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Research article

Optimal dividends in a discrete-time dual risk model with stochastic expenses

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Abstract: Dividend policies play a pivotal role in financial management by aiming to maximize shareholders' interest and effectively managing risk. In this paper, we explore the optimal dividend strategy in a discrete-time compound binomial dual risk framework. This model is suitable for a company whose income comes from occasional operating expenses and settlements only once per unit of time. We assume that expenses are subject to dynamic changes influenced by economic factors, following a Markov chain. With or without a ceiling constraint on dividend payments, we prove that the optimal value function serves as the exclusive solution to a discrete Hamilton-Jacobi-Bellman (HJB) equation through the utilization of the fixed-point theorem. Furthermore, we derive a straightforward computational approach for determining the optimal strategy. Finally, we provide numerical examples to illustrate the theoretical findings and calculation methods.

Keywords: optimal dividend strategy; dual risk model; stochastic expenses; HJB equation; fixed-point theorem
Mathematics Subject Classification: 49M25, 60G51, 93E20

1. Introduction

In finance and insurance, the classic risk model primarily focuses on discussing the ruin probability and dividend strategies for insurance companies (see, e.g., Konstantinides et al. [1], Gerber and Shiu [2]). In the past year, there has been a growing number of studies concentrated on a model that is dual to the classic risk model. Apart from the ruin probabilities of pension funds, dual risk models can also be used to describe the surplus of companies whose income is derived from occasional operating expenses, such as pharmaceutical or petroleum businesses. Some of these studies include Avanzi et al. [3], Gerber and Smith [4], Ng [5], Yao et al. [6], Zhao et al. [7], Yang et al. [8], Fahim and Zhu [9], and Song and Sun [10], among others. In this model, a company's surplus takes the form of

$$U(t) = u - ct + S(t),\ t \geq 0, \tag{1.1}$$

where $U(0) = u(u \geq 0)$ denotes the initial surplus, the constant $c \geq 0$ is the rate of expenses, and the process $\{S(t)\}$ is the aggregate gains or profits.

Dividend problems under the model (1.1) were first considered by Avanzi et al. [3]. Since then, various studies have mainly focused on continuous-time dual risk models, such as compound Poisson dual models (see, e.g., Avanzi et al. [3], Liu et al. [11], Pérez and Yamazaki [12]) and spectrally positive Lévy risk models (see, e.g., Bayraktar et al. [13], Zhao et al. [14], Song and Sun [10]). Contrary to continuous-time dual risk models, the discrete-time risk model has its special features and is closer to reality, as in practice, many risk events often occur in discrete and intermittent forms. In addition, they are also of independent interest since formulas for discrete-time models are recursive in nature and easily programmable in practice, while still reproducing the continuous analog results as limiting cases (see, e.g., Dickson et al. [15] and Cossette et al. [16]). Therefore, it is meaningful to consider the discrete-time dual risk model.

The compound binomial risk model, first proposed by Gerber [17], can serve as an approximation to the continuous-time compound Poisson model, where premiums, claim amounts, and the initial surplus are assumed to be integers. There have been several extensions discussed, including Bao [18], Landriault [19], Yang et al. [20], Drekic et al. [21], Tan and Yuan [22], and Lin [23]. Regarding the dual risk model, most existing research has obtained explicit expressions for value functions and optimal strategies when gains follow exponential or mixed exponential distributions. However, a general solution method is currently lacking to handle cases where gains are distributed according to a general distribution. This situation was one of the main motivations for discussing a compound binomial model in this paper.

Motivated by the works of Yang et al. [20], Dibu and Jacob [24], Tan and Yuan [22], and Greenblatt [25], we delve into the optimal dividend problems with a stochastic expense per each unit of time. In most previous discussions about model (1.1), the assumption was that the expenses rate c remained constant (see, e.g., Avanzi et al. [3], Ng [5], Yao et al. [6], and Liu et al. [11]). However, the expenses in company business activities may change due to the economic environment and the company's cash flow, which may be influenced by different types of information from the market. For example, petroleum companies are often affected by significant changes in investment prices caused by political and economic factors. The idea of incorporating stochastic expenses $C_t(t \geq 0)$ into dual models is inspired by the concept of stochastic premiums in classical risk models. This concept was first proposed by Boucherie et al. [26], who based their work on the composite Poisson classical risk model. Afterward, Yang et al. [20] discussed risk models with stochastic premiums in a Markovian environment and obtained optimal dividend strategies. Dibu and Jacob [24] discussed a dividend strategy in which premiums and claims both follow compound Poisson processes in the risk model. Tan and Yuan [22] provided an algorithm for the optimal control strategy by constructing a nonlinear operator, under the assumption of stochastic premiums. The analysis of expenses, considering that they are subject to random fluctuations influenced by various factors, including the economic environment and market conditions, can help companies make more scientific and rational investment decisions. This is one of the main research objectives of this paper.

The optimal strategy manifests as a barrier strategy when the dividend process is unconstrained. Avanzi et al. [3] presented an extensive analysis on the establishment of the most advantageous dividend barrier within the compound Poisson dual risk model. Then, Gerber and Smith [4] devised some approximate algorithms for calculating optimal dividends based on the research by Avanzi et al. [3]. Due to the lack of constraints imposed by barrier strategies on dividends, companies' ruin occurs with probability one (see Sendova et al. [27]). To alleciate this, Ng [5] considered employing a threshold strategy to restrict dividend payments. They proposed a method for calculating the optimal dividend threshold using two integro-differential equations and the Laplace transform. Several recent contributions on the threshold dividend strategy have been made by Yao et al. [6], Cheung and Wong [28], Liu et al. [11], Yang et al. [8], and Hu et al. [29], among others. Unlike previous studies on dividend strategies within specified dividend methods, we want to find the optimal strategy among all feasible strategies, inspired by Yao et al. [30].

This paper aims to find the optimal dividend strategy for shareholders in a compound binomial dual risk model, under the assumption that the occurrence of a gain $X_t(X_t > 0)$ during a time period $(t-1, t]$ is related to the expenditure of expenses C_t. Compared with the existing research results, we propose several differences, advantages, and novelty. First, this paper formulates the optimal dividend framework for a discrete-time compound binomial dual model, which is different from both the existing continuous-time dual risk model (Avanzi et al. [3], Yang et al. [8]) and the compound binomial risk model (Tan and Yang [31], Bazyari [32]). Currently, there is no universal method for obtaining the optimal value function when gains are governed by compound binomial distributions and there is a lack of rigorous numerical analysis or specific characterization of optimal strategies. Second, we discuss the stochastic expenses in a Markovian environment, where the future state is solely dictated by the present state, regardless of antecedent states. The Markovian environment, known for its probabilistic properties, provides a concise mathematical framework for modeling decision-making processes in economics (Asmussen [33]). Third, considering both the existence and nonexistence of constraints on the upper limit of dividend payments, we employ the fixed-point principle derived from a contraction mapping to establish the uniqueness of the optimal value function as the solution to a discrete HJB equation (see, e.g., Marciniak and Palmowski [34], Xu and Woo [35]). Additionally, we also provide a specific algorithm for optimal strategies. By comparing the two dividend payments scenarios mentioned above, we have concluded that the strategy without a ceiling constraint is feasible and in line with the principle of economic market's distribution of dividends based on investment ratios. However, most literature on optimal dividends mainly focuses on dividend payments with an upper bound constraint, disregarding the importance of situations without any ceiling constraints. Finally, the numerical example illustrates that the optimal strategy under a payment limit constraint is a threshold strategy. In contrast, without this constraint, the optimal strategy becomes a barrier strategy.

The rest of this paper is organized as follows: In Section 2, a surplus process with stochastic expenses will be established. The existence and uniqueness of the optimal strategy are proved in Section 3. Sections 4 and 5 discuss the Bellman recursive algorithm and provide two functions for approximating the optimal value function, considering the presence or absence of a ceiling constraint on dividend payments, respectively. Section 6 presents two numerical examples to validate the algorithm. A conclusion is provided in Section 7.

2. Model setup

Before we proceed, let us introduce some symbols that will be utilized later.

(a) Let $a_1 \vee a_2 = \max\{a_1, a_2\}$ and $a_1 \wedge a_2 = \min\{a_1, a_2\}$.

(b) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^k = \{0, 1, 2, \ldots k\}(k \in \mathbb{N})$, and $\mathbb{N}^+ = \{1, 2, \ldots\}$.

(c) Unless specifically stated otherwise, both i and j denote the elements within set $\{1, 2, \ldots, m\}(m \in \mathbb{N}^+)$.

Suppose there exists only one settlement within the per unit time interval $(t-1, t]$ for the company, occurring at time t. In this section, we will initially explore the dynamics of expense rates C_t within each discrete $(t-1, t]$. We shall presume that the sequence $\{C_t\}$ follows the dynamics of a Markov chain, where every state C_t is measurable with respect to a σ-algebra $\mathcal{F}_t$ within a finite state space consisting of enumerable components $\{c_1, c_2, \cdots, c_m\}$. Specifically, let the initial state C_0 represent the expense in the time interval $(-1, 0]$. The transition probability from state i to j is denoted as $p_{ij} = \Pr(C_t = c_j | C_{t-1} = c_i)$.

In this paper, we are investigating a discrete-time dual risk model that incorporates both dynamic expenses and stochastic gains. In this model, the company has decided to manage funds and make strategic choices exclusively at times $t = 0, 1, \ldots$ Before imposing a dividend strategy, the dual risk model should be considered

$$U(t) = u - \sum_{k=1}^{t} C_k + S(t), \quad t \in \mathbb{N}^+. \tag{2.1}$$

The surplus process (2.1) is a discrete-time model with expenses as random variables, which differs from model (1.1). Therefore, this paper presents an expanded version of a classical dual risk model that incorporates additional factors. We assume that within each discrete time interval $(t-1, t]$, only one potential gain may occur immediately before time t, and the occurrence of gains hinges upon the variable C_t. Then, we use $\varepsilon_t^{C_t} = 0$ to indicate that no gain events occur during the time interval $(t-1, t]$ under the condition $C_t = c_i$. Additionally, $\varepsilon_t^{C_t} = 1$ indicates that a gain event occurrs, with the corresponding amount denoted as $X_t (X_t \in \mathbb{N}^+)$. The probability distribution of $\{\varepsilon_t^{C_t}\}$ is given by

$$\Pr(\varepsilon_t^{C_t} = h | C_t = c_i) = \theta_i^h (1 - \theta_i)^{1-h}, \ h = 0, 1,$$

where $0 < \theta_i < 1$.

The binomial sequence $N(t) = \sum_{k=1}^{t} \varepsilon_k^{C_k}$ represents the cumulative count of gains received by the company until time t. Subsequently, the total gains can be expressed as a compound binomial sequence, as demonstrated in Tan and Yang [31]:

$$S(t) = \sum_{k=1}^{t} X_k \varepsilon_k^{C_k},$$

where the sequence $\{X_t : X_t \in \mathbb{N}^+\}$ represents a collection of gains that are independent and identically distributed. Furthermore, these gains $\{X_t\}$ are independent of the sequence $\{\varepsilon_t^{C_t}\}$. Here, $\{X_t\}$ and $\{\varepsilon_t^{C_t}\}$ are both assumed to be $\{\mathcal{F}_t\}$-measurable (Tan et al. [36]), where $\mathcal{F}_t$ includes all the information at time t and prior. Let us denote the probability function, distribution function, and mean of the random variables X_t by f, F, and μ_{X_t}, respectively.

In model (2.1), we subsequently consider incorporating an admissible dividend strategy at time t, denoted as d_t. The value of d_t should be a nonnegative integer and must not exceed the surplus at time t. Additionally, the dividend sequences $\{d_t\}$ are measurable with respect to $\mathcal{F}_t$. The set consisting of these admissible strategies is denoted as Θ. Consequently, the surplus process, under the control of Θ and with an initial state $C_0 = c_i$, is described as follows:

$$U_i^{\Theta}(t) = U_i^{\Theta}(t-1) - C_t + X_t \varepsilon_t^{C_t} - d_t,\ t \in \mathbb{N}^+, \tag{2.2}$$

where $U_i^{\Theta}(0) = u$ and $d_0 = 0$. In model (2.2), even though fund fluctuations occur continuously, financial settlements take place at the end of each fixed operating period, and shareholder dividends are also distributed. This can be seen as a simplified version of the continuous model, which aligns with the operating mode of most financial companies in the real world. In this model, both the company's expenses and gains are treated as random variables, enhancing its practicality.

Let $\Gamma_{i,u}^{\Theta} = \inf\{t : U_i^{\Theta}(t) < 0\}(\Gamma_{i,u}^{\Theta} = \infty$ if ruin does not occur) be the time of ruin with the initial expenses state $C_0 = c_i$. For a given valuation discount factor $\nu(0 < \nu < 1)$, the expected total discounted dividends until ruin with the initial expenses state $C_0 = c_i$ is

$$V_i(u, \Theta) = E[\sum_{k=0}^{\Gamma_{i,u}^{\Theta}} \nu^k d_k | U_i^{\Theta}(0) = u]. \tag{2.3}$$

Since the model (2.1) exhibits the Markov property, which states that the future state is solely determined by the present state, regardless of antecedent states (Asmussen [33]), this indicates that if the surplus remains the same at any two time periods, then the optimal dividend amounts should also be the same. Therefore, we only need to consider admissible dividend strategies that depend on the surplus at each time t. This type of dividend strategy in Θ is a function of the surplus (as described in Theorem 3.1). In order to facilitate the discussion, we will use $\varphi_i(u)$ to represent the dividend amount when the surplus is u and expenses state is c_i. The dividend in the expenses state $C_t = c_j$ for $t \in \mathbb{N}^+$ is rewritten as $\varphi_j(U^{\Theta}(t-1) - C_t + X_t \varepsilon_t^{C_t})$. To determine the optimal value function $V_i^*(u, \varphi_i)$ and optimal strategy $\varphi_i^*(u)$ in (2.3), we will use appropriate calculations and consider the relevant equation, which satisfies the following conditions:

$$V_i^*(u, \varphi_i) = \sup_{\varphi_i \in \Theta} V_i(u, \varphi_i), \tag{2.4}$$

and

$$\varphi_i^*(u) = \arg\sup_{\varphi_i \in \Theta} V_i(u, \varphi_i). \tag{2.5}$$

3. The optimal strategy

The main goal of this section is to use the associated HJB equation and fixed-point theorem for contraction mapping to find the explicit solution of V_i^* and φ_i^* for compound binomial distributed gains. We will discuss the optimal strategy in two cases: the presence and absence of a ceiling constraint on dividend payments. For any strategy $\varphi_i(u) \in \Theta$, we possess

$$\varphi_i(u) \in [0, u \wedge M_0], \tag{3.1}$$

where M_0 is a nonnegative integer that includes the case of $M_0 < \infty$ and $M_0 = \infty$.

For the sake of convenience, we abbreviate $V_i(u, \varphi_i)$ and $V_i^*(u, \varphi_i)$ as $V_i(u)$ and $V_i^*(u)$, respectively, in the following discussion. By applying the principle of total expectation, the function $V_i(u)$ can be shown to satisfy the following discrete equation:

$$\begin{aligned} V_i(u) = \nu \sum_{j=1}^{m} p_{ij} \Big\{ &(1-\theta_j)\Big[V_j(u-c_j-\varphi_j(u-c_j)) + \varphi_j(u-c_j)\Big] \\ &+ \theta_j \sum_{x=1}^{\infty} \Big[V_j(u-c_j+x-\varphi_j(u-c_j+x)) + \varphi_j(u-c_j+x)\Big] f(x) \Big\}. \end{aligned} \tag{3.2}$$

The optimization problem in this paper aims to find the optimal strategies φ_i^* so that the value function V_i in (3.2) can attain its maximum value V_i^*.

Theorem 3.1. *For every $\varphi_i(u) \in \Theta$, the optimal value function $V_i^*(u)$ for each i satisfies a discrete HJB equation, which can be stated as follows:*

$$\begin{aligned} V_i^*(u) = \nu \sum_{j=1}^{m} p_{ij} \Big\{ &(1-\theta_j) \max_{a \in \mathbb{N}^{((u-c_j)\wedge M_0)\vee 0}} \Big[V_j(u-c_j-a) + a\Big] \\ &+ \theta_j \sum_{x=1}^{\infty} \Big[\max_{a \in \mathbb{N}^{((u-c_j+x)\wedge M_0)\vee 0}} (V_j(u-c_j+x-a) + a) f(x) \Big] \Big\}. \end{aligned} \tag{3.3}$$

Proof. The optimal value of (3.2) can be obtained by

$$\begin{aligned} V_i^*(u) = \sup_{\varphi_j \in \Theta} \Big\{ \nu \sum_{j=1}^{m} p_{ij} \Big[&(1-\theta_j)(V_j(u-c_j-\varphi_j(u-c_j)) + \varphi_j(u-c_j)) \\ &+ \theta_j \sum_{x=1}^{\infty} (V_j(u-c_j+x-\varphi_j(u-c_j+x)) + \varphi_j(u-c_j+x)) f(x) \Big] \Big\}, \end{aligned} \tag{3.4}$$

which is equivalent to (3.3). Hence, (3.3) holds. □

Note that $\boldsymbol{L}^\infty$ represents the set of all bounded sequences of real numbers within the product space $[0, +\infty) \times \{1, 2, \cdots, m\}$. For any two points $\boldsymbol{X} = (X_i(u))$ and $\boldsymbol{Y} = (Y_i(u)) \in \boldsymbol{L}^\infty$, a distance in $\boldsymbol{L}^\infty$ is defined as follows:

$$d(\boldsymbol{X}, \boldsymbol{Y}) = \sup_{u,i} |X_i(u) - Y_i(u)|, \tag{3.5}$$

then $\boldsymbol{L}^\infty = (\boldsymbol{L}^\infty, d)$ is a complete Banach space.

Thus, for any $\boldsymbol{V} = (V_i(u)) \in \boldsymbol{L}^\infty$, we can define a strategy $b_{V_i}(u)$ as follows:

$$b_{V_i}(u) = \arg \max_{x \in \mathbb{N}^{u \wedge M_0}} \{V_i(u-x) + x\}. \tag{3.6}$$

If such a strategy exists, for any $(Z_i(u)) \in \boldsymbol{L}^\infty$, define an operator $\boldsymbol{T} = (T_1, T_2, \cdots, T_m)$, where T_i are m nonlinear operators on $\boldsymbol{L}^\infty$, which satisfy

$$\begin{aligned} T_i Z_i(u) = \nu \sum_{j=1}^{m} p_{ij} \Big\{ &(1-\theta_j)[Z_j(u-c_j-b_{Z_j}(u-c_j)) + b_{Z_j}(u-c_j)] \\ &+ \theta_j \sum_{x=1}^{\infty} [Z_j(u-c_j+x-b_{Z_j}(u-c_j+x)) + b_{Z_j}(u-c_j+x)] f(x) \Big\}, \end{aligned} \tag{3.7}$$

and then operator T_i is a mapping on $\boldsymbol{L}^\infty \to \boldsymbol{L}^\infty$. Let $\boldsymbol{TV}^* = (T_1V_1^*, T_2V_2^*, \cdots, T_mV_m^*)$ and $\boldsymbol{V}^* = (V_i^*(u))$, then (3.3) can be written as

$$V_i^*(u) = T_iV_i^*(u), \tag{3.8}$$

or

$$\boldsymbol{V}^* = \boldsymbol{TV}^*. \tag{3.9}$$

Theorem 3.2. *Based on the condition of* $0 < \nu < 1$, (3.9) *has a solution, which is unique.*

Proof. Without loss of generality, let us suppose that for any $\boldsymbol{X} = (X_i(u))$ and $\boldsymbol{Y} = (Y_i(u)) \in \boldsymbol{L}^\infty$, the following inequality $X_i(u - b_{X_i}(u)) + b_{X_i}(u) - [Y_i(u - b_{Y_i}(u)) + b_{Y_i}(u)] \geq 0$ holds; from (3.5) and (3.6), we have

$$\begin{aligned}
&\left|X_i(u - b_{X_i}(u)) + b_{X_i}(u) - [Y_i(u - b_{Y_i}(u)) + b_{Y_i}(u)]\right| \\
\leq&\left|X_i(u - b_{X_i}(u)) + b_{X_i}(u) - [Y_i(u - b_{X_i}(u)) + b_{X_i}(u)]\right| \\
\leq& d(\boldsymbol{X}, \boldsymbol{Y}).
\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}
d(\boldsymbol{TX}, \boldsymbol{TY}) \leq& \nu \sum_{j=1}^{m} p_{ij}\Big[(1-\theta_j) + \theta_j \sum_{x=1}^{\infty} f(x)\Big] d(\boldsymbol{X}, \boldsymbol{Y}) \\
\leq& \nu d(\boldsymbol{X}, \boldsymbol{Y}).
\end{aligned}$$

Due to the restriction $0 < \nu < 1$, it can be inferred that the operator $\boldsymbol{T}$ is a contraction mapping on $\boldsymbol{L}^\infty$. Therefore, for any $u \in \mathbb{N}$, (3.9) has a solution, which is unique. □

Remark 3.1. *Theorem 3.2 shows that the strategy* b_{V_i} *exists in* Θ *and is a function of the* V_i. *Furthermore, Theorem 3.1 provides the explicit expression of the optimal value function* V_i^*, *while Theorem 3.2 establishes the incontrovertible existence and uniqueness of* V_i^*.

Theorem 3.3. *For every* $\varphi_i(u) \in \Theta$, *the value function* $V_i(u)$ *attains optimality if and only if*

$$\varphi_i(u) = \arg \sup_{x \in \mathbb{N}^{u \wedge M_0}} \{V_i(u - x) + x\}. \tag{3.10}$$

Proof. If $V_i(u)$ is optimal value function, it satisfies both (3.2) and (3.3). By comparing the two equations, we can deduce that

$$V_i(u - \varphi_i(u)) + \varphi_i(u) = \max_{x \in \mathbb{N}^{u \wedge M_0}} [V_i(u - x) + x].$$

Thus, (3.10) holds. Conversely, if we have (3.10), then the corresponding value function $V_i(u)$ becomes the optimal value function. This is due to the existence and uniqueness of the solution of (3.9). □

Conclusion 3.1. *For any* $\varphi_i(u) \in \Theta$, *the strategy that fulfills* (3.10) *is considered optimal, and an explicit solution is available:*

$$\varphi_i^*(u) = b_{V_i}(u), \tag{3.11}$$

and (3.10) is equivalent to

$$\varphi_i^*(u) = u - \arg \sup_{x \in [0 \vee (u - M_0), u]} \{V_i(x) - x\}, \quad x \in \mathbb{N}. \tag{3.12}$$

4. Properties and algorithm for the optimal strategy with a ceiling restriction

In this section, we consider the case where $M_0 < \infty$ in (3.1) to provide a clearer illustration of the algorithm for V_i^* and φ_i^*. We use Θ_0 to denote the set of all admissible dividend strategies with a ceiling constraint of a nonnegative integer $M_0 (M_0 < \infty)$ in Θ.

Lemma 4.1. *For every $\varphi_i(u) \in \Theta_0$ and $0 < \nu < 1$, the value function is bounded and satisfies the following inequality:*

$$0 \le V_i(u) \le \frac{\nu M_0}{1-\nu}. \tag{4.1}$$

Proof. For any $\varphi_i(u) \in \Theta_0$, we have

$$\begin{aligned}\sup_u V_i(u) \le \nu \sum_{j=1}^{m} p_{ij} \Big\{ &(1-\theta_j)[\max_j \sup_u V_j(u) + M_0] \\ &+ \theta_j \sum_{x=1}^{\infty} [\max_j \sup_u V_j(u) + M_0] f(x) \Big\},\end{aligned}$$

hence, (4.1) holds. □

From Eq (3.1), let $u \to \infty$, we can apply the dominated convergence theorem to obtain that

$$\lim_{u\to\infty} V_i(u) = \nu \sum_{j=1}^{m} p_{ij} [\lim_{u\to\infty} V_j(u) + \lim_{u\to\infty} \varphi_j(u)].$$

If $\lim_{u\to\infty} \varphi_i(u) = M_0$, we have

$$\lim_{u\to\infty} V_i(u) = \frac{\nu M_0}{1-\nu}. \tag{4.2}$$

Theorem 4.1. *For each strategy $\varphi_i(u) \in \Theta_0$, it holds that*

$$\liminf_{u\to\infty} \varphi_i^*(u) = M_0. \tag{4.3}$$

Proof. Suppose there is an integer $q \in \{1, 2, \ldots, m\}$ that satisfies

$$\lim_{u\to\infty} \varphi_q^*(u) = M_1 < M_0. \tag{4.4}$$

Then, there must be a nonnegative integer sequence $\{u_n\}$ such that $\lim_{n\to\infty} \varphi_q^*(u_n - c_q) = M_1$, and for any given i, we have

$$\begin{aligned}V_i^*(u_n) = \nu \sum_{j=1}^{m} p_{ij} \Big\{ &(1-\theta_j)\Big[V_j^*(u_n - c_j - \varphi_j^*(u_n - c_j)) + \varphi_j^*(u_n - c_j)\Big] \\ &+ \theta_j \sum_{x=1}^{\infty} \Big[V_j^*(u_n - c_j + x - \varphi_j^*(u_n - c_j + x)) + \varphi_j^*(u_n - c_j + x)\Big] f(x) \Big\}.\end{aligned} \tag{4.5}$$

By taking the limit as $n \to \infty$ on both sides of (4.5), we can obtain:

$$\lim_{u\to\infty} V_i^*(u) \le \frac{\nu M_0}{1-\nu} + \frac{\nu p_{iq}(1-\theta_q)(M_1-M_0)}{1-\nu} < \frac{\nu M_0}{1-\nu}. \tag{4.6}$$

Combining with (4.2), it becomes evident that inequality (4.6) contradicts the optimality of $\phi_i^*(u)$. This indicates that (4.4) is incorrect, while (4.3) is true. □

Remark 4.1. *According to Theorem 4.1, there exists a positive integer N^* such that*

$$\varphi_i^*(u) = M_0, \text{ for all } u \ge N^*. \tag{4.7}$$

If a positive integer value denoted by u_0 satisfies the condition $V(u_0) \ge \lfloor \frac{\nu M_0}{1-\nu} \rfloor$, where $\lfloor x \rfloor$ represents the largest integer less than or equal to x, then

$$\varphi_i^*(u) = M_0, \text{ for all } u \ge N^* = u_0 + M_0. \tag{4.8}$$

That is because

$$\lfloor \frac{\nu M_0}{1-\nu} \rfloor \le V_i(u-M_0) < V_i(u) \le \frac{\nu M_0}{1-\nu},$$

and

$$V_i(u-M_0) + M_0 = \max_{x\in\mathbb{N}^{u\wedge M_0}} \{V_i(u-x)+x\}, \text{ for all } u \ge N^* = u_0 + M_0.$$

Conclusion 4.1. *Based on Theorem 4.1 and Remark 4.1, the following conclusions can be drawn:*

(i) The optimal dividend strategy $\varphi_i^*(u) \in \Theta_0$ is a threshold strategy. The expression of $\varphi_i^*(u) \in \Theta_0$ can be rephrased as

$$\varphi_i^*(u) = \hat{b}_{V_i}(u) = \begin{cases} \arg \sup_{x\in\mathbb{N}^{u\wedge M_0}} \{V_i(u-x)+x\}, & 0 \le u < N^*, \\ M_0, & u \ge N^*. \end{cases} \tag{4.9}$$

(ii) The set Θ includes threshold strategies.

Next, we will provide specific algorithms to calculate φ_i^* and V_i^*. Sometimes, the possible gains X_t are unbounded, leading to a sum of infinite terms in (3.9). In the design of specific algorithms, we approximate the value of V_i^* by utilizing an upper and lower bound through the squeeze theorem. This method allows us to avoid summing an infinite number of terms when calculating V_i^*. We can represent this approximation with the following two equations:

$$\begin{aligned} V_i^{*(1)}(u) =& \nu \sum_{j=1}^m p_{ij}\Big\{(1-\theta_j)\Big[V_j^{*(1)}(u-c_j-\hat{b}_{V_j^{*(1)}}(u-c_j)) + \hat{b}_{V_j^{*(1)}}(u-c_j)\Big] \\ &+\theta_j \sum_{x=1}^{n_0} \Big[V_j^{*(1)}(u-c_j+x-\hat{b}_{V_j^{*(1)}}(u-c_j+x)) + \hat{b}_{V_j^{*(1)}}(u-c_j+x)\Big] f(x)\Big\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} V_i^{*(2)}(u) =& \nu \sum_{j=1}^m p_{ij}\Big\{(1-\theta_j)\Big[V_j^{*(2)}(u-c_j-\hat{b}_{V_j^{*(2)}}(u-c_j)) + \hat{b}_{V_j^{*(2)}}(u-c_j)\Big] \\ &+\theta_j \sum_{x=1}^{n_0} \Big[V_j^{*(2)}(u-c_j+x-\hat{b}_{V_j^{*(2)}}(u-c_j+x)) + \hat{b}_{V_j^{*(2)}}(u-c_j+x)\Big] f(x)\Big\} \\ &+ \frac{\nu M_0 \bar{F}(n_1) J_i}{1-\nu}, \end{aligned} \tag{4.11}$$

where $J_i = \sum_{j=1}^{m} p_{ij}\theta_j$, $\bar{F} = 1 - F$, and n_0 is a positive integer. Define operators $\mathbf{T}^{(1)} = (T_1^{(1)}, \ldots, T_m^{(1)})$ and $\mathbf{T}^{(2)} = (T_1^{(2)}, \ldots, T_m^{(2)})$, where $T_i^{(1)}V_i^{*(1)}$ equals the right-hand side of (4.10), and $T_i^{(2)}V_i^{*(2)}$ equals the right-hand side of (4.11). Clearly, $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ are also contraction mappings. Therefore, based on the discussion in Section 3, we can conclude that Eqs (4.10) and (4.11) both exist and have unique solutions.

Theorem 4.2. *For each strategy that $\varphi_i(u) \in \Theta_0$, the following inequality holds:*

$$V_i^{*(1)}(u) \le V_i^*(u) \le V_i^{*(2)}(u). \tag{4.12}$$

Proof. (I) We first prove the inequality $V_i^{*(1)}(u) \le V_i^*(u)$. According to (4.10), for any given i, we obtain that

$$\begin{aligned} V_i^{*(1)}(u) \le \nu \sum_{j=1}^{m} p_{ij}\Big\{&(1-\theta_j)\Big[V_j^{*(1)}(u - c_j - \hat{b}_{V_j^{*(1)}}(u - c_j)) + \hat{b}_{V_j^{*(1)}}(u - c_j)\Big] \\ &+ \theta_j \sum_{x=1}^{\infty} \Big[V_j^{*(1)}(u - c_j + x - \hat{b}_{V_j^{*(1)}}(u - c_j + x)) + \hat{b}_{V_j^{*(1)}}(u - c_j + x)\Big] f(x)\Big\}. \end{aligned} \tag{4.13}$$

The right side of (4.13) is denoted as $M_{i,1}^{*(1)}(u)$, and

$$\begin{aligned} M_{i,1}^{*(1)}(u) \le \nu \sum_{j=1}^{m} p_{ij}\Big\{&(1-\theta_j)\Big[M_{i,1}^{*(1)}(u - c_j - \hat{b}_{M_{i,1}^{*(1)}}(u - c_j)) + \hat{b}_{M_{i,1}^{*(1)}}(u - c_j)\Big] \\ &+ \theta_j \sum_{x=1}^{\infty} \Big[M_{i,1}^{*(1)}(u - c_j + x - \hat{b}_{M_{i,1}^{*(1)}}(u - c_j + x)) + \hat{b}_{M_{i,1}^{*(1)}}(u - c_j + x)\Big] f(x)\Big\}. \end{aligned} \tag{4.14}$$

We have defined m as the new operator $\bar{T}_i$ to transform $M_{i,1}^{*(1)}(u)$ to the right-hand side of (4.14). Then, we have $V_i^{*(1)}(u) \le M_{i,1}^{(1)}(u) \le \bar{T}_i M_{i,1}^{(1)}(u)$, and the operator $\bar{T}_i$ is a contraction mapping operator in Banach space $\boldsymbol{L}^\infty$.

Let $M_{i,2}^{(1)}(u) = \bar{T}_i M_{i,1}^{(1)}(u)$, and then $M_{i,2}^{(1)}(u) \le \bar{T}_i M_{i,2}^{(1)}(u)$. After proceeding, we obtain a monotonically increasing sequence of functions $\{M_{i,n}^{(1)}(u)\}$ that satisfies $M_{i,n}^{(1)}(u) \le \bar{T}_i M_{i,n}^{(1)}(u) = M_{i,n+1}^{(1)}(u)$, and then we have

$$\lim_{n\to\infty} M_{i,n}^{(1)}(u) = V_i^*(u).$$

Therefore, $V_i^{*(1)}(u) \le V_i^*(u)$.

(II) In the same way, it can be proven that $V_i^*(u) \le V_i^{*(2)}(u)$. For any given i, according to (3.3) and (4.11), we have

$$\begin{aligned} V_i^*(u) \le \nu \sum_{j=1}^{m} p_{ij}\Big\{&(1-\theta_j)\Big[V_j^*(u - c_j - \hat{b}_{V_j^*}(u - c_j)) + \hat{b}_{V_j^*}(u - c_j)\Big] \\ &+ \theta_j \sum_{x=1}^{n_0} \Big[V_j^*(u - c_j + x - \hat{b}_{V_j^*}(u - c_j + x)) + \hat{b}_{V_j^*}(u - c_j + x)\Big] f(x)\Big\} \\ &+ \frac{\nu M_0 \bar{F}(n_0) \sum_{j=1}^{m} p_{ij}\theta_j}{1-\nu}. \end{aligned} \tag{4.15}$$

The new m operators $\hat{T}_i$ are defined to transform the function $V_i^*(u)$ to the right side of (4.15), which is denoted as $M_{i,1}^{(2)}(u)$. Obviously, the operator $\hat{T}_i$ is also a contraction mapping operator in Banach space $\boldsymbol{L}^\infty$. Hence, we have $V_i^*(u) \le \hat{T}_i V_i^*(u) = M_{i,1}^{(2)}(u)$.

Let $M_{i,2}^{(2)}(u) = \hat{T}_i M_{i,1}^{(2)}(u)$, and then $M_{i,1}^{(2)}(u) \le \hat{T}_i M_{i,1}^{(2)}(u) = M_{i,2}^{(2)}(u)$. Similarly, we can obtain a new monotonically increasing sequence of functions $\{M_{i,n}^{(2)}(u), n \in \mathbb{N}^+\}$ that satisfies $M_{i,n}^{(2)}(u) \le \hat{T}_i M_{i,n}^{(2)}(u) = M_{i,n+1}^{(2)}(u)$, and

$$\lim_{n\to\infty} M_{i,n}^{(2)}(u) = V_i^{*(2)}(u).$$

Therefore, we have $V_i^*(u) \le V_i^{*(2)}(u)$. By combining (I) and (II), we can deduce that (4.12) holds. □

Theorem 4.3. *For any* $\varphi_i(u) \in \Theta_0$ *and* $0 < \nu < 1$*, it holds that*

$$d(V_i^{*(1)}(u), V_i^{*(2)}(u)) \le \frac{\nu M_0 \bar{F}(n_0) J_{(m)}}{(1-\nu)[1-\nu+\nu J_{(1)}\bar{F}(n_0)]}. \tag{4.16}$$

where $J_{(m)} = \max_i J_i$, and $J_{(1)} = \min_i J_i$.

Proof. According to the proof of Theorem 3.2 , we can conclude that

$$\begin{aligned} d(V_i^{*(1)}(u), V_i^{*(2)}(u)) \le \nu \sum_{j=1}^{m} p_{ij}\Big[(1-\theta_j) d(V_j^{*(1)}(u), V_j^{*(2)}(u)) \\ + \theta_j \sum_{x=1}^{n_0} d(V_j^{*(1)}(u), V_j^{*(2)}(u)) f(x)\Big] + \frac{\nu M_0 \bar{F}(n_0) J_{(m)}}{1-\nu}, \end{aligned}$$

then, we have

$$\begin{aligned} \max_i d(V_i^{*(1)}(u), V_i^{*(2)}(u)) \le \nu[1 - \sum_{j=1}^{m} p_{ij}\theta_j \bar{F}(n_0)] \max_i d(V_i^{*(1)}(u), V_i^{*(2)}(u)) \\ + \frac{\nu M_0 \bar{F}(n_0) J_{(m)}}{1-\nu}. \end{aligned}$$

Hence, inequality (4.16) holds. □

According to Theorems 4.2 and 4.3, as the value of n_0 approaches infinity, $\bar{F}(n_0)$ tends to zero, and both $V_i^{*(1)}(u)$ and $V_i^{*(2)}(u)$ can indefinitely converge toward the optimal value function $V_i^*(u)$ using the fixed-point principle.

To streamline the computation process, we exclusively present the Bellman recursive algorithm on $V_i^{*(1)}(u)$ as an approximate value of $V_i^*(u)$, based on $V_i^{*(1)}(u)$ being a fixed point of $\bar{T}_i$. The corresponding algorithm on $V_i^{*(2)}(u)$ can be obtained in a similar manner. For any given initial function column $(V_{i,0}^{(1)}(u))$, we can use a recursive formula to calculate subsequent values $(V_{i,s}^{(1)}(u))$, where $s = 1, 2, \cdots$. The recursive formula is as follows:

$$\begin{aligned} V_{i,s}^{(1)}(u) =& \nu \sum_{j=1}^{m} p_{ij}\Big\{(1-\theta_j)\Big[V_{j,s-1}^{(1)}(u - c_j - \hat{b}_{V_{j,s-1}^{(1)}}(u-c_j)) + \hat{b}_{V_{j,s-1}^{(1)}}(u-c_j)\Big] \\ &+ \theta_j \sum_{x=1}^{n_0} \Big[V_{j,s-1}^{(1)}(u - c_j + x - \hat{b}_{V_{j,s-1}^{(1)}}(u-c_j+x)) + \hat{b}_{V_{j,s-1}^{(1)}}(u-c_j+x)\Big] f(x)\Big\}. \end{aligned} \tag{4.17}$$

and the value of $V_i^{*(1)}(u)$ satisfies:

$$V_i^{*(1)}(u) = \lim_{n\to\infty} V_{i,n}^{(1)}(u) = \lim_{n\to\infty} \bar{T}_i^n V_{i,0}^{(1)}(u). \tag{4.18}$$

Therefore, in instances where the value of $n(n \in \mathbb{N}^+)$ exceeds a specific threshold, it becomes permissible to select $V_{i,n}^{(1)}(u)$ as a viable approximation for $V_i^{*(1)}(u)$. According to the proof of Theorem 3.2 and (4.18), the error estimation formula can be expressed as follows:

$$d(V_i^{*(1)}(u), V_{i,n}^{(1)}(u)) \leq \frac{\nu^n}{1-\nu}. \tag{4.19}$$

Correspondingly, the optimal strategy $\varphi_i^*(u)$ can be obtained. More specifically, the Bellman recursive algorithm for $V_i^{*(1)}(u)$ and $\varphi_i^*(u)$ works as follows:

Step 1. Precision control: the value of n_0 is obtained using (4.16) to meet the desired level of accuracy.

Step 2. Determine the iteration steps n using (4.19) in order to reach the desired level of accuracy precisely.

Step 3. Iterative calculation: given an initial function column $(V_{i,0}^{(1)}(u))$, where $u \in \mathbb{N}^+$, calculate $(V_{i,s}^{(1)}(u))$ for values of s ranging from 1 to n. Finally, obtain the approximate values of $(\varphi_i^*(u))$ and $(V_i^{*(1)}(u))$.

5. Optimal strategy without payment restrictions

In this section, we will delve into an algorithm and its properties for achieving the most favorable approach in scenarios where $M_0 = \infty$ as depicted in (3.1). We use a subset Θ_1 within Θ to indicate all admissible strategies without a ceiling constraint for dividend payments. From Theorems 3.1 and 3.2, we are aware that the optimal strategy exists within Θ_1, and it is a function in relation to $V_i(u)$. However, an unbounded surplus leads to an unlimited computational burden when $M_0 = \infty$. To make the computation feasible, we need to confine the surplus u within certain boundaries by analyzing the properties exhibited by the strategies. Let $H = \max_i \sum_{j=1}^m p_{ij}(\theta_j\mu_x - c_j)$. We assume that c_j, μ_x are finite, and $H \geq 0$. This assumption requires finite expenses and expected gains. Additionally, the condition $H \geq 0$ ensures the company's operations are profitable.

Lemma 5.1. *For any $\varphi_i(u) \in \Theta_1$ and $0 < \nu < 1$, it holds that*

$$\sup_{u\in\mathbb{N}} (V_i(u) - u) \leq \frac{\nu H}{1-\nu}. \tag{5.1}$$

Proof. From (3.2), we have

$$\begin{aligned}
V_i(u) - u = {} & (\nu - 1)u + \nu \sum_{j=1}^m p_{ij}(\theta_j\mu_x - c_j) \\
& + \nu \sum_{j=1}^m p_{ij}\Big\{(1-\theta_j)\Big[V_j(u - c_j - \varphi_j(u-c_j)) - (u - c_j - \varphi_j(u-c_j))\Big] \\
& + \theta_j \sum_{x=1}^\infty \Big[V_j(u - c_j + x - \varphi_j(u - c_j + x)) \\
& - (u - c_j + x - \varphi_j(u - c_j + x))\Big] f(x)\Big\}.
\end{aligned}$$

then,

$$\sup_u (V_i(u) - u) \le \nu \sum_{j=1}^{m} p_{ij} \max_j \sup_u (V_j(u) - u) + \nu H,$$

it is obvious that

$$\max_i \sup_u (V_i(u) - u) \le \nu \max_j \sup_u (V_j(u) - u) + \nu H.$$

Hence, inequality (5.1) holds. □

Theorem 5.1. *Assume that* $\varphi_i^*(u) \in \Theta_1$ *and* $0 < \nu < 1$*, it can be concluded that*

$$\arg\sup_{u\in\mathbb{N}} (V_i(u) - u) \le \frac{\nu H}{(1-\nu)^2}, \tag{5.2}$$

The proof process here is similar to that of Theorem 5.1 in Yang et al. [20], so it is omitted here.

Through the above discussion, we can draw the following conclusion: the surplus in company operations is influenced by expenses, expected earnings, and dividends over a unit of time. When the initial surplus u exceeds a positive integer, only a portion of u is used in operations while the remaining surplus remains idle. The value of $V_i^*(u)$, which is influenced by the discount rate, will be lower than u, and the company operates with debt. This lacks practical significance for profit-driven companies. Therefore, according to Theorem 5.1 and the optimal strategy control in Θ_1, the surplus u is assumed to have an upper bound of $\lfloor \frac{\nu H}{(1-\nu)^2} \rfloor$ after dividend payments at each time t, and this upper bound is denoted as H_0.

Let $\boldsymbol{L}^{H_0}$ represent the set of all bounded sequences of real numbers within the product space $[0, H_0] \times \{1, 2, \ldots, m\}$. Consequently, $(V_i(u)) \in \boldsymbol{L}^{H_0}$ whenever the variable u is confined within the interval $[0, H_0]$. For any two points $\boldsymbol{X} = (X_i(u))$, $\boldsymbol{Y} = (Y_i(u)) \in \boldsymbol{L}^{H_0}$, we define a distance in $\boldsymbol{L}^{H_0}$ by

$$d_1(\boldsymbol{X}, \boldsymbol{Y}) = \sup_{u\in\mathbb{N}^{H_0}, i} |X_i(u) - Y_i(u)|, \tag{5.3}$$

then, $\boldsymbol{L}^{H_0} = (\boldsymbol{L}^{H_0}, d_1)$ is a complete Banach space.

For any $(V_i(u)) \in \boldsymbol{L}^{H_0}$, we define an admissible strategy

$$\tilde{b}_{V_i}(u) = u - \arg\sup_{x\in\mathbb{N}^{u\wedge H_0}} \{V_i(x) - x\}, \tag{5.4}$$

and the mappings $\widetilde{\boldsymbol{T}} = (\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_m)$, where $\widetilde{T}_i$ is a nonlinear operator on $\boldsymbol{L}^{H_0}$, which satisfies

$$\begin{aligned} \widetilde{T}_i V_i(u) = \nu \sum_{j=1}^{m} p_{ij} \Big\{ & (1-\theta_j)[V_j(u - c_j - \tilde{b}_{V_j}(u - c_j)) + \tilde{b}_{V_j}(u - c_j)] \\ & + \theta_j \sum_{x=1}^{\infty} [V_j(u - c_j + x - \tilde{b}_{V_j}(u - c_j + x)) + \tilde{b}_{V_j}(u - c_j + x)] f(x) \Big\}. \end{aligned} \tag{5.5}$$

Similar in the arrangement on space $\boldsymbol{L}^{\infty}$, proof of $\widetilde{\boldsymbol{T}}$ being a contraction mapping on $\boldsymbol{L}^{H_0}$ is easily achievable, and the existence of $\tilde{b}_{V_i}(u)$ in Θ_1 is effortlessly attainable. Therefore, an optimal value function $V_i^*(u)(u \in \mathbb{N}^{H_0})$ that satisfies (5.5) exists and is unique, and the corresponding strategy $\tilde{b}_{V_i}(u)$ is an optimal strategy in Θ_1.

Given the constraint that the surplus after each dividend payments should not exceed H_0, we will propose a specific algorithm to determine the optimal strategy $\varphi_i(u) \in \Theta_1$. In cases where the potential gains X_t are an unbounded random variable, we will explore the utilization of the following two functions to approximate $(V_i^*(u)) \in \boldsymbol{L}^{H_0}$, i.e.,

$$\begin{aligned}\widetilde{V}_i^{*(1)}(u) = \nu \sum_{j=1}^{m} p_{ij}\Big\{&(1-\theta_j)\Big[\widetilde{V}_j^{*(1)}(u-c_j-\tilde{b}_{\widetilde{V}_j^{*(1)}}(u-c_j)) + \tilde{b}_{\widetilde{V}_j^{*(1)}}(u-c_j)\\ &+ \theta_j \sum_{x=1}^{n_1}\Big[\widetilde{V}_j^{*(1)}(u-c_j+x-\tilde{b}_{\widetilde{V}_j^{*(1)}}(u-c_j+x)) + \tilde{b}_{\widetilde{V}_j^{*(1)}}(u-c_j+x)\Big]f(x)\Big\},\end{aligned} \tag{5.6}$$

$$\begin{aligned}\widetilde{V}_i^{*(2)}(u) = \nu \sum_{j=1}^{m} p_{ij}\Big\{&(1-\theta_j)\Big[\widetilde{V}_j^{*(2)}(u-c_j-\tilde{b}_{\widetilde{V}_j^{*(2)}}(u-c_j)) + \tilde{b}_{\widetilde{V}_j^{*(2)}}(u-c_j)\Big]\\ &+ \theta_j \sum_{x=1}^{n_1}\Big[\widetilde{V}_j^{*(2)}(u-c_j+x-\tilde{b}_{\widetilde{V}_j^{*(2)}}(u-c_j+x)) + \tilde{b}_{\widetilde{V}_j^{*(2)}}(u-c_j+x)\Big]f(x)\Big\}\\ &+ \nu(Q(n_1)J_i - \bar{F}(n_1)\sum_{j=1}^{m} p_{ij}\theta_j c_j),\end{aligned} \tag{5.7}$$

where $Q(n_1) = (\frac{\nu H}{1-\nu} + H_0)\bar{F}(n_1) + \mu_x - \sum_{x=1}^{n_1} xf(x)$, and $n_1 \in \mathbb{N}^+$.

Similarly, we can easily demonstrate that

$$\widetilde{V}_i^{*(1)}(u) \le V_i^*(u) \le \widetilde{V}_i^{*(2)}(u), \text{ for } u \in \mathbb{N}^{H_0}, \tag{5.8}$$

where $V_i^*(u)$ satisfies the right-hand side of (5.5) with V_j replaced by V_j^* for all j. By Eqs (5.6)–(5.8), the deviation between $\widetilde{V}_i^{(1)}(u)$ and $\widetilde{V}_i^{*(2)}(u)$ is

$$d_1(\widetilde{V}_i^{*(1)}(u), \widetilde{V}_i^{*(2)}(u)) \le \frac{\nu[Q(n_1)J_{(m)} - J_c\bar{F}(n_1)]}{1-\nu+\nu J_{(1)}\bar{F}(n_1)}, \tag{5.9}$$

where $J_c = \min_i \sum_{j=1}^{m} p_{ij}\theta_j c_j$. By assuming that $c_j < \infty$, $\mu_x < \infty$ and considering the formulas in (5.6) and (5.7), we can find that the right-hand side of (5.9) is nonnegative and also approaches zero as $\bar{F}(n_1) \to 0$ when $n_1 \to \infty$.

Therefore, as the value of n_1 increases, $\widetilde{V}_i^{(1)}(u)$ and $\widetilde{V}_i^{*(2)}(u)$ can approach $V_i^*(u)$ infinitely. We define an operator $\widetilde{\boldsymbol{T}}' = (\widetilde{T}_1', \widetilde{T}_2', \cdots, \widetilde{T}_m')$, where $\widetilde{T}_i'\widetilde{V}_i^{(1)}(u)$ is the right-hand side of (5.6). Obviously, $\widetilde{\boldsymbol{T}}'$ also exhibits the characteristics of a contraction mapping when applied to $\boldsymbol{L}^{H_0}$. The Bellman recursive algorithm for $\widetilde{V}_i^{(1)}(u)$ under restrictions on unbounded dividend payments is as follows: Given an initial function column $(\widetilde{V}_{i,0}^{(1)}(u))$ in the range of $[0, \nu H_0 + \frac{\nu H}{1-\nu})]$, we can construct a new recursive formula $(\widetilde{V}_{i,s}^{(1)}(u))(s = 1, 2, \ldots, n)$ that replaces $\hat{b}_{V_{j,s-1}^{(1)}}$ on the right side of (4.17) with $\tilde{b}_{\widetilde{V}_{j,s-1}^{(1)}}$. Then, the estimated error between $\widetilde{V}_{i,n}^{(1)}(u)$ and $\widetilde{V}_i^{*(1)}(u)$ is less than or equal to $\nu^{n+1}/(1-\nu)$.

The calculation steps are similar to the steps of the algorithm provided at the end of Section 4, but with specific conditional assumptions taken into account for surplus u.

6. Numerical illustration

To demonstrate the computation of the value of φ_i^* and V_i^*, we present two numerical examples. For each example, we consider scenarios with and without ceiling constraints on dividend payments, along with various gains probability distributions.

Example 6.1. Let us assume that the discount factor $\nu = 0.97$ and the expenses $\{C_t\}$ follow a Markov chain that comprises three distinct states, with values of 3, 5, and 6. The transition probability is as follows:

$$P = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.15 & 0.6 & 0.25 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.$$

Next, let us assume that the probabilities of gains occurrence $\theta_i (i = 1, 2, 3)$ are 0.5, 0.76, and 0.9, respectively, associated with C_i. The goal of this example is to discuss the optimal strategy under the different payment conditions given in Sections 4 and 5 using a discrete random distribution and the geometric distribution with an expected gain of $\mu_x = 8$.

1) Discrete random distribution. The probability function satisfies:

x	2	3	7	10	28	52	88
$f(x)$	0.17	0.2	0.28	0.2	0.12	0.025	0.005

2) Geometric distribution. The probability function satisfies:

$$f(x) = \frac{1}{10}\left(\frac{9}{10}\right)^{x-1}, \ x = 1, 2, \cdots .$$

Assuming that both the estimated deviation and iterative calculation error are no more than 10^{-4}, we obtained a total of $n = 418$ iterations in the two distributions mentioned above. We obtain $n_0 = 168$ in the geometric distribution when $M_0 = 5$ in Section 4. Similarly, we determine $n_1 = 196$ when M_0 is considered infinite. In the algorithm provided in Section 5, for any positive integer $u_1 \in \mathbb{N}^{H_0}$, we aim to obtain an approximation of $V_i^*(u)(u \in \mathbb{N}^{u_1})$. To achieve this goal, for example, calculate iteratively $\widetilde{V}_{i,s}^{(1)}(u)(u \in \mathbb{N}^{u_1+196\times(418-s)})$ for s ranging from 1 to 418 according to a geometric distribution, using an initial value function $\widetilde{V}_{i,0}^{(1)}(u) = 1(u \in \mathbb{N}^{u_1+196\times 418})$. After the iterative calculations, we choose $\widetilde{V}_{i,418}^{(1)}(u)(u \in \mathbb{N}^{u_1})$ as an approximation of $V_i^*(u)$, where $H_0 = 3039$.

Tables 1 and 2 show approximate values of $\varphi_i^*(u)$ in the two distributions mentioned above. It can be easily seen that the $\varphi_i^*(u)$ are threshold strategies (see Ng [5] and Yang et al. [8]) when the dividend payments have a ceiling restriction, and the corresponding thresholds are 25 and 24 for the two different distributions, respectively. In addition, Tables 1 and 2 demonstrate that $\varphi_i^*(u)$ without ceiling constraints are barrier strategies (see Avanzi [3] and Bayraktar et al. [13]). The corresponding barriers for these example are $\{32, 32, 32\}$ and $\{29, 30, 30\}$, respectively.

Table 1. The optimal strategy $\varphi_i^*(u)$ with discrete random distribution in Example 6.1.

$M_0 = 5$								
u	0-25	26	27	28	29	30	31	others
$\varphi_i^*(u)(i = 1, 2, 3)$	0	1	2	3	4	5	5	5
$M_0 = \infty$								
u	0-31	32	33	34	35	36	37	$\cdots$
$\varphi_i^*(u)(i = 1, 2, 3)$	0	1	2	3	4	5	6	$\cdots$

Table 2. The optimal strategy $\varphi_i^*(u)$ with geometric distribution in Example 6.1.

$M_0 = 5$								
u	0-24	25	26	27	28	29	30	others
$\varphi_1^*(u)$	0	1	2	3	4	5	5	5
u	0-25	26	27	28	29	30	31	others
$\varphi_i^*(u)(i = 2, 3)$	0	1	2	3	4	5	5	5
$M_0 = \infty$								
u	0-29	30	31	32	33	34	35	$\cdots$
$\varphi_1^*(u)$	0	1	2	3	4	5	6	$\cdots$
u	0-30	31	32	33	34	35	36	$\cdots$
$\varphi_i^*(u)(i = 2, 3)$	0	1	2	3	4	5	6	$\cdots$

Figures 1–4 show the partial approximations of $V_i^*(u)$, which are affected by the presence or absence of a ceiling constraint on dividend payments. It can be observed that the optimal value functions $V_i^*(u)$ consistently exhibit an increasing trend across different distributions within the same state. The value of $V_i^*(u)$ initially increases rapidly, which indicates that the company is performing well operationally. However, once the surplus u surpasses a certain threshold due to operational constraints, further increases may not yield any additional benefits. Instead, they could potentially reduce overall profits through discounting and capital accumulation. Particularly when the dividend payments have a ceiling restriction, as $u \to \infty$, $V_i^*(u)$ gradually levels off, which is consistent with the conclusion of (4.2). These phenomena align with real-world financial and economic environments.

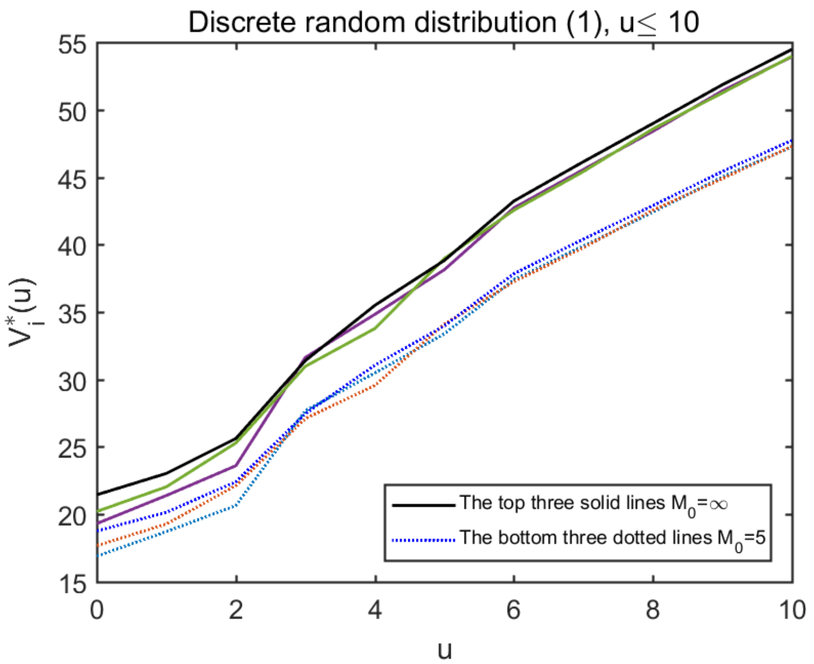

Figure 1. The value of $V_i^*(u)(u \leq 10)$ with discrete random distribution in Example 6.1.

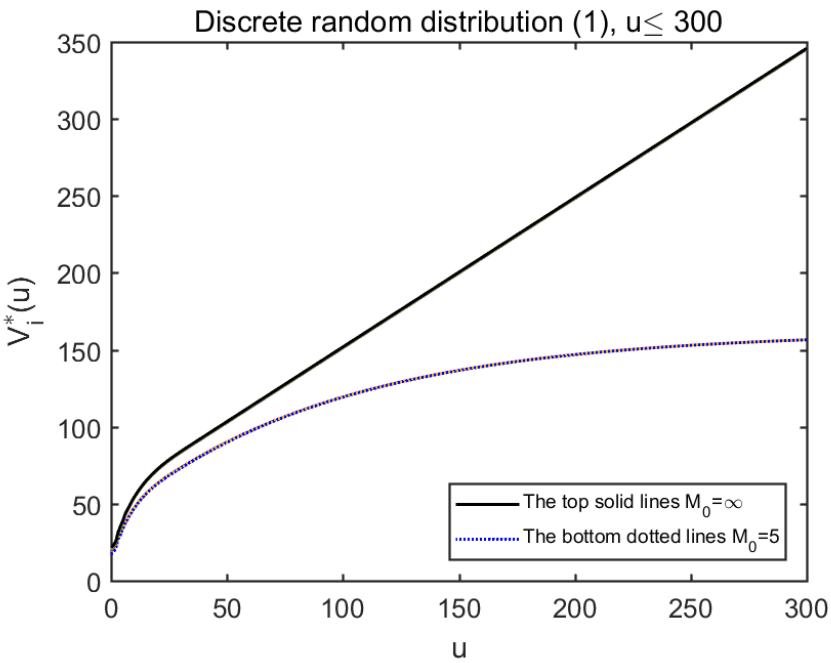

Figure 2. The value of $V_i^*(u)(u \leq 300)$ with discrete random distribution in Example 6.1.

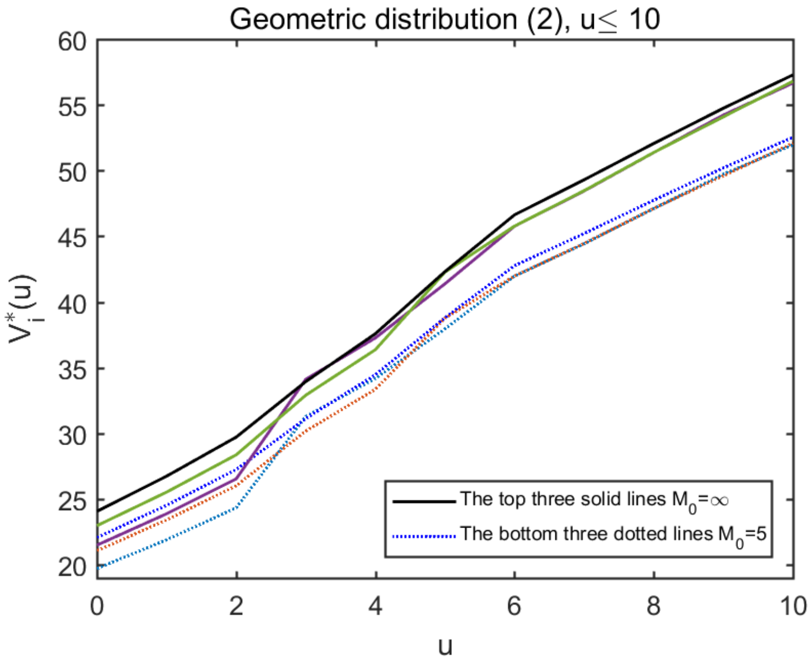

Figure 3. The value of $V_i^*(u)(u \leq 10)$ with geometric distribution in Example 6.1.

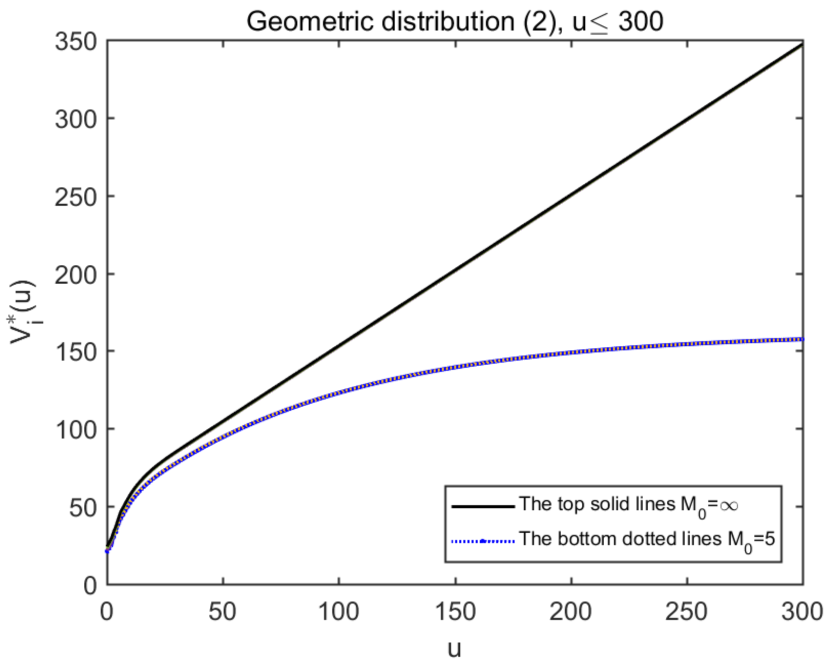

Figure 4. The value of $V_i^*(u)(u \leq 300)$ with geometric distribution in Example 6.1.

Example 6.2. Let us assume that the discount factor $\nu = 0.97$, the expenses $\{C_t\}$ follow a four-state

Markov chain, with values of 2, 3, 4, and 5, and the transition probability matrix is

$$P = \begin{pmatrix} 0.45 & 0.15 & 0.25 & 0.15 \\ 0.2 & 0.35 & 0.15 & 0.3 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.05 & 0.2 & 0.1 & 0.65 \end{pmatrix}.$$

Let us assume that the probabilities of gains occurrence θ_i are 0.4, 0.55, 0.7, and 0.85, respectively, with relation to C_i. In this example, we discuss the discrete uniform distribution and mixed geometric distribution with $\mu_x = 8$.

3) Discrete uniform distribution. The probability function satisfies:

$$f(x) = \frac{1}{15},\ x = 1, 2, \ldots, 15.$$

4) Mixed geometric distribution. The probability function satisfies:

$$f(x) = \frac{4}{5} \times \frac{1}{7} \times \left(\frac{6}{7}\right)^{x-1} + \frac{1}{5} \times \frac{1}{12} \times \left(\frac{11}{12}\right)^{x-1},\ x = 1, 2, \ldots$$

Applying an iterative process similar to Example 6.1, we also assume that estimation accuracy and calculation error do not exceed 10^{-4}. In the mixed geometric distribution, we obtained that $n_0 = 181$ when $M_0 = 5$, and $n_1 = 212$ when $M_0 = \infty$. Then, we choose $V^{(1)}_{i,418}(u)$ and $\widetilde{V}^{(1)}_{i,418}(u)(u \in \mathbb{N}^{H_0})$ as approximations of $V_i^*(u)$, respectively, where $H_0 = 1799$.

Tables 3 and 4 show that the optimal strategies can be categorized as threshold strategies under the constraint $M_0 = 4$. The corresponding thresholds for these strategies are $\{18, 18, 18, 18\}$ and $\{19, 19, 19, 20\}$, respectively. Additionally, the optimal strategies can also be classified as barrier strategies when $M_0 = \infty$. The barriers for these strategies are $\{20, 20, 21, 21\}$ and $\{23, 23, 23, 24\}$. Finally, Tables 5 and 6 show some specific approximations value of $V_i^*(u)$.

Table 3. The optimal strategy $\varphi_i^*(u)$ with discrete uniform distribution in Example 6.2.

$M_0 = 4$							
u	0-18	19	20	21	22	23	others
$\varphi_i^*(u)(i = 1, 2, 3, 4)$	0	1	2	3	4	4	4
$M_0 = \infty$							
u	0-20	21	22	23	24	25	$\cdots$
$\varphi_i^*(u)(i = 1, 2)$	0	1	2	3	4	5	$\cdots$
u	0-21	22	23	24	25	26	$\cdots$
$\varphi_i^*(u)(i = 3, 4)$	0	1	2	3	4	5	$\cdots$

Table 4. The optimal strategy $\varphi_i^*(u)$ with mixed geometric distribution in Example 6.2.

$M_0 = 4$							
u	0-19	20	21	22	23	24	others
$\varphi_i^*(u)(i = 1, 2, 3)$	0	1	2	3	4	4	4
u	0-20	21	22	23	24	25	others
$\varphi_4^*(u)$	0	1	2	3	4	4	4
$M_0 = \infty$							
u	0-23	24	25	26	27	28	$\cdots$
$\varphi_i^*(u)(i = 1, 2, 3)$	0	1	2	3	4	5	$\cdots$
u	0-24	25	26	27	28	29	$\cdots$
$\varphi_4^*(u)$	0	1	2	3	4	5	$\cdots$

Table 5. The value of $V_i^*(u)$ with discrete uniform distribution in Example 6.2.

$M_0 = 4$											
u	0	1	2	3	4	5	17	18	19	20	21
$V_1^*(u)$	13.022	14.507	19.294	21.926	25.486	27.826	45.438	46.439	47.414	48.369	49.312
$V_2^*(u)$	13.991	15.636	18.779	22.583	25.422	28.077	45.584	46.587	47.565	48.523	49.465
$V_3^*(u)$	14.571	16.307	18.820	21.830	25.770	28.104	45.625	46.631	47.612	48.572	49.516
$V_4^*(u)$	15.443	17.363	19.706	22.892	25.493	28.716	45.835	46.840	47.820	48.781	49.724
u	22	23	24	25	26	27	28	100	300	500	700
$V_1^*(u)$	50.247	51.175	52.096	53.006	53.902	54.787	55.661	97.878	126.375	129.055	129.307
$V_2^*(u)$	50.398	51.323	52.242	53.151	54.047	54.931	55.804	97.939	126.381	129.056	129.307
$V_3^*(u)$	50.448	51.371	52.288	53.196	54.093	54.978	55.851	97.959	126.383	129.056	129.307
$V_4^*(u)$	50.656	51.578	52.492	53.397	54.292	55.176	56.047	98.044	126.391	129.057	129.307
$M_0 = \infty$											
u	0	1	2	3	4	5	17	18	19	20	21
$V_1^*(u)$	13.302	14.818	19.709	22.397	26.033	28.424	46.431	47.466	48.479	49.477	50.461
$V_2^*(u)$	14.291	15.971	19.182	23.067	25.968	28.680	46.576	47.613	48.628	49.627	50.614
$V_3^*(u)$	14.884	16.657	19.224	22.299	26.323	28.708	46.617	47.656	48.675	49.677	50.666
$V_4^*(u)$	15.774	17.735	20.129	23.384	26.040	29.332	46.825	47.862	48.879	49.880	50.868
u	22	23	24	25	26	27	28	100	300	500	700
$V_1^*(u)$	51.437	52.410	53.380	54.350	55.320	56.290	57.260	127.100	321.100	515.100	709.100
$V_2^*(u)$	51.592	52.565	53.535	54.505	55.475	56.445	57.415	127.255	321.255	515.255	709.255
$V_3^*(u)$	51.646	52.620	53.590	54.560	55.530	56.500	57.470	127.310	321.310	515.310	709.310
$V_4^*(u)$	51.849	52.823	53.795	54.765	55.735	56.705	57.675	127.515	321.515	515.515	709.515

Table 6. The value of $V_i^*(u)$ with mixed geometric distribution in Example 6.2.

$M_0 = 4$											
u	0	1	2	3	4	5	18	19	20	21	22
$V_{1,463}^{(1)}(u)$	8.985	10.293	13.917	16.200	19.110	21.210	40.023	41.041	42.033	43.003	43.955
$V_{2,463}^{(1)}(u)$	9.546	10.932	13.470	16.516	19.003	21.304	40.134	41.157	42.152	43.125	44.079
$V_{3,463}^{(1)}(u)$	9.891	11.325	13.448	16.042	19.203	21.285	40.166	41.191	42.190	43.165	44.121
$V_{4,463}^{(1)}(u)$	10.363	11.859	13.819	16.491	18.947	21.595	40.326	41.354	42.354	43.331	44.289
u	23	24	25	26	27	28	29	100	300	500	700
$V_{1,463}^{(1)}(u)$	44.898	45.834	46.763	47.685	48.595	49.493	50.380	93.675	125.534	128.929	129.290
$V_{2,463}^{(1)}(u)$	45.021	45.955	46.882	47.803	48.713	49.612	50.499	93.730	125.540	128.929	129.290
$V_{3,463}^{(1)}(u)$	45.063	45.996	46.921	47.841	48.751	49.651	50.538	93.747	125.542	128.929	129.290
$V_{4,463}^{(1)}(u)$	45.232	46.164	47.088	48.006	48.914	49.813	50.700	93.822	125.550	128.930	129.290
$M_0 = \infty$											
u	0	1	2	3	4	5	18	19	20	21	22
$\widetilde{V}_{1,549}^{(1)}(u)$	9.750	11.169	15.101	17.578	20.735	23.013	43.394	44.494	45.565	46.610	47.634
$\widetilde{V}_{2,549}^{(1)}(u)$	10.359	11.863	14.616	17.922	20.620	23.115	43.518	44.622	45.697	46.746	47.774
$\widetilde{V}_{3,549}^{(1)}(u)$	10.734	12.289	14.593	17.407	20.837	23.096	43.553	44.661	45.739	46.791	47.821
$\widetilde{V}_{4,549}^{(1)}(u)$	11.246	12.870	14.996	17.895	20.560	23.433	43.732	44.842	45.923	46.977	48.010
u	23	24	25	26	27	28	29	100	300	500	700
$\widetilde{V}_{1,549}^{(1)}(u)$	48.641	49.634	50.615	51.590	52.562	53.533	54.503	123.373	317.373	511.373	705.373
$\widetilde{V}_{2,549}^{(1)}(u)$	48.784	49.780	50.765	51.742	52.714	53.685	54.655	123.525	317.525	511.525	705.525
$\widetilde{V}_{3,549}^{(1)}(u)$	48.834	49.832	50.819	51.797	52.770	53.741	54.711	123.581	317.581	511.581	705.581
$\widetilde{V}_{4,549}^{(1)}(u)$	49.025	50.025	51.014	51.994	52.969	53.940	54.911	123.781	317.781	511.781	705.781

From the numerical examples provided, it is evident that when surplus has an upper limit constraint $M_0(M_0 < \infty)$, the optimal strategies after surpassing a certain constant will be M_0, which is consistent with (4.9). In cases where there are no constraints on dividend payments, all surplus exceeding the barrier will be used for dividends. Numerical examples show that different forms of dividend payments directly affect the dividend threshold or barrier. In addition, with the same distribution of gains, the optimal value function is larger when there are no payment constraints, and the difference between them becomes more significant as the surplus increases. This phenomenon suggests that company operators should fully consider the optimal dividend problem when making dividend decisions to ensure adequate funding for production activities. The simulation results indicate that in modern enterprises, such as pharmaceutical and petroleum companies, maximizing benefits for shareholders requires a clear understanding of the scale of operations and the expected gains. Subsequently, companies can develop the most effective financing strategy based on the optimal strategies and properties obtained from the model. This approach primarily aims to ensure the effective utilization of company surplus and to optimize dividends for shareholders. These conclusions further enhance the practical application of the model discussed in this paper.

7. Conclusions

In this paper, we delve into the optimal dividend problem within the framework of a compound binomial dual risk model, which is suitable for companies whose income comes from occasional stochastic operating expenses and generates random gains at each unit time interval. When considering whether there is a ceiling on dividend payments or not, we have proven that the optimal value function represents the unique solution of a discrete HJB equation. Furthermore, we have derived an algorithm for determining the optimal strategy. The estimated results in the numerical examples demonstrate that our optimal dividend strategy is a threshold strategy, whether with or without a ceiling payment constraint. In the future, we can consider exploring the optimal dividend problem under the constraint of survival probability (Alcoforado et al. [37]), incorporating delays in dividend payments (Cheung and Wong [28]) in the dual risk model with stochastic expenses, or conducting a real data analysis (Wang et al. [38]).

Author contributions

Li Deng: conceptualization, methodology, formal analysis writing-original draft; Zhichao Chen: formal analysis, software, writing-review editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict of interest.

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