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*Research article*

# Cumulative entropy properties of consecutive systems

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**Abstract:** We investigated certain properties of cumulative entropy related to the lifetime of consecutive $k$-out-of-$n$:F systems. First, we presented a technique to compute the cumulative entropy of the lifetimes of these systems and studied their preservation properties using the established stochastic orders. Furthermore, we derived valuable bounds applicable in cases where the distribution function of component lifetimes is complex or when systems consist of numerous components. To facilitate practical applications, we introduced two nonparametric estimators for the cumulative entropy of these systems. The efficiency and reliability of these estimators were demonstrated using simulated analysis and subsequently validated using real data sets.

**Keywords:** consecutive $k$-out-of-$n$:F systems; cumulative entropy; Shannon entropy; stochastic orders; real data
**Mathematics Subject Classification:** 94A17

## 1. Introduction

Numerous researchers have used Shannon's information theory to evaluate systems based on their uncertainty and available data to predict their lifetime. Over the last three decades, intensive research has been conducted on $k$-out-of-$n$ systems with consecutive structures and various configurations. These systems can be classified on the basis of their component arrangement (linear or circular) and their operational state (failure or function). A linear consecutive $k$-out-of-$n$:F system, which consists of $n$ components with independent and identically distributed (iid) lifetimes arranged in a linear sequence, fails when at least $k$ components fail. To illustrate this concept, let us consider an oil pipeline as a representative case. This pipeline consists of $n$ pumping stations evenly spaced at 100 km intervals, each capable of transporting up to 400 km of oil. The system fails if four stations fail. This scenario illustrates a linear, consecutive 4-out-of-$n$:F system. A conventional parallel system corresponds to a consecutive $n$-out-of-$n$:F system in which all $n$ components fail. Conversely, a series system corresponds to a 1-out-of-$n$:F system in which at least one component fails. Derive analytically the mean operating time between failures for a non-repairable component system studied in [1]. Optimal designs of series consecutive $k$-out-of-$n$: G systems when $k < n \leq 2k$ is obtained in [2]. To show that for any fixed $k$ the lifetime of a (linear or circular) consecutive k-out-of-n:F system is stochastically decreasing in $n$, Boland [3] used recursive relations for the reliability of such systems with independent, identically distributed components. Representations for the reliability of systems with consecutive types as a mixture of the reliability of order statistics when the systems consist of interchangeable components were obtained in [4]. Comprehensive reviews of previous studies in this area can be found in several publications, including [5–7].

The distribution of the lifetimes of linear, consecutive $k$-out-of-$n$ systems is simple if $2k \geq n$ has shown the determination in [8]. Therefore, we focus on scenarios where $2k \geq n$ holds, because this condition simplifies the mathematical analysis and facilitates the derivation of various results. These systems consist of components with lifetimes denoted by $X_i$, $1 \leq i \leq n$, each characterized by a probability density function (pdf) $f(x)$ and a cumulative distribution function (cdf) $F(x)$. The total lifetime of the system is represented by the random variable (rv) $T_{k|n:F}$. The cdf of the consecutive $k$-out-of-$n$:F system for $2k \geq n$ can be expressed as (see e.g., Lemma 2.1 in [9])

$$F_{k|n:F}(x) = (n-k+1)F^k(x) - (n-k)F^{k+1}(x), \ x > 0. \tag{1}$$

An important contribution to the relationship between information theory and reliability by investigating the information properties of order statistics was made in [10]. Shannon's differential entropy, which refers to Shannon's pioneering work [11], has gained widespread adoption as a measure of uncertainty and has become a fundamental concept in probability theory. It is defined by $H(X) = -\mathbb{E}[\log f(X)]$ for a non-negative continuous rv $X$ with pdf $f(x)$, such that $\log(\cdot)$ means natural logarithm and $\mathbb{E}[\cdot]$ stands for the expectation. Although Shannon's differential entropy offers numerous advantages, an alternative measure of uncertainty, the cumulative residual entropy (CRE), was proposed by [12]. In contrast to traditional entropy, CRE utilizes $S(x) = 1 - F(x)$ instead of $f(x)$, as follows:

$$\mathcal{E}(X) = -\int_0^\infty S(x)\log S(x)\,dx. \tag{2}$$

For a detailed study of the preliminary aspects of (2), the associated dynamical form, and its various generalizations, we refer the readers to [13,14]. In the spirit of (2), the cumulative entropy (CE) introduced in [15] by replacing $S(x)$ with the cdf $F(x)$, as

$$\begin{aligned}\mathcal{CE}(X) &= -\int_0^\infty F(x)\log F(x)dx \\ &= \int_0^1 \frac{\xi(u)}{f(F^{-1}(u))}du,\end{aligned} \tag{3}$$

where $F^{-1}(u) = \inf\{x; F(x) \geq u\}$ and $\xi(u) = -u\log\, u,\ 0 \leq u \leq 1$.

A key advantage of the CE measure is its connection to the mean inactivity time (MIT) function, given by $\tilde{m}(x) = \mathbb{E}(x - X|X \leq x)$. Moreover, the CE is the expected value of the MIT function, expressed as $\mathbb{E}(\widetilde{m}(X)) = \mathcal{CE}(X)$, as [15] shows. This relationship underscores the CE's utility in reliability theory, given that the MIT function is commonly used to characterize the aging properties of systems or components. CE fulfills the condition $0 \leq \mathcal{CE}(X) \leq \infty$, since the argument of the logarithm is a probability measure. Moreover, CE is zero only if $X$ is a degenerate random variable. It is worth noting that if $Y = aX + b$, with $0 \neq a \in R$, and $b \in R$, then $\mathcal{CE}(Y) = a\mathcal{CE}(X)$ if $a > 0$ and $\mathcal{CE}(Y) = a\mathcal{E}(X)$ if $a < 0$. Consequently, CE acts as a dispersion measure. Ahmadi et al. [16] investigated the properties of CE in two dimensions. Di Crescenzo and Toomaj [17] have defined the reversed relevation transform as a dual to the relevance transformation for two absolutely continuous, non-negative, independent random variables and apply such transformations to the lifetimes of the components of parallel and series systems under suitable proportionality assumptions for the hazard rates. Kayal [18] introduced a generalization of the proposed CE. In addition, CE has been extended to more general cases, as shown in [19–21] and related studies.

Several studies have investigated the information properties of order statistics and coherent systems. Recently, Toomaj and Doostparast [22] integrated the concepts of reliability theory and information theory and used a system signature to analyze the entropy criteria of mixed systems. The CRE of mixed systems under the assumption that the lifetimes of the components are iid was investigated in [23]. A system signature to study the fractional cumulative residual entropy of coherent systems was used in [24]. In a separate study, the CRE of a coherent system with multiple components under the assumption that all components fail at a given time was investigated in [25]. They presented several properties, including formulations, bounds, and orderings for this measure, as well as a method for evaluating a superior system based on the cumulative Kullback-Leibler information set, which serves as a discriminating feature. Moreover, the Rényi entropy for coherent systems with $n$ components assuming that all components fail at a given time $t$ has been studied by [26]. They presented numerous results showing computational formulas for this entropic measure and establishing certain bounds on this measure as well as stochastic order results.

We build on this research to carry out existing research on information measures in reliability. We investigate the uncertainty properties of CE, particularly in the context of consecutive $k$-out-of-$n$: F systems. The simple and adaptable nature of the reliability function for these systems has led us to investigate their CE further.

The remainder of this article is organized as follows. In Section 2, we derive a representation of the CE for successive $k$-out-of-$n$: F systems with lifetime $T_{k|n:F}$ based on samples from an arbitrary continuous distribution function $F$. This representation is related to the CE of samples from a uniform distribution. We also analyze the preservation of the stochastic order properties of this system. This section provides useful bounds for the CE of consecutive $k$-out-of-$n$: F systems. In Section 3, we present several characterization results, and in Section 4, we present computational results that confirm our derived results. To this end, we present two non-parametric estimators for the CE of consecutive systems and demonstrate their effectiveness using real and simulated data. In Section 5, we conclude the study by summarizing the major results and contributions.

## 2. CE of consecutive $k$-out-of-$n$:F System

This section is divided into two parts. First, we derive a mathematical expression for the CE of a consecutive $k$-out-of-$n$:F system and analyze the preservation properties of its stochastic order. Then, we establish a set of essential bounds to study consecutive $k$-out-of-$n$:F systems.

*2.1. Expression and stochastic orders*

In the following, we derive an explicit expression for the CE of a consecutive $k$-out-of-$n$:F system with a lifetime $T_{k|n:F}$, where the lifetime of the components follows a joint continuous distribution function $F$. We use the probability integral transformation $U_{k|n:F} = F(T_{k|n:F})$ to obtain a useful formula. The transformations of the system components, $U_i = F(X_i)$ for $i = 1, \dots, n$, are iid random rvs uniformly distributed on $[0, 1]$. Using (1), if $2k \geq n$, the cdf of $U_{k|n:F}$ is given by

$$G_{k|n:F}(u) = (n-k+1)u^k - (n-k)u^{k+1}, \tag{4}$$

for all $0 < u < 1$. We are now prepared to present the following theorem based on these transformations.
**Theorem 2.1.** *For $2k \geq n$, the CE of $T_{k|n:F}$, can be expressed as follows:*

$$\mathcal{CE}(T_{k|n:F}) = \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))} du, \tag{5}$$

where $\xi(x) = -x\log x, 0 < x < 1$, and $G_{k|n:F}(u)$ is define in (4).

*Proof.* Note that, since $\xi(x) = -x\log x, 0 < x < 1$, we have

$$\begin{aligned}
\xi\left(F_{k|n:F}(x)\right) &= -F_{k|n:F}(x)\log F_{k|n:F}(x) \\
&= -\Big((n-k+1)F^k(x) \\
&\quad - (n-k)F^{k+1}(x)\Big)\log\Big((n-k+1)F^k(x) - (n-k)F^{k+1}(x)\Big) \\
&= \xi\Big((n-k+1)F^k(x) - (n-k)F^{k+1}(x)\Big),
\end{aligned}$$

for all $x > 0$. By employing the change of $u = F(x)$ and referring to (1) and (3), we can derive

$$\begin{aligned}
\mathcal{CE}(T_{k|n:F}) &= \int_0^\infty \xi\left(F_{k|n:F}(x)\right) dx \\
&= \int_0^\infty \xi((n-k+1)F^k(x) - (n-k)F^{k+1}(x))dx \\
&= \int_0^1 \frac{\xi((n-k+1)u^k - (n-k)u^{k+1})}{f(F^{-1}(u))} du, \text{(taking } u = F(x)) \\
&= \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))} du,
\end{aligned}$$

where

$$\begin{aligned}
\xi\left(G_{k|n:F}(u)\right) &= -G_{k|n:F}(x)\ \log G_{k|n:F}(x) \\
&= -\Big((n-k+1)u^k - (n-k)u^{k+1}\Big)\log\Big((n-k+1)u^k - (n-k)u^{k+1}\Big) \\
&= \xi\Big((n-k+1)u^k - (n-k)u^{k+1}\Big),
\end{aligned}$$

for all $0 < u < 1$, and this completes the proof.

Following Eq (5), we present the following illustrative example.
**Example 2.1.** Consider a linear consecutive 3-out-of-5:F system with a lifetime $T_{3|5:F} = \min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4), \max(X_3, X_4, X_5))$ as shown in Figure 1.

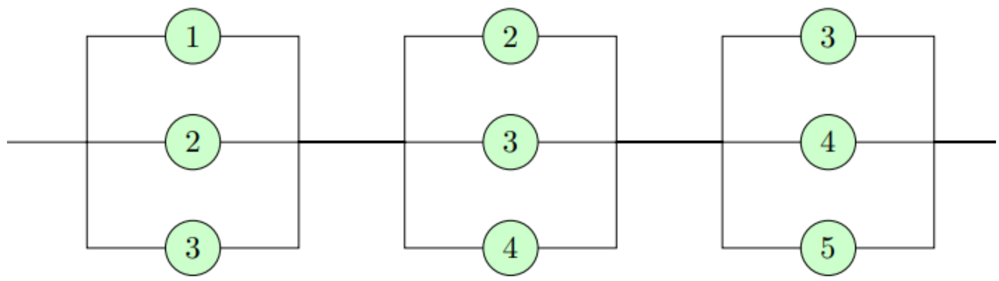


**Figure 1.** A linear consecutive 3-out-of-5:F system.

As shown in Figure 1, this system can be considered a hybrid system with shared components. This concept is further explored in Sections 5 and 6 of [27] and [28]. The precise value of CE of the lifetime of the linear consecutive 3-out-of-5:G system can be calculated using Eq (5) for certain standard component lifetime distributions. For instance, consider the following models:

- Consider that the lifetimes of the components are iid having the common exponential distribution characterized by the cdf a

$$F(x) = 1 - e^{-\lambda x}, x, \lambda > 0.$$

The exponential distribution plays a vital role in reliability and survival analysis, commonly used to represent the lifetimes of components in systems that do not exhibit aging over time due to its memoryless property. Therefore, by assuming an exponential distribution for the lifetimes of components, there is no benefit in replacing components that have been used, as they continue to perform reliably. Since $f(F^{-1}(u)) = \lambda(1-u)$ for $0 < u < 1$, recalling (5), we get

$$\mathcal{CE}(T_{k|6:F}) = \int_0^1 \frac{\xi(G_{k|n:F}(u))}{\lambda(1-u)} du.$$

It is evident that the CE decreases as the parameter λ increases. Therefore, an increase in λ leads to a decrease in the uncertainty of the system's lifetime in terms of cumulative entropy.

- Assume further that the lifetimes of the components are iid following the common Fréchet distribution, also known as inverse Weibull distribution, with the cdf as

$$F(x) = e^{-x^{-\alpha}}, \qquad x > 0,$$

where $\alpha > 0$ is a shape parameter. It is worth noting that Fréchet distribution is a special case of the generalized extreme value distribution. Moreover, it is utilized to analyze extreme events, including the highest one-day rainfall amounts and peak river discharges recorded annually in hydrology. In decline curve analysis, the decreasing trend in the time series data of oil or gas production rates for a well can be modeled using the Fréchet distribution [29,30]. The pdf of this distribution is $f(x) = \alpha x^{-(\alpha+1)} e^{-x^{-\alpha}}$, $x > 0$. It is not hard to see that $f(F^{-1}(u)) = \alpha u(-\log(u))^{\frac{\alpha+1}{\alpha}}$ for all $0 < u < 1$. Thus, from Eq (5), we can derive the following expression:

$$\mathcal{CE}\left(T_{k|6:F}\right) = \int_0^1 \frac{\xi\left(G_{k|n:F}(u)\right)}{\alpha u(-\log(u))^{\frac{\alpha+1}{\alpha}}} du. \qquad (6)$$

It is worth mentioning that, since it is difficult to obtain an explicit analytical expression, we used a computational approach to study the relationship between $\mathcal{CE}(T_{3|5:F})$ and the parameter $\alpha > 1$ (for $\alpha < 1$ the integral is divergent). This method provides information on how the parameters of the Fréchet distribution influence the CE of a consecutive 3-out-of-5:F system. Figure 2 summarizes the numerical analysis and shows the relationship between $\mathcal{CE}(T_{3|5:F})$ and $\alpha$. As the shape parameter $\alpha$ increases, the system uncertainty with respect to the CE decreases. These results show the significant influence of the Fréchet distribution parameter $\alpha$ on the CE and uncertainty of the consecutive 3-out-of-5:F system.

In what follows, we show that the CE of the consecutive $k$-out-of-$n$:F system retains both the dispersive order and the location-independent riskier order. Before discussing these peculiarities, we introduce the definitions for these stochastic orders. In this discussion, $\mathfrak{R}_+ = \{X; X \geq 0\}$ stands for the collection of all non-negative random variables with support $(0, \infty)$ that have an absolutely continuous distribution.

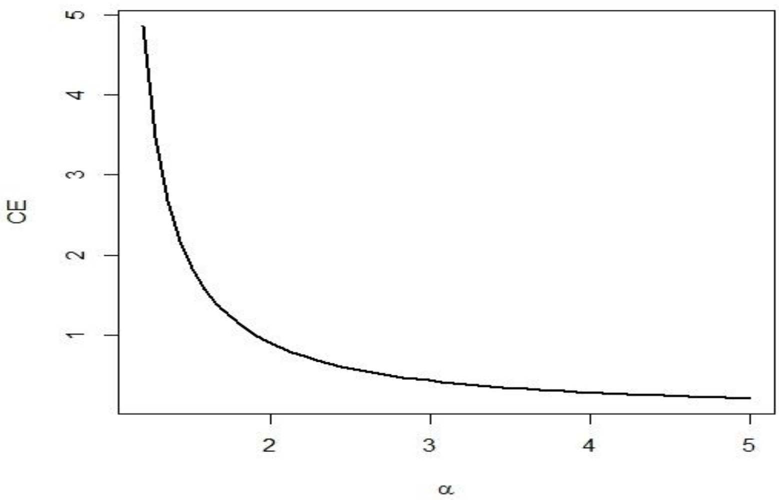


**Figure 2.** The plot of $\mathcal{CE}(T_{3|5:F})$ with respect to $\alpha$ as demonstrated in Example 2.1.

**Definition 2.1.** *Let* $X \in \mathfrak{R}_+$ *and* $Y \in \mathfrak{R}_+$ *with pdfs* $f_X$ *and* $f_Y$, *cdfs* $F_X$ *and* $F_Y$, *survival functions* $S_X$ *and* $S_Y$ *and hazard rate (hr) functions* $\lambda_X(x) = \frac{f_X(x)}{S_X(x)}$ *and* $\lambda_Y(x) = \frac{f_Y(x)}{S_Y(x)}$, *respectively. Then,*

1) $X$ belongs to increasing [resp. decreasing] failure rate (abbreviated by IFR [resp. DFR]) if $\lambda_X$ is an increasing (a decreasing) function;
2) $X$ is said to be less than or equal with $Y$ in the hazard rate order (written as $X \leq_{hr} Y$) whenever $\lambda_X(t) \geq \lambda_Y(t)$ for all $t > 0$;
3) $X$ is said to be less than or equal with $Y$ in the dispersive order (written as $X \leq_d Y$) whenever $F_X^{-1}(v) - F_X^{-1}(u) \leq F_Y^{-1}(v) - F_Y^{-1}(u),\ 0 < u \leq v < 1$;
4) $X$ is said to be less than or equal with $Y$ in the location-independent riskier order (written as $X \leq_{lir} Y$) whenever $\int_0^{F_X^{-1}(p)} F_X(x)dx \leq \int_0^{F_Y^{-1}(p)} F_Y(x)dx,\ p \in (0,1)$.

It is important to note that Bickel and Lehmann [31] first used the order $\leq_d$ for certain non-parametric inferences, while Jewiitt [32] introduced the order $\leq_{lir}$ for use in expected utility theory and its insurance-related applications. According to [33], $X \leq_d Y$ if, and only if,

$$f_Y\left(F_Y^{-1}(v)\right) \leq f_X\left(F_X^{-1}(v)\right),\ for\ \ all\ \ 0 < v < 1. \tag{7}$$

The following implications are known:

$$\text{If } X \leq_{\rm hr} Y \text{ and either X or Y is DFR} \Longrightarrow X \leq_d Y \Longrightarrow X \leq_{lir} Y. \qquad (8)$$

Taking into account Eqs (3) and (8) and the fact that $\xi(u)$ is non-negative for all $0 \leq u \leq 1$, it follows that $\mathcal{CE}(X) \leq \mathcal{CE}(Y)$, if $X \leq_d Y$. This conclusion can be further substantiated by applying (7).

**Corollary 2.1.** *If $X \leq_{hr} Y$ and $X$ or $Y$ is DFR, then $\mathcal{CE}(X) \leq \mathcal{CE}(Y)$.*

Assume that $Z$ be a random variable with cdf H. Then, the cumulative reversed hazard function is defined as

$$\eta_Z(x) = \int_0^x H(z)dz, \qquad x > 0.$$

Landsberger and Meilijson [34] showed that

$$X \leq_{lir} Y \Leftrightarrow \eta_Y^{-1}(x) - \eta_X^{-1}(x) \text{ is increasing in } x > 0. \qquad (9)$$

In the following, we consider $T_{k|n:F}^X$ and $T_{k|n:F}^Y$ as the lifetimes of two consecutive $k$-out-of-$n$:F systems with iid absolutely continuous component lifetimes with the common pdfs $f_X$ and $f_Y$ and cdfs $F_X$ and $F_Y$, respectively. We present a theorem that shows that the CE of a series system with $k$ components is smaller than that of a consecutive $k$-out-of-$n$:F system, provided that both have components with the DFR property.

**Theorem 2.2.** *Let $Z_{1:m}$ be the lifetime of a series system consisting of iid components with common hr function $h$. Let $lim_{t\to\infty} r(t) = \lambda$ and $1 \leq k \leq n$, where $r$ is the common hr of $X$ and let $T_{k|n:F}$ belong to IFR class such that $h(t) \geq \lambda \left[\frac{n}{k}\right]/m$ for all $t \geq 0$. If $Z$ belongs to DFR class, then $\mathcal{CE}(Z_{1:m}) \leq \mathcal{CE}(T_{k|n:F})$.*

*Proof.* Since $Z$ is DFR, then $Z_{1:m}$ is also DFR. Moreover, under the conditions $T_{k|n:F}$ is IFR such that $h(t) \geq \lambda[n/k]/m$ for all $t \geq 0$, we have $Z_{1:m} \leq_{hr} T_{k|n:F}$ due to Theorem 3.2 of [27]. Thus, Corollary 2.1 concludes the proof.

The next example illustrates the application of Theorem 2.2.

**Example 2.2.** Consider a Gamma distribution whose cdf is given by $F(t) = 1 - \lambda t e^{-\lambda t} - e^{-\lambda t}$. We find that $r(t) = \frac{\lambda^2 t}{1+\lambda t} \to \lambda$ as $t \to \infty$. With $n = 4$ and $k = 2$, since $X$ is IFR and a linear consecutive 2-out-of-4:F system preserves the IFR property (see Theorem 4.3.13 of [5]), we conclude that $Z_{1:4} \leq_{hr} T_{2|4:F}$ for $h(t) \geq \frac{\lambda}{2}$ for all $t \geq 0$. So, Theorem 2.2 thus implies that $\mathcal{CE}(Z_{1:m}) \leq \mathcal{CE}(T_{k|n:F})$ provided that $h(t)$ is a decreasing function in $t$.

The following theorem outlines the conditions under which the dispersive order is preserved under the formation of consecutive systems.

**Theorem 2.3.** *If $X \leq_d Y$, then $\mathcal{CE}(T_{k|n:F}^X) \leq \mathcal{CE}(T_{k|n:F}^Y)$.*

*Proof.* The result can be easily derived from Eqs (5) and (7).

As an application of Theorem 2.3, consider the following example.

**Example 2.3.** Let us consider two consecutive 4-out-of-5:F systems with lifetimes $T_{4|5:F}^X$ and $T_{4|5:F}^Y$. System $T_{4|5:F}^X$ has iid component lifetimes $X_1, X_2, X_3, X_4, X_5$, which follow the Makeham distribution with cdf $F(x) = 1 - e^{-2x + e^{-x} - 1}$ for $x > 0$. Moreover, system $T_{4|5:F}^Y$ consists of iid component lifetimes $Y_1, Y_2, Y_3, Y_4$ that follow an exponential distribution with cdf $F_Y(x) = 1 - e^{-x}$ for $x > 0$. The hr functions are $\lambda_X(x) = 2 - e^{-x}$ and $\lambda_Y(x) = 1$, showing that $\lambda_X(x) > \lambda_Y(x)$ for $x > 0$ i.e., $X \leq_{hr} Y$. Since $Y$ possesses the DFR property, relation (8) indicates $X \leq_d Y$. Consequently, by Theorem 2.3, we have $\mathcal{CE}(T_{4|5:F}^X) \leq \mathcal{CE}(T_{4|5:F}^Y)$, meaning the uncertainty associated with $T_{4|5:F}^X$ is less than or equal to that of $T^Y_{4|5:F}$ in terms of the CE measure.

The next theorem outlines the conditions for the preservation of the location-independent riskier order in consecutive systems.

**Theorem 2.4.** *If* $X \leq_{lir} Y$, *and*

$$\frac{\xi(G_{k|n:F}(t))}{t}, \qquad 0 \leq t \leq 1$$

is a decreasing function of $t$, then $\mathcal{CE}(T^X_{k|n:F}) \leq \mathcal{CE}(T^Y_{k|n:F})$.

*Proof.* First note that Eq (1) can be rewritten as $F_{k|n:F}(x) = G_{k|n:F}(F(x))$. Assumption $X \leq_{lir} Y$ and Eq (9) imply

$$\frac{d}{dx}(\eta_Y^{-1}(x) - \eta_X^{-1}(x)) = \frac{1}{F_Y(\eta_Y^{-1}(x))} - \frac{1}{F_X(\eta_X^{-1}(x))} \geq 0,$$

which means

$$F_X(x) \geq F_Y\left(\eta_Y^{-1}\left(\eta_X(x)\right)\right) \tag{10}$$

for all $x > 0$. Now, we get

$$\begin{aligned}
-\int_0^\infty F^X_{k|n:F}(x)\log F^X_{k|n:F}(x)dx &= -\int_0^\infty \frac{F^X_{k|n:F}(x)\log F^X_{k|n:F}(x)}{F_X(x)} F_X(x)\mathrm{d}x \\
&= \int_0^\infty \frac{\xi(F^X_{k|n:F}(x))}{F_X(x)} F_X(x)\mathrm{d}x \\
&= \int_0^\infty \frac{\xi(G_{k|n:F}(F_X(x)))}{F_X(x)} F_X(x)\mathrm{d}x \\
&\leq \int_0^\infty \frac{\xi\left(G_{k|n:F}\left(F_Y\left(\eta_Y^{-1}(\eta_X(x))\right)\right)\right)}{F_Y\left(\eta_Y^{-1}(\eta_X(x))\right)} F_X(x)\mathrm{d}x,
\end{aligned} \tag{11}$$

where the inequality arises from the fact that $\xi(G_{k|n:F}(t))/t$ decreases for $0 \leq t \leq 1$ and using (10). Setting $u = \eta_Y^{-1}(\eta_X(x))$, we have

$$dx = \frac{F_Y(u)}{F_X(\eta_X^{-1}(\eta_Y(u)))}du.$$

Upon using this, (11) reduces to

$$\begin{aligned}
&\int_{\eta_Y^{-1}(\eta_X(0))}^\infty \frac{\xi\left(G_{k|n:F}\left(F_Y(u)\right)\right)}{F_Y(u)} \frac{F_X\left(\eta_X^{-1}\left(\eta_Y(u)\right)\right)F_Y(u)}{F_X\left(\eta_X^{-1}\left(\eta_Y(u)\right)\right)}\mathrm{d}u \\
&= \int_{\eta_Y^{-1}(\eta_X(0))}^\infty \xi(G_{k|n:F}(F_Y(u)))\mathrm{d}u \\
&= -\int_0^\infty F^Y_{k|n:F}(u)\log F^Y_{k|n:F}(u)du.
\end{aligned}$$

The final equality in the above relation is derived by observing that $\eta_Y^{-1}(\eta_X(0)) = 0$, which implies $\mathcal{CE}(T^X_{k|n:F}) \leq \mathcal{CE}(T^Y_{k|n:F})$. Thus, the proof is finished.

*2.2. Bounds*

Since there are no closed-form expressions for the CE of consecutive systems in different distributions with complex distribution functions or multiple components, it is crucial to establish certain bounds for these scenarios. Given this problem, we investigated the effectiveness of these bounds in describing the CE of consecutive systems. Our initial results showed that the CE of the system is bounded by the joint CE of its components.

**Theorem 2.5.** *For* $2k \geq n$, *the CE of* $T_{k|n:F}$ *are bounded as follows:*

$$\mathfrak{B}_1\mathcal{CE}(X_1) \leq \mathcal{CE}(T_{k|n:F}) \leq \mathfrak{B}_2\mathcal{CE}(X_1),$$

where $\mathfrak{B}_1 = \inf_{u\in(0,1)}\frac{\xi(G_{k|n:F}(u))}{\xi(v)}$, $\mathfrak{B}_2 = \sup_{u\in(0,1)}\frac{\xi(G_{k|n:F}(u))}{\xi(u)}$.

*Proof.* The upper bound can be determined from (5) as shown below

$$\begin{aligned}
\mathcal{CE}(T_{k|n:F}) &= \int_0^1 \frac{\xi\left(G_{k|n:F}(u)\right)}{f\left(F^{-1}(u)\right)}du \\
&= \int_0^1 \frac{\xi(G_{k|n:F}(u))}{\xi(u)}\frac{\xi(u)}{f(F^{-1}(u))}du \\
&\leq \sup_{u\in(0,1)} \frac{\xi\left(G_{k|n:F}(u)\right)}{\xi(u)}\int_0^1 \frac{\xi(u)}{f\left(F^{-1}(u)\right)}du \\
&= \mathfrak{B}_2\mathcal{CE}(X_1).
\end{aligned}$$

The lower bound can be derived using a similar approach.

The following theorem provides further simple and practical bounds for the function $\xi(u)$ and the pdf extremes.

**Theorem 2.6.** *Let* $T_{k|n:F}$ *be the lifetime of consecutive* $k$*-out-of-*$n$*:F system having the common pdf* $f_X(x)$ *and cdf* $F_X(x)$. *If* $S$ *is the support of* $f$, $m = inf_{x\in S}f(x)$ *and* $M = sup_{x\in S}f(x)$, *then*

$$\frac{\mathcal{CE}(U_{k|n:F})}{M} \leq \mathcal{CE}(T_{k|n:F}) \leq \frac{\mathcal{CE}(U_{k|n:F})}{m}, \tag{12}$$

where $\mathcal{CE}(U_{k|n:F}) = \int_0^1 \xi(G_{k|n:F}(u))du$ and $\xi(u) = -u\log u$.

*Proof.* Since $m \leq f(F^{-1}(u)) \leq M,\ 0 < u < 1$, from (5), we have

$$\mathcal{CE}(T_{k|n:F}) = \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))}du \geq \frac{1}{M}\int_0^1 \xi(G_{k|n:F}(u))du.$$

The upper bound can be obtained similarly.

It is important to realize that $\mathcal{CE}(U_{k|n:F})$ represents the cumulative entropy of a consecutive $k$-out-of-$n$:F system with a joint uniform distribution on $(0,1)$. The constraints in Eq (12) depend on the extremes of the pdf $f$. If the lower bound m is zero, there is no upper bound. If the upper bound $M$ is infinite, there is no lower bound either. The bounds in Theorems 2.5 and 2.6 are particularly useful when the cumulative distribution function does not have a closed form, which makes it difficult to evaluate the cumulative distribution function in (1). For many known distributions, the CE expressions simplify the calculation of the bounds in Theorem 2.5. In cases with complex cumulative distribution functions, such as mixed distributions or systems with multiple components, Theorems 2.5 and 2.6 can help predict the uncertainty in the lifetime of the system using cumulative entropy. If both theorems provide computable bounds, then the maximum of the two lower bounds can be used.

To demonstrate the application of the bounds from Theorems 2.6 and 2.7, we consider an example with a consecutive $k$-out-of-$n$:F system.

**Example 2.4.** Assume a linear consecutive 6-out-of-12:F system with lifetime $T_{6|12:F} = \max\left(X_{[1:6]}, X_{[2:7]}, \dots, X_{[6:12]}\right)$, where $X_{[j:m]} = \min(X_j, \dots, X_m)$ for $1 \leq j < m \leq 12$. It is straightforward to calculate that $\mathcal{CE}(U_{6|12:F}) = 0.1386067$ and $\mathfrak{B}_1 = 0, \mathfrak{B}_2 = 1.783154$. The bounds in Theorems 2.5 and 2.6 can be calculated for common component lifetime distributions. To illustrate, we consider the following models as examples:

1) Assuming a half-normal distribution with pdf

$$f_X(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}, \qquad x > 0, \qquad \sigma > 0.$$

It is easy to see that $m = 0$ and $M = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}$. Applying the result from Theorem 2.6, we can obtain the lower bound $\mathcal{CE}(T_{6|12:F}) \geq \frac{0.1960195}{\sigma\sqrt{\pi}}$. Furthermore, using the bound provided in Theorem 2.5, we can derive $\mathcal{CE}(T_{6|12:F}) \leq 1.783154\mathcal{CE}(X_1)$. By combining these two bounds, we can conclude that $\frac{0.1960195}{\sigma\sqrt{\pi}} \leq \mathcal{CE}\left(T_{6|12:F}\right) \leq 1.783154\mathcal{CE}(X_1)$.

2) Suppose that $X$ follows a Fréchet distribution with cdf given in (6). Then $m = 0$ and

$$M = \alpha\left(\frac{\alpha}{\alpha+1}\right)^{-\frac{\alpha+1}{\alpha}} e^{-(1+\frac{1}{\alpha})}.$$

Furthermore, using the bound provided in Theorem 2.5, we can derive $\mathcal{CE}(T_{6|12:F}) \leq 1.783154\mathcal{CE}(X_1)$. By combining these two bounds, we can conclude that

$$0.1386067\alpha\left(\frac{\alpha}{\alpha+1}\right)^{-\frac{\alpha+1}{\alpha}} e^{-\left(1+\frac{1}{\alpha}\right)} \leq \mathcal{CE}\left(T_{6|12:F}\right) \leq 1.783154\mathcal{CE}(X_1).$$

## 3. Characterization results

In this section, we present characterization results related to the cumulative entropy of consecutive $k$*-out-of-n:* F systems. Characterizations of symmetric continuous distributions using extropy and related measures, such as cumulative residual extropy and cumulative past extropy, have been studied in [35,36]. These researchers found that a key feature of symmetric distributions is the equality of these measures for upper and lower order statistics. Additionally, using concomitants of order statistics from the Farlie-Gumbel-Morgenstern family, they demonstrated that this characteristic holds for these measures as well. We now demonstrate that the cumulative entropy of the lifetime of consecutive $k$-out-of-$n$:G system uniquely characterizes the parent distribution of the component lifetime.

**Theorem 3.1.** *Let* $T^X_{k|n:F}$ *and* $T^Y_{k|n:F}$ *be lifetimes of two consecutive* $k$*-out-of-n:G systems having the common pdfs* $f_X(x)$ *and* $f_Y(x)$ *and cdfs* $F_X(x)$ *and* $F_Y(x)$*, respectively. Then,* $F_X$ *and* $F_Y$ *belong to the same family of distributions if and only if* $X \leq_d Y$ *and*

$$\mathcal{CE}\left(T^X_{k|n:F}\right) = \mathcal{CE}\left(T^Y_{k|n:F}\right),$$

*for all k and n such that* $2k \geq n$.

*Proof.* The necessity is trivial, so we must demonstrate the sufficiency. First, observe that Eq (5) can be rewritten as follows:

$$\mathcal{CE}\left(T^{X}_{k|n:F}\right)=\int_0^1 \frac{\phi(u)}{f_X(F_X^{-1}(u))}du, \tag{13}$$

where $\phi(u)=-G_{k|n:F}(u)\log\left(G_{k|n:F}(u)\right)$, for all $0<u<1$. The same argument applies to $\mathcal{CE}\left(T^{Y}_{k|n:F}\right)$. From (13) and by the assumption that $\mathcal{CE}\left(T^{X}_{k|n:F}\right)=\mathcal{CE}\left(T^{Y}_{k|n:F}\right)$, one can write

$$\int_0^1 \phi(u)\left[\frac{1}{f_Y(F_Y^{-1}(u))}-\frac{1}{f_X(F_X^{-1}(u))}\right]du=0. \tag{14}$$

When $2k\geq n$, we find that $0<G_{k|n:F}(u)<1$, for all $0<u<1$, so it follows that $-G_{k|n:F}(u)\ \log\left(G_{k|n:F}(u)\right)>0$, leading to the conclusion that $\phi(u)>0$, for $0<u<1$. Assumption $X\leq_d Y$ implies $f_X\left(F_X^{-1}(u)\right)\geq f_Y(F_Y^{-1}(u))$ for all $0<u<1$ due to relation (7). Thus, the expression within the brackets in the integrand (14) becomes positive. This implies

$$f_X(F_X^{-1}(u))=f_Y(F_Y^{-1}(u)), \qquad a.e. \qquad u\in(0,1).$$

It follows that $F_X^{-1}(u)=F_Y^{-1}(u)+d$, where $d$ is a constant. Since $\lim_{u\to 0}F_X^{-1}(u)=\lim_{u\to 0}F_Y^{-1}(u)=0$ for all $u\in(0,1)$, we conclude that $F_X^{-1}(u)=F_Y^{-1}(u)$. This indicates that $F_X$ and $F_Y$ have the same family of distributions.

As mentioned, a sequential $n$-out-of-$n$:F system is a series system. The properties of this system are described in the following corollary.

**Corollary 3.1.** *Under the conditions of Theorem 3.1,* $F_X$ *and* $F_Y$ *belong to the same family of distributions if and only if* $X\leq_d Y$ *and*

$$\mathcal{CE}\left(T^{X}_{n|n:F}\right)=\mathcal{CE}\left(T^{Y}_{n|n:F}\right), \qquad \text{for all } n\geq 1.$$

The subsequent theorem provides a further characterization.

**Theorem 3.2.** *Under the conditions of Theorem 3.1,* $F_X$ *and* $F_Y$ *belong to the same family of distributions, but for a change in scale, if and only if* $X\leq_d Y$ *and*

$$\frac{\mathcal{CE}\left(T^{X}_{k|n:F}\right)}{\mathcal{CE}(Y)}=\frac{\mathcal{CE}\left(T^{Y}_{k|n:F}\right)}{\mathcal{CE}(X)}, \tag{15}$$

*for all k and n such that* $2k\geq n$.

*Proof.* The necessity is trivial and hence it remains to prove the sufficiency. From (15), we can write

$$\frac{\mathcal{CE}\left(T^{X}_{k|n:F}\right)}{\mathcal{CE}(Y)}=\int_0^1 \frac{\phi(u)}{\mathcal{CE}(Y)f_X(F_X^{-1}(u))}du, \tag{16}$$

where $\phi(u)$ is defined as in the proof of Theorem 3.1. The same argument applies to $\mathcal{CE}\left(T^{Y}_{k|n:F}\right)/\mathcal{CE}(X)$. From (15) and (16), we have

$$\int_0^1 \frac{\phi(u)}{\mathcal{CE}(Y)f_X(F_X^{-1}(u))}du=\int_0^1 \frac{\phi(u)}{\mathcal{CE}(X)f_Y(F_Y^{-1}(u))}du. \tag{17}$$

Let us set $c=\mathcal{CE}(Y)/\mathcal{CE}(X)$. By assumption $X\leq_d Y$, from Theorem 3.1 of [37], it holds that $c\geq 1$. Moreover, relation (17) can be expressed as

$$\int_0^1 \left[\frac{1}{f_Y\left(F_Y^{-1}(u)\right)}-\frac{1}{cf_X\left(F_X^{-1}(u)\right)}\right]\phi(u)du=0. \tag{18}$$

The assumption $X \leq_d Y$ implies that $\frac{f_X\left(F_X^{-1}(u)\right)}{f_Y(F_Y^{-1}(u))} \geq 1$, or equivalently $\frac{cf_X\left(F_X^{-1}(u)\right)}{f_Y(F_Y^{-1}(u))} \geq 1$, for all $0 < u < 1$, since $c \geq 1$. Consequently, the expression within the brackets in the integrand (18) is positive. This implies that

$$f_X(F_X^{-1}(u)) = cf_Y(F_Y^{-1}(u)), \qquad a.e. \qquad z \in (0,1).$$

Thus, it follows that $F_X^{-1}(u) = cF_Y^{-1}(u) + d$, where $d$ is a constant. Since $\lim_{u\to 0} F_X^{-1}(u) = \lim_{u\to 0} F_Y^{-1}(u) = 0$ for all $u \in (0,1)$, we conclude that $F_X^{-1}(u) = cF_Y^{-1}(u)$. This indicates that $F_X$ and $F_Y$ belong to the same family of distributions but for a change of scale.

Using Theorem 3.2, we get the following corollary.

**Corollary 3.2.** *Suppose the assumptions of Theorem 3.2 hold. Then,* $F_X$ *and* $F_Y$ *belong to the same family of distributions, but for a change in scale, if and only if* $X \leq_d Y$ *and*

$$\frac{\mathcal{CE}\left(T_{n|n:F}^X\right)}{\mathcal{CE}(Y)} = \frac{\mathcal{CE}\left(T_{n|n:F}^Y\right)}{\mathcal{CE}(X)},$$

*for all* $n \geq 1$.

## 4. Nonparametric estimation

In this section, we develop two non-parametric methods for estimating the cumulative entropy of consecutive $k$-*out-of*-$n$:F systems. Let us assume a sequence of iid continuous, non-negative rvs $X_1, X_2, \dots, X_N$, where $X_{1:N} \leq X_{2:N} \leq \cdots \leq X_{N:N}$ denotes their order statistics. Applying Eq (5), the CE of $T_{k|n:F}$ can be reformulated for the case $2k \geq n$ as follows:

$$\mathcal{CE}(T_{k|n:F}) = \int_0^1 \frac{\xi\left(G_{k|n:F}(u)\right)}{f(F^{-1}(u))} du = \int_0^1 \xi\left(G_{k|n:F}(u)\right) \left[\frac{dF^{-1}(u)}{du}\right] du$$

$$= \int_0^1 \xi((n-k+1)u^k - (n-k)u^{k+1}) \left[\frac{dF^{-1}(u)}{du}\right] du. \tag{18}$$

Using Eq (18), we estimate $\mathcal{CE}(T_{k|n:F})$ by approximating the derivative of the inverse distribution function at sample points. Following [38], we estimate this derivative as

$$\frac{dF^{-1}(u)}{du} = \frac{N(X_{i+m:N} - X_{i-m:N})}{2m},$$

where $X_{i:N} = X_{1:N}$ for $i < 1$ and $X_{i:N} = X_{N:N}$ for $i > N$, $N$ is the sample size and $m$ is a positive integer referred to as the window size, satisfying $m \leq N/2$. Consequently, an estimator for $\mathcal{CE}\left(T_{k|n:F}\right)$ is obtained as follows:

$$\widehat{\mathcal{CE}}_1(T_{k|n:F}) = \frac{1}{N}\sum_{i=1}^{N} \xi\left(G_{k|n:F}\left(\frac{i}{N+1}\right)\right)\left(\frac{N(X_{i+m:N} - X_{i-m:N})}{2m}\right)$$

$$= \frac{1}{N}\sum_{i=1}^{N} \xi\left((n-k+1)\left(\frac{i}{N+1}\right)^k - (n-k)\left(\frac{i}{N+1}\right)^{k+1}\right)$$

$$\times \left(\frac{N(X_{i+m:N} - X_{i-m:N})}{2m}\right). \tag{19}$$

The second estimator is constructed using the empirical cumulative distribution function associated with $F(x)$ of the sample, as follows:

$$F_N(x) = \sum_{i=1}^{N-1} \frac{i}{N} I_{[x_{i:N}, x_{(i+1)}]}, \ x \geq 0,$$

where $I_A(x) = 1$ if $x \in A$. Based on Eq (5), the empirical CE estimator for the consecutive $k$-out-of-$n$:F system is given by

$$\widehat{\mathcal{CE}}_2(T_{k|n:F}) = \int_0^\infty \xi((n-k+1)F_N^k(x) - (n-k)F_N^{k+1}(x))dx$$

$$= \sum_{i=1}^{N-1} \int_{X_{i:N}}^{X_{(i+1)}} \xi\left((n-k+1)F_N^k(x) - (n-k)F_N^{k+1}(x)\right)dx$$

$$= \sum_{i=1}^{N-1} \xi\left((n-k+1)\left(\frac{i}{N}\right)^k - (n-k)\left(\frac{i}{N}\right)^{k+1}\right)D_{i+1}, \tag{20}$$

where $D_{i+1} = X_{i+1:N} - X_{i:N}$, $i = 1,2,\dots,N-1$, denotes the sample spacings.

In the following, we examine Monte Carlo simulations to evaluate the performance of the proposed estimators $\widehat{\mathcal{CE}}_1(T_{k|n:F})$ and $\widehat{\mathcal{CE}}_2(T_{k|n:F})$ defined in (19) and (20), respectively. Conducted using R software, the simulation is repeated 5,000 times across sample sizes N=20, 30, 40, 50, and 100, utilizing a conventional exponential distribution. We calculate the average bias and root mean square error (RMSE) for different sample sizes *N* and parameter combinations (k, n). The smoothing parameter m can be determined using the heuristic formula $m = [\sqrt{N} + 0.5]$, where $[x]$ is the integer part of $x$. The bias and RMSE of the estimators $\widehat{\mathcal{CE}}_1(T_{k|n:F})$ and $\widehat{\mathcal{CE}}_2(T_{k|n:F})$ are given in Tables 1 and 2 for 5,000 times. After examining these tables, we arrive at the following conclusions:

1) For all $k$ and $n$, as the sample size $N$ increases, the RMSE of the estimators decreases while the bias has an opposite manner.
2) For fixed $n$ and $N$, as the number of consecutive working components $k$ increases, the RMSE of the estimators decreases while the bias has an opposite manner.
3) For fixed $k$ and $N$, as the number of components of the system *n* increases, the RMSE of the estimators decreases.

Generally, the second estimator has better performance with respect to the first estimator. Thus, the results show that the efficiency of the estimator is influenced by the number of components $n$ and the number of consecutive working components $k$.

**Table 1.** The Bias and RMSE of the first estimator $\widehat{\mathcal{CE}}_1(T_{k|n:F})$ for different choices of $k$ and $n$.

| | | N = 20 | | N = 30 | | N = 40 | | N = 50 | | N = 100 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n | k | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| 5 | 3 | –0.124138 | 0.230344 | –0.100742 | 0.191860 | –0.095708 | 0.171305 | –0.093843 | 0.160603 | -0.094036 | 0.131869 |
| | 4 | –0.038756 | 0.263316 | 0.000577 | 0.231150 | 0.036344 | 0.218155 | 0.051543 | 0.201405 | 0.093028 | 0.173277 |
| | 5 | –0.091323 | 0.287947 | –0.009914 | 0.253112 | 0.038630 | 0.247169 | 0.066143 | 0.237489 | 0.147523 | 0.237116 |
| 6 | 3 | 0.072069 | 0.167615 | 0.070267 | 0.138012 | 0.066130 | 0.124728 | 0.061647 | 0.109277 | 0.040760 | 0.075014 |
| | 4 | –0.031567 | 0.239911 | 0.010755 | 0.209871 | 0.022254 | 0.184367 | 0.035022 | 0.174554 | 0.055193 | 0.134261 |
| | 5 | –0.151385 | 0.310038 | –0.079245 | 0.260249 | –0.048472 | 0.232668 | –0.015527 | 0.219851 | 0.041610 | 0.175196 |
| | 6 | –0.272339 | 0.380990 | –0.184579 | 0.315350 | –0.127539 | 0.277684 | –0.088733 | 0.253952 | 0.009005 | 0.199077 |
| 7 | 4 | 0.021823 | 0.208960 | 0.042960 | 0.183286 | 0.054940 | 0.161538 | 0.054626 | 0.150306 | 0.057243 | 0.113776 |
| | 5 | –0.092157 | 0.269532 | –0.034616 | 0.238141 | –0.008311 | 0.214772 | 0.008427 | 0.198731 | 0.051931 | 0.159579 |
| | 6 | –0.213470 | 0.350137 | –0.135670 | 0.289292 | –0.086826 | 0.260511 | –0.055488 | 0.242480 | 0.031440 | 0.186683 |
| | 7 | –0.331163 | 0.417481 | –0.230386 | 0.349043 | –0.161969 | 0.305570 | –0.130153 | 0.279632 | -0.007831 | 0.216406 |
| 8 | 4 | 0.075346 | 0.191621 | 0.080118 | 0.161910 | 0.074436 | 0.143996 | 0.076248 | 0.134626 | 0.059875 | 0.092973 |
| | 5 | –0.035972 | 0.244008 | 0.000687 | 0.214961 | 0.023995 | 0.194971 | 0.038143 | 0.183108 | 0.061133 | 0.145446 |
| | 6 | –0.168064 | 0.312048 | –0.088598 | 0.260961 | –0.048191 | 0.238333 | –0.020311 | 0.224606 | 0.037779 | 0.178471 |
| | 7 | –0.291871 | 0.384639 | –0.185520 | 0.314696 | –0.129251 | 0.281766 | –0.093509 | 0.254625 | 0.004763 | 0.198204 |
| | 8 | –0.385751 | 0.455766 | –0.276726 | 0.378007 | –0.211355 | 0.332942 | –0.163147 | 0.300208 | –0.034566 | 0.218227 |

**Table 2.** The Bias and RMSE of the second estimator $\widehat{\mathcal{CE}}_2(T_{k|n:F})$ for different choices of $k$ and $n$.

| | | N = 20 | | N = 30 | | N = 40 | | N = 50 | | N = 100 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $k$ | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| 5 | 3 | –0.025125 | 0.175512 | –0.014116 | 0.141257 | –0.015236 | 0.123774 | –0.012079 | 0.109911 | –0.005279 | 0.078412 |
| | 4 | –0.072958 | 0.267537 | –0.050056 | 0.219368 | –0.029108 | 0.190125 | –0.032334 | 0.172732 | –0.012504 | 0.123541 |
| | 5 | –0.123044 | 0.332033 | –0.084990 | 0.272151 | –0.055920 | 0.239984 | –0.048001 | 0.219015 | –0.023000 | 0.156351 |
| 6 | 3 | –0.001871 | 0.132976 | 0.000809 | 0.109061 | 0.000794 | 0.092947 | –0.000688 | 0.082732 | –0.000010 | 0.058974 |
| | 4 | –0.045391 | 0.228448 | –0.028728 | 0.182676 | –0.024040 | 0.157982 | –0.016516 | 0.145011 | –0.011101 | 0.102678 |
| | 5 | –0.100189 | 0.299852 | –0.064542 | 0.250162 | –0.049838 | 0.217331 | –0.038395 | 0.195598 | –0.018189 | 0.139785 |
| | 6 | –0.131134 | 0.357338 | –0.099022 | 0.297588 | –0.077961 | 0.262055 | –0.057160 | 0.238616 | –0.028988 | 0.170018 |
| 7 | 4 | –0.026347 | 0.182529 | –0.018355 | 0.150178 | –0.011043 | 0.133090 | –0.010400 | 0.117107 | –0.006821 | 0.083850 |
| | 5 | –0.075611 | 0.266131 | –0.041516 | 0.221699 | –0.037394 | 0.192767 | –0.029404 | 0.171981 | –0.014986 | 0.122495 |
| | 6 | –0.124722 | 0.331439 | –0.081209 | 0.278495 | –0.064191 | 0.245961 | –0.050115 | 0.219265 | –0.026817 | 0.155459 |
| | 7 | –0.169382 | 0.377862 | –0.106689 | 0.319331 | –0.086256 | 0.286752 | –0.068001 | 0.256148 | –0.032641 | 0.182052 |
| 8 | 4 | –0.002768 | 0.152239 | –0.003317 | 0.119965 | 0.000883 | 0.106233 | 0.001972 | 0.094212 | –0.000304 | 0.066692 |
| | 5 | –0.052062 | 0.233434 | –0.032683 | 0.190547 | –0.027234 | 0.163031 | –0.021617 | 0.147743 | –0.008801 | 0.104927 |
| | 6 | –0.096677 | 0.298313 | –0.065127 | 0.255635 | –0.048338 | 0.221591 | –0.041602 | 0.196996 | –0.019996 | 0.140928 |
| | 7 | –0.141255 | 0.362541 | –0.094531 | 0.302222 | –0.081784 | 0.265432 | –0.062192 | 0.240006 | –0.026913 | 0.168080 |
| | 8 | –0.181689 | 0.401899 | –0.131193 | 0.343492 | –0.091136 | 0.300505 | –0.070395 | 0.267088 | –0.035192 | 0.200356 |

### 4.1. Real data analysis

To determine how well the CE estimators of the successive $k$-out-of-$n$:F systems match the theoretical entropy value, we applied the estimator to the actual data.

**Example 4.1.** The data are the active repair times (in hours) given in [39] for an airborne communication device. The observations are listed below:

**Dataset:** 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22, 24.5. This data is modeled using the Weibull distribution with the pdf

$$f(x) = \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}},\ x > 0,$$

where $\lambda > 0$ and $\beta > 0$ are scale and shape parameters, respectively. As noted in [29], the datasets were fitted using the Weibull distribution via the maximum likelihood method for parameter estimation. The resulting parameters are $\hat{\lambda} = 3.391$ and $\hat{\beta} = 0.899$. The Kolmogorov-Smirnov statistic is $0.120$ with a p-value of $0.517$, confirming a good fit between the observed data and the fitted exponential distribution.

Table 3 lists the different combinations of $k$ and $n$. The results show that there is a correlation between the theoretical entropy value and its estimate when the functional components make up more than half of the total ($n$).

**Table 3.** Comparison of theoretical values and estimates of CE of $T_{k|n:F}$ based on Weibull distribution for active repair times (in hours) for an airborne communication transceiver.

| $k$ | $\mathcal{CE}(T_{k\|5:F})$ | $\widehat{\mathcal{CE}}_1(T_{k\|5:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|5:F})$ | $\mathcal{CE}(T_{k\|6:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|6:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|6:F})$ |
|---|---|---|---|---|---|---|
| 3 | 0.188772 | 3.202371 | 2.708464 | 0.139103 | 2.307699 | 1.786721 |
| 4 | 0.252975 | 4.412466 | 4.098577 | 0.225504 | 3.978375 | 3.511787 |
| 5 | 0.282173 | 4.786405 | 4.098577 | 0.267998 | 4.650849 | 4.532230 |
| 6 | | | | 0.289143 | 4.779245 | 5.167557 |
| $k$ | $\mathcal{CE}(T_{k\|7:F})$ | $\widehat{\mathcal{CE}}_1(T_{k\|7:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|7:F})$ | $\mathcal{CE}(T_{k\|8:F})$ | $\widehat{\mathcal{CE}}_1(T_{k\|8:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|8:F})$ |
| 4 | 0.191428 | 3.414607 | 2.827453 | 0.151613 | 2.737446 | 2.057078 |
| 5 | 0.247997 | 4.397852 | 4.073618 | 0.223027 | 4.043979 | 3.538539 |
| 6 | 0.278365 | 4.738680 | 4.867858 | 0.263090 | 4.605239 | 4.497665 |
| 7 | 0.294482 | 4.708580 | 5.384117 | 0.285978 | 4.731164 | 5.137700 |
| 8 | | | | 0.298730 | 4.597261 | 5.563095 |

**Example 4.2** Let us now analyze the data from [40], which pertains to vinyl chloride measurements collected from monitoring wells during cleanup gradient assessments. The data sets are presented as follows:

**Dataset:** 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

In Table 4, we present the log-likelihood, AIC, BIC, and results from the K-S goodness of fit test. The findings indicate that the exponential distribution, with a maximum likelihood estimate of 0.53, closely aligns with the actual data distribution. We further examine the proximity of the CE estimators for consecutive $k$-out-of-8:G systems to the theoretical CE value, assuming an exponential distribution for component lifetimes with a rate parameter of $\hat{\lambda} = 0.53$.

**Table 4.** Criteria for selecting models for the vinyl chloride data.

| Distribution | Log-likelihood | AIC | BIC | K-S | P-value |
|---|---|---|---|---|---|
| Exponential | –55.452 | 112.905 | 114.431 | 0.088 | 0.950 |
| Weibull | –55.449 | 114.899 | 117.952 | 0.091 | 0.936 |
| Log-normal | –55.204 | 114.408 | 117.461 | 0.086 | 0.959 |
| Gamma | –55.413 | 114.826 | 117.879 | 0.097 | 0.904 |
| Log-logistic | –55.945 | 115.891 | 118.943 | 0.086 | 0.959 |

Table 5 illustrates various combinations of of $k$ and $n$. The results show that there is a correlation between the theoretical entropy value and its estimate when the functional components make up more than half of the total ($n$).

**Table 5.** Comparison of theoretical values and estimates of CE of $T_{k|n:F}$ based on exponential distribution for vinyl chloride measurements collected from monitoring wells during cleanup gradient assessments.

| $k$ | $\mathcal{CE}(T_{k\|5:F})$ | $\widehat{\mathcal{CE}}_1(T_{k\|5:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|5:F})$ | $\mathcal{CE}(T_{k\|6:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|6:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|6:F})$ |
|---|---|---|---|---|---|---|
| 3 | 0.188772 | 3.202371 | 1.279895 | 0.139103 | 2.307699 | 0.96451 |
| 4 | 0.252975 | 4.412466 | 1.667539 | 0.225504 | 3.978375 | 1.51285 |
| 5 | 0.282173 | 4.786405 | 1.793849 | 0.267998 | 4.650849 | 1.736112 |
| 6 | | | | 0.289143 | 4.779245 | 1.800122 |
| $k$ | $\mathcal{CE}(T_{k\|7:F})$ | $\widehat{\mathcal{CE}}_1(T_{k\|7:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|7:F})$ | $\mathcal{CE}(T_{k\|8:F})$ | $\widehat{\mathcal{CE}}_1(T_{k\|8:F})$ | $\widehat{\mathcal{CE}}_2(T_{k\|8:F})$ |
| 4 | 0.191428 | 3.414607 | 1.311014 | 0.151613 | 2.737446 | 1.068160 |
| 5 | 0.247997 | 4.397852 | 1.636633 | 0.223027 | 4.043979 | 1.501486 |
| 6 | 0.278365 | 4.73868 | 1.768091 | 0.263090 | 4.605239 | 1.703695 |
| 7 | 0.294482 | 4.70858 | 1.793099 | 0.285978 | 4.731164 | 1.778476 |
| 8 | | | | 0.298730 | 4.597261 | 1.777036 |

## 5. Conclusions

We investigate the application of the CE concept to consecutive $k$-out-of-$n$:F systems. An important finding is the strong correlation between the CE of these systems derived from continuous and uniform distributions. This finding simplifies CE calculations in numerous practical scenarios. However, deriving CE expressions in closed form remains a challenge for systems with large or complex component distributions. To solve this problem, we establish advantageous bounds on the CE of consecutive $k$-out-of-$n$:F systems. These bounds serve as important tools for researchers and practitioners to analyze CE behavior. In addition, we present two non-parametric estimators developed specifically for consecutive $k$-out-of-$n$:F systems and demonstrate their practical utility using real data applications. Furthermore, CE estimation provides valuable insights into system uncertainty and facilitates informed decision-making and meaningful data analysis. Our results can also be applied to other measures of information, such as fractional cumulative entropy, cumulative residual Tsallis entropy, and cumulative Tsallis entropy. To summarize, this study provides expressions for the system lifetimes of consecutive $k$-out-of-$n$:F systems and CE preservation of the stochastic order. Moreover, we have developed useful bounds that are particularly important when the cdf does not have a closed form, as in the case of complex cumulative distribution functions, including mixed distributions or multicomponent systems. Additionally, we present several valuable characterization results and propose two non-parametric estimators for the CE of the system lifetime.

## Author contributions

Mashael A. Alshehri: Methodology, software, validation, formal analysis, resources, writing-review and editing, visualization, project administration; Mohamed Kayid: Investigation, writing-original draft, writing-review and editing, visualization, supervision. All authors have read and agreed to the published version of the manuscript.

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## Conflict of interest

There are no conflict of interest.

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