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Research article

Crafting optimal cardiovascular treatment strategy in Pythagorean fuzzy dynamic settings

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Abstract: The prevalence of cardiovascular disease (CVD) is a major issue in world health. There is a compelling desire for precise and effective methods for making decisions to determine the most effective technique for treating CVD. Here, we focused on the urgent matter at hand. Pythagorean fuzzy dynamic settings are exceptionally proficient at capturing ambiguity because they can handle complex problem specifications that involve both Pythagorean uncertainty and periodicity. In this article, we introduced a pair of novel aggregation operators: The Pythagorean fuzzy dynamic ordered weighted averaging (PFDOWA) operator and the Pythagorean fuzzy dynamic ordered weighted geometric (PFDOWG) operator, and we proved various structural properties of these concepts. Using these operators, we devised a systematic methodology to handle multiple attribute decision-making (MADM) scenarios incorporating Pythagorean fuzzy data. Moreover, we endeavored to address a MADM problem, where we discerned the most efficacious strategy for the management of CVD through the application of the proposed operators. Finally, we undertook an exhaustive comparative analysis to evaluate the ability of the suggested methods in connection with several developed procedures, therefore demonstrating the reliability of the generated methodologies.

Keywords: Pythagorean fuzzy set; dynamic aggregation operators; cardiovascular disease
Mathematics Subject Classification: 90B50, 94D05

The abbreviations used in this study are summarized as below:

Description	Abbreviation
Cardiovascular disease	CVD
Fuzzy set	FS
Intuitionistic fuzzy set	IFS
Pythagorean fuzzy set	PFS
Pythagorean fuzzy sets	PFSs
Pythagorean fuzzy dynamic ordered weighted averaging operator	PFDOWA
Pythagorean fuzzy dynamic ordered weighted geometric operator	PFDOWG
Multi-attribute decision-making	MADM
Pythagorean fuzzy ordered weighted averaging operator	PFOWA
Pythagorean fuzzy ordered weighted geometric operator	PFOWG
Pythagorean fuzzy weighted averaging operator	PFWA
Pythagorean fuzzy weighted geometric operator	PFWG
Aggregation operators	AOs

1. Introduction

MADM constitutes a pivotal component of the decision-making process (DM). Its objective is to choose the most rational option from a spectrum of choices, taking into account a multitude of conflicting criteria. The MADM method is recognized as a notable approach due to its immediate application, disregarding the reluctance of individuals with decision-making difficulties and the complexity of decision-making scenarios. The significance of this technique is increasing because of its ability to effectively address real-world issues in several fields. In recent years, healthcare issues have become a central focus for researchers employing decision-making approaches [1–3]. The growing importance of MADM is due to its capacity to effectively address real-world problems across a variety of domains through the use of AOs. These operators are explicitly designed to scrutinize each value within the original set and combine all values into a singular entity within the same set. Prior to the advent of AOs, selections were solely based on entire collections. Nevertheless, the concept of belonging to a set is often imprecise, particularly in disciplines such as biology, social sciences, language and linguistics, psychology, economics, and other social sciences. In these fields, traditional mathematical techniques appear somewhat insufficient. To address this challenge, Zadeh [4] proposed the notion of partial membership in a set, which he termed a fuzzy set. Kahne [5] developed a significant decision-making approach that can be used to assess potential solutions by considering numerous factors with different levels of importance. An approach for identifying the optimal choice using fuzzy logic was presented [6]. Fuzzy sets' fundamental operations were investigated in [7]. The AOs for fuzzy sets were first introduced by Yager in [8]. Atanassov [9] introduced the notion of IFS to further develop the concept of fuzzy sets. Szmidt and Kaeprzyk [10] developed a method for solving group decision problems using the intuitionistic fuzzy (IF) environment. Li [11] did a study on fundamental linear programming techniques and methodologies for MADM in an uncertain environment. Xu and Yager [12] developed many geometric AOs for IFS. In 2007, several basic arithmetic aggregation algorithms were created [13] using an IFS framework. [14] suggested the utilization of generalized AOs on IFS as a possible solution to the MADM problem. The researchers in

[15] aimed to tackle challenges related to group decision-making in IFS environments by utilizing AOs. Huang [16] proposed a DM method for IFS using Hamacher AOs. Verma [17] was the first to use the Bonferroni mean operators for IFS. In order to tackle MADM issues, the Aczel-Alsina AOs of IFS were introduced by [18].

The condition $\mu + \eta \leq 1$ restricts the possibility of having membership and non-membership grades in IFS. To prevent this scenario, Yager [19,20] proposed (PFSs) as a successor to IFSs. PFSs are designed to handle imprecision and complicated uncertainty by imposing the constraint $\mu^2 + \eta^2 \leq 1$, where μ represents membership grade and η represents non-membership grade. Moreover, the Pythagorean fuzzy number (PFN) was proposed by Zhang and Xu [21] to describe an object's dual characteristics. PFSs generally permit and incorporate a higher amount of ambiguity than IFSs. Yager [22] demonstrated the development of several AOs, such as the weighted power geometric and PFOWA operators. A similarity measure and two AOs form the basis of an innovative method suggested by Zhang [23] in a Pythagorean fuzzy (PF) setting. Peng and Yaun [24] explored the basic characteristics of Pythagorean fuzzy AOs. Wei [25] introduced a collection of generalized Pythagorean fuzzy AOs and highlighted their applications in decision-making. Zeng et al. [26] later integrated AOs and distance measurements to develop the PF weighted and ordered weighted distance operators. For more developments about PFSs, we refer to [27–29]. A set of generalized Pythagorean fuzzy geometric AOs was proposed by Garg [30], which incorporated Einstein's operations. Furthermore, Garg [31,32] examined the utilization of the PF environment in diverse decision-making scenarios. Liang et al. [33] devised a PF Bonferroni mean AO and presented a proficient method for its computation to address decision-making problems. To go deeper into this specific area of expertise, one might examine [34–37].

All the aforementioned decision-making challenges involve the consideration of decision information with simultaneous inputs. However, certain domains of decision-making, such as multi-period dynamic investment, medical diagnosis, dynamic personal selection, and dynamic military system efficiency evaluation, entail the collection of data at various time intervals. Xu and Yager [38] first discovered this particular kind of information in their study on dynamic MADM problems. They presented an application of MADM problems utilizing the uncertain IF dynamic weighted averaging (UIFDWA) operator and the IF dynamic weighted averaging (IFDWA) operator to obtain dynamic or uncertain dynamic IF information. In addition, the IFDWA and UIFDWA operators were employed to address two dynamic MADM challenges when the attributes' arguments are utilized in IFNs or IVIFNs. Dynamic MADM problems involving the utilization of IVIFNs or IFN attribute values were examined by Wei [39]. He introduced weighted geometric AOs comprised of various periods to gather dynamic IFS data, including the IF dynamic weighted geometric (IFDWG) operator and the uncertain IF dynamic weighted geometric (UIFDWG) operator. The scholarly discourse on PFSs has rapidly gained popularity among numerous researchers, as evidenced by the considerable focus they have garnered across diverse studies [40–44].

1.1. The research gap, motivation, and advantages of this study

PFSs sustain higher levels of uncertainty than IFSs, underscoring their suitability for addressing scenarios marked by heightened complexity and ambiguity in MADM challenges. Ordered weighted AOs do not depend on preassigned weights designated to certain attributes. These operators allow decision-makers to include the uncertainty and imprecision associated with real-world scenarios by rearranging the input values to accurately reflect their importance in the decision-making process. Moreover, ordered weighted AOs with varying time intervals offer a strong tool for temporal aggregation and decision-making. Researchers have focused mainly on issues where all of the initially selected data was collected at the same time for decision-making. Nonetheless, it is common in many decision-making environments to collect relevant data at several time points for the underlying selection. In the context of the MADM, the term "time interval" refers to how the decision-maker prefers to receive information throughout various time periods. It is accomplished by employing a time varying function. By combining components into the dynamic paradigm, it becomes feasible to analyze variations in membership degrees as time passes and examine changes within particular time frames. This aspect improves decision-making accuracy, provides insight into modifications, and evaluates the dynamics of fuzzy sets. To address these problems, a variety of techniques are needed. The PFDOWA and PFDOWG operators are essential components of decision-making models because they allow for dynamic modifications and accurately depict complex interactions among systems affected by imprecision and uncertainty. These operators aid in prioritizing pertinent, current data and provide insights into both short- and long-term trends by varying weights over time. In addition, the ordered weighted AOs have already been proposed for PF settings. However, these operators are not capable of handling uncertainty at different time periods. As a result, it is critical to provide a detailed explanation of these operators for PF sets in order to effectively address such scenarios and fill the gap. This motivates us to study dynamic ordered weighted AOs in the framework of PFSs. The dynamic nature of the operator, together with the PFS, is an essential part of our strategy. The key motivations for this study are listed below:

a) Dynamic operators provide exceptional flexibility when dealing with ambiguous data.
b) These operators exhibit exceptional proficiency in switching diverse data into a single value, consequently effectively addressing the complicated aspects of decision-making in an ever-changing environment.
c) In dynamic contexts characterized by ever-changing conditions, PFSs present an advanced representation that can accommodate emerging preferences or circumstances.

The major contributions of this study are explained as follows:

i. We introduce two innovative AOs, the PFDOWA operator and the PFDOWG operator. We design these operators to apply to procedures that use PF data in decision-making.
ii. We thoroughly examine the essential characteristics of PFDOWA and PFDOWG operators, such as idempotency, boundedness, and monotonicity.
iii. By utilizing the PFDOWA and PFDOWG operators, an organized approach for addressing MADM problems is proposed. This demonstrates the practical significance of these operators.
iv. The suggested method is used to tackle a specific MADM issue that is focused on treating CVD. It underscores the significance of PFDOWA and PFDOWG operators in the context of decision-making processes.
v. To determine the efficacy of the proposed methods in comparison to a variety of established techniques, we perform a comprehensive comparative analysis. These comparison results demonstrate the consistency and validity of the developed methodology.

To accomplish the work presented in this article, the remaining portion of this study is organized as follows: In Section 2, we provide a few basic definitions that are crucial for understanding the major findings presented in this research work. Section 3 contains an introduction to the study of PFDOWA and PFDOWG operators and establishes the key structural aspects of these phenomena. Section 4 presents a step-by-step methodology for tackling the MADM problem through the utilization of PF dynamic information, employing the suggested operators. In Section 5, we illustrate the validity of defined approaches in the framework of solving the MADM problem of selecting an efficient treatment method for CVD. Additionally, we include a comparative study to demonstrate the effectiveness and practicality of these novel approaches in contrast with established approaches. In Section 6, we describe the conclusions drawn from the complete research.

2. Preliminaries

In this section, we give key definitions that are required for a complete understanding of the subject matter covered in this article.

Definition 1. ([9]) Assuming that $\mathcal{Z}$ is a universe of discourse, an IFS $\mathcal{I}$ is described as follows:

$$\mathcal{I} = \{z, \mu_{\mathcal{I}}(z), \eta_{\mathcal{I}}(z) | z \in \mathcal{Z}\},$$

where $\mu_{\mathcal{I}} : \mathcal{Z} \to [0,1]$ and $\eta_{\mathcal{I}}: \mathcal{Z} \to [0,1]$ represent the membership function and non-membership function, respectively, that admit the criteria $0 \leq \mu_{\mathcal{I}}(z) + \eta_{\mathcal{I}}(z) \leq 1$. The hesitation margin of the IFS $\mathcal{I}$ can be characterized as follows: $\pi_{\mathcal{I}}(z) = 1 - \mu_{\mathcal{I}}(z) - \eta_{\mathcal{I}}(z)$.

Definition 2. ([22]). Let $\mathcal{Z}$ be a universal set. A PFS $\mathcal{P}$ can be characterized in the following manner:

$$\mathcal{P} = \{z, \mu_{\mathcal{P}}(z)\ , \eta_{\mathcal{P}}(z) | z \in \mathcal{Z}\},$$

where, $\mu_{\mathcal{P}}: \mathcal{Z} \to [0,1]$ and $\eta_{\mathcal{P}}: \mathcal{Z} \to [0,1]$, are called membership and non-membership functions respectively, under a few restrictions: $0 \leq \mu_{\mathcal{P}}(z) \leq 1$, $0 \leq \eta_{\mathcal{P}}(z) \leq 1$ and $0 \leq \mu_{\mathcal{P}}^2(z) + \eta_{\mathcal{P}}^2(z) \leq 1\ \forall\, z \in \mathcal{Z}$. Additionally, the degree of indeterminacy of (PFS) $\mathcal{P}$ is specified as $\pi_{\mathcal{P}}(z) = \sqrt{1 - \mu_{\mathcal{P}}^2(z) - \eta_{\mathcal{P}}^2(z)}$, such that $0 \leq \pi_{\mathcal{P}}(z) \leq 1\ \forall\, z \in \mathcal{Z}$.

Now, we specify the membership and non-membership degrees of $z \in \mathcal{Z}$ as $z = (\mu, \eta)$. The aforementioned description of z is referred to as a PF number.

Definition 3. ([22]) Assume that ϕ is a collection of PFNs represented as $\tilde{\alpha}_i = (\mu_i, \eta_i)$ where, $i = 1,2, \ldots, n$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector linked to $\tilde{\alpha}_i$ with condition $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The Pythagorean fuzzy weighted averaging AO is a mapping $PFWA: \phi^n \to \phi$, defined by the rule:

$$PFWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \oplus_{i=1}^{n} \omega_i \tilde{\alpha}_i$$

$$= \left(\sqrt{1 - \prod_{i=1}^{n} (1 - \mu_i^2)^{\omega_i}}, \prod_{i=1}^{n} \eta_i^{\omega_i} \right).$$

Definition 4. ([27]) Assume that ϕ is a collection of PFNs represented as $\tilde{\alpha}_i = (\mu_i, \eta_i)$ where, $i = 1,2, \ldots, n$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_i$ with that that conditions $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The Pythagorean fuzzy weighted geometric AO is a function $PFWG: \phi^n \to$

ϕ, defined by the rule:

$$PFWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \otimes_{i=1}^{n} (\tilde{\alpha}_i)^{\omega_i}$$

$$= \left(\prod_{i=1}^{n} \mu_i^{\omega_i}, \sqrt{1 - \prod_{i=1}^{n} (1 - \eta_i^2)^{\omega_i}} \right).$$

Definition 5. ([20]) Assume that ϕ is a collection of PFNs represented as $\tilde{\alpha}_i = (\mu_i, \eta_i)$ where, $i = 1,2,\dots,n$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_i$ with the conditions that $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The Pythagorean fuzzy ordered weighted averaging AO is a mapping $PFOWA: \phi^n \to$ ϕ, defined by the rule:

$$PFOWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \oplus_{i=1}^{n} \omega_i \tilde{\alpha}_{\varrho(i)}$$

$$= \left(\sqrt{1 - \prod_{i=1}^{n} \left(1 - \mu_{\varrho(i)}^2\right)^{\omega_i}}, \prod_{i=1}^{n} \eta_{\varrho(i)}^{\omega_i} \right).$$

Notice, $\varrho(1), \varrho(2), \dots, \varrho(n)$ is the permutation of $\{1, 2, \dots, n\}$ in a certain manner, satisfying condition $\tilde{\alpha}_{\varrho(i-1)} \geq \tilde{\alpha}_{\varrho(i)}$.

Definition 6. ([45]) Assume that ϕ is a collection of PFNs represented as $\tilde{\alpha}_i = (\mu_i, \eta_i)$ where, $i = 1,2,\dots,n$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector linked to $\tilde{\alpha}_i$ with the conditions $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The PFOWG operator is a function $PFOWG$: $\phi^n \to \phi$, defined by the rule:

$$PFOWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \otimes_{i=1}^{n} \left(\tilde{\alpha}_{\varrho(i)}\right)^{\omega_i}$$

$$= \left(\prod_{i=1}^{n} \mu_{\varrho(i)}^{\omega_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \eta_{\varrho(i)}^2\right)^{\omega_i}} \right).$$

Definition 7. ([21]) For any PFN $\tilde{\alpha} = (\mu, \eta)$ two vital functions are outlined as follows:

i. The score function $S(\tilde{\alpha})$ is computed in the following manner: $S(\tilde{\alpha}) = \mu^2 - \eta^2$, where, $S(\tilde{\alpha}) \in [-1,1]$.

ii. The definition of the accuracy function $H(\tilde{\alpha})$ is described as follows: $H(\tilde{\alpha}) = \mu^2 + \eta^2$, where, $H(\tilde{\alpha}) \in [0,1]$.

Definition 8. ([38]) Let t be a time variable, then $\mathcal{I}_t = (\mu_t, \eta_t)$ is an IF variable where, $\mu_t \in [0,1]$, $\eta_t \in [0,1]$ with restriction $\mu_t + \eta_t \leq 1$.

In the context of IF variable $\mathcal{I}_t = (\mu_t, \eta_t)$ if a time sequence is known $t = (t_1, t_2, \dots, t_p)$, then notation $\mathcal{I}_{t_1}, \mathcal{I}_{t_2}, \dots, \mathcal{I}_{t_p}$ represents a collection of $\wp$ IF numbers considered at $\wp$ different time periods.

3. Dynamic ordered weighted aggregation operators for PFNs

In this section, we examine the dynamic operations that occur inside the PF framework. Furthermore, we present the concepts of dynamic ordered weighted AOs for PFNs, namely PFDOWA and PFDOWG, and analyze their key features.

Definition 9. Let t symbolize a variable that represents time. Within this framework, we establish $\tilde{\alpha}_t = (\mu_t\ ,\ \eta_t)$ as a PF variable where, $\mu_t \in [0,1]\ , \eta_t \in [0,1]$ with constraint $\mu_t^2 + \eta_t^2 \leq 1$.

For a PF variable $\tilde{\alpha}_t = (\mu_t, \eta_t)$ if we have a time sequence $t = (t_1, t_2, \dots, t_p)$ then $\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}$ represents p PFNs, associated with distinct time periods.

The following definition introduces us to dynamic operational laws applied to PFNs.

Definition 10. Assume two PFNs $\tilde{\alpha}_{t_1} = (\mu_{t_1}, \eta_{t_1})$ and $\tilde{\alpha}_{t_2} = (\mu_{t_2}, \eta_{t_2})$. The fundamental ordering principles governing the interrelationship of operational laws in PFNs can be characterized as follows:

1- $\tilde{\alpha}_{t_1} \leq \tilde{\alpha}_{t_2}$, if $\mu_{t_1} \leq \mu_{t_2}$ and $\eta_{t_1} \leq \eta_{t_2}$

2- $\tilde{\alpha}_{t_1} = \tilde{\alpha}_{t_2}$, if and only if $\tilde{\alpha}_{t_1} \leq \tilde{\alpha}_{t_2}$ and $\tilde{\alpha}_{t_2} \leq \tilde{\alpha}_{t_1}$

Definition 11. Consider $\tilde{\alpha}_t = (\mu_t, \eta_t)$, $\tilde{\alpha}_{t_1} = (\mu_{t_1}, \eta_{t_1})$ and $\tilde{\alpha}_{t_2} = (\mu_{t_2}, \eta_{t_2})$ represent three PFNs and $\Upsilon > 0$. The dynamic operations for these PFNs are described as follows:

i. $\tilde{\alpha}_{t_1} \oplus \tilde{\alpha}_{t_2} = \left(\sqrt{\mu_{t_1}^2 + \mu_{t_2}^2 - \mu_{t_1}^2 \mu_{t_2}^2}\ , \eta_{t_1}\, \eta_{t_2}\right)$

ii. $\tilde{\alpha}_{t_1} \otimes \tilde{\alpha}_{t_2} = \left(\mu_{t_1}\mu_{t_2}\ , \sqrt{\eta_{t_1}^2 + \eta_{t_2}^2 - \eta_{t_1}^2 \eta_{t_2}^2}\right)$

iii. $\lambda\tilde{\alpha}_t = \left(\sqrt{1 - (1 - \mu_t^2)^{\Upsilon}}, \eta_t^{\Upsilon}\right)$

iv. ${\tilde{\alpha}_t}^{\Upsilon} = \left(\mu_t^{\Upsilon}, \sqrt{1 - (1 - \eta_t^2)^{\Upsilon}}\right)$

Definition 12. Assume that ϕ is a collection of PFNs represented as $\tilde{\alpha}_{t_{ř}} = (\mu_{t_{ř}}\ , \eta_{t_{ř}})$ where, $ř = 1,2.3, \dots, p$ and $\Upsilon_t = \left[\Upsilon_{t_1}, \Upsilon_{t_2}, \dots, \Upsilon_{t_p}\ \right]^T$ is the weight vector linked to the time periods $t_{ř}$ such that $\sum_{ř=1}^{p} \Upsilon_{t_{ř}} = 1$ and $\Upsilon_{t_{ř}} \in [0,1]$. In this context, the PFDOWA is a mapping $PFDOWA\colon \Phi^p \to \phi$ defined by the rule:

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) = \oplus_{ř=1}^{p} \Upsilon_{t_{ř}} \tilde{\alpha}_{\varrho(t_{ř})}$$

$$= \left(\sqrt{1-\prod_{ř=1}^{p}\left(1-\mu_{\varrho(t_ř)}^{2}\right)^{\Upsilon_{t_ř}}}, \prod_{ř=1}^{p} \eta_{\varrho(t_ř)}^{\Upsilon_{t_ř}}\right)$$

In the upcoming theorem, we will demonstrate that the aggregated value of PFNs at different time periods, using the PFDOWA operator, results in a new PFN.

Theorem 1. Consider p numbers of PFNs expressed as $\tilde{\alpha}_{t_ř} = \left(\mu_{t_ř}, \eta_{t_ř}\right)$ at p different time periods $t_ř$, where $ř = 1,2,3,\dots,p$, and the associated weight vector is $\Upsilon_t = \left[\Upsilon_{t_1}, \Upsilon_{t_2}, \dots., \Upsilon_{t_p}\right]^T$ with some restrictions $\sum_{ř=1}^{p} \Upsilon_{t_ř} = 1$ and $\Upsilon_{t_ř} \in [0,1]$. The aggregated value of these PFNs using the PFDOWA operator is also a PFN. It can be described as follows:

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) = \left(\sqrt{1-\prod_{ř=1}^{p}\left(1-\mu_{\varrho(t_ř)}^{2}\right)^{\Upsilon(t_ř)}}, \prod_{ř=1}^{p} \eta_{\varrho(t_ř)}^{\Upsilon(t_ř)}\right)$$

Proof. The technique of mathematical induction is used to establish the proof of this theorem. The proof begins by examining the basic case when $p = 2$

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}\right) = \Upsilon_{t_1}\tilde{\alpha}_{\varrho(t_1)} \oplus \Upsilon_{t_2}\tilde{\alpha}_{\varrho(t_2)}$$

By separating the components $\Upsilon_{t_1}\tilde{\alpha}_{\varrho(t_1)}$ and $\Upsilon_{t_2}\tilde{\alpha}_{\varrho(t_2)}$ utilizing definition 12, we derive the subsequent equations:

$$\Upsilon_{t_1}\tilde{\alpha}_{\varrho(t_1)} = \left[\sqrt{1-\left(1-\mu_{\varrho(t_1)}^{2}\right)^{\Upsilon_{t_1}}}, \eta_{\varrho(t_1)}^{\Upsilon_{t_1}}\right]$$

$$\Upsilon_{t_2}\tilde{\alpha}_{\varrho(t_2)} = \left[\sqrt{1-\left(1-\mu_{\varrho(t_2)}^{2}\right)^{\Upsilon_{t_2}}}, \eta_{\varrho(t_2)}^{\Upsilon_{t_2}}\right]$$

Thus,

$$\Upsilon_{t_1}.\tilde{\alpha}_{\varrho(t_1)} \oplus \Upsilon_{t_2}.\tilde{\alpha}_{\varrho(t_2)} = \left[\sqrt{1-\left(1-\mu_{\varrho(t_1)}^{2}\right)^{\Upsilon_{t_1}}}, \eta_{\varrho(t_1)}^{\Upsilon_{t_1}}\right] \oplus \left[\sqrt{1-\left(1-\mu_{\varrho(t_2)}^{2}\right)^{\Upsilon_{t_2}}}, \eta_{\varrho(t_2)}^{\Upsilon_{t_2}}\right]$$

$$= \left[\sqrt{1-\left(1-\mu_{\varrho(t_1)}^{2}\right)^{\Upsilon_{t_1}}\left(1-\mu_{\varrho(t_2)}^{2}\right)^{\Upsilon_{t_2}}}, \eta_{\varrho(t_1)}^{\Upsilon_{t_1}} \eta_{\varrho(t_2)}^{\Upsilon_{t_2}}\right]$$

Consequently,

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}\right) = \left[\sqrt{1 - \prod_{ř=1}^{2}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}}}, \prod_{ř=1}^{2}\eta_{\varrho(t_ř)}^{\Upsilon_{t_ř}}\right]$$

Hence, the result is true for $p = 2$.

Now, we assume that the given result is true for $p = n > 2$, then we have,

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_n}\right) = \left[\sqrt{1 - \prod_{ř=1}^{n}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}}}, \prod_{ř=1}^{n}\eta_{\varrho(t_ř)}^{\Upsilon_{t_ř}}\right]$$

Now we prove the result for $p = n + 1$

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_n}, \tilde{\alpha}_{t_{n+1}}\right) = \Upsilon_{t_1}\tilde{\alpha}_{\varrho(t_1)} \oplus \Upsilon_{t_2}\tilde{\alpha}_{\varrho(t_2)} \oplus \dots \oplus \Upsilon_{t_n}\tilde{\alpha}_{\varrho(t_n)} \oplus \Upsilon_{t_{n+1}}\tilde{\alpha}_{\varrho(t_{n+1})}$$

$$= \left[\sqrt{1 - \prod_{ř=1}^{n}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}}}, \prod_{ř=1}^{n}\eta_{\varrho(t_ř)}^{\Upsilon_{t_ř}}\right] \oplus \left[\sqrt{1 - \left(1 - \mu^2_{\varrho(t_{n+1})}\right)^{\Upsilon_{t_{n+1}}}}, \left(\eta_{\varrho(t_{n+1})}\right)^{\Upsilon_{t_{n+1}}}\right]$$

It follows that

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_n}, \tilde{\alpha}_{t_{n+1}}\right) = \left[\sqrt{1 - \prod_{ř=1}^{n+1}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}}}, \prod_{ř=1}^{n+1}\eta_{\varrho(t_ř)}^{\Upsilon_{t_ř}}\right]$$

Hence, we can deduce that the result is applicable to all positive integer values of p.

Example 1. Consider $\tilde{\alpha}_{t_1} = (0.30, 0.80)$, $\tilde{\alpha}_{t_2} = (0.40, 0.70)$, $\tilde{\alpha}_{t_3} = (0.20, 0.80)$ and $\tilde{\alpha}_{t_4} = (0.70, 0.60)$ represents four PFNs and $\Upsilon_t = (0.2, 0.1, 0.3, 0.4)^T$ is the weight vector linked to time periods $t_ř$ where $ř = 1,2,3,4$. First we compute the scores values of $\tilde{\alpha}_{t_ř}$ by applying definition 7 as follows:

$$S\left(\tilde{\alpha}_{t_1}\right) = -0.55,\ S\left(\tilde{\alpha}_{t_2}\right) = -0.33,$$

$$S\left(\tilde{\alpha}_{t_3}\right) = -0.6,\ S\left(\tilde{\alpha}_{t_4}\right) = 0.13$$

then the permuted values of PFNs are organized as follows:

$$\tilde{\alpha}_{\varrho(t_1)} = (0.70, 0.60), \tilde{\alpha}_{\varrho(t_2)} = (0.40, 0.70), \tilde{\alpha}_{\varrho(t_3)} = (0.30, 0.80), \tilde{\alpha}_{\varrho(t_4)} = (0.20, 0.80)$$

In view of definition 12, the aggregated values of the above PFNs is calculated as follows:

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\tilde{\alpha}_{t_3},\tilde{\alpha}_{t_4}\right)=\left(\sqrt{1-\prod_{\check{r}=1}^{4}\left(1-\mu_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}}},\prod_{\check{r}=1}^{4}\eta_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}}\right)$$

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\tilde{\alpha}_{t_3},\tilde{\alpha}_{t_4}\right)$$

$$=\left(\sqrt{1-(1-0.7^2)^{0.2}(1-0.4^2)^{0.1}(1-0.3^2)^{0.3}(1-0.2^2)^{0.4}},\ (0.6)^{0.2}(0.7)^{0.1}(0.8)^{0.3}(0.8)^{0.4}\right)$$

It follows that

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\tilde{\alpha}_{t_3},\tilde{\alpha}_{t_4}\right)=(0.423,0.745)$$

In the upcoming theorem, we establish the idempotency property of the PFDOWA operator, showing that if all PFNs are equal for a given time period, the aggregated result will also equal that PFN.

Theorem 2. Consider the p number of PFNs represented as $\tilde{\alpha}_{(t_{\check{r}})}=\left(\mu_{(t_{\check{r}})},\eta_{(t_{\check{r}})}\right)$, existing at p different time periods $t_{\check{r}}$, and let $\Upsilon_t=\left(\Upsilon_{t_1},\Upsilon_{t_2},\dots,\Upsilon_{t_p}\right)^T$ be the weight vector linked to time periods $t_{\check{r}}$ with the condition $\sum_{\check{r}=1}^{p}\Upsilon_{t_{\check{r}}}=1$ and $\Upsilon_{t_{\check{r}}}\in[0,1]$, where $\check{r}=1,2,\dots,p$. If $\tilde{\alpha}_{\varrho(t_{\check{r}})}=\tilde{\alpha}_{\varrho(t_j)}$ are mathematically equal for all $\check{r}$ and for some $j\in\{1,2,\dots,p\}$ where, $\tilde{\alpha}_{\varrho(t_j)}=\left(\mu_{\varrho(t_j)},\eta_{\varrho(t_j)}\right)$. Then

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\dots,\tilde{\alpha}_{t_p}\right)=\tilde{\alpha}_{\varrho(t_j)}$$

Proof. Given that $\tilde{\alpha}_{\varrho(t_{\check{r}})}=\tilde{\alpha}_{\varrho(t_j)}$ for all $\check{r}=1,2,\dots,p$ and for some $j\in\{1,2,\dots,p\}$, which implies that $\mu_{\varrho(t_{\check{r}})}=\mu_{\varrho(t_j)}$ and $\eta_{\varrho(t_{\check{r}})}=\eta_{\varrho(t_j)}$. Thus, we have

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\dots,\tilde{\alpha}_{t_p}\right)=\left(\sqrt{1-\left(1-\mu_{\varrho(t_j)}^{2}\right)^{\sum_{\check{r}=1}^{p}\Upsilon_{t_{\check{r}}}}},\eta_{\varrho(t_j)}^{\sum_{\check{r}=1}^{p}\Upsilon_{t_{\check{r}}}}\right)$$

$$=\left(\sqrt{1-\left(1-\mu_{\varrho(t_j)}^{2}\right)},\eta_{\varrho(t_j)}\right)=\left(\sqrt{\mu_{\varrho(t_j)}^{2}},\eta_{\varrho(t_j)}\right)=\left(\mu_{\varrho(t_j)},\eta_{\varrho(t_j)}\right)$$

Consequently,

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\dots,\tilde{\alpha}_{t_p}\right)=\tilde{\alpha}_{\varrho(t_j)}$$

In the upcoming theorem, we establish the boundedness property of the PFDOWA operator, demonstrating that the aggregated value of PFNs lies between the defined lower and upper bounds.

Theorem 3. Consider $\tilde{\alpha}^- = \left(\min_{t_ř}\left(\mu_{\varrho(t_ř)}\right), \max_{t_ř}\left(\eta_{\varrho(t_ř)}\right)\right)$ and $\tilde{\alpha}^+ = \left(\max_{t_ř}\left(\mu_{\varrho(t_ř)}\right), \min_{t_ř}\left(\eta_{\varrho(t_ř)}\right)\right)$ represent the lower bound and upper bound of PFNs $\tilde{\alpha}_{t_ř} = \left(\mu_{t_ř}, \eta_{t_ř}\right)$ where, $ř = 1,2,\dots,p$. Let $\Upsilon_t = \left(\Upsilon_{t_1}, \Upsilon_{t_2}, \dots, \Upsilon_{t_p}\right)^T$ be the corresponding weight vector of time periods $t_ř$ such that $\Upsilon_{t_ř} \in [0,1]$ satisfying the condition $\sum_{ř=1}^{p} \Upsilon_{t_ř} = 1$. Then

$$\tilde{\alpha}^- \leq PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) \leq \tilde{\alpha}^+.$$

Proof. Consider the outcome when employing the PFDOWA operator on a PFNs collection**,** indicated as $PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) = (\mu_t, \eta_t)$,

In view of the given conditions, we have

$$\min_{t_ř}\left\{\mu_{\varrho(t_ř)}\right\} \leq \mu_{\varrho(t_ř)} \leq \max_{t_ř}\left\{\mu_{\varrho(t_ř)}\right\}$$

$$\Rightarrow \min_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\} \leq \mu^2_{\varrho(t_ř)} \leq \max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}$$

$$\Rightarrow 1 - \max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\} \leq 1 - \mu^2_{\varrho(t_ř)} \leq 1 - \min_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}$$

$$\Rightarrow \prod_{ř=1}^{p}\left(1 - \max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}\right)^{\Upsilon_{t_ř}} \leq \prod_{ř=1}^{p}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}} \leq \prod_{ř=1}^{p}\left(1 - \min_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}\right)^{\Upsilon_{t_ř}}$$

$$\Rightarrow \left(1 - \max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}\right)^{\sum_{ř=1}^{p}\Upsilon_{t_ř}} \leq \prod_{ř=1}^{p}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}} \leq \left(1 - \min_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}\right)^{\sum_{ř=1}^{p}\Upsilon_{t_ř}}$$

$$\Rightarrow \left(1 - \max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}\right) \leq \prod_{ř=1}^{p}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}} \leq \left(1 - \min_{\mathrm{t_k}}\left\{\mu^2_{\varrho(\mathrm{t_k})}\right\}\right)$$

$$\Rightarrow \min_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\} \leq 1 - \prod_{ř=1}^{p}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}} \leq \max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}$$

$$\Rightarrow \sqrt{\min_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}} \leq \sqrt{1 - \prod_{ř=1}^{p}\left(1 - \mu^2_{\varrho(t_ř)}\right)^{\Upsilon_{t_ř}}} \leq \sqrt{\max_{t_ř}\left\{\mu^2_{\varrho(t_ř)}\right\}}$$

Consequently,

$$\min_{t_{ř}}\{\mu_{\varrho(t_{ř})}\} \leq \mu_t \leq \max_{t_{ř}}\{\mu_{\varrho(t_{ř})}\} \tag{3.1}$$

Moreover, in view of given conditions we have

$$\max_{t_{ř}}\{\eta_{\varrho(t_{ř})}\} \leq \eta_{\varrho(t_{ř})} \leq \min_{t_{ř}}\{\eta_{\varrho(t_{ř})}\}$$

$$\Rightarrow \prod_{ř=1}^{p}\left(\max_{t_{ř}}\{\eta_{\varrho(t_{ř})}\}\right)^{\Upsilon_{t_{ř}}} \leq \prod_{ř=1}^{p}\left(\eta_{\varrho(t_{ř})}\right)^{\Upsilon_{t_{ř}}} \leq \prod_{ř=1}^{p}\left(\min_{t_{ř}}\{\eta_{\varrho(t_{ř})}\}\right)^{\Upsilon_{t_{ř}}}$$

$$\Rightarrow \left(\max_{t_{ř}}\{\eta_{\varrho(t_{ř})}\}\right)^{\sum_{ř=1}^{p}\Upsilon_{t_{ř}}} \leq \prod_{ř=1}^{p}\left(\eta_{\varrho(t_{ř})}\right)^{\Upsilon_{t_{ř}}} \leq \left(\min_{t_{ř}}\{\eta_{\varrho(t_{ř})}\}\right)^{\sum_{ř=1}^{n}\Upsilon_{t_{ř}}}$$

$$\Rightarrow \max_{t_{ř}}\{\eta_{\varrho(t_{ř})}\} \leq \eta_t \leq \min_{t_{ř}}\{\eta_{\varrho(t_{ř})}\}. \tag{3.2}$$

In view of definition 12 and by comparing (3.1) and (3.2) we obtain the following expression:

$$\tilde{\alpha}^{-} \leq PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \ldots, \tilde{\alpha}_{t_p}\right) \leq \tilde{\alpha}^{+}.$$

In the upcoming theorem, we will establish the monotonicity property of the PFDOWA operator, showing that if one set of PFNs dominates another, their aggregated values will preserve this ordering.

Theorem 4. Let $\tilde{\alpha}_{t_{ř}} = \left(\mu_{t_{ř}}, \eta_{t_{ř}}\right)$ and $\tilde{\alpha}'_{t_{ř}} = \left(\mu'_{t_{ř}}, \eta'_{t_{ř}}\right)$ where, $ř = 1,2,\ldots,p$, be any two collection of PFNs. Consider $\Upsilon_t = \left(\Upsilon_{t_1}, \Upsilon_{t_2}, \ldots, \Upsilon_{t_p}\right)^T$ signifies the weight vector linked to time periods $t_{ř}$, such that $\Upsilon_{t_{ř}} \in [0,1]$ and $\sum_{ř=1}^{p}\Upsilon_{ř} = 1$. If $\mu_{\varrho(t_{ř})} \leq \mu'_{\varrho(t_{ř})}$ and $\eta_{\varrho(t_{ř})} \geq \eta'_{\varrho(t_{ř})}$ then we can establish that:

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \ldots, \tilde{\alpha}_{t_p}\right) \leq PFDOWA\left(\tilde{\alpha}'_{t_1}, \tilde{\alpha}'_{t_2}, \ldots, \tilde{\alpha}'_{t_p}\right)$$

Proof. The application of PFDOWA operator yields:

$$PFDOWA\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \ldots, \tilde{\alpha}_{t_p}\right) = (\mu_t, \eta_t) \text{ and}$$

$$PFDOWA\left(\tilde{\alpha}'_{t_1}, \tilde{\alpha}'_{t_2}, \ldots, \tilde{\alpha}'_{t_p}\right) = (\mu'_t, \eta'_t)$$

Since $\mu_{\varrho(t_{ř})} \leq \mu'_{\varrho(t_{ř})}$, which shows that $\mu^2_{\varrho(t_{ř})} \leq \mu^{2\,\prime}_{\varrho(t_{ř})}$, this implies that

$$1 - \mu^2_{\varrho(t_{ř})} \geq 1 - \mu^{2\,\prime}_{\varrho(t_{ř})}$$

$$\Rightarrow \prod_{\check{r}=1}^{p}\left(1-\mu_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}} \geq \prod_{\check{r}=1}^{p}\left(1-\mu_{\varrho(t_{\check{r}})}^{2\,\prime}\right)^{\Upsilon_{t_{\check{r}}}}$$

$$\Rightarrow 1-\prod_{\check{r}=1}^{p}\left(1-\mu_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}} \leq 1-\prod_{\check{r}=1}^{p}\left(1-\mu_{\varrho(t_{\check{r}})}^{2\,\prime}\right)^{\Upsilon_{t_{\check{r}}}}$$

$$\Rightarrow \sqrt{1-\prod_{\check{r}=1}^{p}\left(1-\mu_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}}} \leq \sqrt{1-\prod_{\check{r}=1}^{p}\left(1-\mu_{\varrho(t_{\check{r}})}^{2\,\prime}\right)^{\Upsilon_{t_{\check{r}}}}}$$

It follows that

$$\mu_t \leq \mu_t' \tag{3.3}$$

Moreover, by taking into account $\eta_{\varrho(t_{\check{r}})} \geq \eta'_{\varrho(t_{\check{r}})}$, we have

$$\prod_{\check{r}=1}^{p}\eta_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}} \geq \prod_{\check{r}=1}^{p}\eta_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}\,\prime}$$

It follows that

$$\Rightarrow \eta_t \geq \eta_t' \tag{3.4}$$

Consequently, in view of definition 12 and by comparing (3.3) and (3.4), we obtain the following:

$$PFDOWA\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\ldots,\tilde{\alpha}_{t_p}\right) \leq PFDOWA\left(\tilde{\alpha}'_{t_1},\tilde{\alpha}'_{t_2},\ldots,\tilde{\alpha}'_{t_p}\right).$$

Definition 13. Assume that ϕ is a collection of PFNs represented as $\tilde{\alpha}_{t_{\check{r}}} = \left(\mu_{t_{\check{r}}}, \eta_{t_{\check{r}}}\right)$ where, $\check{r} = 1,2.3,\ldots,p$ and $\Upsilon_t = \left[\Upsilon_{t_1},\Upsilon_{t_2},\ldots,\Upsilon_{t_p}\right]^T$ is the weight vector linked to the time periods $t_{\check{r}}$ such that $\sum_{\check{r}=1}^{p}\Upsilon_{t_{\check{r}}} = 1$ and $\Upsilon_{t_{\check{r}}} \in [0,1]$. The PFDOWG operator is a mapping $PFDOWG: \phi^p \to \phi$ specified by the rule:

$$PFDOWG\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\ldots,\tilde{\alpha}_{t_p}\right) = \otimes_{\check{r}=1}^{p}\left(\tilde{\alpha}_{\varrho(t_{\check{r}})}\right)^{\Upsilon_{t_{\check{r}}}}$$

$$= \left(\prod_{\check{r}=1}^{p}\mu_{\varrho(t_{\check{r}})}^{\Upsilon(t_{\check{r}})}, \quad \sqrt{1-\prod_{\check{r}=1}^{p}\left(1-\eta_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon(t_{\check{r}})}}\right)$$

In the upcoming theorem, we will demonstrate that the aggregated value of PFNs at different time periods, using the PFDOWG operator, results in a new PFN.

Theorem 5. Consider p numbers of PFNs expressed as $\tilde{\alpha}_{t_{ř}} = (\mu_{t_{ř}}, \eta_{t_{ř}})$ at p different time periods $t_{ř}$ where, $ř = 1,2,3,\ldots,p$, and the associated weight vector is $\Upsilon_t = [\Upsilon_{t_1}, \Upsilon_{t_2}, \ldots, \Upsilon_{t_p}]^T$ with some restrictions $\sum_{ř=1}^{p} \Upsilon_{t_{ř}} = 1$ and $\Upsilon_{t_{ř}} \in [0,1]$. The aggregated value of these PFNs using PFDOWG operator is also a PFN. It can be described as follows:

$$PFDOWG(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \ldots, \tilde{\alpha}_{t_p}) = \left[\prod_{ř=1}^{p} \mu_{\varrho(t_{ř})}^{\Upsilon_{t_{ř}}}, \sqrt{1 - \prod_{ř=1}^{p} \left(1 - \eta_{\varrho(t_{ř})}^2\right)^{\Upsilon_{t_{ř}}}}\right]$$

Example 2. Consider $\tilde{\alpha}_{t_1} = (0.70,0.40)$, $\tilde{\alpha}_{t_2} = (0.80,0.30)$, $\tilde{\alpha}_{t_3} = (0.50,0.70)$ and $\tilde{\alpha}_{t_4} = (0.50,0.40)$ represent four PFNs and $\Upsilon_t = (0.2,0.1,0.3,0.4)^T$ is the weight vector linked to time periods $t_{ř}$ where $ř = 1,2,3,4$. First we compute the scores values of $\tilde{\alpha}_{t_{ř}}$ by applying definition 7 as follows:

$$S(\tilde{\alpha}_{t_1}) = 0.33, \quad S(\tilde{\alpha}_{t_2}) = 0.55,$$

$$S(\tilde{\alpha}_{t_3}) = -0.24, \; S(\tilde{\alpha}_{t_4}) = 0.09$$

Then the permuted values of PFNs are given as:

$$\tilde{\alpha}_{\varrho(t_1)} = (0.80,0.30), \tilde{\alpha}_{\varrho(t_2)} = (0.70,0.40), \tilde{\alpha}_{\varrho(t_3)} = (0.50,0.40), \tilde{\alpha}_{\varrho(t_4)} = (0.50,0.70)$$

In view of definition 13, the aggregated values of above PFNs in the context of PFDOWG operator is calculated as follows:

$$PFDOWG(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \tilde{\alpha}_{t_3}, \tilde{\alpha}_{t_4}) = \left(\prod_{ř=1}^{4} \mu_{\varrho(t_{ř})}^{\Upsilon(t_{ř})}, \quad \sqrt{1 - \prod_{ř=1}^{4} \left(1 - \eta_{\varrho(t_{ř})}^2\right)^{\Upsilon(t_{ř})}}\right)$$

$$PFDOWG(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \tilde{\alpha}_{t_3}, \tilde{\alpha}_{t_4})$$

$$= ((0.8)^{0.2}(0.7)^{0.1}(0.5)^{0.3}(0.5)^{0.4}), \sqrt{1 - (1 - 0.3^2)^{0.2}(1 - 0.4^2)^{0.1}(1 - 0.4^2)^{0.3}(1 - 0.7^2)^{0.4}}$$

It follows that

$$PFDOWG(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \tilde{\alpha}_{t_3}, \tilde{\alpha}_{t_4}) = (0.568,0.548).$$

In the upcoming theorem, we establish the idempotency property of the PFDOWG operator, showing that if all PFNs are equal for a given time period, the aggregated result will also equal that PFN.

Theorem 6. Consider the collection of p number of PFNs represented as $\tilde{\alpha}_{(t_{ř})} = \left(\mu_{(t_{ř})}, \eta_{(t_{ř})}\right)$, existing at p different time periods $t_{ř}$, and let $\Upsilon_t = \left(\Upsilon_{t_1}, \Upsilon_{t_2}, \dots, \Upsilon_{t_p}\right)^T$ be the weight vector linked to time periods $t_{ř}$ with the condition $\sum_{ř=1}^{p} \Upsilon_{t_{ř}} = 1$ and $\Upsilon_{t_{ř}} \in [0,1]$, where $ř = 1,2,\dots,p$. If $\tilde{\alpha}_{\varrho(t_{ř})} = \tilde{\alpha}_{\varrho(t_j)}$ are mathematically equal for all $ř$ and for some $j \in \{1,2,\dots,p\}$ where, $\tilde{\alpha}_{\varrho(t_j)} = \left(\mu_{\varrho(t_j)}, \eta_{\varrho(t_j)}\right)$. Then

$$PFDOWG\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) = \tilde{\alpha}_{\varrho(t_j)}$$

In the upcoming theorem, we establish the boundedness property of the PFDOWG operator, demonstrating that the aggregated value of PFNs lies between the defined lower and upper bounds.

Theorem 7. Consider $\tilde{\alpha}^{-} = \left(\min_{ř}\left(\mu_{\varrho(t_{ř})}\right), \max_{ř}\left(\eta_{\varrho(t_{ř})}\right)\right)$ and $\tilde{\alpha}^{+} = \left(\max_{ř}\left(\mu_{\varrho(t_{ř})}\right), \min_{ř}\left(\eta_{\varrho(t_{ř})}\right)\right)$ be the lower bound and upper bound of PFNs $\tilde{\alpha}_{\varrho(t_{ř})} = \left(\mu_{\varrho(t_{ř})}, \eta_{\varrho(t_{ř})}\right)$ where, $ř = 1,2,\dots,p$. Let $\Upsilon_t = \left(\Upsilon_{t_1}, \Upsilon_{t_2}, \dots, \Upsilon_{t_p}\right)^T$ be the weight vector associated with time periods $t_{ř}$ such that $\Upsilon_{t_{ř}} \in [0,1]$ meeting the restriction $\sum_{ř=1}^{p} \Upsilon_{t_{ř}} = 1$. Then

$$\tilde{\alpha}^{-} \leq PFDOWG\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) \leq \tilde{\alpha}^{+}.$$

In the upcoming theorem, we establish the monotonicity property of the PFDOWG operator, showing that if one set of PFNs dominates another, their aggregated values will preserve this ordering.

Theorem 8. Let $\tilde{\alpha}_{t_{ř}} = \left(\mu_{t_{ř}}, \eta_{t_{ř}}\right)$ and $\tilde{\alpha}'_{t_{ř}} = \left(\mu'_{t_{ř}}, \eta'_{t_{ř}}\right)$ where, $ř = 1,2,\dots,p$, be any two collection of PFNs. Let $\Upsilon_t = \left(\Upsilon_{t_1}, \Upsilon_{t_2}, \dots, \Upsilon_{t_p}\right)^T$ represents the weight vector linked to time periods $t_{ř}$, such that $\Upsilon_{t_{ř}} \in [0,1]$ and $\sum_{ř=1}^{p} \Upsilon_{ř} = 1$. If $\mu_{\varrho(t_{ř})} \leq \mu'_{\varrho(t_{ř})}$ and $\eta_{\varrho(t_{ř})} \geq \eta'_{\varrho(t_{ř})}$ then we can establish that:

$$PFDOWG\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \dots, \tilde{\alpha}_{t_p}\right) \leq PFDOWG\left(\tilde{\alpha}'_{t_1}, \tilde{\alpha}'_{t_2}, \dots, \tilde{\alpha}'_{t_p}\right).$$

4. Application of the suggested aggregation operators for MADM problems

In this section, we present a new approach for solving MADM issues. This method utilizes PF dynamic information and applies the recently defined PF dynamic ordered weighted AOs.

1) Let us consider a finite set of alternatives represented as $\bar{\Upsilon} = \{\bar{\Upsilon}_1, \bar{\Upsilon}_2, \dots, \bar{\Upsilon}_m\}$.
2) Let a collection of attributes be indicated as $\mathfrak{Q} = \{\mathfrak{Q}_1, \mathfrak{Q}_2, \dots, \mathfrak{Q}_n\}$ and the related weight vector is $\varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_n)^T$ such that $\varsigma_j \in [0,1]$ and $\sum_{j=1}^{n} \varsigma_j = 1$ where $j = 1,2, \dots, n$.
3) In addition, we incorporate the notion of distinct time periods $t_{\check{r}}$ where, $\check{r} = 1,2, \dots, p$, and these time periods are characterized by an associated weight vector $\Upsilon_t = \left[\Upsilon_{t_1}, \Upsilon_{t_2}, \dots, \Upsilon_{t_p}\right]^T$, where $\Upsilon_{t_{\check{r}}} \in [0,1]$ and $\sum_{\check{r}=1}^{p} \Upsilon_{t_{\check{r}}} = 1$.
4) Consider $R_{t_{\check{r}}} = \left[\varepsilon_{ij}(t_{\check{r}})\right]_{m\times n} = \left(\mu_{ij(t_{\check{r}})}, \eta_{ij(t_{\check{r}})}\right)_{m\times n}$ indicates the PF dynamic decision matrices of time periods $t_{\check{r}}$, where $\mu_{ij(t_{\check{r}})}$ represents the ratings to which the alternative $\bar{\Upsilon}_i$ fulfills the attribute $\mathfrak{Q}_j$ at time period $t_{\check{r}}$, and $\eta_{ij(t_{\check{r}})}$ reflects the degree that the alternative $\bar{\Upsilon}_i$ fails to satisfy with the attribute $\mathfrak{Q}_j$ at time period $t_{\check{r}}$. In addition, the following restrictions apply to these values:

$$\mu_{ij\varrho(t_{\check{r}})} \in [0,1],\ \eta_{ij\varrho(t_{\check{r}})} \in [0,1] \text{ and } \left(\mu_{ij\varrho(t_{\check{r}})}\right)^2 + \left(\eta_{ij\varrho(t_{\check{r}})}\right)^2 \le 1.$$

Based on the aforementioned decision information, we formulate a methodology for assessing and selecting the most favorable alternative. The process involves the following steps:

4.1. Procedure for PFDOWA operator

Step 1. Obtain the PF dynamic decision matrices $R_{t_{\check{r}}} = \left[\varepsilon_{ij(t_{\check{r}})}\right]_{m\times n} = \left(\mu_{ij(t_{\check{r}})}, \eta_{ij(t_{\check{r}})}\right)_{m\times n}$ that indicate various alternatives associated with attributes.

Step 2. To acquire the permuted PF dynamic decision matrices $R_{\varrho(t_{\check{r}})} = \left[\varepsilon_{ij\varrho(t_{\check{r}})}\right]_{m\times n} = \left(\mu_{ij\varrho(t_{\check{r}})}, \eta_{ij\varrho(t_{\check{r}})}\right)_{m\times n}$ we implement the subsequent two stages:

- Obtain the score values for each attribute $\mathfrak{Q}_j$ relating to each alternative $\bar{\Upsilon}_i$ within each matrix $R_{t_{\check{r}}}$ during the time period $t_{\check{r}}$ using definition 7.
- Formulate the permuted PF dynamic decision matrices by organizing the calculated values from the preceding step for each attribute $\mathfrak{Q}_j$, related to each alternative $\bar{\Upsilon}_i$ within each matrix $R_{t_{\check{r}}}$, in a descending order during the time period $t_{\check{r}}$.

Step 3. Utilize the recently developed PFDOWA operator in the following manner:

$$PFDOWA\left(\varepsilon_{ij\varrho(t_1)}, \varepsilon_{ij\varrho(t_2)}, \dots, \varepsilon_{ij\varrho(t_p)}\right) = \varepsilon_{ij\varrho(t_{\check{r}})} = \left(\mu_{ij\varrho(t_{\check{r}})}, \eta_{ij\varrho(t_{\check{r}})}\right)$$

It implies that

$$PFDOWA\left(\mathcal{E}_{ij\varrho(t_1)}, \mathcal{E}_{ij\varrho(t_2)}, \dots, \mathcal{E}_{ij\varrho(t_p)}\right) = \left(\sqrt{1 - \prod_{\check{r}=1}^{p}\left(1 - \mu^2_{ij\varrho(t_{\check{r}})}\right)^{\Upsilon_{t_{\check{r}}}}}, \prod_{\check{r}=1}^{p} \eta_{ij\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}}\right)$$

Here, i varies from 1 to m and and j varies from 1 to n. This technique integrates all the PF decision matrices to a unified PF decision matrix as $R = \left[\mathcal{E}_{ij}\right]_{m\times n} = \left(\mu_{ij}, \eta_{ij}\right)_{m\times n}$.

Step 4. Determine the combined value $\mathcal{E}_i = (\mu_i, \eta_i)$ of the alternatives $\bar{Y}_i$ by applying the PFWA operator to the collective decision matrix as follows:

$$PFWA(\mathcal{E}_{i1}, \mathcal{E}_{i2}, \dots, \mathcal{E}_{in}) = \mathcal{E}_i = (\mu_i, \eta_i)$$

It follows that:

$$PFWA(\mathcal{E}_{i1}, \mathcal{E}_{i2}, \dots, \mathcal{E}_{in}) = \left(\sqrt{1 - \prod_{j=1}^{n}\left(1 - \mu^2_{ij}\right)^{\varsigma_j}}, \prod_{j=1}^{n} \eta_{ij}^{\varsigma_j}\right)$$

Step 5. Applying definition 7, compute the scores $S(\mathcal{E}_i)$ of the PF preference values $\mathcal{E}_i$ for all the alternatives $\bar{Y}_i$.

Step 6. Obtain the score values $S(\mathcal{E}_i)$, write all $\bar{Y}_i$ in decreasing sequence and select the best one.

Figure 1 demonstrates the flowchart of developed approach for addressing MADM issues through the PFDOWA operator.

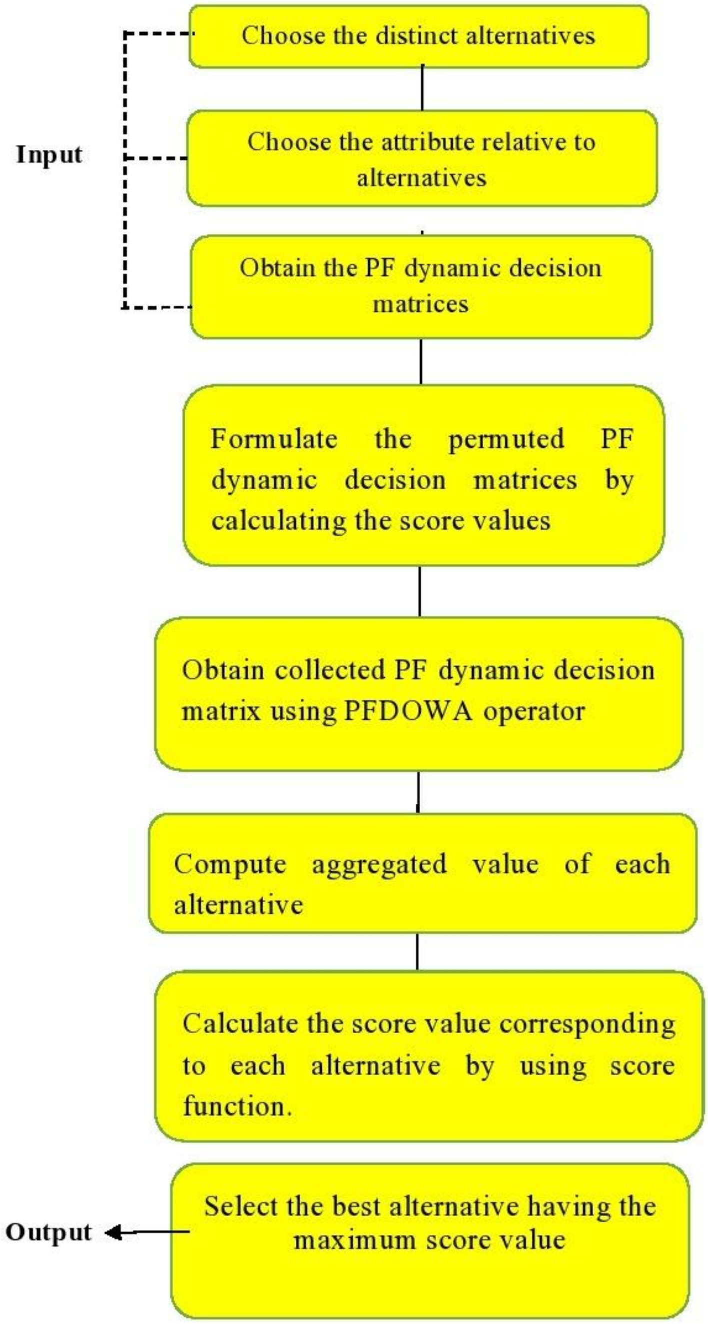

Figure 1. Algorithmic workflow for addressing MADM issues through the PFDOWA operator.

4.2. Procedure for PFDOWG operator

Step 1. Obtain the PF dynamic decision matrices $R_{t_ř} = [\varepsilon_{ij(t_ř)}]_{m\times n} = (\mu_{ij(t_ř)}, \eta_{ij(t_ř)})_{m\times n}$ that indicate various alternatives associated with attributes.

Step 2. To acquire the permuted PF dynamic decision matrices $R_{\varrho(t_{ř})} = \left[\mathcal{E}_{ij\varrho(t_{ř})}\right]_{m\times n} = \left(\mu_{ij\varrho(t_{ř})}, \eta_{ij\varrho(t_{ř})}\right)_{m\times n}$ we implement the subsequent two stages:

- Acquire the score values for each attribute Ω_j related to each alternative $\bar{Y}_i$ within each matrix $R_{t_{ř}}$ during the time period $t_{ř}$ using definition 7.
- Formulate the permuted PF dynamic decision matrices by arranging the calculated values from the preceding step for each attribute Ω_j, related to each alternative $\bar{Y}_i$ within each matrix $R_{t_{ř}}$, in a descending order during the time period $t_{ř}$.

Step 3. Utilize the recently developed PFDOWG operator in the following manner:

$$PFDOWG\left(\mathcal{E}_{ij\varrho(t_1)}, \mathcal{E}_{ij\varrho(t_2)}, \dots, \mathcal{E}_{ij\varrho(t_p)}\right) = \mathcal{E}_{ij\varrho(t_{ř})} = \left(\mu_{ij\varrho(t_{ř})}, \eta_{ij\varrho(t_{ř})}\right)$$

It implies that

$$PFDOWG\left(\mathcal{E}_{ij\varrho(t_1)}, \mathcal{E}_{ij\varrho(t_2)}, \dots, \mathcal{E}_{ij\varrho(t_p)}\right) = \left(\prod_{ř=1}^{p}\left(\mu_{ij\varrho(t_{ř})}\right)^{\Upsilon_{t_{ř}}}, \sqrt{1-\prod_{ř=1}^{p}\left(1-\eta^2_{ij\varrho(t_{ř})}\right)^{\Upsilon_{t_{ř}}}}\right)$$

Here, i varies from 1 to m and and j varies from 1 to n. This technique integrates all the PF decision matrices to a unified PF decision matrix as $R = \left[\mathcal{E}_{ij}\right]_{m\times n} = \left(\mu_{ij}, \eta_{ij}\right)_{m\times n}$.

Step 4. Calculate the aggregated value $\mathcal{E}_i = (\mu_i, \eta_i)$ of the alternatives $\bar{Y}_i$ by applying PFWG operator on collective decision matrix as follows:

$$PFWG(\mathcal{E}_{i1}, \mathcal{E}_{i2}, \dots, \mathcal{E}_{in}) = \mathcal{E}_i = (\mu_i, \eta_i)$$

It follows that:

$$PFWG(\mathcal{E}_{i1}, \mathcal{E}_{i2}, \dots, \mathcal{E}_{in}) = \left(\prod_{ř=1}^{n}(\mu_{iř})^{\varsigma_{ř}}, \sqrt{1-\prod_{ř=1}^{n}\left(1-\eta^2_{iř}\right)^{\varsigma_{ř}}}\right).$$

Step 5. Applying definition 7, compute the scores $S(\mathcal{E}_i)$ of the PF preference values $\mathcal{E}_i$ for all the alternatives $\bar{Y}_i$.

Step 6. Obtain the score values $S(\mathcal{E}_i)$, write all $\bar{Y}_i$ in decreasing sequence and select the best one.

Figure 2 demonstrates the flowchart of developed approach for addressing MADM issues through the PFDOWG operator.

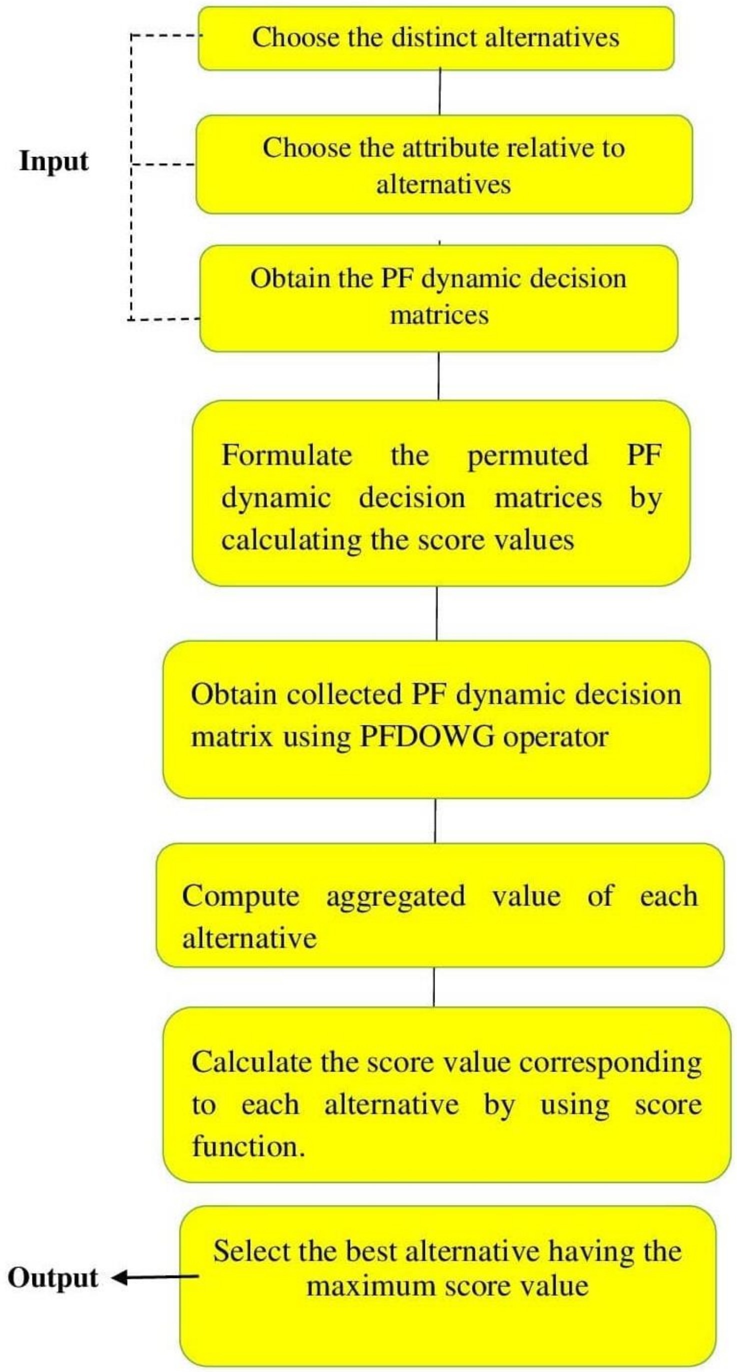

Figure 2. Algorithmic workflow for addressing MADM issues through the PFDOWG operator.

5. Utilizing the proposed aggregation operators in MADM problems

In this section of the article, we solve a MADM problem within the context of algorithm developed in section 4 and establish a comparative analysis to showcase the validity of this novel approach with the existing methodologies.

5.1. Case study

Illnesses affecting the heart and blood vessels are commonly known as cardiovascular illnesses. In the United States, abnormal saturated fat is present in over half of the population. There are two possibilities for managing CVD: Changing your lifestyle or getting medication from your doctor. CVD is easier to treat the earlier it is identified. In the US as well as throughout the world, CVD is the main cause of death. Heart disease is responsible for 655,000 deaths in America each year. In the United States, about 50% of people have some form of cardiovascular disease. Both men and women are impacted. Indeed, one in every three women dies from cardiovascular disease. People of different age groups, races, and socioeconomic backgrounds are affected by it. Tobacco, excessive drinking, insufficient nutrition, and a sedentary lifestyle are the main psychological risk factors linked to cardiovascular illness and death. Behavioral risk factors may result in the development of high cholesterol levels, high blood sugar levels, elevated blood fatty acids, weight gain, or obesity in adults. These intermediate risk variables suggest a heightened likelihood of experiencing a heart attack, bleeding, or cardiac arrest, along with related outcomes. In order to mitigate the risks linked to CVD, it is advisable to quit smoking, reduce salt intake, increase consumption of fruits and vegetables, engage in regular exercise, and avoid excessive alcohol use. To encourage the adoption and maintenance of healthy habits, it is critical to enact health policies that make healthy options more accessible and affordable. CVDs are influenced by additional underlying causes. Globalization, urbanization, and population aging are the major components that exert influence on financial, social, and cultural aspects. Genetic factors, anxiety, and poverty are additional risk factors for CVD. Treatment for hypertension, diabetes, and high blood cholesterol is essential to mitigate cardiovascular risk and prevent strokes and heart attacks in individuals with these conditions.

Studies have highlighted significant advancements in CVD detection and treatment. Innovative imaging techniques for liver and cardiovascular conditions have enhanced disease management [46], while histone modifications have emerged as potential therapeutic targets in calcific aortic valve disease [47]. Intravascular ultrasonography has also shown promise in treating carotid stenosis [48]. Research into hypoxia-induced signaling [49] and new approaches in electrocardiogram classifications [50] further underscore the need for comprehensive strategies to reduce CVD risks and improve outcomes.

Various methodologies exist for the management and prevention of CVD all over the world. A brief description of some important approaches to preventing CVD. These approaches include.

1) An array of healthy lifestyle choices can contribute to the enhancement of cardiovascular health and the mitigation of illness risk. These choices encompass refraining from smoking, sustaining an optimal weight by adhering to a balanced diet, and engaging in consistent physical activity. Consistent physical exercise aids in weight management while concurrently enhancing cardiovascular health. Many health advantages may be obtained from a diet abundant in fruits, vegetables, lean meats, and whole grains, which can also successfully decrease blood pressure. The cessation of smoking markedly diminishes the chance of developing cardiovascular disease.
2) Pharmacological interventions play a vital role in preventing and managing CVD risks, encompassing antihypertensive agents, statins, and antiplatelet medications. These techniques of therapy are widely accepted by physicians, who need strict attention to the suggested regimens to assure their effectiveness.

3) Primary safeguarding represents a proactive strategy that aims to prevent the development of cardiovascular disease in asymptomatic individuals. This all-encompassing approach entails the implementation of a healthy nutritional plan, consistent engagement in physical activity, and periodic screening for potential risk factors, such as high cholesterol levels and blood pressure.
4) Secondary mitigation stands as a pivotal aspect within the comprehensive cardiovascular disease approach. This requirement involves impeding the progression of CVD among individuals who are already suffering from the disease. Such endeavors necessitate the enactment of prudent lifestyle changes, the discerning application of pharmaceutical treatments, and vigilant oversight by healthcare professionals to observe progress and mitigate unexpected adverse reactions.
5) Within the domain of cardiovascular disease prevention and management, the meticulous management of risk factors is of the highest priority. Individuals can markedly diminish their vulnerability to CVD by efficiently managing situations such as diabetes, hypertension, and elevated cholesterol levels. The orchestration of this preventive regimen involves pharmaceutical adjustments, modifications to one's lifestyle, and vigilant medical supervision.
6) An integral facet of the multidimensional approach to combating CVD encompasses community-based initiatives. These endeavors encompass a diverse array of activities, comprising public health campaigns, support groups and educational programs, all aimed at enhancing public awareness of CVD and fostering healthier lifestyles

Based on data available in 2019, cardiovascular disease appeared as the predominant cause of death in Asia, accounting for approximately eleven million fatalities. Around 39 percent of these possible fatalities from CVD could have been prevented. The amount of premature deaths from CVD was greater than the number of premature deaths from CVD in the US (23%), Europe (22%), and worldwide (34%). Attributable to CVDs, the global mortality toll witnessed a substantial escalation, rising from approximately 12.1 million in 1990, evenly distributed between genders, to 18.6 million in 2019.

The exacerbation of patient treatment expenses and challenges within Pakistan’s pharmaceutical industry in recent decades can be attributed to an inequitable distribution of medical supplies, inefficiencies within the healthcare sector, and deficiencies in the health framework. Diseases are evolving towards increased severity, and the twenty-first century’s socioeconomic globalization has brought about significant and enduring enhancements in individuals’ living standards. The burgeoning discord between ecological considerations and human expansion is increasingly apparent. Numerous urban areas in Pakistan are grappling with severe climatic conditions, posing heightened challenges for the medical sector. The prevailing environmental issues are currently impeding the healthcare industry in Pakistan. Presently, the number of patients diagnosed with CVD is escalating at an accelerated rate. The inclination of individuals to seek medical counsel and aid at larger hospitals is notably elevated in comparison to the inferior care settings and treatment offerings found at smaller clinics. Lahore hospital, being the largest medical facility in Pakistan, is equipped with state-of-the-art medical equipment and ample resources. Given a considerable increase in workload over the past several decades, the Lahore hospital has been unable to adequately meet the increased demands. Within the framework of Lahore hospital, the establishment of a medical system with hierarchies is regarded as a viable strategy to alleviate the strain induced by the substantial influx of patients. The objective is to systematically categorize the degree of treatment complexity according to the type of disease. The attainment of medical qualifications from diverse institutions equips practitioners to adeptly address a diverse spectrum of ailments. A fundamental aspect of the hierarchical healthcare system is the classification of diseases into different severity categories. There is a hierarchical medical care system in place, and instead of sending all patients to a Grade III or Class A institution, people with different medical conditions can choose from numerous tiers of institutions. Determining diverse levels of illness severity is an essential initial step in building the hierarchical framework. In order to help build the medical system's hierarchical structure, we aim to classify the different stages of CVD.

5.2. Illustrated example

The crucial step is to carefully choose a group of experts who possess a deep comprehension of the problem's importance. Considering that this article mainly entails medical therapy, we have primarily focused on selecting physicians who have experience in this subject. The consultants have been assigned the responsibility of managing CVD by employing a range of different methods and key features in the treatment process.

Let $\{\bar{\Upsilon}_1, \bar{\Upsilon}_2, \bar{\Upsilon}_3, \bar{\Upsilon}_4, \bar{\Upsilon}_5\}$ represent the set of alternatives for the treatment of CVD;

i. $\bar{\Upsilon}_1$: Angioplasty;
ii. $\bar{\Upsilon}_2$: Lifestyle modifications;
iii. $\bar{\Upsilon}_3$: Blood pressure management;
iv. $\bar{\Upsilon}_4$: Medication;
v. $\bar{\Upsilon}_5$: Coronary Artery Bypass Grafting (ABG);

Let $\{\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \mathfrak{Q}_4\}$ represent the set of attributes, each of which contributes to the medical care of CVD.

i. $\mathfrak{Q}_1$: Efficiency;
ii. $\mathfrak{Q}_2$: Reliability;
iii. $\mathfrak{Q}_3$: Expertise required;
iv. $\mathfrak{Q}_4$: Sensitivity;

A medical team wants to analyze the five possible alternative $\bar{\Upsilon}_i$ values, where i ranges from 1 to 5, utilizing PF information. In accordance with the aforementioned four attributes $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$ and $\mathfrak{Q}_4$ at the time periods t_1, t_2, and t_3. Here t_1 designates the period 1990–1999, t_2 is pans from 2000 to 2009, and t_3 portrays the progression from 2010 to 2019. The medical team has assigned weight values to the three time periods, represented by the weight vector $\Upsilon_t = [0.2,0.3,0.5]^T$, where $\sum_{\check{r}=1}^{3} \Upsilon_{t_{\check{r}}} = 1$ and the weight vector of attributes is $\varsigma = [0.15,0.16,0.5,0.19]^T$ where, $\sum_{j=1}^{4} \varsigma_j = 1$. The medical expert opinion to legitimate the alternative $\bar{\Upsilon}_i$ with respect to the time period $t_{\check{r}}$ is summarized in the f PF dynamic decision matrices R_{t_1}, R_{t_2} and R_{t_3} (See Tables 2–4).

Table 2. PF dynamic decision matrix R_{t_1}.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{\Upsilon}_1$	(0.8,0.3)	(0.4,0.2)	(0.9,0.2)	(0.5,0.6)
$\bar{\Upsilon}_2$	(0.6,0.5)	(0.7,0.3)	(0.4,0.6)	(0.5,0.4)
$\bar{\Upsilon}_3$	(0.5,0.5)	(0.8,0.3)	(0.6,0.3)	(0.5,0.6)
$\bar{\Upsilon}_4$	(0.9,0.2)	(0.6,0.5)	(0.7,0.3)	(0.4,0.5)
$\bar{\Upsilon}_5$	(0.3,0.7)	(0.5,0.4)	(0.8,0.2)	(0.6,0.5)

Table 3. PF dynamic decision matrix R_{t_2}.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{\text{Y}}_1$	(0.6,0.3)	(0.5,0.3)	(0.8,0.3)	(0.7,0.2)
$\bar{\text{Y}}_2$	(0.7,0.3)	(0.8,0.2)	(0.5,0.4)	(0.6,0.5)
$\bar{\text{Y}}_3$	(0.9,0.2)	(0.6,0.5)	(0.7,0.3)	(0.4,0.5)
$\bar{\text{Y}}_4$	(0.5,0.5)	(0.8,0.3)	(0.5,0.6)	(0.6,0.3)
$\bar{\text{Y}}_5$	(0.4,0.5)	(0.5,0.3)	(0.6,0.3)	(0.7,0.4)

Table 4. PF dynamic decision matrix R_{t_3}.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{\text{Y}}_1$	(0.6,0.3)	(0.5,0.2)	(0.7,0.4)	(0.3,0.4)
$\bar{\text{Y}}_2$	(0.4,0.5)	(0.6,0.5)	(0.7,0.3)	(0.9,0.2)
$\bar{\text{Y}}_3$	(0.6,0.5)	(0.3,0.7)	(0.5,0.4)	(0.8,0.2)
$\bar{\text{Y}}_4$	(0.7,0.4)	(0.5,0.3)	(0.4,0.5)	(0.6,0.3)
$\bar{\text{Y}}_5$	(0.7,0.4)	(0.7,0.3)	(0.6,0.5)	(0.4,0.8)

Within the context of the PFDOWA and PFDOWG operators, the given MADM problem is solved as follows:

5.3. Procedure I (PFDOWA operator)

Step 1. The permuted PF dynamic decision matrix $R_{\varrho(t_1)} = \left[\mathcal{E}_{ij\varrho(t_1)}\right]_{5\times4} = \left(\mu_{ij\varrho(t_1)}, \eta_{ij\varrho(t_1)}\right)_{5\times4}$ is computed as follows:

i. Apply definition 7 to ascertain the score values of all four attributes of each alternative of decision matrix R_{t_1} at the time period t_1.
   - For alternative $\bar{\text{Y}}_1$, we have

$$S(\mathcal{E}_{11}) = 0.55,\ \ S(\mathcal{E}_{12}) = 0.12,$$

$$S(\mathcal{E}_{13}) = 0.77,\ \ S(\mathcal{E}_{14}) = -0.11$$

   - For alternative $\bar{\text{Y}}_2$, we have

$$S(\mathcal{E}_{21}) = 0.11,\ \ S(\mathcal{E}_{22}) = 0.40,$$

$$S(\mathcal{E}_{23}) = -0.2,\ S(\mathcal{E}_{24}) = 0.09$$

   - For alternative $\bar{\text{Y}}_3$, we have

$$S(\mathcal{E}_{31}) = 0, S(\mathcal{E}_{32}) = 0.55,$$

$$S(\mathcal{E}_{33}) = 0.27,\ \ S(\mathcal{E}_{34}) = -0.11$$

   - For alternative $\bar{\text{Y}}_4$, we have

$$S(\mathcal{E}_{41}) = 0.77,\ \ S(\mathcal{E}_{42}) = 0.11,$$

$$S(\mathcal{E}_{43}) = 0.40,\ \ S(\mathcal{E}_{44}) = -0.09$$

- For alternative $\bar{Y}_5$, we have

$$S(\mathcal{E}_{51}) = -0.40, \quad S(\mathcal{E}_{52}) = 0.09,$$

$$S(\mathcal{E}_{53}) = 0.60,\ \ S(\mathcal{E}_{54}) = 0.11$$

ii. List the values obtained in the previous stage for each option in in decreasing sequence, displayed below:

- For alternative $\bar{Y}_1$;

$$S(\mathcal{E}_{13}) > S(\mathcal{E}_{11}) > S(\mathcal{E}_{12}) > S(\mathcal{E}_{14})$$

- For alternative $\bar{Y}_2$;

$$S(\mathcal{E}_{22}) > S(\mathcal{E}_{21}) > S(\mathcal{E}_{24}) > S(\mathcal{E}_{23})$$

- For alternative $\bar{Y}_3$;

$$S(\mathcal{E}_{32}) > S(\mathcal{E}_{33}) > S(\mathcal{E}_{31}) > S(\mathcal{E}_{34})$$

- For alternative $\bar{Y}_4$;

$$S(\mathcal{E}_{41}) > S(\mathcal{E}_{43}) > S(\mathcal{E}_{42}) > S(\mathcal{E}_{44})$$

- For alternative $\bar{Y}_5$;

$$S(\mathcal{E}_{53}) > S(\mathcal{E}_{54}) > S(\mathcal{E}_{52}) > S(\mathcal{E}_{51})$$

Moreover, the permuted PF decision matrix $R_{\varrho(t_2)} = \left[\mathcal{E}_{ij\varrho(t_2)}\right]_{5\times4} = \left(\mu_{ij\varrho(t_2)}, \eta_{ij\varrho(t_2)}\right)_{5\times4}$ is computed as follows:

i. Apply definition 7 to ascertain the score values of all four attributes of each alternative of decision matrix R_{t_2} at the time period t_2.

- For alternative $\bar{Y}_1$, we have

$$S(\mathcal{E}_{11}) = 0.27,\ \ S(\mathcal{E}_{12}) = 0.16,$$

$$S(\mathcal{E}_{13}) = 0.55,\ \ S(\mathcal{E}_{14}) = 0.45$$

- For alternative $\bar{Y}_2$, we have

$$S(\mathcal{E}_{21}) = 0.40,\ \ S(\mathcal{E}_{22}) = 0.60,$$

$$S(\mathcal{E}_{23}) = 0.09,\ \ S(\mathcal{E}_{24}) = 0.11$$

- For alternative $\bar{Y}_3$, we have

$$S(\mathcal{E}_{31}) = 0.77,\ \ S(\mathcal{E}_{32}) = 0.11,$$

$$S(\mathcal{E}_{33}) = 0.40,\ \ S(\mathcal{E}_{34}) = -0.09$$

- For alternative $\bar{Y}_4$, we have

$$S(\mathcal{E}_{41}) = 0, S(\mathcal{E}_{42}) = 0.55,$$

$$S(\mathcal{E}_{43}) = -0.11, \quad S(\bar{Y}_{44}) = 0.27$$

- For alternative $\bar{Y}_5$, we have

$$S(\mathcal{E}_{51}) = -0.09, \quad S(\mathcal{E}_{52}) = 0.16,$$
$$S(\mathcal{E}_{53}) = 0.27, \quad S(\mathcal{E}_{54}) = 0.33$$

ii. List the values obtained in the previous stage for each option in in decreasing sequence, displayed below:

- For alternative $\bar{Y}_1$;

$$S(\mathcal{E}_{13}) > S(\mathcal{E}_{14}) > S(\mathcal{E}_{11}) > S(\mathcal{E}_{12})$$

- For alternative $\bar{Y}_2$;

$$S(\mathcal{E}_{22}) > S(\mathcal{E}_{21}) > S(\mathcal{E}_{24}) > S(\mathcal{E}_{23})$$

- For alternative $\bar{Y}_3$;

$$S(\mathcal{E}_{31}) > S(\mathcal{E}_{33}) > S(\mathcal{E}_{32}) > S(\mathcal{E}_{34})$$

- For alternative $\bar{Y}_4$;

$$S(\mathcal{E}_{42}) > S(\mathcal{E}_{44}) > S(\mathcal{E}_{41}) > S(\mathcal{E}_{43})$$

- For alternative $\bar{Y}_5$;

$$S(\mathcal{E}_{54}) > S(\mathcal{E}_{53}) > S(\mathcal{E}_{52}) > S(\mathcal{E}_{51})$$

Furthermore, the permuted PF decision matrix $R_{\varrho(t_3)} = \left[\mathcal{E}_{ij\varrho(t_3)}\right]_{5\times 4} = \left(\mu_{ij\varrho(t_3)}, \eta_{ij\varrho(t_3)}\right)_{5\times 4}$ is computed as follows:

i. Apply definition 7 to ascertain the score values of all four attributes of each alternative of decision matrix R_{t_3} at the time period t_3.

- For alternative $\bar{Y}_1$, we have

$$S(\mathcal{E}_{11}) = 0.27, \quad S(\mathcal{E}_{12}) = 0.21,$$
$$S(\mathcal{E}_{13}) = 0.33, \quad S(\mathcal{E}_{14}) = -0.07$$

- For alternative $\bar{Y}_2$, we have

$$S(\mathcal{E}_{21}) = -0.09, \quad S(\mathcal{E}_{22}) = 0.11,$$
$$S(\mathcal{E}_{23}) = 0.40, \quad S(\mathcal{E}_{24}) = 0.77$$

- For alternative $\bar{Y}_3$, we have

$$S(\mathcal{E}_{31}) = 0.11, \quad S(\mathcal{E}_{32}) = -0.40,$$
$$S(\mathcal{E}_{33}) = 0.09, \quad S(\mathcal{E}_{34}) = 0.60$$

- For alternative $\bar{Y}_4$, we have

$$S(\mathcal{E}_{41}) = 0.33, \quad S(\mathcal{E}_{42}) = 0.16,$$
$$S(\mathcal{E}_{43}) = -0.09, \quad S(\mathcal{E}_{44}) = 0.27$$

- For alternative $\bar{Y}_5$, we have

$$S(\mathcal{E}_{51}) = 0.33,\ \ S(\mathcal{E}_{52}) = 0.40,$$

$$S(\mathcal{E}_{53}) = 0.11,\ \ S(\mathcal{E}_{54}) = 0.48$$

ii. List the values obtained in the previous stage for each option in in decreasing sequence, displayed below:

- For alternative $\bar{Y}_1$;

$$S(\mathcal{E}_{13}) > S(\mathcal{E}_{11}) > S(\mathcal{E}_{12}) > S(\mathcal{E}_{14})$$

- For alternative $\bar{Y}_2$;

$$S(\mathcal{E}_{24}) > S(\mathcal{E}_{23}) > S(\mathcal{E}_{22}) > S(\mathcal{E}_{21})$$

- For alternative $\bar{Y}_3$;

$$S(\mathcal{E}_{34}) > S(\mathcal{E}_{31}) > S(\mathcal{E}_{33}) > S(\mathcal{E}_{32})$$

- For alternative $\bar{Y}_4$;

$$S(\mathcal{E}_{41}) > S(\mathcal{E}_{44}) > S(\mathcal{E}_{42}) > S(\mathcal{E}_{43})$$

- For alternative $\bar{Y}_5$;

$$S(\mathcal{E}_{54}) > S(\mathcal{E}_{52}) > S(\mathcal{E}_{51}) > S(\mathcal{E}_{53})$$

Step 2. Formulate the permuted PF decision matrices $R_{\varrho(t_1)}$, $R_{\varrho(t_2)}$, and $R_{\varrho(t_2)}$ (See Tables 5–7) using the data acquired from step 1.

Table 5. Permuted PF dynamic decision matrix $R_{\varrho(t_1)}$.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{Y}_1$	(0.9,0.2)	(0.8,0.3)	(0.4,0.2)	(0.5,0.6)
$\bar{Y}_2$	(0.7,0.3)	(0.6,0.5)	(0.5,0.4)	(0.4,0.6)
$\bar{Y}_3$	(0.8,0.3)	(0.6,0.3)	(0.5,0.5)	(0.5,0.6)
$\bar{Y}_4$	(0.9,0.2)	(0.7,0.3)	(0.6,0.5)	(0.4,0.5)
$\bar{Y}_5$	(0.8,0.2)	(0.6,0.5)	(0.5,0.4)	(0.3,0.7)

Table 6. Permuted PF dynamic decision matrix $R_{\varrho(t_2)}$.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{Y}_1$	(0.8,0.3)	(0.7,0.2)	(0.6,0.3)	(0.5,0.3)
$\bar{Y}_2$	(0.8,0.2)	(0.7,0.3)	(0.6,0.5)	(0.5,0.4)
$\bar{Y}_3$	(0.9,0.2)	(0.7,0.3)	(0.6,0.5)	(0.4,0.5)
$\bar{Y}_4$	(0.8,0.3)	(0.6,0.3)	(0.5,0.5)	(0.5,0.6)
$\bar{Y}_5$	(0.7,0.4)	(0.6,0.3)	(0.5,0.3)	(0.4,0.5)

Table 7. Permuted PF dynamic decision matrix $R_{\varrho(t_3)}$.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{Y}_1$	(0.7,0.4)	(0.6,0.3)	(0.5,0.2)	(0.3,0.5)
$\bar{Y}_2$	(0.9,0.2)	(0.7,0.3)	(0.6,0.5)	(0.4,0.5)
$\bar{Y}_3$	(0.8,0.2)	(0.6,0.5)	(0.5,0.4)	(0.3,0.7)
$\bar{Y}_4$	(0.7,0.4)	(0.6,0.3)	(0.5,0.3)	(0.4,0.5)
$\bar{Y}_5$	(0.7,0.3)	(0.7,0.4)	(0.6,0.5)	(0.4,0.8)

Step 3. We aggregate all of the PF permuted decision matrices R_{t_1}, $R_{t_2}$and R_{t_3} using the PFDOWA operator to obtain a collective PF decision matrix R, as shown in Table 8.

Table 8. Collective PF decision matrix R using PFDOWA operator.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{Y}_1$	(0.78,0.32)	(0.68,0.27)	(0.52,0.23)	(0.42,0.45)
$\bar{Y}_2$	(0.85,0.22)	(0.68,0.33)	(0.58,0.48)	(0.43,0.48)
$\bar{Y}_3$	(0.84,0.22)	(0.63,0.39)	(0.53,0.45)	(0.38,0.61)
$\bar{Y}_4$	(0.79,0.32)	(0.62,0.3)	(0.52,0.39)	(0.43,0.53)
$\bar{Y}_5$	(0.72,0.30)	(0.65,0.38)	(0.55,0.41)	(0.42,0.68)

Step 4. Compute the aggregated value $\mathcal{E}_i = (\mu_i, \eta_i)$ of the alternatives $\bar{Y}_i$ where, $i = 1,2,3,4,5$ by employing the PFWA operator on Table 8. The outcomes of this procedure are listed in Table 9.

Table 9. Results of alternatives aggregation using the PFWA operator.

	$\mathcal{E}_i$
$\bar{Y}_1$	(0.596,0.278)
$\bar{Y}_2$	(0.644,0.402)
$\bar{Y}_3$	(0.607,0.419)
$\bar{Y}_4$	(0.586,0.385)
$\bar{Y}_5$	(0.582,0.426)

Step 5. To compute the scores $S(\mathcal{E}_i)$ of the entire PF preference values $\mathcal{E}_i$ for each alternative $\bar{Y}_i$, utilize definition 7.

$$S(\mathcal{E}_1) = (0.277)$$

$$S(\mathcal{E}_2) = (0.253)$$

$$S(\mathcal{E}_3) = (0.193)$$

$$S(\mathcal{E}_4) = (0.195)$$

$$S(\mathcal{E}_5) = (0.157)$$

Step 6. In the light of the score values $S(\mathcal{E}_i)$ obtained from the previous step the alternatives $\bar{Y}_i$ are to be ranked as follows: $\bar{Y}_1 > \bar{Y}_2 > \bar{Y}_4 > \bar{Y}_3 > \bar{Y}_5$.

Step 7. The aforementioned discussion concludes that "Angioplasty" is the efficient strategy to cure the CVD.

Figure 3 shows the hierarchical ordering of alternatives as determined by the PFDOWA operator.

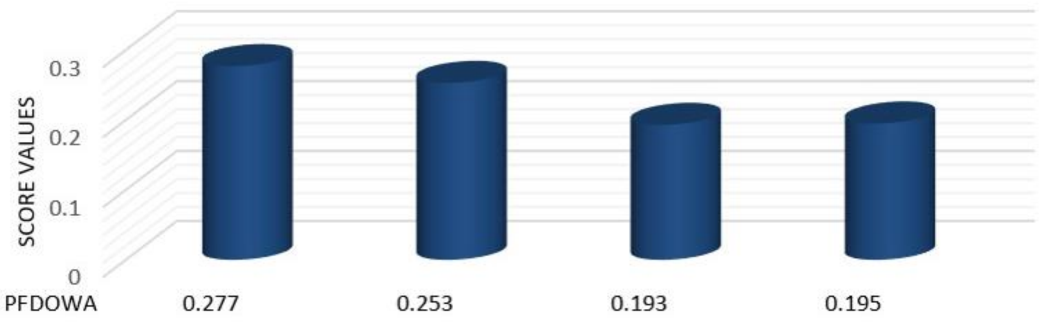

Figure 3. Hierarchical ordering of alternatives determined by the PFDOWA operator.

5.4. Procedure II (PFDOWG operator)

Similarly, the PFDOWG operator resolves the previously mentioned MADM problem employing the subsequent method:

Step 1. To get a collective PF decision matrix R (see Table 10), we aggregate all the PF permuted decision matrices $R_{t_{\check{r}}}$ employing the PFDOWG operator.

Table 10. Collective decision matrix by applying PFDOWG operator.

	$\mathfrak{Q}_1$	$\mathfrak{Q}_2$	$\mathfrak{Q}_3$	$\mathfrak{Q}_4$
$\bar{Y}_1$	(0.77,0.34)	(0.67,0.27)	(0.51,0.24)	(0.39,0.47)
$\bar{Y}_2$	(0.83,0.22)	(0.68,0.35)	(0.58,0.48)	(0.43,0.50)
$\bar{Y}_3$	(0.83,0.22)	(0.63,0.42)	(0.53,0.45)	(0.36,0.63)
$\bar{Y}_4$	(0.77,0.34)	(0.62,0.3)	(0.52,0.42)	(0.43,0.53)
$\bar{Y}_5$	(0.72,0.32)	(0.64,0.40)	(0.55,0.43)	(0.38,0.72)

Step 2. Compute the aggregated value $\mathcal{E}_i = (\mu_i, \eta_i)$ of the alternatives $\bar{Y}_i$ where, $i = 1,2,3,4,5$ by employing the PFWG operator on Table 10. The Table 11 presents the computed values.

Table 11. Aggregated values of alternatives under PFWG operator.

	$\mathcal{E}_i$
$\bar{Y}_1$	(0.534,0.318)
$\bar{Y}_2$	(0.593,0.439)
$\bar{Y}_3$	(0.541,0.468)
$\bar{Y}_4$	(0.547,0.419)
$\bar{Y}_5$	(0.547,0.498)

Step 3. Utilizing definition 7 to calculate the scores $S(\mathcal{E}_i)$ of the overall PF preference values $\mathcal{E}_i$ for each of the alternatives $\bar{Y}_i$.

$$S(\mathcal{E}_1) = (0.184)$$

$$S(\mathcal{E}_2) = (0.159)$$

$$S(\mathcal{E}_3) = (0.074)$$

$$S(\mathcal{E}_4) = (0.124)$$

$$S(\mathcal{E}_5) = (0.051)$$

Step 4. In view of the score values $S(\mathcal{E}_i)$ the alternatives $\bar{Y}_i$ are to be ranked in descending order as follows: $\bar{Y}_1 > \bar{Y}_2 > \bar{Y}_4 > \bar{Y}_3 > \bar{Y}_5$ and the optimal alternative is to be chosen.
Step 5. The aforementioned discussion concludes that "Angioplasty" is the best strategy to cure the CVD.

Figure 4 shows the hierarchical ordering of alternatives as determined by the PFDOWG operator.

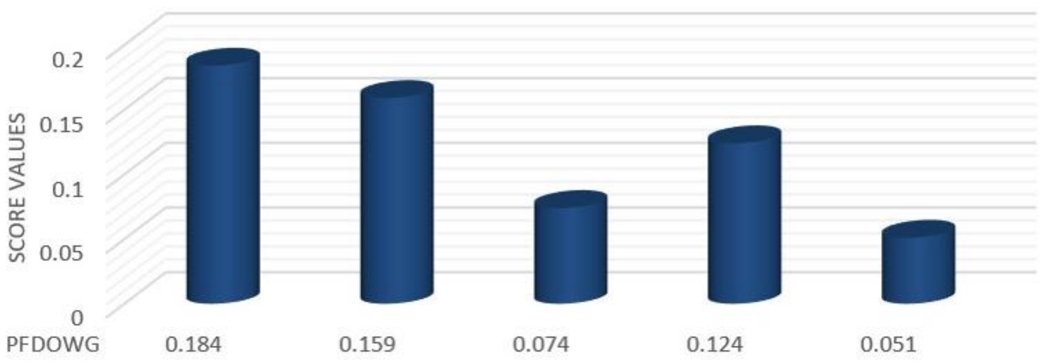

Figure 4. Hierarchical ordering of alternatives determined by the PFDOWG operator.

5.5. Comparative evaluation

In the subsequent section, we compare our suggested operators to those developed in [22,38,39,45,51] to evaluate their effectiveness and reliability for the MADM issue. We utilize many approaches, such as the IFDWA, IFDWG, PFDOWA, and PFDOWG operators, to gather and consolidate similar data. The results produced by these operators are compiled in Table 12, and the outcomes provided according to their ranking are presented in Table 13.

Table 12. Aggregated outcomes of alternatives across various existing operators.

	IFDWA [38]	IFDWG [39]	PFDOWA	PFDOWG
$\mathcal{E}(\bar{Y}_1)$	(0.688,0.407)	(0.618,0.326)	(0.596,0.278)	(0.534,0.318)
$\mathcal{E}(\bar{Y}_2)$	(0.654,0.378)	(0.597,0.395)	(0.644,0.402)	(0.593,0.439)
$\mathcal{E}(\bar{Y}_3)$	(0.592,0.396)	(0.581,0.464)	(0.607,0.419)	(0.541,0.468)
$\mathcal{E}(\bar{Y}_4)$	(0.577,0.405)	(0.535,0.434)	(0.586,0.385)	(0.547,0.419)
$\mathcal{E}(\bar{Y}_5)$	(0.615,0.403)	(0.550,0.422)	(0.582,0.426)	(0.547,0.498)

Table 13. Scoring data and rankings of alternatives for current and newly suggested techniques.

	$S(\mathcal{E}_1)$	$S(\mathcal{E}_2)$	$S(\mathcal{E}_3)$	$S(\mathcal{E}_4)$	$S(\mathcal{E}_5)$	Ranking order
IFDWA	0.281	0.276	0.196	0.172	0.212	$\bar{Y}_1 > \bar{Y}_2 > \bar{Y}_5 > \bar{Y}_3 > \bar{Y}_4$
IFDWG	0.292	0.202	0.117	0.101	0.128	$\bar{Y}_1 > \bar{Y}_2 > \bar{Y}_5 > \bar{Y}_3 > \bar{Y}_4$
PFDOWA	0.277	0.253	0.193	0.195	0.157	$\bar{Y}_1 > \bar{Y}_2 > \bar{Y}_4 > \bar{Y}_3 > \bar{Y}_5$
PFDOWG	0.184	0.159	0.074	0.124	0.051	$\bar{Y}_1 > \bar{Y}_2 > \bar{Y}_4 > \bar{Y}_3 > \bar{Y}_5$

Comparison 1. The suggested techniques are genuine and suitable for MADM challenges. The primary reasoning for the superiority of our suggested methods lies in the fact that these models exhibit a more comprehensive framework, which serves as an expansion of the techniques developed in [22]. In addition, the methods developed in [22] are inadequate for the analysis of the data in Tables 2–4 because they cannot handle the problems related to decision-making that are reliant on time, whereas the models proposed in this study can handle data from many time intervals, making them more adaptable.

Comparison 2. The incorporation of the techniques presented in this study within the context of the dynamic PF environment enhances their efficacy. This is due to the fact that they assess many time periods and provide a more precise assessment of the data under consideration. The approaches delineated in [38,39] can be considered a specific case of the innovative methods introduced in our current investigation because they have a more restricted area of application compared to the methods presented in this study within the framework of PF knowledge. Thus, the newly proposed approaches provide a wider array of comprehensive alternatives for identifying and mitigating ambiguity than the methods outlined in [38,39].

Comparison 3. A significant quantity of data is lost because the aggregation techniques explained in [45,51] do not include time intervals, whereas the proposed aggregation techniques are dynamic because they can handle data from many time intervals, making them more adaptable.

The above discussion highlights that the mathematical frameworks of the proposed operators utilizing PF knowledge with time intervals are highly effective and substantial. This establishes the novel techniques' efficiency over other strategies.

6. Conclusions

We seek to unveil innovative strategies for addressing complex decision-making challenges through the application of a dynamic PF framework. Although the literature has already established various useful operators, none of these specifically take into account the issue of time duration in PF settings. Consequently, utilizing a dynamic PF model proves to be a more efficient method for elucidating information associated with time-dependent considerations, as it adeptly handles two-dimensional data within an integrated framework. In light of these factors, we have implemented a new collection of operators, specifically PFDOWA and PFDOWG, within the framework of the PF environment. We have examined several features of these operators. Furthermore, we have suggested step-by-step mathematical mechanisms to address dynamic PF MADM issues using innovative strategies. Moreover, we have demonstrated the practical application of these newly devised methodologies in discerning the optimal strategy for treating cardiovascular disease in a patient. Finally, we have compared these novel approaches with existing methods to show their reliability and importance.

6.1. Limitations of present research

Despite the advantages offered by the methodologies proposed in this article, several limitations remain:

i. These techniques are incapable of dealing with situations in which the sum of the squares of the membership and non-membership values surpasses 1.
ii. Due to their limitation to accepting just two parameters, these techniques are incapable of managing model cases that involve picture and spherical fuzzy information.

6.2. Potential future research directions of the current study

The principal objective of forthcoming research is to address the constraints recognized in the present study by formulating methodologies applicable to generalized settings, specifically within the contexts of interval-valued Pythagorean fuzzy, complex Pythagorean fuzzy, bipolar fuzzy, spherical fuzzy, and picture fuzzy frameworks. Another goal of future research will be to create a comprehensive decision-analysis tool that employs PF dynamic aggregation operators in order to maximize their significance and efficiency. The strategies suggested in this article will be flexible and applicable to a range of situations, such as the creation of more flexible financial plans, real-time social media activity monitoring online, dynamic military management assessment, dynamic and private shortlisting procedures, addressing the lack of energy in developing countries, and addressing time-dependent MADM issues.

Author contributions

Mehwish Shehzadi: Original draft preparation, Formal analysis, Investigation; Hanan Alolaiyan: Writing and review, Validation of results, Investigation; Umer Shuaib: Conceptualization, Supervision, Methodology; Abdul Razaq: Investigation, Writing and review, Formal analysis; Qin Xin: Investigation, Writing and review, Methodology.

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Conflict of interest

All the authors declare no conflict of interest.

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Appendix

Proof (Theorem 5). The technique of mathematical induction is used to establish the proof of this theorem.

For $p = 2$. We have

$$PFDOWG\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}\right) = \tilde{\alpha}_{\varrho(t_1)}{}^{\Upsilon_{t_1}} \otimes \tilde{\alpha}_{\varrho(t_2)}{}^{\Upsilon_{t_2}}$$

Utilizing definition 12, we get

$$\tilde{\alpha}_{\varrho(t_1)}{}^{\Upsilon_{t_1}} = \left[\mu_{\varrho(t_1)}^{\Upsilon_{t_1}}, \sqrt{1-\left(1-\eta_{\varrho(t_1)}^2\right)^{\Upsilon_{t_1}}}\right]$$

$$\tilde{\alpha}_{\varrho(t_2)}{}^{\Upsilon_{t_2}} = \left[\mu_{\varrho(t_2)}^{\Upsilon_{t_2}}, \sqrt{1-\left(1-\eta_{\varrho(t_2)}^2\right)^{\Upsilon_{t_2}}}\right]$$

Thus,

$$\begin{aligned}
&\tilde{\alpha}_{\varrho(t_1)}{}^{\Upsilon_{t_1}} \otimes \tilde{\alpha}_{\varrho(t_2)}{}^{\Upsilon_{t_2}} \\
&= \left[\mu_{\varrho(t_1)}^{\Upsilon_{t_1}}, \sqrt{1-\left(1-\eta_{\varrho(t_1)}^2\right)^{\Upsilon_{t_1}}}\right] \otimes \left[\mu_{\varrho(t_2)}^{\Upsilon_{t_2}}, \sqrt{1-\left(1-\eta_{\varrho(t_2)}^2\right)^{\Upsilon_{t_2}}}\right] \\
&= \left[\mu_{\varrho(t_1)}^{\Upsilon_{t_1}}\mu_{\varrho(t_2)}^{\Upsilon_{t_2}}, \sqrt{1-\left(1-\eta_{\varrho(t_1)}^2\right)^{\Upsilon_{t_1}}\left(\left(1-\eta_{\varrho(t_2)}^2\right)^{\Upsilon_{t_2}}\right)}\right]
\end{aligned}$$

In consequence,

$$PFDOWG\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}\right) = \left[\prod_{\check{r}=1}^{2} \mu_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}}, \sqrt{1-\prod_{\check{r}=1}^{2}\left(1-\eta_{\varrho(t_{\check{r}})}^2\right)^{\Upsilon_{t_{\check{r}}}}}\right]$$

Hence, the result is true for $p = 2$.

Furthermore, we assume that the given result is true for $= n > 2$, then we have:

$$PFDOWG\left(\tilde{\alpha}_{t_1}, \tilde{\alpha}_{t_2}, \ldots, \tilde{\alpha}_{t_n}\right) = \otimes_{\check{r}=1}^{n} \tilde{\alpha}_{\varrho(t_{\check{r}})}{}^{\Upsilon_{t_{\check{r}}}}$$

$$\left[\prod_{\check{r}=1}^{n} \mu_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}}, \sqrt{1-\prod_{\check{r}=1}^{n}\left(1-\eta_{\varrho(t_{\check{r}})}^2\right)^{\Upsilon_{t_{\check{r}}}}}\right]$$

Now we prove the result $p = n + 1$. It can be expressed as

$$PFDOWG(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\ldots,\tilde{\alpha}_{t_n},\tilde{\alpha}_{t_{n+1}}) = \otimes_{\check{r}=1}^{n}\tilde{\alpha}_{\varrho(t_{\check{r}})}{}^{\Upsilon_{t_{\check{r}}}}\otimes\tilde{\alpha}_{\varrho(t_{n+1})}{}^{\Upsilon_{t_{n+1}}}$$

$$=\left[\prod_{\check{r}=1}^{n}\mu_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}},\sqrt{1-\prod_{\check{r}=1}^{n}\left(1-\eta_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}}}\right]\otimes\left[\left(\mu_{\varrho(t_{n+1})}\right)^{\Upsilon_{t_{n+1}}},\sqrt{1-\left(1-\eta_{\varrho(t_{n+1})}^{2}\right)^{\Upsilon_{t_{n+1}}}}\right]$$

It follows that

$$PFDOWG(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\ldots,\tilde{\alpha}_{t_n},\tilde{\alpha}_{t_{n+1}}) = \left[\prod_{\check{r}=1}^{n+1}\mu_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}},\sqrt{1-\prod_{\check{r}=1}^{n+1}\left(1-\eta_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}}},\right]$$

Hence, the result holds for each $p \in \mathbb{Z}^{+}$.

Proof (Theorem 6). Given that $\tilde{\alpha}_{\varrho(t_{\check{r}})} = \tilde{\alpha}_{\varrho(t_j)}$ for all $\check{r} = 1,2,\ldots,p$ and for some $j \in \{1,2,\ldots,p\}$ which implies that $\mu_{\varrho(t_{\check{r}})} = \mu_{\varrho(t_j)}$ and $\eta_{\varrho(t_{\check{r}})} = \eta_{\varrho(t_j)}$.

$$PFDOWG\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\ldots,\tilde{\alpha}_{t_p}\right) = \left(\prod_{\check{r}=1}^{p}\mu_{\varrho(t_{\check{r}})}^{\Upsilon_{t_{\check{r}}}},\sqrt{1-\prod_{\check{r}=1}^{p}\left(1-\eta_{\varrho(t_{\check{r}})}^{2}\right)^{\Upsilon_{t_{\check{r}}}}}\right)$$

$$=\left(\mu_{\varrho(t_j)}^{\sum_{\check{r}=1}^{p}\Upsilon_{t_{\check{r}}}},\sqrt{1-\left(1-\eta_{\varrho(t_j)}^{2}\right)^{\sum_{\check{r}=1}^{p}\Upsilon_{t_{\check{r}}}}}\right)$$

$$=\left(\mu_{\varrho(t_j)},\sqrt{1-\left(1-\eta_{\varrho(t_j)}^{2}\right)}\right)=\left(\mu_{\varrho(t_j)},\sqrt{\eta_{\varrho(t_j)}^{2}}\right)=\left(\mu_{\varrho(t_j)},\eta_{\varrho(t_j)}\right)$$

Consequently,

$$PFDOWG\left(\tilde{\alpha}_{t_1},\tilde{\alpha}_{t_2},\ldots,\tilde{\alpha}_{t_p}\right) = \tilde{\alpha}_{\varrho(t_j)}$$

The proofs of Theorems 7 and 8 can be derived by employing analogous reasoning to that used in the proofs of Theorems 3 and 4, respectively.