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Research article

Generalized Bayesian inference study based on type-II censored data from the class of exponential models

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Abstract: Generalized Bayesian (GB) is a Bayesian approach based on the learning rate parameter (LRP) ($0 < \eta < 1$) as a fraction of the power of the likelihood function. In this paper, we consider the GB method to perform inference studies for a class of exponential distributions. Generalized Bayesian estimators (GBE) and generalized empirical Bayesian estimators (GEBE) for the parameters of the considered distributions are obtained based on the censored type II samples. In addition, generalized Bayesian prediction (GBP) and generalized empirical Bayesian prediction (GEBP) are considered using a one-sample prediction scheme. Monte Carlo simulations and illustrative example are performed for one parameter models to compare the performance of the GBE and GEBE estimation results and the GBP and GEBP prediction results for different values of the LRP.

Keywords: generalized Bayesian; learning rate parameter; generalized Bayesian estimators; generalized empirical Bayesian estimators; generalized Bayesian prediction; generalized empirical Bayesian prediction; simulation
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Abbreviations

GB	Generalized Bayesian
LRP	Learning rate parameter
GBE	Generalized Bayesian estimator
GEBE	Generalized empirical Bayesian estimator
GBP	Generalized Bayesian prediction
GEBP	Generalized empirical Bayesian prediction

MLE	Maximum likelihood estimator
EB	Empirical Bayes
EBE	Empirical Bayes estimator
EBP	Empirical Bayes prediction

1. Introduction

In the Bayesian inference techniques, GB analysis was introduced and studied based on the learning rate parameter ($0 < \eta < 1$). The traditional Bayesian framework for $\eta = 1$ is a fraction of the power of the likelihood function $L(\theta) \equiv L(\theta;\ data)$ for the parameter $\theta \in \Theta$. This means that if the prior distribution of the parameter θ is $\pi(\theta)$, then the GB posterior distribution for θ is

$$\pi^*(\theta \mid data) \propto L^\eta(\theta)\pi(\theta), \theta \in \Theta,\ 0 < \eta < 1. \quad (1)$$

In this study, the GBE, GEBE, GBP, and GEBP distributions from the class of exponential distributions are examined using type II censored samples. Thus, the aim of this study is to examine all GB and GEB results for different LRP values, including $\eta = 1$, which describe the traditional Bayes. For more information on the GB approach and how to select the value for the rate parameter, see [1–13]. Specifically, the choice of the learning rate was studied in [3–6] using the Safe Bayes algorithm based on the minimization of a sequential risk measure. In [7] and [8], another learning rate selection method was proposed, which included two different information adaptation strategies. The authors in [11] investigated GBE based on a joint type-II censored sample from multiple exponential populations, using various values of the learning rate parameter. The same study was presented in [13] but was based on joint hybrid censoring. In [11,13] a range of values for the learning rate parameter have been chosen to obtain the best estimators for the parameters of the corresponding distributions, then GB results were compared with the traditional Bayesian results. Here we conduct our study based on different values of LRP to find out the effect of different values of LRP on the estimation and prediction results.

A one-sample prediction scheme is a Bayesian prediction method that determines the point predictor or prediction interval for unknown future values in the same sample based on the currently available observations. A two-sample prediction scheme or a multiple-sample prediction scheme are two other ways in which Bayesian prediction can utilize currently available observations to predict one or more future samples. Numerous authors have addressed the prediction of future failures or samples using different censoring techniques in the context of different prediction methods. We highlight some points that are relevant to our research. For instance, [12] investigated the GBP using a combined type-II censored sample drawn from multiple exponential populations. A study using a joint type-II censored sample from two exponential populations for Bayes estimation and prediction was published in [14]. Based on a generalized order statistic and multiple type II censoring, a Bayesian prediction for the future values of distributions from the class of exponential distributions was constructed in [15,16].

In the Bayesian study, the parameter of the distribution under investigation is a random variable, i.e., this unknown parameter is distributed according to the prior distribution. Empirical Bayes (EB) is a Bayesian study in which the parameters of the prior distribution (hyperparameters) are also unknown. By combining the density function of the distribution and the prior distributions, we obtain the marginal density function of the hyperparameters, which is used to estimate the hyperparameters. Therefore, the data of the original distribution are used to find the maximum likelihood estimators (MLEs) of these hyperparameters. EB has been introduced by many authors; for example, [17] studied the empirical Bayes estimator (EBE) of reliability performances with progressive type-II censoring of the Lomax model. The reliability and hazard function of the Kumaraswamy distribution were determined by [18] using progressive censored type II samples to estimate the EBE of the parameters. The Rayleigh distribution was studied in [19] to determine EBE and empirical Bayes prediction (EBP).

The rest of this article is organized as follows: Section 2 introduces the class of exponential models and then describes the problem of GB, GEB, GBP, and GEBP for this class. Section 3 applies the investigation from Section 2 to the exponential and Rayleigh models, which are given as examples of the one parameter exponential class. In Section 4, we present simulation study besides an illustrative example based on real data for the exponential and Rayleigh models to obtain the GBE, GEBE, GBP and GEBP for different LRP values and compare the results. Finally, Section 5 discusses the results and concludes the paper.

2. Estimation and prediction

In this section, we introduce the exponential class of models and examine the problems of the GB, GEB, GBP, and GEBP for this class.

2.1. The model

Let $\boldsymbol{\theta}$ be the vector of parameters, define a function $g(x;\boldsymbol{\theta}) \equiv g(x)$, and its derivative $g'(x)$ where $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to 0^+} g(x) = 0$. The probability density function (pdf), the cumulative probability density function (cdf) and the survival function (sf) of the exponential class are each given by:

$$f(x;\boldsymbol{\theta}) = g'(x)\exp[-g(x)],\ x > 0,\ \boldsymbol{\theta} > 0; \tag{2}$$

$$F(x;\boldsymbol{\theta}) = 1 - \exp[-g(x)], \tag{3}$$

and

$$\overline{F}(x;\boldsymbol{\theta}) = \exp[-g(x)]. \tag{4}$$

The likelihood function under type-II censored data from the class is given by,

$$\begin{aligned} L(\underline{x};\boldsymbol{\theta}) &= c\,\overline{F}(x_r;\boldsymbol{\theta})^{n-r}\prod_{i=1}^{r} f(x_i;\boldsymbol{\theta}) \\ &\propto A(\underline{x};\boldsymbol{\theta})\exp[-B(\underline{x};\boldsymbol{\theta})], \end{aligned} \tag{5}$$

where, $c = \frac{n!}{(n-r)!}$, $A(\underline{x};\boldsymbol{\theta}) = \prod_{i=1}^{r} g'(x_i)$, $B(\underline{x};\boldsymbol{\theta}) = \sum_{i=1}^{r} g(x_i) + (n-r)g(x_r)$, $\underline{x} = (x_1,\dots,x_r)$.

Consider the prior distribution of $\boldsymbol{\theta}$ in the following general form:

$$\pi(\boldsymbol{\theta};\boldsymbol{\delta}) = I_{\boldsymbol{\delta}}{}^{-1}C(\boldsymbol{\theta};\boldsymbol{\delta})\exp[-D(\boldsymbol{\theta};\boldsymbol{\delta})], \tag{6}$$

where, $I_{\boldsymbol{\delta}} = \int_{\boldsymbol{\theta}} C(\boldsymbol{\theta};\boldsymbol{\delta})\exp[-D(\boldsymbol{\theta};\boldsymbol{\delta})]\,d\boldsymbol{\theta}$, $\boldsymbol{\delta}$ is a vector of hyperparameters.

Combining (5) and (6), after raising (5) to the fractional power η, the GB posterior distribution of $\boldsymbol{\theta}$ is given by,

$$\begin{aligned} \pi_G^*(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x}) &= I_{\boldsymbol{\delta}}^{*\,-1}L^{\eta}(\underline{x};\boldsymbol{\theta})\pi(\boldsymbol{\theta};\boldsymbol{\delta}) \\ &= I_{\boldsymbol{\delta}}^{*\,-1}\,G(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x})\exp[-H(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x})], \end{aligned} \tag{7}$$

where $G_{\boldsymbol{\delta}} = A^{\eta}(\underline{x};\boldsymbol{\theta})C(\boldsymbol{\theta};\boldsymbol{\delta})$, $H_{\boldsymbol{\delta}} = \eta B(\underline{x};\boldsymbol{\theta}) + D(\boldsymbol{\theta};\boldsymbol{\delta})$, and $I_{\boldsymbol{\delta}}^* = \int_{\boldsymbol{\theta}} G_{\boldsymbol{\delta}}\exp[-H_{\boldsymbol{\delta}}]\,d\boldsymbol{\theta}$.

Under the squared error loss function, then GBE is given by,

$$\widehat{\boldsymbol{\theta}}_{GB} = E(\boldsymbol{\theta}) = \int_{\boldsymbol{\theta}} \boldsymbol{\theta}\,\pi_G^*(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x})\,d\boldsymbol{\theta}. \tag{8}$$

2.2. Generalized empirical Bayesian estimation

Combining (1) and (6), to obtain the marginal pdf $f(x;\boldsymbol{\delta})$ as follows:

$$f(x;\boldsymbol{\delta}) = \int_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta};\boldsymbol{\delta})\, f(x;\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$= I_{\boldsymbol{\delta}}{}^{-1} \int_{\boldsymbol{\theta}} C(\boldsymbol{\theta};\boldsymbol{\delta}) g'(x) \exp[-\{D(\boldsymbol{\theta};\boldsymbol{\delta}) + g(x)\}]\, d\boldsymbol{\theta}. \quad (9)$$

From pdf in (9) we obtain the cdf $F(x;\boldsymbol{\delta})$**,** then the likelihood function under type-II censored data is given by,

$$L_E(\underline{x};\boldsymbol{\delta}) = c\, \overline{F}(x_r;\boldsymbol{\delta})^{n-r} \prod_{i=1}^{r} f(x_i;\boldsymbol{\delta}). \quad (10)$$

Using the loglikelihood function $\mathcal{L}_E(\underline{x};\boldsymbol{\delta}) = \log L_E(\underline{x};\boldsymbol{\delta})$, to find the maximum likelihood estimator (MLE) $\widehat{\boldsymbol{\delta}}$ as follows:

$$\widehat{\boldsymbol{\delta}} = \underset{\boldsymbol{\delta}}{\operatorname{argmax}}\, \mathcal{L}_E(\underline{x};\boldsymbol{\delta}). \quad (11)$$

By solving the following equation,

$$\frac{\partial \mathcal{L}_E(\underline{x};\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} = 0. \quad (12)$$

Substituting by $\hat{\delta}$ in (7), we obtain the posterior GE as follows:

$$\pi^*_{GE}(\boldsymbol{\theta};\widehat{\boldsymbol{\delta}},\underline{x}) = I^{*-1}_{\widehat{\boldsymbol{\delta}}}\, G_{\widehat{\boldsymbol{\delta}}} \exp[-H_{\widehat{\boldsymbol{\delta}}}]. \quad (13)$$

Under the squared error loss function, GEBE is given by

$$\widehat{\boldsymbol{\theta}}_{GE} = E(\boldsymbol{\theta}) = \int_{\boldsymbol{\theta}} \boldsymbol{\theta}\, \pi^*_{GE}(\boldsymbol{\theta};\widehat{\boldsymbol{\delta}},\underline{x})\, d\boldsymbol{\theta}. \quad (14)$$

2.3. One sample prediction scheme

To determine the GBP and GEBP intervals using a one-sample prediction scheme under the type-II censored sample from the class, the first r ordered statistics $\underline{x}$ are observed from a random sample of size n; $r < n$. A one-sample prediction scheme is considered to predict the rest of unobserved values $x_s,\ s = r+1, \dots, n$. The conditional density function of x_s given $\underline{x}$ is given by

$$f(x_s|\underline{x}) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \left[\overline{F}(x_r) - \overline{F}(x_s)\right]^{s-r-1} \overline{F}(x_s)^{n-s}\, \overline{F}(x_r)^{-(n-r)} f(x_s). \quad (15)$$

Substituting by (2), (4) in (15), the conditional density function of x_s given $\underline{x}$ is,

$$f(x_s|\underline{x}) = \sum_{j=0}^{s-r-1} c_j\, g'(x_s) \exp\left[-n_j\{g(x_s) - g(x_r)\}\right], \quad (16)$$

where, $c_j = \frac{(-1)^{s-r-j-1}\,(n-r)!}{j!(s-r-j-1)!(n-s)!}$, $n_j = n - r - j$.

Combining (7) and (16), then integrating with respect to $\boldsymbol{\theta}$, the GB predictive density function is given by,

$$f^*_G(x_s|\underline{x}) = I^{*-1}_{\boldsymbol{\delta}} \sum_{j=0}^{s-r-1} c_j \int_{\boldsymbol{\theta}} g'(x_s) G_{\boldsymbol{\delta}} \exp\left[-\left\{n_j\left(g(x_s) - g(x_r)\right) + H_{\boldsymbol{\delta}}\right\}\right] d\boldsymbol{\theta}. \quad (17)$$

The predictive reliability function of $x_s,\ s = r+1, \dots, n$ is given by:

$$\overline{F}_{\boldsymbol{\delta}}(t|\underline{x}) = I^{*-1}_{\boldsymbol{\delta}} \sum_{j=0}^{s-r-1} \frac{c_j}{n_j} \int_{\boldsymbol{\theta}} G_{\boldsymbol{\delta}} \exp\left[-\left\{n_j\left(g(t) - g(x_r)\right) + H_{\boldsymbol{\delta}}\right\}\right] d\boldsymbol{\theta}. \quad (18)$$

Equation (18) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds $(L_{\boldsymbol{\delta}}, U_{\boldsymbol{\delta}})$. GEBP bounds $(L_{\hat{\delta}}, U_{\hat{\delta}})$, can be obtained by substituting by $\hat{\delta}$ in (18) then equating to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$.

3. Applications

In this section, we apply the results in the previous section to one parameter models; therefore, the models that are discussed here are exponential $\text{Exp}(\theta)$ and Rayliegh $\text{Ray}(\theta)$. For these two distributions, the parameter θ assumed to be unknown, we may consider the conjugate prior distribution of θ as a gamma prior distribution, $\theta \sim Gam(\delta_1, \delta_2)$, hence,

$$C(\theta;\boldsymbol{\delta}) = \theta^{\delta_1 - 1},\ D(\theta;\boldsymbol{\delta}) = \delta_2\theta;\ I_{\boldsymbol{\delta}}^{-1} = \frac{{\delta_2}^{\delta_1}}{\Gamma(\delta_1)},\ \delta_1, \delta_2 > 0. \tag{19}$$

3.1. Exponential model

Here we give the essential functions and important forms derived in Section 2 and Eq (19) for the exponential model as follows:

$$g(x) = \theta x, x > 0. \tag{20}$$

For the likelihood function, we have

$$A(\underline{x};\theta) = \theta^r, T_E = \sum_{i=1}^{r} x_i + (n-r)x_r \text{ and } B(\underline{x};\theta) = \theta T_E.$$

Generalized posterior function can be formed from the following:

$$G_{\boldsymbol{\delta}} = \theta^{\eta r + \delta_1 - 1},\ H_{\boldsymbol{\delta}} = \theta(\eta T_E + \delta_2) \text{ and } I_{\boldsymbol{\delta}}^{*} = \frac{\Gamma(\eta r + \delta_1)}{(\eta T_E + \delta_2)^{\eta r + \delta_1}}.$$

The GBE of the parameter θ is given by,

$$\hat{\theta}_{GB} = \frac{\eta r + \delta_1}{\eta T_E + \delta_2}. \tag{21}$$

The predictive reliability function of x_s is given by,

$$\overline{F}_{\boldsymbol{\delta}}(t|\underline{x}) = \sum_{j=0}^{s-r-1} \frac{c_j}{n_j}\left[1 + \frac{n_j(t-x_r)}{\eta T_E + \delta_2}\right]^{-(\eta r + \delta_1)}. \tag{22}$$

Equating (22) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds $(L_{\boldsymbol{\delta}}, U_{\boldsymbol{\delta}})$.

The functions and forms under GEB study can be illustrated as follows:

The marginal pdf $f(x;\boldsymbol{\delta})$ is given in the following form,

$$f(x;\boldsymbol{\delta}) = \frac{\delta_1 {\delta_2}^{\delta_1}}{(x+\delta_2)^{(\delta_1+1)}}. \tag{23}$$

Using pdf $f(x;\boldsymbol{\delta})$ and cdf $F(x;\boldsymbol{\delta})$, we obtain the likelihood function based on type-II censored data, as follows:

$$L_E(\underline{\boldsymbol{x}};\boldsymbol{\delta}) \propto \frac{\left(\delta_1 {\delta_2}^{\delta_1}\right)^r \prod_{i=1}^{r}(x_i+\delta_2)^{-(\delta_1+1)}}{\left(1+\frac{x_r}{\delta_2}\right)^{(n-r)\delta_1}}. \tag{24}$$

By differentiating the loglikelihood function $\mathcal{L}_E(\underline{x};\boldsymbol{\delta})$ w. r. to δ_1 and δ_2 and equating each equation to zero, then solving them numerically, we obtain the estimators $\hat{\delta}_1$ and $\hat{\delta}_2$.

The GEBE of the parameter θ is given by,

$$\hat{\theta}_{GE}=\frac{\eta r+\hat{\delta}_1}{\eta T_E+\hat{\delta}_2}. \tag{25}$$

The predictive reliability function of x_s is given by,

$$\bar{F}_{\hat{\delta}}(t|\underline{x})=\sum_{j=0}^{s-r-1}\frac{c_j}{n_j}\left[1+\frac{n_j(t-x_r)}{\eta T_E+\hat{\delta}_2}\right]^{-(\eta r+\hat{\delta}_1)}. \tag{26}$$

Equating (26) to $(1+\alpha)/2$ and $(1-\alpha)/2$, respectively, we obtain $(1-\alpha)\%$ GEBP bounds $(L_{\hat{\delta}}, U_{\hat{\delta}})$.

3.2. Rayleigh model

The essential functions and important forms derived in Section 2 and Eq (19) for Rayliegh distribution are derived as follows:

$$g(x)=\frac{\theta x^2}{2}, x>0. \tag{27}$$

For the likelihood function we have

$$A(\underline{x};\theta)=\theta^r\prod_{i=1}^r x_i,\ T_R=\sum_{i=1}^r x_i^2/2+(n-r)x_r^2/2 \text{ and } B(\underline{x};\theta)=\theta T_R.$$

Generalized posterior function, can be formed from the following:

$$G_{\boldsymbol{\delta}}=\theta^{\eta r+\delta_1-1}\left(\prod_{i=1}^r x_i\right)^\eta,\ H_{\boldsymbol{\delta}}=\theta(\eta T_R+\delta_2) \text{ and } I_{\boldsymbol{\delta}}^*=\frac{\left(\prod_{i=1}^r x_i\right)^\eta\Gamma(\eta r+\delta_1)}{(\eta T_R+\delta_2)^{\eta r+\delta_1}}.$$

The GBE of the parameter θ is given by,

$$\hat{\theta}_{GB}=\frac{\eta r+\delta_1}{\eta T_R+\delta_2}. \tag{28}$$

The predictive reliability function of x_s is given by,

$$\bar{F}_{\boldsymbol{\delta}}(t|\underline{x})=\sum_{j=0}^{s-r-1}\frac{c_j}{n_j}\left[1+\frac{n_j(t^2-x_r^2)/2}{\eta T_R+\delta_2}\right]^{-(\eta r+\delta_1)}. \tag{29}$$

Equating (29) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds $(L_{\boldsymbol{\delta}}, U_{\boldsymbol{\delta}})$.

The marginal pdf $f(x;\boldsymbol{\delta})$ is given in the following form:

$$f(x;\boldsymbol{\delta})=\frac{\delta_1{\delta_2}^{\delta_1}}{(x+\delta_2)^{(\delta_1+1)}}. \tag{30}$$

Using pdf $f(x;\boldsymbol{\delta})$ and cdf $F(x;\boldsymbol{\delta})$ we obtain the likelihood function based on type-II censored data, as follows:

$$L_{\boldsymbol{E}}(\underline{\boldsymbol{x}};\boldsymbol{\delta})\propto\frac{(\delta_1{\delta_2}^{\delta_1})^r\prod_{i=1}^r(x_i+\delta_2)^{-(\delta_1+1)}}{(1+x_r/\delta_2)^{(n-r)\delta_1}}. \tag{31}$$

By differentiating the loglikelihood function $\mathcal{L}_E(\underline{x};\boldsymbol{\delta})$ with respect to δ_1 and δ_2 and equating each equation to zero, then solving them numerically, we obtain the estimators $\hat{\delta}_1$ and $\hat{\delta}_2$.

The GEBE of the parameter θ is given by,

$$\hat{\theta}_{GE}=\frac{\eta r+\hat{\delta}_1}{\eta T_E+\hat{\delta}_2}. \tag{32}$$

The predictive reliability function of x_s is given by,

$$\bar{F}_{\hat{\boldsymbol{\delta}}}(t|\underline{x})=\sum_{j=0}^{s-r-1}\frac{c_j}{n_j}\left[1+\frac{n_j(t^2-x_r^2)/2}{\eta T_R+\hat{\delta}_2}\right]^{-(\eta r+\hat{\delta}_1)}. \tag{33}$$

Equating (32) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GEBP bounds $(L_{\hat{\delta}}, U_{\hat{\delta}})$.

4. Numerical analysis

In this section, a simulation study for the exponential and Rayleigh distributions is presented to obtain the GBE, GEBE, GBP, and GEBP for different LRP values and compare the results. An example based on real data for exponential and Rayleigh distributions is given to illustrate the results.

4.1. Simulation study

In this subsection, the results of the Monte Carlo simulation are presented to evaluate the performance of the inference methods derived in the previous sections. The simulation study is designed and carried out for the two models as follows:

- Generate one sample from each distribution with size $n = 50$ and choosing $r = 30, 40, 50$.
- Based on the chosen values of the hyperparameters $(\delta_1, \delta_2) = (4, 2)$, the suggested value for the parameter is $\theta = 2$, where θ is obtained as the mean of gamma distribution in (18).
- For EB, we use MLE $(\hat{\delta}_1, \hat{\delta}_2)$ to compute $\hat{\theta}_{GE}$, where the results of MLE $(\hat{\delta}_1, \hat{\delta}_2)$ based on exponential and Rayliegh distributions are shown in Table 1.

Table 1. The MLEs of the hyperparameters $\hat{\delta}_1, \hat{\delta}_2$ under different data from the two distributions.

	$\boldsymbol{Exp(\theta)}$	$\boldsymbol{Ray(\theta)}$
$\boldsymbol{(n, r)}$	$(\hat{\delta}_1, \hat{\delta}_2)$	
(20, 15)	(10.497, 4.1)	(10.95,4.625)
(50, 30)	(11.31, 4.926)	(11, 4.4252)
(50, 40)	(10.9, 5.218)	(10.995, 5.1456)
(50, 50)	(10.3, 5.289)	(10.5, 5.2727)

- For the Monte Carlo simulations, we use $M = 10{,}000$ replicates; therefore, the estimator $\hat{\theta} = \frac{\sum_{i=1}^{M} \hat{\theta}_i}{M}$, and the estimated risk, $ER = \sqrt{\frac{\sum_{i=1}^{M} (\theta_{\mathrm{i}} - \hat{\theta}_{\mathrm{i}})^2}{M}}$.
- Using (21), (25), (28) and (32), the estimation results are obtained and expressed by the estimator $\hat{\theta}$ and ER for different values of LRP, where $\eta = 0.1, 0.5,$ and 1.
- The results of GBE and GEBE for exponential and Rayliegh distributions are shown in Tables 2 and 4.
- Prediction results are based on one sample from each distribution with size $n = 20$, the number of observations is $r = 15$. We then compute the GBP, GEBP bounds, and their lengths at $\alpha = 0.05$, for the future values with $s = 16, 18, 20$ using (22), (26), (29), and (33).
- The results of GBP and GEBP for exponential and Rayliegh distributions are shown in Tables 3 and 5.

From Table 2. According to $\hat{\theta}$ and *ER,* GBE becomes better for small values of LRP but for large values of $\boldsymbol{r}$, which means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$ (complete sample). GEBE becomes better for large values of LRP and for large values of $\boldsymbol{r}$, which means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$. In general, the result of GBE is better than that of GEBE.

From Table 3. According to the length of the interval, GBP and GEBP become better for large values of LRP, which means getting the best result at $\eta = 1$ and $s = 16$. In general, the result of GEBP is better than that of GBP.

Table 2. GBE and GEBE for the parameter of exponential distribution.

$\boldsymbol{r}$	$\boldsymbol{\eta}$	$\hat{\boldsymbol{\theta}}_{GB}$	$\boldsymbol{ER_{GB}}$	$\hat{\boldsymbol{\theta}}_{GE}$	$\boldsymbol{ER_{GE}}$
30	0.1	2.0491	0.0033	2.2308	0.0070
40		2.0436	0.0027	2.0687	0.0064
50		2.0295	0.0023	1.9677	0.0067
30	0.5	2.0611	0.0078	2.1430	0.0058
40		2.0461	0.0052	2.0526	0.0053
50		2.0401	0.0046	2.0052	0.0047
30	1	2.0655	0.0127	2.1119	0.0051
40		2.0495	0.0111	2.0495	0.0046
50		2.0414	0.0078	2.0187	0.0035

Table 3. GBP and GEBP bound for exponential future values.

s	$\boldsymbol{\eta}$	$(\boldsymbol{L}, \boldsymbol{U})_{GB}$	*length*	$(\boldsymbol{L}, \boldsymbol{U})_{GE}$	*length*
16	0.1	(0.6596, 1.1826)	0.5230	(0.6591, 1.0062)	0.3471
18		(0.7282, 2.1919)	1.4637	(0.7178, 1.6180)	0.9002
20		(0.9338, 4.9415)	4.0077	(0.8992, 3.3181)	2.4189
16	0.5	(0.6597, 1.0920)	0.4323	(0.6592, 1.0142)	0.3550
18		(0.7321,1.8567)	1.1246	(0.7237, 1.6225)	0.8988
20		(0.9557, 3.9800)	3.0243	(0.9251, 3.3245)	2.3994
16	1	(0.6596, 1.0642)	0.4046	(0.6593, 1.0181)	0.3588
18		(0.7337, 1.7554)	1.0217	(0.7273, 1.6225)	0.8952
20		(0.9653, 3.6914)	2.7261	(0.9411, 3.3220)	2.3809

From Table 4. GBE becomes better for small values of LRP but for large values of $\boldsymbol{r}$, which means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$ (complete sample). GEBE becomes better for large values of LRP and for large values of $\boldsymbol{r}$, except for the complete sample, the result becomes better for small values of LRP which means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. The result of GBE is better than that of GEBE at $\boldsymbol{r} = 30, 40$, but GEBE is better than GBE for the complete sample.

From Table 5. According to the length of the interval, GBP and GEBP become better for large values of LRP, which means getting the best result at $\eta = 1$ and $s = 16$. In general, the result of GEBP is better than that of GBP.

Table 4. GBE and GEE for the parameter of Rayleigh distribution.

r	η	$\hat{\theta}_{GB}$	ER_{GB}	$\hat{\theta}_{GE}$	ER_{GE}
30	0.1	2.0131	0.0028	2.3676	0.0129
40		2.0124	0.0023	2.1026	0.0082
50		2.0122	0.0013	1.9992	0.0013
30	0.5	2.0415	0.0060	2.2088	0.0095
40		2.0321	0.0026	2.0677	0.0048
50		2.0311	0.0024	2.0166	0.0015
30	1	2.0514	0.0075	2.1532	0.0074
40		2.0436	0.0059	2.0603	0.0020
50		2.0349	0.0041	2.0260	0.0018

Table 5. GBP and GEBP bound for Rayleigh future values.

s	η	$(L,U)_{GB}$	*length*	$(L,U)_{GE}$	*length*
16	0.1	(1.1379, 1.5297)	0.3918	(1.1376, 1.4252)	0.2876
18		(1.1967, 2.0880)	0.8913	(1.1914, 1.8238)	0.6324
20		(1.3577, 3.1400)	1.7823	(1.3448, 2.6322)	1.2874
16	0.5	(1.1379, 1.4695)	0.3316	(1.1377, 1.4253)	0.2876
18		(1.2000, 1.9209)	0.7209	(1.1954, 1.8145)	0.6191
20		(1.3738, 2.8175)	1.4437	(1.3597, 2.6121)	1.2524
16	1	(1.1379, 1.4507)	0.3128	(1.1378, 1.4251)	0.2873
18		(1.2013, 1.8674)	0.6661	(1.976, 1.8084)	0.6108
20		(1.3807, 2.7132)	1.3325	(1.3689, 2.5993)	1.2304

4.2. Illustrative example

To illustrate the results, two examples based on real data are given for exponential and Rayleigh distributions, respectively. For the GB study, there is no information about the prior, and noninformative prior should be used; therefore, we suggest the hyperparameters as $(\delta_1, \delta_2) = (0.0001, 0.0001)$, which results in the MLE for the parameter, which means there is no effect for the LRP in the case of noninformative prior.

The data in Table 6, contains times to breakdown of an insulating fluid between electrodes recorded at 34kv (see, [20]). Table 7 provide the MLEs of $\hat{\delta}_1, \hat{\delta}_2$ under breakdown data from the two distributions.

Table 6. Breakdown time data ($n = 19$).

0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89

Table 7. The MLEs of $\hat{\delta}_1, \hat{\delta}_2$ under breakdown data from the two distributions.

	$\boldsymbol{Exp(\theta)}$	$\boldsymbol{Ray(\theta)}$
$\boldsymbol{(n, r)}$	$(\hat{\delta}_1, \hat{\delta}_2)$	
(19, 10)	(7.427, 1.1)	(7.211, 0.64)
(19, 15)	(6.658, 1)	(5.767, 0.55)
(19, 19)	(5.794, 0.95)	(4.685, 0.5)

From Table 8, the results of GBE are nearly the MLE for the parameter in the case of noninformative prior, as we see there is no effect for the LRP in both distributions, and the best result for exponential distribution is $\hat{\theta}_{GB} = 0.0696$, while the best result for Rayliegh distribution is $\hat{\theta}_{GB} = 0.0036$. GEBE becomes better for large values of LRP and for large values of $\boldsymbol{r}$, which means the best result for the exponential distribution is $\hat{\theta}_{GE} = 0.0905$, while the best result for the Rayliegh distribution is $\hat{\theta}_{GE} = 0.0045$. Because of using noninformative prior, the result of GEBE is better than that of GBE.

Table 8. GBE and GEBE for the parameters of the two distributions.

		$\boldsymbol{Exp(\theta)}$		$\boldsymbol{Ray(\theta)}$	
r	η	$\hat{\theta}_{GB}$	$\hat{\theta}_{GE}$	$\hat{\theta}_{GB}$	$\hat{\theta}_{GE}$
10	0.1	0.1138	0.8525	0.0395	0.31651
15		0.0670	0.3489	0.0054	0.0263
19		<u>0.0696</u>	0.2725	<u>0.0036</u>	0.0127
10	0.5	0.1138	0.2760	0.0395	0.0960
15		0.0670	0.1254	0.0054	0.0096
19		0.0696	0.1113	0.0036	0.0055
10	1	0.1138	0.1959	0.0395	0.0678
15		0.0670	0.0963	0.0054	0.0075
19		0.0696	<u>0.0905</u>	0.0036	<u>0.0045</u>

From Tables 9 and 10, according to the length of the interval, GBP becomes better for large values of LRP, which means getting the best result at $\eta = 1$ and $s = 16$, but GEBP becomes better for small value of LRP, that means getting the best result at $\eta = 0.1$ and $s = 16$. In general, the result of GEBP is better than that of GBP.

Table 9. GBP and GEBP bound for exponential future values.

s	$\boldsymbol{\eta}$	$\boldsymbol{(L, U)_{GB}}$	*length*	$\boldsymbol{(L, U)_{GE}}$	*length*
16	0.1	(31.845, 91.575)	59.723	(31.768, 35.092)	3.324
19		(36.975, 488.5)	451.525	(33.067, 51.565)	18.498
16	0.5	(31.845, 49.523)	17.678	(31.85, 40.151)	8.301
19		(38.56, 137.8)	99.240	(35.561, 80.011)	44.450
16	1	(31.845, 47.349)	15.504	(31.816, 42.185)	10.369
19		(38.9, 121.15)	82.250	(36.807, 90.784)	53.977

Table 10. GBP and GEBP bound for Rayliegh future values.

s	$\boldsymbol{\eta}$	$(\boldsymbol{L},\boldsymbol{U})_{\boldsymbol{GB}}$	*length*	$(\boldsymbol{L},\boldsymbol{U})_{\boldsymbol{GE}}$	*length*
16	0.1	(31.787, 49.785)	17.998	(31.758, 33.154)	1.396
19		(33.711, 110.561)	76.85	(32.289,39.415)	7.126
16	0.5	(31.787, 38.009)	6.222	(31.771,35.048)	3.277
19		(34.285, 60.122)	25.837	(33.269, 47.731)	14.462
16	1	(31.786, 37.30)	5.514	(31.776,35.712)	3.936
19		(34.406, 56.605)	22.199	(33.718,50.222)	16.504

5. Discussion and conclusions

In this study, one-parameter models belonging to the class of exponential models are considered. Two well-known models, $Exp(\theta)$ and $Ray(\theta)$, are examined based on a censored type-II sample. GB, GEB, GBP, and GEBP are discussed for the two distributions with different values of LRP η. In the following subsections, we discuss simulation and illustrative example results.

5.1. Simulation results

From the results in Table 2 to Table 5, we can summarize the results of the two distributions as follows:

5.1.1. The result of exponential model

- GBE becomes better for small values of LRP but for large values of $\boldsymbol{r}$, which means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. GEBE becomes better for large values of LRP and for large values of $\boldsymbol{r}$, which means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$.
- GBP and GEBP become better for large values of $\boldsymbol{r}$ and LRP, which means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$.
- The result of GBE is better than that of GEBE, but the result of GEBP is better than that of GBP.
- Small values of LRP give the best result for GBE but vice versa for GEBP.

5.1.2. The result of Rayleigh model

- GBE becomes better for small values of LRP but for large values of $\boldsymbol{r}$, which means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. GEBE becomes better for large values of LRP and for large values of $\boldsymbol{r}$, except for the complete sample, the result becomes better for small values of LRP, which means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. The result of GBE is better than that of GEBE at $\boldsymbol{r} = 30, 40$, but GEBE is better than GBE for the complete sample.
- GBP and GEBP become better for large values of $\boldsymbol{r}$ and LRP, which means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$.
- The result of GBE is better than that of GEBE, but the result of GEBP is better than that of GBP.
- Small values of LRP for the complete sample give the best result for GBE but vice versa for GEBP.

5.2. Illustrative example

In this subsection, because of using noninformative prior, we can say that GBE is MLE. From the results in Table 6 to Table 10, we can summarize the results of estimation and prediction for the two distributions as follows:

5.2.1. The result of estimation

- There is no effect for the LRP on the results of GBE in the case of noninformative prior.
- The best result of GBE for exponential distribution is $\hat{\theta}_{GB} = 0.0696$.
- The best result of GEBE for exponential distribution is $\hat{\theta}_{GE} = 0.0905$.
- The best result of GBE for Rayliegh distribution is $\hat{\theta}_{GB} = 0.0036$.
- The best result of GEBE for Rayliegh distribution is $\hat{\theta}_{GB} = 0.0045$.
- In both distributions, GEBE becomes better for large values of LRP and for large values of $\boldsymbol{r}$, which means getting the best result for the complete sample at $\eta = 1$.
- Because of using the noninformative prior, the result of GEBE is better than that of GBE (MLE).

5.2.2. The result of prediction

- According to the length of the interval, GBP becomes better for large values of LRP, which means getting the best result at $\eta = 1$ and $s = 16$.
- GEBP becomes better for small values of LRP, which means getting the best result at $\eta = 0.1$ and $s = 16$.
- The result of GEBP is better than that of GBP (because of using noninformative prior).

Generally, we can conclude that the result of GBE is better than that of GEBE, but the result of GEBP is better than GBP. Small values of LRP for the complete sample mostly give the best result for GBE and GEBE but LRP differ for the best result of GBP and GEBP.

The study here is based on one-parameter models; in future work, the study can be extended to two or more parameters.

Author contributions

Yahia Abdel-Aty: Project administration, methodology, investigation; Mohamed Kayid: Writing – original draft, formal analysis, data curation, conceptualization; Ghadah Alomani: Writing – review & editing, supervision, software, resources, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

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