://doi.org/10.1016/j.jalgebra.2008.05.033
4. W. Rump, Right l-groups, geometric Garside groups, and solutions of the quantum Yang–Baxter equation, *J. Algebra*, **439** (2015), 470–510. https://doi.org/10.1016/j.jalgebra.2015.04.045
5. W. Rump, The L-algebra of Hurwitz primes, *J. Number Theory*, **190** (2018), 394–413. https://doi.org/10.1016/j.jnt.2018.03.004
6. W. Rump, Von Neumann algebras, L-algebras, Baer $*$-monoids, and Garside groups, *Forum Math.*, **30** (2018), 973–995. https://doi.org/10.1515/forum-2017-0108
7. Y. L. Wu, Y. C. Yang, Orthomodular lattices as L-algebras, *Soft Comput.*, **24** (2020), 14391–14400. https://doi.org/10.1007/s00500-020-05242-7
8. G. Birkhoff, *Lattice theory*, Providence: American Mathematical Society, 1967.

9. R. Cignoli, Deductive systems and congruence relations in ortholattices, *Bulletin of the Section of Logic*, **7** (1978), 87–88.

10. J. D. Monk, B. Robert, *Handbook of Boolean algebras*, Netherlands: North-Holland, 1989.

11. W. Rump, L-Algebras and topology, *J. Algebra Appl.*, **22** (2023), 2350034. https://doi.org/10.1142/S0219498823500342

12. W. Rump, Symmetric quantum sets and L-algebras, *Int. Math. Res. Notices*, **2022** (2022), 1770–1810. https://doi.org/10.1093/imrn/rnaa135

13. W. Rump, Semidirect products in algebraic logic and solutions of the quantum Yang-Baxter equation, *J. Algebra Appl.*, **7** (2008), 471–490. https://doi.org/10.1142/S0219498808002904

14. W. Rump, X. Zhang, L-Effect algebras, *Stud. Logica*, **108** (2020), 725–750. https://doi.org/10.1007/s11225-019-09873-2

15. W. Rump, L. Vendramin, The prime spectrum of an L-algebra, *Proc. Amer. Math. Soc.*, **152** (2024), 3197–3207. https://doi.org/10.1090/proc/16802

16. Y. L. Wu, J. Wang, Y. C. Yang, Lattice-ordered effect algebras and L-algebras, *Fuzzy Set. Syst.*, **369** (2019), 103–113. https://doi.org/10.1016/j.fss.2018.08.013

17. W. Rump, Prime L-algebras and right-angled Artin groups, *Semigroup Forum*, **106** (2023), 481–503. https://doi.org/10.1007/s00233-023-10343-4

18. P. T. Johnstone, *Stone spaces*, Cambridge: Cambridge University Press, 1982.