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Research article

A novel two-grid Crank-Nicolson mixed finite element method for nonlinear fourth-order sin-Gordon equation

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Abstract: A new nonlinear fourth-order sin-Gordon equation with actual physical background is first created. Then, by introducing an auxiliary function, the nonlinear fourth-order sin-Gordon equation is decomposed into the nonlinear system of equations of second-order derivatives of spatial variables. Subsequently, the time derivative is discretized by using the Crank-Nicolson (CN) scheme to construct a new time semi-discretized mixed CN (TSDMCN) scheme. Thereafter, the spatial variables in the TSDMCN scheme are discretized by using a two-grid mixed finite element (MFE) method to construct a new two-grid CN MFE (TGCNMFE) method with unconditional stability and second-order time precision, which consists of a system of nonlinear MFE equations defined on coarser grids and a system of linear MFE equations defined on finer grids with sufficiently high precision, and is very easy to solve. The existence, stability, and error estimates of the TSDMCN and TGCNMFE solutions are strictly proved theoretically, and the superiorities of the TGCNMFE method and the correctness of theoretical results are verified by two sets of numerical experiments.

Keywords: nonlinear fourth-order sin-Gordon equation; time semi-discretize mixed Crank-Nicolson scheme; two-grid Crank-Nicolson mixed finite element method; existence; stability; error estimate
Mathematics Subject Classification: 65M15, 65N12, 65N35

1. Introduction

Let $\Omega \subset R^d$ $(d = 2, 3)$ be a connected bounded domain with the boundary $\partial\Omega$. We propose a new nonlinear fourth-order sin-Gordon (NFOSG) equation, which is different from the standard sin-Gordon equation in [1, 2] and is defined as follows:

Problem 1. *Find* $\varpi : [0, t_e] \to C^4(\bar{\Omega})$*, satisfying*

$$\begin{cases} \varpi_{tt}(\boldsymbol{x}, t) - \Delta\varpi(\boldsymbol{x}, t) + \Delta^2\varpi(\boldsymbol{x}, t) + \sin(\varpi(\boldsymbol{x}, t)) = 0, & (\boldsymbol{x}, t) \in \Omega \times (0, t_e), \\ \varpi(\boldsymbol{x}, t) = \Delta\varpi(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \partial\Omega \times (0, t_e), \\ \varpi(\boldsymbol{x}, 0) = \varpi_0(\boldsymbol{x}), \quad \varpi_t(\boldsymbol{x}, 0) = \varpi_1(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

where t_e *is a specified time upper bound,*

$$\varpi_{tt}(\boldsymbol{x}, t) = \partial^2\varpi(\boldsymbol{x}, t)/\partial t^2, \quad \boldsymbol{x} = (x_1, x_2, \cdots, x_d), \quad \Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$$

is the Laplace operator, both $\varpi_0(\boldsymbol{x})$ *and* $\varpi_1(\boldsymbol{x})$ *are sufficiently smooth known initial functions, and*

$$\varpi_t(\boldsymbol{x}, t) = \partial\varpi(\boldsymbol{x}, t)/\partial t.$$

The NFOSG equation is obtained by adding a fourth-order term $\Delta^2\varpi(\boldsymbol{x}, t)$ to the standard sin-Gordon equation. It is well known (see, e.g., [3–5]) that the fourth-order term $\Delta^2\varpi(\boldsymbol{x}, t)$ can describe the deformation of an object under the action of an external force such as seismic waves and flying aircraft, while the nonlinear term $\sin(\omega(\boldsymbol{x}, t))$ can describe the nonlinear behavior of a moving body/wave. Therefore, the NFOSG equation can not only describe the fluid mechanics of porous media [6–8], groundwater dynamics [9], seepage mechanics and groundwater hydraulics [10–12], etc., like the standard sin-Gordon equation, but also describe the nonlinear deformation behavior of moving bodies/waves, such as the movement of seismic waves or the movement of flying aircraft. Therefore, the NFOSG equation is an important mathematical physics model with a real application background and has a wider range of applications than the standard sin-Gordon equation.

However, as a result of the NFOSG equation including a strong nonlinear term $\sin(\omega(\boldsymbol{x}, t))$, it is very difficult to solve analytically. A numerical method is the most effective choice to solve the NFOSG equation. Fortunately, the numerical methods of the NFOSG equation play a pivotal role in simulation of the earthquake wave, the motion of flying aircraft, as well as the fluid mechanics of porous media, groundwater dynamics, seepage mechanics, groundwater hydraulics, and other phenomena. Therefore, it is very important to study the numerical methods of the NFOSG equation.

A large number of numerical examples have showed that the two-grid finite element (FE) algorithm is one of the best numerical methods for solving nonlinear partial differential equations (PDEs), which consists of a nonlinear FE system of equations defined on coarser grids and a linear FE system of equations defined on finer grids with sufficiently high precision. Hence, it can simplify computation and enhance calculation efficiency. It was originally used to solve quasi-linear elliptic equations [13]. More recently, Shi's and Liu's teams have used it to solve some of the more complex nonlinear PDEs (see [14–16]).

However, to our knowledge, at the moment, there has been no report on the Crank-Nicolson (CN) mixed FE (MFE) (CNMFE) format for the NFOSG equation reduced with two-grid FE technique. Therefore, the main task of this paper is to develop a new two-grid CNMFE (TGCNMFE) method for the NFOSG equation (i.e., Problem 1). The TGCNMFE method has at least three benefits. First, by introducing an auxiliary function $\varphi = \Delta\varpi$, the NFOSG equation is split into two second-order equations, which can be easily solved by lower degree FEs such as linear or quadratic FEs. Second, the

TGCNMFE method has unconditional stability and second-order time accuracy. Third, the TGCNMFE method also consists of a nonlinear MFE equation defined on coarser grids and a linear MFE equation defined on finer grids, which greatly simplifies the calculation and improves the calculation efficiency.

Although some single-grid FE methods for the standard sin-Gordon equation with only second-order derivatives of spatial variables have been proposed in [10–12], they are completely different from the TGCNMFE method for the NFOSG equation with fourth-order derivatives of spatial variables. Both the TGCNMFE method and the NFOSG equation are far more complex than those in [10–12]. Therefore, both the establishment of the TGCNMFE method and the theoretical analysis of existence, stability, and errors of the TGCNMFE solutions herein are more difficult and require more skills than the single-grid FE methods in [10–12], but the NFOSG equation has a wider range of applications than the standard sin-Gordon equation, as mentioned above. Hence, it is very valuable to study the TGCNMFE method for solving the NFOSG equation.

The rest of this paper consists of the following four sections. In Section 2, by introducing an auxiliary function $\varphi = \Delta\varpi$, we split the NFOSG equation into two second-order equations, and by using the time CN scheme to discretize them, we design a new time semi-discretization mixed CN (TSDMCN) scheme and discuss the existence, stability, and error estimates of the TSDMCN solutions. In Section 3, by using the two-grid MFE method to discretize the TSDMCN scheme, we construct a new TGCNMFE method for the NFOSG equation and analyze the existence, stability, and errors of the TGCNMFE solutions. In Section 4, we use two sets of numerical experiments to confirm the correctness of the obtained theoretical results and show the superiorities of the TGCNMFE method. Finally, we give the main conclusions of this article and future study prospects in Section 5.

2. A new time semi-discretization CN scheme

The Sobolev spaces and norms used in this context are classical (see [5, 17, 18]). Let

$$\mathbb{W} = H_0^1(\Omega) \quad \text{and} \quad \varphi = -\Delta\varpi.$$

Thus, by Green's formula, we can derive the following mixed variational format of Problem 1.

Problem 2. *For any $t \in (0,\ t_e)$, find $(\varpi, \varphi) \in \mathbb{W} \times \mathbb{W}$, meeting*

$$\begin{cases} (\varpi_{tt}, \upsilon) + (\nabla\varphi, \nabla\upsilon) + (\varphi, \upsilon) = (f(\varpi), \upsilon), & \forall\upsilon \in \mathbb{W}, \\ (\nabla\varpi, \nabla\vartheta) = (\varphi, \vartheta), & \forall\vartheta \in \mathbb{W}, \\ \varpi(\boldsymbol{x}, 0) = \varpi_0(\boldsymbol{x}),\ \ \varpi_t(\boldsymbol{x}, 0) = \varpi_1(\boldsymbol{x}),\ \varphi(\boldsymbol{x}, 0) = -\Delta\varpi_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \end{cases} \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the L^2 inner product, and

$$f(\varpi) = -\sin(\varpi).$$

Using the proof method of Theorem 3.3.1 in [5] or the proof method of the following Theorem 1, we can demonstrate the existence and stability of the generalized solutions for Problem 2.

In order to establish the TGCNMFE method, we first set up a new TSDMCN scheme. Therefore, we assume that $K > 0$ is an integer,

$$\Delta t = t_e/K$$

is the time step, and ϖ^k and φ^k are the approximations to $\varpi(\boldsymbol{x}, t)$ and $\varphi(\boldsymbol{x}, t)$ at

$$t_k = k\Delta t \ (0 \leqslant k \leqslant K),$$

respectively.

Using an implicit scheme of time to discretize the first equation in Problem 2 yields

$$\frac{1}{\Delta t^2}(\varpi^{k+1} - 2\varpi^k + \varpi^{k-1}, \upsilon) + (\nabla\varphi^{k+1}, \nabla\upsilon) + (\varphi^{k+1}, \upsilon) = (f(\varpi^{k+1}), \upsilon), \quad \forall \upsilon \in \mathbb{W}. \tag{2.2}$$

Using an explicit scheme of time to discretize the first equation in Problem 2 yields

$$\frac{1}{\Delta t^2}(\varpi^{k+1} - 2\varpi^k + \varpi^{k-1}, \upsilon) + (\nabla\varphi^{k-1}, \nabla\upsilon) + (\varphi^{k-1}, \upsilon) = (f(\varpi^{k-1}), \upsilon), \quad \forall \upsilon \in \mathbb{W}. \tag{2.3}$$

By adding formulas (2.2) and (2.3), we obtain the following brand-new TSDMCN format, which is different from the existing time semi-discrete formats, including that in [1].

Problem 3. *Find* $\{\varpi^k, \varphi^k\} \in \mathbb{W} \times \mathbb{W}$ $(1 \leqslant k \leqslant K)$*, meeting*

$$\begin{cases} \dfrac{1}{\Delta t^2}(\varpi^{k+1} - 2\varpi^k + \varpi^{k-1}, \upsilon) + \dfrac{1}{2}(\nabla(\varphi^{k+1} + \varphi^{k-1}), \nabla\upsilon) + \dfrac{1}{2}(\varphi^{k+1} + \varphi^{k-1}, \upsilon) \\ = \dfrac{1}{2}(f(\varpi^{k+1}) + f(\varpi^{k-1}), \upsilon), \quad \forall \upsilon \in \mathbb{W}, \quad 1 \leqslant k \leqslant K-1, \\ (\nabla\varpi^k, \nabla\vartheta) = (\varphi^k, \vartheta), \quad \forall \vartheta \in \mathbb{W}, \quad 0 \leqslant k \leqslant K, \\ \varpi^0 = \varpi_0, \quad \varpi^1 = \varpi^0 + \Delta t\varpi_1, \quad \varphi^0 = -\Delta\varpi_0, \quad \varphi^1 = \varphi_0 - \Delta t\Delta\varpi_1, \quad in \ \Omega. \end{cases} \tag{2.4}$$

For Problem 3, we obtain the following results:

Theorem 1. *Problem 3 has a unique series of solutions*

$$\{\varpi^k, \varphi^k\}_{k=1}^K \subset \mathbb{W} \times \mathbb{W}$$

meeting the following boundness, i.e., stability:

$$\|\nabla\varpi^k\|_0 + \|\nabla\varphi^k\|_0 \leqslant c(\|\varpi_0\|_1 + \|\varpi_1\|_1), \qquad 1 \leqslant k \leqslant K. \tag{2.5}$$

The c used in this context is a generical positive constant independent of Δt. And, when $\varpi_0(\boldsymbol{x})$ and $\varpi_1(\boldsymbol{x})$ are adequately smooth, the series of solutions $\{\varpi^k, \varphi^k\}_{k=1}^K$ meets the error estimates

$$\|\nabla(\varpi(t_k) - \varpi^k)\|_0 + \|\nabla(\varphi(t_k) - \varphi^k)\|_0 \leqslant c\Delta t^2, \qquad 1 \leqslant k \leqslant K, \tag{2.6}$$

where

$$\varpi(t_k) = \varpi(\boldsymbol{x}, t_k) \quad and \quad \varphi(t_k) = \varphi(\boldsymbol{x}, t_k) \ (1 \leqslant k \leqslant K).$$

Proof. The demonstration of Theorem 1 consists of the following three parts.

(1) The existence and uniqueness of series of solutions of Problem 3.

Taking

$$\upsilon = \varpi^{k+1} - \varpi^{k-1}$$

in the first subsystem of equations of (2.4), by the second subsystem of equations of (2.4), the Hölder and Cauchy inequalities, and the Lagrange differential mean formula (LDMF), we obtain

$$\begin{aligned}
&\|\varpi^{k+1}-\varpi^{k}\|_0^2-\|\varpi^{k}-\varpi^{k-1}\|_0^2+\frac{\Delta t^2}{2}(\|\varphi^{k+1}\|_0^2-\|\varphi^{k-1}\|_0^2)+\frac{\Delta t^2}{2}(\|\nabla\varpi^{k+1}\|_0^2-\|\nabla\varpi^{k-1}\|_0^2)\\
&=\frac{\Delta t^2}{2}(f(\varpi^{k+1})+f(\varpi^{k-1}),\varpi^{k+1}-\varpi^{k-1})\\
&\leqslant c\Delta t^2(\|\varpi^{k+1}-\varpi^{k}\|_0^2+\|\varpi^{k}-\varpi^{k-1}\|_0^2),\quad 1\leqslant k\leqslant K-1.
\end{aligned}\tag{2.7}$$

By summing (2.7) from 1 to k ($k \leqslant K-1$) and using the third equation of (2.4), we obtain

$$\begin{aligned}
&\|\varpi^{k+1}-\varpi^{k}\|_0^2+\frac{\Delta t^2}{2}(\|\varphi^{k+1}\|_0^2+\|\varphi^{k}\|_0^2)+\frac{\Delta t^2}{2}(\|\nabla\varpi^{k+1}\|_0^2+\|\nabla\varpi^{k}\|_0^2)\\
&\leqslant\|\varpi^{1}-\varpi^{0}\|_0^2+\frac{\Delta t^2}{2}(\|\varphi^{1}\|_0^2+\|\varphi^{0}\|_0^2)+\frac{\Delta t^2}{2}(\|\nabla\varpi^{1}\|_0^2+\|\nabla\varpi^{0}\|_0^2)+c\Delta t^2\sum_{j=0}^{k}\|\varpi^{j+1}-\varpi^{j}\|_0^2\\
&\leqslant c\Delta t^2(\|\varpi_0\|_1^2+\|\varpi_1\|_1^2)+c\Delta t^2\sum_{j=0}^{k}\|\varpi^{j+1}-\varpi^{j}\|_0^2,\ 1\leqslant k\leqslant K-1.
\end{aligned}\tag{2.8}$$

When Δt is adequately small satisfying $c\Delta t^2 \leqslant 1/4$, by simplifying (2.8), we obtain

$$\begin{aligned}
&\|\varpi^{k+1}-\varpi^{k}\|_0^2+\Delta t^2\|\varphi^{k+1}\|_0^2+\Delta t^2\|\nabla\varpi^{k+1}\|_0^2\\
&\leqslant c\Delta t^2(\|\varpi_0\|_1^2+\|\varpi_1\|_1^2)+c\Delta t^2\sum_{j=0}^{k-1}\|\varpi^{j+1}-\varpi^{j}\|_0^2,\quad 1\leqslant k\leqslant K-1.
\end{aligned}\tag{2.9}$$

By applying Gronwall's inequality (see [5, Lemme 3.1.9]) to (2.9), we obtain

$$\begin{aligned}
&\|\varpi^{k+1}-\varpi^{k}\|_0^2+\Delta t^2\|\varphi^{k+1}\|_0^2+\Delta t^2\|\nabla\varpi^{k+1}\|_0^2\\
&\leqslant c\Delta t^2(\|\varpi_0\|_1^2+\|\varpi_1\|_1^2)\exp(ck\Delta t^2)\leqslant c\Delta t^2(\|\varpi_0\|_1^2+\|\varpi_1\|_1^2),\quad 1\leqslant k\leqslant K-1.
\end{aligned}\tag{2.10}$$

Thereupon, we obtain

$$\|\varphi^{k}\|_0+\|\nabla\varpi^{k}\|_0\leqslant c(\|\varpi_0\|_1+\|\varpi_1\|_1),\quad 1\leqslant k\leqslant K.\tag{2.11}$$

Further, by the first and second subsystems of equations of (2.4), the Hölder and Cauchy inequalities, and LDMF, we obtain

$$\begin{aligned}
&\|\nabla(\varpi^{k+1}-\varpi^{k})\|_0^2-\|\nabla(\varpi^{k}-\varpi^{k-1})\|_0^2+\frac{\Delta t^2}{2}(\|\nabla\varphi^{k+1}\|_0^2-\|\nabla\varphi^{k-1}\|_0^2)\\
&\quad+\frac{\Delta t^2}{2}(\|\varphi^{k+1}\|_0^2-\|\varphi^{k-1}\|_0^2)=(\varpi^{k+1}-2\varpi^{k}+\varpi^{k-1},\varphi^{k+1}-\varphi^{k-1})\\
&\quad+\frac{\Delta t^2}{2}(\nabla(\varphi^{k+1}+\varphi^{k-1}),\nabla(\varphi^{k+1}-\varphi^{k-1}))+\frac{\Delta t^2}{2}(\varphi^{k+1}+\varphi^{k-1},\varphi^{k+1}-\varphi^{k-1})\\
&=\frac{\Delta t^2}{2}(f(\varpi^{k+1})+f(\varpi^{k-1}),\varphi^{k+1}-\varphi^{k-1})\leqslant c\Delta t(\|\varpi^{k+1}-\varpi^{k}\|_0^2+\|\varpi^{k}-\varpi^{k-1}\|_0^2)
\end{aligned}$$

$$+ c\Delta t^3(\|\varphi^{k+1}\|_0^2 + \|\varphi^{k-1}\|_0^2), \quad 1 \leqslant k \leqslant K-1. \tag{2.12}$$

By summing (2.12) from 1 to k ($k \leqslant K-1$) and using the third equation of (2.4), we obtain

$$\begin{aligned}&\|\nabla(\varpi^{k+1} - \varpi^k)\|_0^2 + \frac{\Delta t^2}{2}(\|\nabla\varphi^{k+1}\|_0^2 + \|\nabla\varphi^k\|_0^2) + \frac{\Delta t^2}{2}(\|\varphi^{k+1}\|_0^2 + \|\varphi^k\|_0^2)\\ &\leqslant c\Delta t\sum_{j=0}^{k}\|\varpi^{j+1} - \varpi^k\|_0^2 + c\Delta t^3\sum_{j=0}^{k}\|\varphi^{j+1}\|_1^2 + c\Delta t^2(\|\varpi_0\|_1^2 + \|\varpi_1\|_1^2), \quad 1 \leqslant k \leqslant K-1.\end{aligned} \tag{2.13}$$

When Δt is adequately small satisfying $c\Delta t^2 \leqslant 1/4$, by simplifying (2.13), we obtain

$$\begin{aligned}&\|\nabla(\varpi^{k+1} - \varpi^k)\|_0^2 + \Delta t^2\|\nabla\varphi^{k+1}\|_0^2 + \Delta t^2\|\varphi^{k+1}\|_0^2\\ &\leqslant c\Delta t\sum_{j=0}^{k-1}\|\varpi^{j+1} - \varpi^k\|_0^2 + c\Delta t^3\sum_{j=0}^{k-1}\|\varphi^{j+1}\|_1^2 + c\Delta t^2(\|\varpi_0\|_1^2 + \|\varpi_1\|_1^2), \quad 1 \leqslant k \leqslant K-1.\end{aligned} \tag{2.14}$$

By applying Gronwall's inequality (see [5, Lemme 3.1.9]) to (2.14), we obtain

$$\|\nabla(\varpi^{k+1} - \varpi^k)\|_0^2 + \Delta t^2\|\nabla\varphi^{k+1}\|_0^2 + \Delta t^2\|\varphi^{k+1}\|_0^2 \leqslant c\Delta t^2(\|\varpi_0\|_1^2 + \|\varpi_1\|_1^2)\exp(ck\Delta t), \quad 1 \leqslant k \leqslant K-1. \tag{2.15}$$

Thereupon, we obtain

$$\|\nabla\varphi^{k+1}\|_0 + \|\varphi^{k+1}\|_0 \leqslant c(\|\varpi_0\|_1 + \|\varpi_1\|_1), \quad 1 \leqslant k \leqslant K-1. \tag{2.16}$$

Thus, when

$$\varpi_0 = \varpi_1 = 0,$$

by (2.11) and (2.16), we obtain

$$\varphi^k = \varpi^k = 0.$$

This signifies that Problem 3 has at least one series of solutions $\{\varpi^k, \varphi^k\}_{k=1}^N$.

If Problem 3 has another series of solutions $\{\tilde{\varpi}^k, \tilde{\varphi}^k\}_{k=1}^N$, it should satisfy the following system of equations:

$$\begin{cases}\frac{1}{\Delta t}(\tilde{\varpi}^{k+1} - 2\tilde{\varpi}^k + \tilde{\varpi}^{k-1}, \upsilon) + \frac{1}{2}(\nabla(\tilde{\varphi}^{k+1} + \tilde{\varphi}^{k-1}), \nabla\upsilon) + \frac{1}{2}(\tilde{\varphi}^{k+1} + \tilde{\varphi}^{k-1}, \upsilon)\\ = \frac{1}{2}(f(\tilde{\varpi}^{k+1}) + f(\tilde{\varpi}^{k-1}), \upsilon), \quad \forall\upsilon \in \mathbb{W}, \quad 1 \leqslant k \leqslant K-1,\\ \left(\nabla\tilde{\varpi}^k, \nabla\vartheta\right) = (\tilde{\varphi}^k, \vartheta), \quad \forall\vartheta \in \mathbb{W}, \quad 0 \leqslant k \leqslant K,\\ \tilde{\varpi}^0 = \varpi_0, \ \ \tilde{\varphi}^0 = -\Delta\varpi_0, \ \ \tilde{\varpi}^1 = \varpi_1, \ \ \tilde{\varphi}^1 = \varphi_0 - \Delta t\Delta\varpi_1, \ \text{ in } \Omega.\end{cases} \tag{2.17}$$

Let

$$E^k = \varpi^k - \tilde{\varpi}^k \quad \text{and} \quad e^k = \varphi^k - \tilde{\varphi}^k.$$

Subtracting (2.17) from (2.4) yields

$$\begin{cases} \dfrac{1}{\Delta t}\left(E^{k+1}-2E^{k}+E^{k-1},\upsilon\right)+\dfrac{1}{2}\left(\nabla(e^{k+1}+e^{k-1}),\nabla\upsilon\right)+\dfrac{1}{2}\left(e^{k+1}+e^{k-1},\upsilon\right) \\ =\dfrac{1}{2}\left(f(\varpi^{k+1})+f(\varpi^{k-1})-f(\tilde{\varpi}^{k+1})-f(\tilde{\varpi}^{k-1}),\upsilon\right),\ \ \forall\upsilon\in\mathbb{W}, 1\leqslant k\leqslant K-1, \\ \left(\nabla E^{k},\nabla\vartheta\right)=(e^{k},\vartheta),\quad \forall\vartheta\in\mathbb{W},\quad 0\leqslant k\leqslant K, \\ E^{0}=E^{1}=0,\ \ e^{0}=e^{1}=0,\quad \text{in }\ \Omega. \end{cases} \tag{2.18}$$

Taking

$$\upsilon=E^{k}-E^{k-1}$$

in the first equation of (2.18), by the second equation of (2.18), the Hölder and Cauchy inequalities, and the LDMF, we obtain

$$\begin{aligned} &2(\|E^{k+1}-E^{k}\|_0^2-\|E^{k}-E^{k-1}\|_0^2)+\Delta t^2(\|e^{k+1}\|_0^2-\|e^{k-1}\|_0^2)+\Delta t^2\left(\|\nabla E^{k+1}\|_0^2-\|\nabla E^{k-1}\|_0^2\right) \\ &=\Delta t^2\left(f(\varpi^{k+1})-f(\tilde{\varpi}^{k+1})+f(\varpi^{k-1})-f(\tilde{\varpi}^{k-1}),E^{k+1}-E^{k-1}\right) \\ &=\Delta t^2\left(E^{k+1}f'(\xi_k)+E^{k-1}f'(\xi_{k-1}),E^{k+1}-E^{k-1}\right) \\ &\leqslant c\Delta t^2(\|E^{k+1}-E^{k}\|_0^2+\|E^{k}-E^{k-1}\|_0^2),\quad 1\leqslant k\leqslant K-1, \end{aligned} \tag{2.19}$$

where ξ_i lies between ϖ^i and $\tilde{w}^i$ $(i=k,k-1)$.

Summing (2.19) from 1 to k $(k\leqslant K-1)$ and noting that

$$E^1=e^1=E^0=e^0=0,$$

we obtain

$$\begin{aligned} &2\|E^{k+1}-E^{k}\|_0^2+\Delta t^2(\|e^{k+1}\|_0^2+\|e^{k}\|_0^2)+\Delta t^2(\|\nabla E^{k+1}\|_0^2+\|\nabla E^{k}\|_0^2) \\ &\leqslant c\Delta t^2\sum_{i=0}^{k}\|E^{i+1}-E^{i}\|_0^2,\quad 1\leqslant k\leqslant K-1. \end{aligned} \tag{2.20}$$

When Δt is adequately small, satisfying $c\Delta t^2\leqslant 1$, by simplifying (2.20), we obtain

$$\|E^{k+1}-E^{k}\|_0^2+\Delta t^2\|e^{k+1}\|_0^2+\Delta t^2\|\nabla E^{k+1}\|_0^2\leqslant c\Delta t^2\sum_{i=0}^{k-1}\|E^{i+1}-E^{i}\|_0^2,\ \ 1\leqslant k\leqslant K-1. \tag{2.21}$$

By applying Gronwall's inequality to (2.21), we obtain

$$\|E^{k+1}-E^{k}\|_0^2+\Delta t^2\|e^{k+1}\|_0^2+\Delta t^2\|\nabla E^{k+1}\|_0^2\leqslant 0,\ \ 1\leqslant k\leqslant K-1. \tag{2.22}$$

Thereupon, we obtain

$$\tilde{\varpi}^k=\tilde{\varpi}^k\quad\text{and}\quad\varphi^k=\varphi^k\ (1\leqslant k\leqslant K).$$

Hence, Problem 3 has a unique series of solutions $\{\varpi^k,\varphi^k\}_{k=1}^K$.

(2) Discuss the boundness, i.e., stability of solutions $\{\varpi^k,\varphi^k\}_{k=1}^K$ of Problem 3.

When Problem 3 has a unique series of $\{\varpi^k, \varphi^k\}_{k=1}^K$, by (2.11) and (2.16), we claim that it is bounded, i.e., stable, namely (2.5) holds.

(3) Estimate the errors of solutions $\{\varpi^k, \varphi^k\}_{k=1}^K$ of Problem 3.

Via the Taylor expansion, we obtain

$$v(t_{k+1}) = v(t_k) + \Delta t v'(t_k) + \frac{\Delta t^2}{2} v''(t_k) + \frac{\Delta t^3}{6} v'''(t_k) + \frac{\Delta t^4}{24} v'''(t_k) + \cdots, \tag{2.23}$$

$$v(t_{k-1}) = v(t_k) - \Delta t v'(t_k) + \frac{\Delta t^2}{2} v''(t_k) - \frac{\Delta t^3}{6} v'''(t_k) + \frac{\Delta t^4}{24} v'''(t_k) + \cdots, \tag{2.24}$$

$$f(v(t_{k+1})) = f(v(t_k)) + [v(t_{k+1}) - v(t_k)] f'(v(t_k)) + \frac{1}{2}(v(t_{k+1}) - v(t_k))^2 f''(v(t_k)) + \cdots, \tag{2.25}$$

$$f(v(t_{k-1})) = f(v(t_k)) + [v(t_{k-1}) - v(t_k)] f'(v(t_k)) + \frac{1}{2}(v(t_{k-1}) - v(t_k))^2 f''(v(t_k)) + \cdots. \tag{2.26}$$

Therefore, we obtain

$$\varpi''(t_k) = \frac{\varpi(t_{k+1}) - 2\varpi(t_k) + \varpi(t_{k-1})}{\Delta t^2} + \frac{\Delta t^2}{24}\varpi'''(\eta_k), \quad t_{k-1} \leqslant \zeta_k \leqslant t_{k+1}, \tag{2.27}$$

$$\varphi(t_k) = \frac{\varphi(t_{k+1}) + \varphi(t_{k-1})}{2} - \Delta t^2 \varphi''(\varsigma_k), \quad t_{k-1} \leqslant \varsigma_k \leqslant t_{k+1}, \tag{2.28}$$

$$f(\varpi(t_k)) = \frac{f(\varpi(t_{k+1})) + f(\varpi(t_{k-1}))}{2} - \Delta t^2 R(\boldsymbol{x}, t), \tag{2.29}$$

where $R(\boldsymbol{x}, t)$ is a bounded remainder function, which is obtained by (2.23)–(2.26).

Thereupon, subtracting the first equation of (2.4) from the first equation of (2.1) after taking $t = t_k$ and setting

$$\varrho^k = \varpi(\boldsymbol{x}, t_k) - \varpi^k \quad \text{and} \quad \tilde{\varrho}^k = \varphi(\boldsymbol{x}, t_k) - \varphi^k,$$

we obtain the following system of error equations:

$$\begin{aligned}
&\frac{1}{\Delta t^2}\left(\varrho^{k+1} - 2\varrho^k + \varrho^{k-1}, \upsilon\right) + \frac{1}{2}\left(\nabla(\tilde{\varrho}^{k+1} + \tilde{\varrho}^{k-1}), \nabla\upsilon\right) + \frac{1}{2}\left(\tilde{\varrho}^{k+1} + \tilde{\varrho}^{k-1}, \upsilon\right) \\
&= \frac{1}{2}\left(f(\varpi(t_{k+1})) - f(\varpi^{k+1}) + f(\varpi(t_{k-1})) - f(\varpi^{k-1}), \upsilon\right) \\
&\quad + \Delta t^2 \left(R(\boldsymbol{x}, t), \upsilon\right), \forall \upsilon \in \mathbb{W}, \quad 1 \leqslant k \leqslant K-1,
\end{aligned} \tag{2.30}$$

$$\left(\nabla\varrho^k, \nabla\vartheta\right) = (\tilde{\varrho}^k, \vartheta), \quad \forall\vartheta \in \mathbb{W}, \quad 0 \leqslant k \leqslant K, \tag{2.31}$$

$$\varrho^0 = \varrho^1 = \tilde{\varrho}^0 = \tilde{\varrho}^1 = 0, \tag{2.32}$$

where $\tilde{R}(\boldsymbol{x}, t)$ is also a bounded remainder function, which is determined by substituting (2.27)–(2.29) into (2.1).

Taking

$$\upsilon = \varrho^{k+1} - \varrho^{k-1}$$

in (2.30) and using Eq (2.31), the Hölder and Cauchy inequalities, Green's formula, and the LDMF, we obtain

$$\|\varrho^{k+1} - \varrho^k\|_0^2 - \|\varrho^k - \varrho^{k-1}\|_0^2 + \frac{\Delta t^2}{2}(\|\tilde{\varrho}^{k+1}\|_0^2 - \|\tilde{\varrho}^{k-1}\|_0^2) + \frac{\Delta t^2}{2}(\|\nabla\varrho^{k+1}\|_0^2 - \|\nabla\varrho^{k-1}\|_0^2)$$

$$= \frac{\Delta t^2}{2}(f(\varpi(t_{k+1})) - f(\varpi^{k+1}) + f(\varpi(t_{k-1})) - f(\varpi^{k-1}), \varrho^{k+1} - \varrho^{k-1}) + \Delta t^4 \left(R(\boldsymbol{x}, t), \varrho^{k+1} - \varrho^{k-1}\right)$$
$$\leqslant c\Delta t(\|\varrho^{k+1} - \varrho^k\|_0^2 + \|\varrho^k - \varrho^{k-1}\|_0^2) + c\Delta t^7, \quad 1 \leqslant k \leqslant K-1. \tag{2.33}$$

Summing (2.33) from 1 to k ($k \leqslant K-1$), and noting that

$$\varrho^0 = \varrho^1 = \tilde{\varrho}^0 = \tilde{\varrho}^1 = 0,$$

we get

$$\|\varrho^{k+1} - \varrho^k\|_0^2 + \frac{\Delta t^2}{2}(\|\tilde{\varrho}^{k+1}\|_0^2 + \|\tilde{\varrho}^k\|_0^2) + \frac{\Delta t^2}{2}(\|\nabla \varrho^{k+1}\|_0^2 + \|\nabla \varrho^k\|_0^2)$$
$$\leqslant c\Delta t \sum_{i=0}^{k} \|\varrho^{i+1} - \varrho^i\|_0^2 + ck\Delta t^7, \quad 1 \leqslant k \leqslant K-1. \tag{2.34}$$

Thus, when Δt is adequately small, satisfying $c\Delta t \leqslant 1/4$, by simplifying (2.34), we get

$$\|\varrho^{k+1} - \varrho^k\|_0^2 + \Delta t^2\|\tilde{\varrho}^{k+1}\|_0^2 + \Delta t^2\|\nabla \varrho^{k+1}\|_0^2 \leqslant c\Delta t \sum_{i=0}^{k-1} \|\varrho^{i+1} - \varrho^i\|_0^2 + c\Delta t^6, \ \ 1 \leqslant k \leqslant K-1. \tag{2.35}$$

By applying Gronwall's inequality to (2.35), we obtain

$$\|\varrho^{k+1} - \varrho^k\|_0^2 + \Delta t^2\|\tilde{\varrho}^{k+1}\|_0^2 + \Delta t^2\|\nabla \varrho^{k+1}\|_0^2 \leqslant c\Delta t^6 \exp(ck\Delta t) \leqslant c\Delta t^6, \ \ 1 \leqslant k \leqslant K-1. \tag{2.36}$$

Thereupon, we obtain

$$\|\varphi(t_k) - \varphi^k\|_0 + \|\nabla(\varpi(t_k) - \varpi^k)\|_0 \leqslant c\Delta t^2, \ \ 1 \leqslant k \leqslant K. \tag{2.37}$$

Further, by (2.30) and (2.31), the Hölder and Cauchy inequalities, and the LDMF, we obtain

$$2(\|\nabla(\tilde{\rho}^{k+1} - \tilde{\rho}^k)\|_0^2 - \|\nabla(\tilde{\rho}^k - \tilde{\rho}^{k-1})\|_0^2) + \Delta t^2(\|\nabla\tilde{\rho}^{k+1}\|_0^2 - \|\nabla\tilde{\rho}^{k-1}\|_0^2) + \Delta t^2(\|\tilde{\rho}^{k+1}\|_0^2 - \|\tilde{\rho}^{k-1}\|_0^2)$$
$$= (\rho^{k+1} - 2\rho^k + \rho^{k-1}, \tilde{\rho}^{k+1} - \tilde{\rho}^{k-1})$$
$$+ \Delta t^2(\nabla(\tilde{\rho}^{k+1} + \tilde{\rho}^{k-1}), \nabla(\tilde{\rho}^{k+1} - \tilde{\rho}^{k-1})) + \Delta t^2(\tilde{\rho}^{k+1} + \tilde{\rho}^{k-1}, \tilde{\rho}^{k+1} - \tilde{\rho}^{k-1})$$
$$= \Delta t^2(f(\varpi(t_{k+1})) - f(\varpi^{k+1}) + f(\varpi(t_{k-1})) - f(\varpi^{k-1}), \tilde{\rho}^{k+1} - \tilde{\rho}^{k-1}) + \Delta t^4(R(\boldsymbol{x}, t), \tilde{\rho}^{k+1} - \tilde{\rho}^{k-1})$$
$$\leqslant c\Delta t(\|\nabla(\tilde{\rho}^{k+1} - \tilde{\rho}^k)\|_0^2 + \|\nabla(\tilde{\rho}^k - \tilde{\rho}^{k-1})\|_0^2) + \Delta t^3(\|\tilde{\rho}^{k+1}\|_0^2 + \|\tilde{\rho}^{k-1}\|_0^2) + c\Delta t^7, \quad 1 \leqslant k \leqslant K-1. \tag{2.38}$$

By summing (2.38) from 1 to k ($k \leqslant K-1$) and noting that

$$\varrho^0 = \varrho^1 = \tilde{\varrho}^0 = \tilde{\varrho}^1 = 0,$$

we obtain

$$2\|\nabla(\tilde{\rho}^{k+1} - \tilde{\rho}^k)\|_0^2 + \Delta t^2(\|\nabla\tilde{\rho}^{k+1}\|_0^2 + \|\nabla\tilde{\rho}^k\|_0^2) + \Delta t^2(\|\tilde{\rho}^{k+1}\|_0^2 + \|\tilde{\rho}^k\|_0^2)$$
$$\leqslant c\Delta t \sum_{j=0}^{k} \|\nabla(\tilde{\rho}^{j+1} - \tilde{\rho}^j)\|_0^2 + c\Delta t^3 \sum_{j=0}^{k} \|\tilde{\rho}^{j+1}\|_0^2 + ck\Delta t^7, \quad 1 \leqslant k \leqslant K-1. \tag{2.39}$$

Thus, when Δt is adequately small, satisfying $c\Delta t \leqslant 1/2$, by simplifying (2.39), we get

$$\begin{aligned}&\|\nabla(\tilde{\rho}^{k+1}-\tilde{\rho}^{k})\|_0^2+\Delta t^2\|\nabla\tilde{\rho}^{k+1}\|_0^2+\Delta t^2\|\tilde{\rho}^{k+1}\|_0^2\\ &\leqslant c\Delta t\sum_{j=0}^{k-1}\|\nabla(\tilde{\rho}^{j+1}-\tilde{\rho}^{j})\|_0^2+c\Delta t^3\sum_{j=0}^{k-1}\|\tilde{\rho}^{j+1}\|_0^2+c\Delta t^6,\quad 1\leqslant k\leqslant K-1.\end{aligned}\tag{2.40}$$

By applying Gronwall's inequality to (2.40), we obtain

$$\|\nabla(\tilde{\rho}^{k+1}-\tilde{\rho}^{k})\|_0^2+\Delta t^2\|\nabla\tilde{\rho}^{k+1}\|_0^2+\Delta t^2\|\tilde{\rho}^{k+1}\|_0^2\leqslant c\Delta t^6\exp(ck\Delta t)\leqslant c\Delta t^6,\quad 1\leqslant k\leqslant K-1.\tag{2.41}$$

Thereupon, we obtain

$$\|\nabla(\varphi(t_k)-\varphi^k)\|_0+\|\varphi(t_k)-\varphi^k\|_0\leqslant c\Delta t^4,\quad 1\leqslant k\leqslant K-1.\tag{2.42}$$

Combining (2.37) with (2.42) yields (2.6). Theorem 1 is proved. □

Remark 1. *Theorem 1 shows that the TSDMCN solutions are stable and their H^1 norm error estimates can reach second-order accuracy, which is the optimal order error estimates.*

3. The TGCNMFE method

In order to construct the TGCNMFE format, it is necessary further to discretize the spatial variables in Problem 3 by using the two-grid MFE method. To this end, we assume that $\Im_H$ is a coarse grid of quasi-uniform partition on $\bar{\Omega}$, which is formed by two-dimensional triangles or quadrangles and three-dimensional tetrahedrons or hexahedrons, and H denotes the maximum diameter of all elements in $\Im_H$. Thus, the FE space defined on the coarse grids is expressed by

$$\mathbb{W}_H=\left\{v_H\in C(\bar{\Omega})\cap\mathbb{W}: v_H|_E\in\mathbb{P}_l(E),\ \forall E\in\Im_H\right\},$$

where $\mathbb{P}_l(E)$ $(l\geqslant 1)$ denotes the space of polynomials with degree $\leqslant l$ defined on the coarse grid element $E\in\Im_H$.

Further, we assume that $\Im_h$ is a fine grid of quasi-uniform partition on $\bar{\Omega}$ and h denotes the maximum diameter of all elements in $\Im_h$ $(h\ll H)$. Likewise, the FE space defined on the fine grids $\Im_h$ is denoted by

$$\mathbb{W}_h=\left\{v_h\in C(\bar{\Omega})\cap\mathbb{W}: v_h|_e\in\mathbb{P}_l(e),\ \forall e\in\Im_h\right\}.$$

Thereupon, a new TGCNMFE formulation can be created as follows.

Problem 4. Step 1. *Find*

$$(\varpi_H^k,\varphi_H^k)\in\mathbb{W}_H\times\mathbb{W}_H\ (1\leqslant k\leqslant K)$$

defined on the coarse grid $\Im_H$, satisfying the nonlinear system of equations:

$$\begin{cases}(\varpi_H^{k+1}-2\varpi_H^k+\varpi_H^{k-1},\upsilon_H)+\dfrac{\Delta t^2}{2}(\nabla(\varphi_H^{k+1}+\varphi_H^{k-1}),\nabla\upsilon_H)+\dfrac{\Delta t^2}{2}(\varphi_H^{k+1}+\varphi_H^{k-1},\upsilon_H)\\ =\dfrac{\Delta t}{2}(f(\varpi_H^{k+1})+f(\varpi_H^{k-1}),\upsilon_H),\forall\upsilon_H\in\mathbb{W}_H,1\leqslant k\leqslant K-1,\\ \left(\nabla\varpi_H^k,\nabla\vartheta_H\right)=(\varphi_H^k,\vartheta_H),\quad\forall\vartheta_H\in\mathbb{W}_H,\quad 0\leqslant k\leqslant K,\\ \varpi_H^0=R_H\varpi_0,\quad\varpi_H^1=R_H\varpi_1,\quad\varphi_H^0=R_H\varphi^0,\quad\varphi_H^1=R_H\varphi^1,\quad in\ \Omega.\end{cases}\tag{3.1}$$

Step 2. *Find*

$$(\varpi_h^k, \varphi_h^k) \in \mathbb{W}_h \times \mathbb{W}_h \ (1 \leqslant k \leqslant K)$$

defined on the fine grid $\Im_h$, satisfying the linear system of equations:

$$\begin{cases} \left(\varpi_h^{k+1} - 2\varpi_h^k + \varpi_h^{k-1}, \upsilon_h\right) + \dfrac{\Delta t}{2}(\nabla(\varphi_h^{k+1} + \varphi_h^{k-1}), \nabla \upsilon_h) + \dfrac{\Delta t}{2}(\varphi_h^{k+1} + \varphi_h^{k-1}, \upsilon_h) \\ = \dfrac{\Delta t}{2}(f(\varpi_H^{k+1}) + f'(\varpi_H^{k+1})(\varpi_h^{k+1} - \varpi_H^{k+1}) + f(\varpi_h^{k-1}), \upsilon_h), \ \forall \upsilon_h \in \mathbb{W}_h, 1 \leqslant k \leqslant K-1, \\ \left(\nabla \varpi_h^k, \nabla \vartheta_h\right) = (\varphi_h^k, \vartheta_h), \quad \forall \vartheta_h \in \mathbb{W}_h, \quad 0 \leqslant k \leqslant K, \\ \varpi_h^0 = R_h \varpi_0, \quad \varpi_h^1 = R_h \varpi_1, \quad \varphi_h^0 = R_h \varphi^0, \quad \varphi_h^1 = R_h \varphi^1, \quad in \ \Omega. \end{cases} \tag{3.2}$$

The above operators R_δ: $\mathbb{W} \to \mathbb{W}_\delta$ ($\delta = H, h$) denote the Ritz projection; i.e., for any $\varphi \in \mathbb{W}$, there exist two unique $R_\delta \varphi \in \mathbb{W}_\delta$ satisfying

$$(\nabla(\varphi - R_\delta \varphi), \nabla \vartheta_\delta) = 0, \ \ \forall \vartheta_\delta \in \mathbb{W}_\delta, \quad \delta = H, h, \tag{3.3}$$

and the following error estimates:

$$|\varphi - R_\delta \varphi|_r \leqslant C\delta^{l+1-r}, \ \ \text{if } \varphi \in \mathbb{W} \cap H^{l+1}(\Omega), \quad \delta = H, \ \ h, r = -1, 0, 1. \tag{3.4}$$

For Problem 4, we obtain the following results:

Theorem 2. *Problem 4 has a unique set of solutions*

$$\{(\varpi_H^k, \varphi_H^k)\}_{k=1}^K \subset \mathbb{W}_H \times \mathbb{W}_H$$

defined on the coarse grid $\Im_H$ and a unique set of solutions

$$\{(\varpi_h^k, \varphi_h^k)\}_{k=1}^K \subset \mathbb{W}_h \times \mathbb{W}_h$$

defined on the fine grid $\Im_h$, respectively, meeting the following unconditional boundness, i.e., unconditional stability:

$$\|\varpi_H^k\|_1 + \|\varpi_h^k\|_1 + \|\varphi_H^k\|_0 + \|\varphi_h^k\|_0 \leqslant c(\|\varpi_0\|_1 + \|\varpi_1\|_1), \quad 1 \leqslant k \leqslant K. \tag{3.5}$$

The c that appears here and after is also a positive constant independent of H, h, and Δt. Further, when

$$h = O(H^{1+1/l}),$$

they meet the following error estimates:

$$\|\varpi(t_k) - \varpi_H^k\|_0 + \|\varphi(t_k) - \varphi_H^k\|_0 + H\|\nabla(\varpi(t_k) - \varpi_H^k)\|_0 + H\|\nabla(\varphi(t_k) - \varphi_H^k)\|_0 \leqslant c(\Delta t^2 + H^{l+1}), \tag{3.6}$$

$$\|\varpi(t_k) - \varpi_h^k\|_0 + \|\varphi(t_k) - \varphi_h^k\|_0 + h\|\nabla(\varpi(t_k) - \varpi_h^k)\|_0 + h\|\nabla(\varphi(t_k) - \varphi_h^k)\|_0 \leqslant c(\Delta t^2 + h^{l+1}), \tag{3.7}$$

where $1 \leqslant k \leqslant K$.

Proof. The demonstration of Theorem 2 consists of the following two parts:

(1) Prove the existence and unconditional stability of TGCNMFE solutions.

(i) Consider the existence and unconditional stability of the TGCNMFE solutions defined on the coarse grid $\Im_H$.

Noting that the system of Eq (3.1) has the same form as the system of Eq (2.4), by using the same approach as proving Theorem 2, we can demonstrate that the nonlinear system of Eq (3.1) has a unique set of solutions

$$\{(\varpi_H^k, \varphi_H^k)\}_{k=1}^K \subset \mathbb{W}_H \times \mathbb{W}_H,$$

satisfying

$$\|\nabla\varpi_H^k\|_0 + \|\nabla\varphi_H^k\|_0 \leqslant c(\|\varpi_0\|_1 + \|\varpi_1\|_1), \quad 1 \leqslant k \leqslant K. \tag{3.8}$$

(ii) Consider the existence and unconditional stability of the TGCNMFE solutions defined on the fine grid $\Im_h$.

Let

$$A((\varpi,\varphi),(\vartheta,\upsilon)) = (\varpi,\upsilon) - (\varphi,\vartheta) + (\nabla\varpi,\nabla\vartheta) + \frac{\Delta t^2}{2}(\nabla\varphi,\nabla\upsilon) - \frac{\Delta t^2}{2}(f'(\varpi_H^{k+1})\varpi,\upsilon) + \frac{\Delta t^2}{2}(\varphi,\upsilon),$$

$$\begin{aligned} F(\upsilon,\vartheta) = &\left(2\varpi_h^k - \varpi_h^{k-1},\upsilon\right) - \frac{\Delta t^2}{2}(\nabla\varphi_h^{k-1},\nabla\upsilon) - \frac{\Delta t^2}{2}(\varphi_h^{k-1},\upsilon) \\ &+ \frac{\Delta t^2}{2}(f(\varpi_H^{k+1}) - f'(\varpi_H^{k+1})\varpi_H^{k+1} + f(\varpi_h^{k-1}),\upsilon). \end{aligned}$$

The linear system of equations (3.2) can be rewritten into as follows:

Find

$$(\varpi_h^k, \varphi_h^k) \in \mathbb{W}_h \times \mathbb{W}_h \ (1 \leqslant k \leqslant K)$$

satisfying the following linear system of equations:

$$\begin{cases} A((\varpi_h^{k+1},\varphi_h^{k+1}),(\vartheta_h,\upsilon_h)) = F(\vartheta_h,\upsilon_h), \ \ \forall(\vartheta_h,\upsilon_h) \in \mathbb{W}_h \times \mathbb{W}_h, \ 1 \leqslant k \leqslant K-1, \\ \varpi_h^0 = R_h\varpi_0(\boldsymbol{x}), \ \ \varpi_h^1 = R_h\varpi_1(\boldsymbol{x}), \ \ \varphi_h^0 = R_h\varphi^0(\boldsymbol{x}), \ \ \varphi_h^1 = R_h\varphi^1(\boldsymbol{x}), \ \ \boldsymbol{x}\in\Omega. \end{cases} \tag{3.9}$$

By Poincaré's inequality, we claim that there is a constant $\theta_0 > 0$ such that

$$\|\vartheta\|_0 \leqslant \|\vartheta\|_1 \leqslant \theta_0\|\nabla\vartheta\|_0 \ \ (\forall\vartheta \in \mathbb{W} = H_0^1(\Omega))$$

and

$$|f'(\varpi_H^{k+1})| = |-\cos(\varpi_H^{k+1})| \leqslant 1.$$

Thereupon, when Δt is adequately small meeting

$$\Delta t^2\theta_0 < 4,$$

there exists a constant

$$\alpha_0 = \min\{\Delta t^2/2, 1 - \Delta t^2\theta_0/4\}/\theta_0^2$$

satisfying

$$A((\varpi,\varphi),(\varpi,\varphi)) = (\varpi,\varphi) - (\varphi,\varpi) + (\nabla\varpi,\nabla\varpi) + \frac{\Delta t^2}{2}(\nabla\varphi,\nabla\varphi) + \frac{\Delta t^2}{2}(\varphi,\varphi) - \frac{\Delta t^2}{2}(\cos(\varpi_H^k)\varpi,\varphi)$$
$$\geqslant \left(1 - \frac{\Delta t\theta_0}{4}\right)\|\nabla\varpi\|_0^2 + \frac{\Delta t}{2}\|\nabla\varphi\|_0^2 + \frac{\Delta t}{2}\|\varphi\|_0^2 - \frac{\Delta t}{4}\|\varphi\|_0^2$$
$$\geqslant \alpha_0\|(\varpi,\varphi)\|_1^2,\ \forall(\varpi,\varphi)\in\mathbb{W}_h\times\mathbb{W}_h, \tag{3.10}$$

where

$$\|(\varpi,\varphi)\|_1 = (\|\varpi\|_1^2 + \|\varphi\|_1^2)^{1/2}$$

is the norm in $\mathbb{W}\times\mathbb{W}$. Thus, $A((\varpi,\varphi),(\vartheta,\upsilon))$ is positive definite, and the bilinear functional $A((\varpi,\varphi),(\vartheta,\upsilon))$ and the linear functional $F(\upsilon,\vartheta)$ are evidently bounded in $\mathbb{W}_h\times\mathbb{W}_h$ for given ϖ_H^{k+1}, ϖ_h^k, ϖ_h^{k-1}, and φ_h^{k-1}. Thereupon, by the Lax-Milgram theorem (see [5, Theorem 1.2.1]), we assert that the linear system of Eq (3.9), namely Step 2 for Problem 4, has a unique set of solutions

$$\left\{(\varpi_h^k,\varphi_h^k)\right\}_{k=1}^K \subset \mathbb{W}_h\times\mathbb{W}_h$$

satisfying

$$\|\nabla\varpi_h^k\|_0 + \|\nabla\varphi_h^k\|_0 \leqslant c(\|\varpi_0\|_1 + \|\varpi_1\|_1),\quad 1\leqslant k\leqslant K. \tag{3.11}$$

This signifies that the series of solutions

$$\{(\varpi_h^k,\varphi_h^k)\}_{k=1}^K \subset \mathbb{W}_h\times\mathbb{W}_h$$

for Problem 4 defined on the fine grid $\Im_h$ is unconditionally bounded, namely it is unconditionally stable. Combining (3.8) with (3.11) yields (3.5).

(2) Estimate the errors of the TGCNMFE solutions of Problem 4.

(a) Estimate the errors of the solutions $\{(\varpi_H^k,\varphi_H^k)\}_{k=1}^K$ of Problem 4 defined on $\Im_H$.

By subtracting (3.1) from (2.4), and taking

$$\upsilon = \upsilon_H \quad\text{and}\quad \vartheta = \vartheta_H,$$

as well as setting

$$E_H^k = \varpi^k - \varpi_H^k,\ \ \rho_H^k = \varpi^k - R_H\varpi^k,\ \ \varrho_H^k = R_H\varpi^k - \varpi_H^k,\ \ \tilde{E}_H^k = \varphi^k - \varphi_H^k,\ \ \tilde{\rho}_H^k = \varphi^k - R_H\varphi^k$$

and

$$\tilde{\varrho}_H^k = R_H\varphi^k - \varphi_H^k,$$

we obtain

$$\frac{1}{\Delta t^2}(E_H^{k+1} - 2E_H^k + E_H^{k-1},\upsilon_H) + \frac{1}{2}(\nabla(\tilde{E}_H^{k+1} + \tilde{E}_H^{k-1}),\nabla\upsilon_H) + \frac{1}{2}(\tilde{E}_H^{k+1} + \tilde{E}_H^{k-1},\upsilon_H)$$
$$= \frac{1}{2}(f(\varpi^{k+1}) - f(\varpi_H^{k+1}) + f(\varpi^{k-1}) - f(\varpi_H^{k-1}),\upsilon_H),\quad \forall\upsilon_H\in\mathbb{W}_H, 1\leqslant k\leqslant K-1, \tag{3.12}$$

$$(\nabla E_H^k, \nabla \vartheta_H) = (\tilde{E}_H^k, \vartheta_H), \quad \forall \vartheta_H \in \mathbb{W}_H, \ \ 1 \leqslant k \leqslant K, \tag{3.13}$$

$$E_H^0 = \varpi_0 - R_H\varpi_0, \ \ E_H^1 = \varpi_1 - R_H\varpi_1, \ \ \tilde{E}_H^0 = \varphi_0 - R_H\varphi_0, \ \ \tilde{E}_H^1 = \varphi_1 - R_H\varphi_1. \tag{3.14}$$

By (3.2)–(3.4), (3.12), (3.13), Taylor's formula, LDMF, and the Hölder and Cauchy inequalities, when

$$h = O(H^{1+1/l}),$$

we obtain

$$\begin{aligned}
&\frac{1}{\Delta t^2}(\|\nabla(E_H^{k+1} - E_H^k)\|_0^2 - \|\nabla(E_H^k - E_H^{k-1})\|_0^2) + \frac{1}{2}(\|\nabla\tilde{E}_H^{k+1}\|_0^2 - \|\nabla\tilde{E}_H^{k-1}\|_0^2)\\
&+ \frac{1}{2}(\|\tilde{E}_H^{k+1}\|_0^2 - \|\tilde{E}_H^{k-1}\|_0^2) = \frac{1}{\Delta t^2}(\nabla(\rho_H^{k+1} - 2\rho_H^k + \rho_H^{k-1}), \nabla(\rho_H^{k+1} - \rho_H^{k-1}))\\
&\quad + \frac{1}{\Delta t^2}(\tilde{E}_H^{k+1} - 2\tilde{E}_H^k + \tilde{E}_H^{k-1}, \varrho_H^{k+1} - \varrho_H^{k-1})\\
&\quad + \frac{1}{2}(\nabla(\tilde{\rho}_H^{k+1} + \tilde{\rho}_H^{k-1}), \nabla(\tilde{\rho}_H^{k+1} - \tilde{\rho}_H^{k-1})) + \frac{1}{2}(\nabla(\tilde{E}_H^{k+1} + \tilde{E}_H^{k-1}), \nabla(\tilde{\varrho}_H^{k+1} - \tilde{\varrho}_H^{k-1}))\\
&\quad + \frac{1}{2}(\tilde{E}_H^{k+1} + \tilde{E}_H^{k-1}, \tilde{\rho}_H^{k+1} - \tilde{\rho}_H^{k-1}) + \frac{1}{2}(\tilde{E}_H^{k+1} + \tilde{E}_H^{k-1}, \tilde{\varrho}_H^{k+1} - \tilde{\varrho}_H^{k-1})\\
&= \frac{1}{\Delta t^2}(\nabla(\rho_H^{k+1} - 2\rho_H^k + \rho_H^{k-1}), \nabla(\rho_H^{k+1} - \rho_H^{k-1}))\\
&\quad + \frac{1}{2}(\nabla(\tilde{\rho}_H^{k+1} + \tilde{\rho}_H^{k-1}), \nabla(\tilde{\rho}_H^{k+1} - \tilde{\rho}_H^{k-1})) + \frac{1}{2}(\tilde{E}_H^{k+1} + \tilde{E}_H^{k-1}, \tilde{\rho}_H^{k+1} - \tilde{\rho}_H^{k-1})\\
&\quad + \frac{1}{2}(f(\varpi^{k+1}) - f(\varpi_H^{k+1}) + f(\varpi^{k-1}) - f(\varpi_H^{k-1}), \tilde{\varrho}_H^{k+1} - \tilde{\varrho}_H^{k-1})\\
&\leqslant \frac{1}{\Delta t}(\|\nabla(E_H^{k+1} - E_H^k)\|_0^2 + \|\nabla(E_H^k - E_H^{k-1})\|_0^2) + c\Delta t H^{2l}\\
&\quad + c\Delta t(\|\tilde{E}_H^{k+1}\|_0^2 + \|\tilde{E}_H^{k-1}\|_0^2), \quad 1 \leqslant k \leqslant K-1.
\end{aligned} \tag{3.15}$$

By summing (3.15) from 1 to k ($k \leqslant K-1$), and using (3.4) and (3.14), we obtain

$$\begin{aligned}
&\frac{1}{\Delta t^2}\|\nabla(E_H^{k+1} - E_H^k)\|_0^2 + \frac{1}{2}(\|\nabla\tilde{E}_H^{k+1}\|_0^2 + \|\nabla\tilde{E}_H^k\|_0^2) + \frac{1}{2}(\|\tilde{E}_H^{k+1}\|_0^2 + \|\tilde{E}_H^k\|_0^2)\\
&\leqslant \frac{c}{\Delta t}\sum_{i=0}^{k}\|\nabla(E_H^{i+1} - E_H^i)\|_0^2 + cH^{2l} + c\Delta t\sum_{i=0}^{k}\|\tilde{E}_H^{i+1}\|_0^2, \ \ 1 \leqslant k \leqslant K-1.
\end{aligned} \tag{3.16}$$

Thus, when Δt is adequately small, satisfying $c\Delta t \leqslant 1/4$, by simplifying (3.16), we get

$$\begin{aligned}
&\frac{1}{\Delta t^2}\|\nabla(E_H^{k+1} - E_H^k)\|_0^2 + \|\nabla\tilde{E}_H^{k+1}\|_0^2 + \|\tilde{E}_H^{k+1}\|_0\\
&\leqslant \frac{c}{\Delta t}\sum_{i=0}^{k-1}\|\nabla(E_H^{i+1} - E_H^i)\|_0^2 + cH^{2l} + c\Delta t\sum_{i=0}^{k-1}\|\tilde{E}_H^{i+1}\|_0^2, \ \ 1 \leqslant k \leqslant K-1.
\end{aligned} \tag{3.17}$$

By applying Gronwall's inequality to (3.17), we obtain

$$\frac{1}{\Delta t^2}\|\nabla(E_H^{k+1} - E_H^k)\|_0^2 + \|\nabla\tilde{E}_H^{k+1}\|_0^2 + \|\tilde{E}_H^{k+1}\|_0^2 \leqslant cH^{2l}\exp(ck\Delta t), \ \ 1 \leqslant k \leqslant K-1. \tag{3.18}$$

Thereupon, we obtain

$$\|\nabla\tilde{E}_H^k\|_0+\|\tilde{E}_H^k\|_0\leqslant cH^l,\quad 1\leqslant k\leqslant K. \tag{3.19}$$

By the Nitsche technique (see [5, Theorem 1.3.9]) and (3.19), we immediately obtain the following error estimates:

$$\|\varphi^k-\varphi_H^k\|_0+H\|\nabla(\varphi^k-\varphi_H^k)\|_0\leqslant cH^{l+1},\ \ 1\leqslant k\leqslant K. \tag{3.20}$$

Further, by (3.2), (3.3), (3.12), (3.13), Taylor's formula, the LDMF, the Hölder and Cauchy inequalities, we obtain

$$\begin{aligned}
&\frac{1}{\Delta t^2}(\|E_H^{k+1}-E_H^k\|_0^2-\|E_H^k-E_H^{k-1}\|_0^2)+\frac{1}{2}(\|\tilde{E}_H^{k+1}\|_0^2-\|\tilde{E}_H^{k-1}\|_0^2)+\frac{1}{2}(\|\nabla E_H^{k+1}\|_0^2-\|\nabla E_H^{k-1}\|_0^2)\\
&=\frac{1}{\Delta t^2}(E_H^{k+1}-2E_H^k+E_H^{k-1},\rho_H^{k+1}-\rho_H^{k-1})+\frac{1}{\Delta t}(E_H^{k+1}-2E_H^k+E_H^{k-1},\varrho_H^{k+1}-\varrho_H^{k-1})\\
&\quad+\frac{1}{2}(\tilde{E}_H^{k+1}+\tilde{E}_H^{k-1},\tilde{\rho}_H^{k+1}-\tilde{\rho}_H^{k-1})+\frac{1}{2}(\nabla(\rho_H^k+\rho_H^{k-1}),\nabla(\rho_H^{k+1}-\rho_H^{k-1}))\\
&\quad+\frac{1}{2}(\tilde{E}_H^{k+1}+\tilde{E}_H^{k-1},\tilde{\varrho}_H^{k+1}-\tilde{\varrho}_H^{k-1})+\frac{1}{2}(\nabla(E_H^k+E_H^{k-1}),\nabla(\varrho_H^{k+1}-\varrho_H^{k-1}))\\
&=\frac{1}{\Delta t^2}(E_H^{k+1}-2E_H^k+E_H^{k-1},\rho_H^{k+1}-\rho_H^{k-1})+\frac{1}{2}(\nabla(\rho_H^k+\rho_H^{k-1}),\nabla(\rho_H^{k+1}-\rho_H^{k-1}))\\
&\quad+\frac{1}{2}(\nabla(\rho_H^k+\rho_H^{k-1}),\nabla(\rho_H^{k+1}-\rho_H^{k-1}))+\frac{1}{2}(f(\varpi^k)-f(\varpi_H^k)+f(\varpi^{k-1})-f(\varpi_H^{k-1}),\varrho_H^k-\varrho_H^{k-1})\\
&\leqslant\frac{c}{\Delta t}(\|E_H^{k+1}-E_H^k\|_0^2+\|E_H^k-E_H^{k-1}\|_0^2)+c\Delta tH^{2l}+c\Delta t(\|\tilde{E}_H^{k+1}\|_0^2+\|\tilde{E}_H^{k-1}\|_0^2),\quad 1\leqslant k\leqslant K-1.
\end{aligned}\tag{3.21}$$

By summing (3.21) from 1 to k, and using (3.4) and (3.14), we obtain

$$\begin{aligned}
&\frac{1}{\Delta t^2}\|E_H^{k+1}-E_H^k\|_0^2+\frac{1}{2}(\|\tilde{E}_H^{k+1}\|_0^2+\|\tilde{E}_H^k\|_0^2)+\frac{1}{2}(\|\nabla E_H^{k+1}\|_0^2+\|\nabla E_H^k\|_0^2)\\
&\leqslant\frac{c}{\Delta t}\sum_{i=0}^{k}\|E_H^{i+1}-E_H^i\|_0^2+cH^{2l}+c\Delta t\sum_{i=0}^{k}\|\tilde{E}_H^{i+1}\|_0^2,\ \ 1\leqslant k\leqslant K-1.
\end{aligned}\tag{3.22}$$

Thus, when Δt is adequately small, satisfying $c\Delta t\leqslant 1/4$, by simplifying (3.22), we get

$$\begin{aligned}
&\frac{1}{\Delta t^2}\|E_H^{k+1}-E_H^k\|_0^2+\|\tilde{E}_H^{k+1}\|_0^2+\|\nabla E_H^{k+1}\|_0^2\\
&\leqslant\frac{c}{\Delta t}\sum_{i=0}^{k-1}\|E_H^{i+1}-E_H^i\|_0^2+cH^{2l}+c\Delta t\sum_{i=0}^{k-1}\|\tilde{E}_H^{i+1}\|_0^2,\ \ 1\leqslant k\leqslant K-1.
\end{aligned}\tag{3.23}$$

By applying Gronwall's inequality to (3.23), we obtain

$$\frac{1}{\Delta t^2}\|E_H^{k+1}-E_H^k\|_0^2+\|\tilde{E}_H^{k+1}\|_0^2+\|\nabla E_H^{k+1}\|_0^2\leqslant cH^{2l}\exp(ck\Delta t),\ \ 1\leqslant k\leqslant K-1. \tag{3.24}$$

Thereupon, we obtain

$$\|\nabla(\varpi^k-\varpi_H^k)\|_0\leqslant cH^l,\ \ 1\leqslant k\leqslant K. \tag{3.25}$$

Further, by the Nitsche technique, we can obtain the following error estimates:

$$\|\varpi^k - \varpi_H^k\|_0 + H\|\nabla(\varpi^k - \varpi_H^k)\|_0 \leqslant cH^{l+1}, \ \ 1 \leqslant k \leqslant K. \tag{3.26}$$

Combining (3.20) and (3.26) with Theorem 1 yields (3.6).

(b) Estimate the errors of solutions $(\varpi_h^k, \varphi_h^k)$ $(1 \leqslant k \leqslant K)$ of Problem 4 defined on the fine grid $\Im_h$. Subtracting (3.2) from (2.4), taking

$$\upsilon = \upsilon_h \quad \text{and} \quad \vartheta = \vartheta_h,$$

and setting

$$E_h^k = \varpi^k - \varpi_h^k, \quad \rho_h^k = \varpi^k - R_h\varpi^k, \quad \varrho_h^k = R_h\varpi^k - \varpi_h^k, \quad \tilde{E}_h^k = \varphi^k - \varphi_h^k, \quad \tilde{\rho}_h^k = \varphi^k - R_h\varphi^k$$

and

$$\tilde{\varrho}_h^k = R_h\varphi^k - \varphi_h^k,$$

we obtain

$$\begin{aligned}
&\frac{1}{\Delta t^2}(E_h^{k+1} - 2E_h^k + E_h^{k-1}, \upsilon_h) + \frac{1}{2}(\nabla(\tilde{E}_h^{k+} + \tilde{E}_h^{k-1}), \nabla\upsilon_h) + \frac{1}{2}(\tilde{E}_h^{k+1} + \tilde{E}_h^{k-1}, \upsilon_h)\\
&= \frac{1}{2}(f(\varpi^{k+1}) - f(\varpi_H^{k+1}) + f(\varpi^{k-1}) - f(\varpi_h^{k-1}) - f'(\varpi_H^{k+1})(\varpi_h^k k + 1 - \varpi_H^{k+1}), \upsilon_h),\\
&\forall\upsilon_h \in \mathbb{W}_h, \ 1 \leqslant k \leqslant K-1,
\end{aligned} \tag{3.27}$$

$$(\nabla E_h^k, \nabla\vartheta_h) = (\tilde{E}_h^k, \vartheta_h), \quad \forall\vartheta_H \in \mathbb{W}_h, \ \ 1 \leqslant k \leqslant K, \tag{3.28}$$

$$E_h^0 = \varpi_0 - R_h\varpi_0, \ \ E_h^1 = \varpi_1 - R_h\varpi_1, \ \ \tilde{E}_h^0 = \varphi_0 - R_h\varphi_0, \ \ \tilde{E}_h^1 = \varphi^1 - R_h\varphi^1. \tag{3.29}$$

By (3.3), (3.4), (3.27), (3.28), Taylor's formula, the LDMF, the Hölder and Cauchy inequalities, and (3.6) or (3.26), when

$$h = O(H^{1+1/l}),$$

we obtain

$$\begin{aligned}
&\frac{1}{\Delta t^2}(\|\nabla(E_h^{k+1} - E_h^k)\|_0^2 - \|\nabla(E_h^k - E_h^{k-1})\|_0^2) + \frac{1}{2}(\|\nabla\tilde{E}_h^{k+1}\|_0^2 - \|\nabla\tilde{E}_h^{k-1}\|_0^2) + \frac{1}{2}(\|\tilde{E}_h^{k+1}\|_0^2 - \|\tilde{E}_h^{k-1}\|_0^2)\\
&= \frac{1}{\Delta t^2}(\nabla(\rho_h^{k+1} - 2\rho_h^k + \rho_h^{k-1}), \nabla(\rho_h^{k+1} - \rho_h^{k-1})) + \frac{1}{2}(\nabla(\tilde{\rho}_h^{k+1} + \tilde{\rho}_h^{k-1}), \nabla(\tilde{\rho}_h^k - \tilde{\rho}_h^{k-1}))\\
&\quad + \frac{1}{2}(\tilde{E}_h^{k+1} + \tilde{E}_h^{k-1}, \tilde{\rho}_h^{k+1} - \tilde{\rho}_h^{k-1}) + \frac{1}{\Delta t^2}(\tilde{E}_h^{k+1} - 2\tilde{E}_h^k + \tilde{E}_h^{k-1}, \varrho_h^{k+1} - \varrho_h^{k-1})\\
&\quad + \frac{1}{2}(\nabla(\tilde{E}_h^{k+1} + \tilde{E}_h^{k-1}), \nabla(\tilde{\varrho}_h^{k+1} - \tilde{\varrho}_h^{k-1})) + \frac{1}{2}(\nabla(\tilde{E}_h^k + \tilde{E}_h^{k-1}), \nabla(\tilde{\varrho}_h^{k+1} - \tilde{\varrho}_h^{k-1}))\\
&= \frac{1}{\Delta t^2}(\nabla(\rho_h^{k+1} - 2\rho_h^k + \rho_h^{k-1}), \nabla(\rho_h^{k+1} - \rho_h^{k-1})) + \frac{1}{2}(\nabla(\tilde{\rho}_h^{k+1} + \tilde{\rho}_h^{k-1}), \nabla(\tilde{\rho}_h^k - \tilde{\rho}_h^{k-1}))\\
&\quad + \frac{1}{2}(\tilde{E}_h^{k+1} + \tilde{E}_h^{k-1}, \tilde{\rho}_h^{k+1} - \tilde{\rho}_h^{k-1}) - \frac{1}{2}(f'(\varpi_H^{k+1})(\varpi_h^{k+1} - \varpi_H^{k+1}), \tilde{\varrho}_h^{k+1} - \tilde{\varrho}_h^{k-1})\\
&\quad + \frac{1}{2}(f(\varpi^{k+1}) - f(\varpi_H^{k+1}) + f(\varpi^{k-1}) - f(\varpi_h^{k-1}), \tilde{\varrho}_h^{k+1} - \tilde{\varrho}_h^{k-1})
\end{aligned}$$

$$\leqslant \frac{2}{\Delta t}(\|\nabla(E_h^{k+1}-E_h^k)\|_0^2+\|\nabla(E_h^k-E_h^{k-1})\|_0^2)+c\Delta th^{2l}$$
$$+c\Delta t(\|\tilde{E}_h^{k+1}\|_0^2+\|\tilde{E}_h^{k-1}\|_0^2),\quad 1\leqslant k\leqslant K-1. \tag{3.30}$$

By summing (3.30) from 1 to k ($k\leqslant K-1$), and using (3.4) and (3.29), we obtain

$$\frac{1}{\Delta t^2}\|\nabla(E_h^{k+1}-E_h^k)\|_0^2+\frac{1}{2}(\|\nabla\tilde{E}_h^{k+1}\|_0^2+\|\nabla\tilde{E}_h^k\|_0^2)+\frac{1}{2}(\|\tilde{E}_h^{k+1}\|_0^2+\|\tilde{E}_h^k\|_0^2)$$
$$\leqslant \frac{2}{\Delta t}\sum_{i=0}^{k}\|\nabla(E_h^{i+1}-E_h^k)\|_0^2+cH^{2l+2}+c\Delta t\sum_{i=0}^{k}\|\tilde{E}_h^{i+1}\|_0^2,\quad 1\leqslant k\leqslant K-1. \tag{3.31}$$

Thus, when Δt is adequately small, satisfying $c\Delta t\leqslant 1/4$, by simplifying (3.31), we obtain

$$\frac{1}{\Delta t^2}\|\nabla(E_h^{k+1}-E_h^k)\|_0^2+\|\nabla\tilde{E}_h^{k+1}\|_0^2+\|\tilde{E}_h^{k+1}\|_0^2$$
$$\leqslant \frac{2}{\Delta t}\sum_{i=0}^{k-1}\|\nabla(E_h^{i+1}-E_h^k)\|_0^2+cH^{2l+2}+c\Delta t\sum_{i=0}^{k-1}\|\tilde{E}_h^{i+1}\|_0^2,\quad 1\leqslant k\leqslant K-1. \tag{3.32}$$

By applying Gronwall's inequality to (3.32), we obtain

$$\frac{1}{\Delta t^2}\|\nabla(E_h^{k+1}-E_h^k)\|_0^2+\|\nabla\tilde{E}_h^{k+1}\|_0^2+\|\tilde{E}_h^{k+1}\|_0^2\leqslant ch^{2l}\exp(ck\Delta t)\leqslant ch^{2l},\ \ 1\leqslant k\leqslant K-1. \tag{3.33}$$

Thereupon, we obtain

$$\|\nabla(\varphi^k-\varphi_h^k)\|_0\leqslant ch^l,\ \ 1\leqslant k\leqslant K. \tag{3.34}$$

By the Nitsche technique (see [5, Theorem 1.3.9]), we easily obtain the following error estimates:

$$\|\varphi^k-\varphi_h^k\|_0+h\|\nabla(\varphi^k-\varphi_h^k)\|_0\leqslant ch^{l+1},\ 1\leqslant k\leqslant K. \tag{3.35}$$

Further, by (3.3), (3.27), (3.28), Taylor's formula, the LDMF, the Hölder and Cauchy inequalities, and (3.6) or (3.26), we get

$$\frac{1}{\Delta t^2}(\|E_h^{k+1}-E_h^k\|_0^2-\|E_h^k-E_h^{k-1}\|_0^2)+\frac{1}{2}(\|\tilde{E}_h^{k+1}\|_0^2-\|\tilde{E}_h^{k-1}\|_0^2)+\frac{1}{2}(\|\nabla E_h^{k+1}\|_0^2-\|\nabla E_h^{k-1}\|_0^2)$$
$$=\frac{1}{\Delta t^2}(E_h^{k+1}-2E_h^k+E_h^{k-1},\rho_h^{k+1}-\rho_h^{k-1})+\frac{1}{2}(\tilde{E}_h^{k+1}+\tilde{E}_h^{k-1},\tilde{\rho}_h^{k+1}-\tilde{\rho}_h^{k-1})$$
$$+\frac{1}{2}(\nabla(\rho_h^k+\rho_h^{k-1}),\nabla(\rho_h^{k+1}-\rho_h^{k-1}))+\frac{1}{\Delta t^2}(E_h^{k+1}-2E_h^k+E_h^{k-1},\varrho_h^{k+1}-\varrho_h^{k-1})$$
$$+\frac{1}{2}(\tilde{E}_h^{k+1}+\tilde{E}_h^{k-1},\tilde{\varrho}_h^{k+1}-\tilde{\varrho}_h^{k-1})+\frac{1}{2}(\nabla(E_h^{k+1}+E_h^{k-1}),\nabla(\varrho_h^{k+1}-\varrho_h^{k-1}))$$
$$=\frac{1}{\Delta t^2}(E_h^{k+1}-2E_h^k+E_h^{k-1},\rho_h^{k+1}-\rho_h^{k-1})+\frac{1}{2}(\tilde{E}_h^{k+1}+\tilde{E}_h^{k-1},\tilde{\rho}_h^{k+1}-\tilde{\rho}_h^{k-1})$$
$$+\frac{1}{2}(\nabla(\rho_h^k+\rho_h^{k-1}),\nabla(\rho_h^{k+1}-\rho_h^{k-1}))-\frac{1}{2}(f'(\varpi_H^{k+1})(\varpi_h^{k+1}-\varpi_H^{k+1}),\varrho_h^{k+1}-\varrho_h^{k-1})$$

$$+\frac{1}{2}(f(\varpi^{k+1})-f(\varpi_H^{k+1})+f(\varpi^{k-1})-f(\varpi_h^{k-1}),\varrho_h^{k+1}-\varrho_h^{k-1})$$
$$\leqslant\frac{c}{\Delta t}(\|E_H^{k+1}-E_H^k\|_0^2+\|E_H^k-E_H^{k-1}\|_0^2)+c\Delta th^{2l}+c\Delta t(\|\tilde{E}_h^{k+1}\|_0^2+\|\tilde{E}_h^{k-1}\|_0^2),\quad 1\leqslant k\leqslant K-1. \quad (3.36)$$

By summing (3.36) from 1 to k ($k \leqslant K-1$), and using (3.4) and (3.29), we obtain

$$\frac{1}{\Delta t^2}\|E_h^{k+1}-E_h^k\|_0^2+\frac{1}{2}(\|\tilde{E}_h^{k+1}\|_0^2+\|\tilde{E}_h^k\|_0^2)+\frac{1}{2}(\|\nabla E_h^{k+1}\|_0^2+\|\nabla E_h^k\|_0^2)$$
$$\leqslant\frac{c}{\Delta t}\sum_{i=0}^{k}\|\nabla(E_h^{i+1}-E_h^k)\|_0^2+cH^{2l+2}+c\Delta t\sum_{i=0}^{k}\|\tilde{E}_h^{i+1}\|_0^2,\quad 1\leqslant k\leqslant K-1. \quad (3.37)$$

Thus, when Δt is adequately small, satisfying $c\Delta t \leqslant 1/4$, by simplifying (3.37), we obtain

$$\frac{1}{\Delta t^2}\|\nabla(E_h^{k+1}-E_h^k)\|_0^2+\|\nabla E_h^{k+1}\|_0^2+\|\tilde{E}_h^{k+1}\|_0^2$$
$$\leqslant\frac{c}{\Delta t}\sum_{i=0}^{k-1}\|\nabla(E_h^{i+1}-E_h^k)\|_0^2+cH^{2l+2}+c\Delta t\sum_{i=0}^{k-1}\|\tilde{E}_h^{i+1}\|_0^2,\quad 1\leqslant k\leqslant K-1. \quad (3.38)$$

By applying Gronwall's inequality to (3.38), we obtain

$$\frac{1}{\Delta t^2}\|\nabla(E_h^{k+1}-E_h^k)\|_0^2+\|\nabla E_h^{k+1}\|_0^2+\|\tilde{E}_h^{k+1}\|_0^2\leqslant ch^{2l}\exp(ck\Delta t)\leqslant ch^{2l},\ \ 1\leqslant k\leqslant K-1. \quad (3.39)$$

Thereupon, we obtain

$$\|\nabla(\varpi^k-\varpi_h^k)\|_0\leqslant ch^l,\quad 1\leqslant k\leqslant K. \quad (3.40)$$

By the Nitsche technique (see [5, Theorem 1.3.9]), we easily deduce the following error estimates

$$\|\varpi^k-\varpi_h^k\|_0+h\|\nabla(\varpi^k-\varpi_h^k)\|_0\leqslant ch^{l+1},\quad 1\leqslant k\leqslant K. \quad (3.41)$$

Thereupon, inequality (3.7) is obtained by combining (3.35) and (3.41) with Theorem 1. This completes the proof for Theorem 2. □

Remark 2. *Theorem 2 shows that the TGCNMFE solutions are unconditionally stable and their theoretical errors reach optimal order. In Section 4, we use the numerical tests to verify the correctness of the obtained theoretical error estimates.*

4. Two sets of numerical experiments

In this section, we provide two sets of numerical experiments to verify the correctness of our theoretical results and show the superiorities of the TGCNMFE method.

4.1. Two-dimensional numerical experiments

For the two-dimensional case, we take

$$\bar{\Omega} = [0,\ 1] \times [0,\ 1] \subset \mathbb{R}^2$$

and the initial functions

$$\varpi_0(\boldsymbol{x}) = \varpi_1(\boldsymbol{x}) = 4\sin(\pi x_1)\sin(\pi x_2)$$

in the NFOSG equation (i.e., Problem 1), in which it may be considered that there is an earthquake focus at center $(0.5, 0.5)$ of the region Ω.

The fine grid division $\Im_h$ is composed of squares with diagonal

$$h = \sqrt{2}/1000$$

and all squares parallel to the coordinate axis. When $l = 1$, in order to satisfy

$$h = O(H^{1+1/l}),$$

i.e.,

$$h = O(H^2),$$

the coarse grid division $\Im_H$ is composed of the squares with diagonal

$$H = \sqrt[4]{2}/\sqrt{1000}$$

and all squares also parallel to the coordinate axis. According to Theorem 2, the L^2 norm error estimates of the TGCNMFE solutions of the NFOSG equation can theoretically reach $O(10^{-6})$ when

$$\Delta t = 1/1000.$$

We first find the TGCNMFE solutions ϖ_h^k and φ_h^k by the TGCNMFE format at $t = 10$, and their contours are respectively shown in Figures 1a and 2a.

In order to show the advantages of the TGCNMFE format, we also find single-grid CNMFE (SGCNMFE) solutions $\hat{\varpi}_h^k$ and $\hat{\varphi}_h^k$ at $t = 10$ by the following SGCNMFE format, and their contours are respectively shown in Figures 1b and 2b.

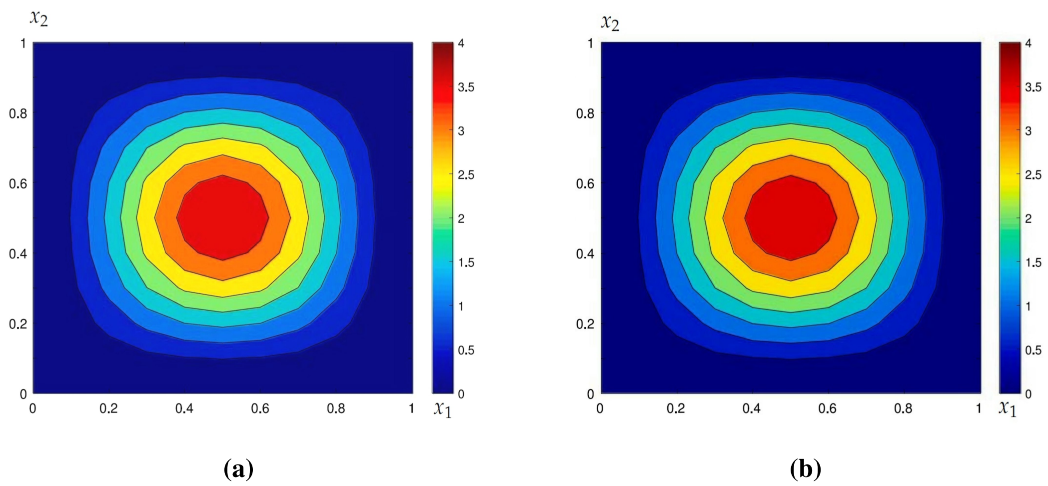

Figure 1. (a) The contour of the TGCNMFE solution of ϖ at $t = 10$; (b) The contour of the SGCNMFE solution of ϖ at $t = 10$.

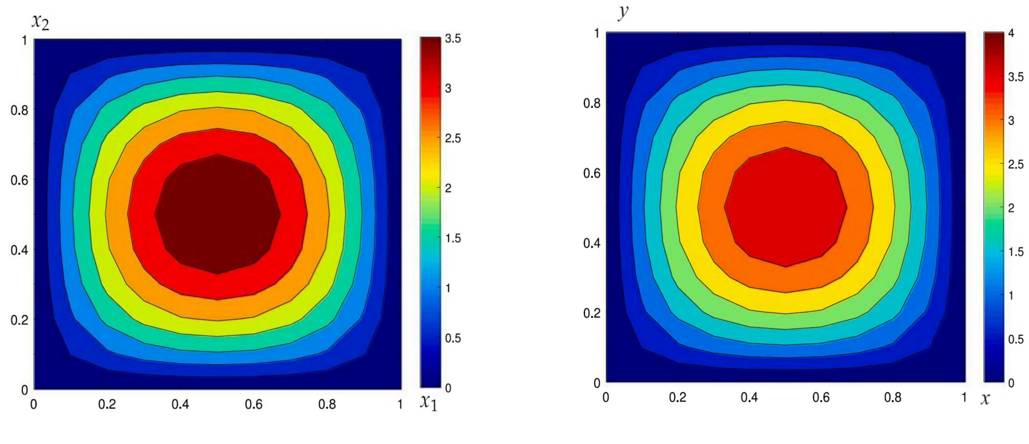

Figure 2. (a) The contour of the TGCNMFE solution of φ at $t = 10$; (b) The contour of the SGCNMFE solution of φ at $t = 10$.

Problem 5. *Find*

$$(\hat{\varpi}_h^k, \hat{\varphi}_h^k) \in \mathbb{W}_h \times \mathbb{W}_h \ (1 \leqslant k \leqslant K)$$

defined on the single fine grid $\Im_h$, satisfying the nonlinear system of equations:

$$\begin{cases} \left(\hat{\varpi}_h^{k+1} - 2\hat{\varpi}_h^k + \hat{\varpi}_h^{k-1}, \upsilon_h\right) + \dfrac{\Delta t}{2}(\nabla(\hat{\varphi}_h^{k+1} + \hat{\varphi}_h^{k-1}), \nabla \upsilon_h) + \dfrac{\Delta t}{2}(\hat{\varphi}_h^{k+1} + \hat{\varphi}_h^{k-1}, \upsilon_h) \\ = \dfrac{\Delta t}{2}(f(\hat{\varpi}_h^{k+1}) + f(\hat{\varpi}_h^{k-1}), \upsilon_h), \ \forall \upsilon_h \in \mathbb{W}_h, 1 \leqslant k \leqslant K-1, \\ \left(\nabla \hat{\varpi}_h^k, \nabla \vartheta_h\right) = (\hat{\varphi}_h^k, \vartheta_h), \quad \forall \vartheta_h \in \mathbb{W}_h, \quad 0 \leqslant k \leqslant K, \\ \hat{\varpi}_h^0 = R_h \varpi_0(\boldsymbol{x}), \quad \hat{\varphi}_h^0 = R_h \varphi^0, \quad \hat{\varpi}_h^1 = R_h \varpi_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \end{cases} \tag{4.1}$$

By comparing the contours of each pair of graphs in Figures 1 and 2, it can be easy to see that the TGCNMFE solutions are very close to the SGCNMFE solutions at $t = 10$.

To truly showcase the benefits of the TGCNMFE format, we record the CPU running time for finding the TGCNMFE and SGCNMFE solutions and their errors when $t = 2.0, 4.0, 6.0, 8.0$, and 10.0, where errors are estimated respectively by $\|\varpi_h^k - \varpi_h^{k-1}\|_0 + \|\varphi_h^k - \varphi_h^{k-1}\|_0$ and $\|\hat{\varpi}_h^k - \hat{\varpi}_h^{k-1}\|_0 + \|\hat{\varphi}_h^k - \hat{\varphi}_h^{k-1}\|_0$, shown in Table 1.

Table 1. The errors of the SGCNMFE and TGCNMFE solutions and CPU running-time at $t = 2, 4, 6, 8$, and 10.

t	SGCNMFE solutions errors	TGCNMFE solutions errors	SGCNMFE method CPU running-time	TGCNMFE method CPU running-time
2.0	2.2316×10^{-6}	3.0273×10^{-6}	215.332 s	112.056 s
4.0	2.4187×10^{-6}	3.1438×10^{-6}	216.662 s	113.112 s
6.0	2.5665×10^{-6}	3.2662×10^{-6}	217.153 s	114.224 s
8.0	2.7782×10^{-6}	3.3861×10^{-6}	218.709 s	115.451 s
10.0	2.8864×10^{-6}	3.4861×10^{-6}	219.618 s	116.872 s

The data in Table 1 show that the numerical errors of TGCNMFE and SGCNMFE solutions reach $O(10^{-6})$ when $t = 2.0, 4.0, 6.0, 8.0$, and 10.0, which coincide with the obtained theoretical errors, but the CPU running time for finding SGCNMFE solutions is almost twice as long as that for finding SGCNMFE solutions. Therefore, the TGCNMFE method is visibly superior to the SGCNMFE method, and the TGCNMFE method is feasible and effective to solve the NFOSG equation.

4.2. Three-dimensional numerical experiments

For the three-dimensional NFOSG equation, we take

$$\bar{\Omega} = [0,\ 1] \times [0,\ 1] \times [0,\ 1] \subset \mathbb{R}^3$$

and the initial functions

$$\varpi_0(\boldsymbol{x}) = \varpi_1(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3),$$

where it may be considered that there is an earthquake focus at center $(0.5, 0.5, 0.5)$ of the region Ω.

The fine grid division $\Im_h$ is composed of cubes with diagonal

$$h = \sqrt{3}/1000$$

and all cubes parallel to the coordinate axis. When $l = 1$, in order to satisfy

$$h = O(H^{1+1/l}),$$

i.e.,

$$h = O(H^2),$$

the coarse grid division $\Im_H$ is composed of cubes with diagonal

$$H = \sqrt[4]{3}/\sqrt{1000}$$

and all cubes also parallel to the coordinate axis. According to Theorem 2, the L^2 norm errors of the TGCNMFE solutions of the NFOSG equation can also theoretically reach $O(10^{-6})$ when

$$\Delta t = 1/1000.$$

We find the TGCNMFE solutions ϖ_h^k and φ_h^k by the TGCNMFE format at $t = 1$, and their contours are respectively shown in Figures 3a and 4a.

To show that the TGCNMFE method is superior to the SGCNMFE method, we also find the SGCNMFE solutions $\hat{\varpi}_h^k$ and $\hat{\varphi}_h^k$ at $t = 1$ by the SGCNMFE format above (Problem 5), which are respectively shown in Figures 3b and 4b.

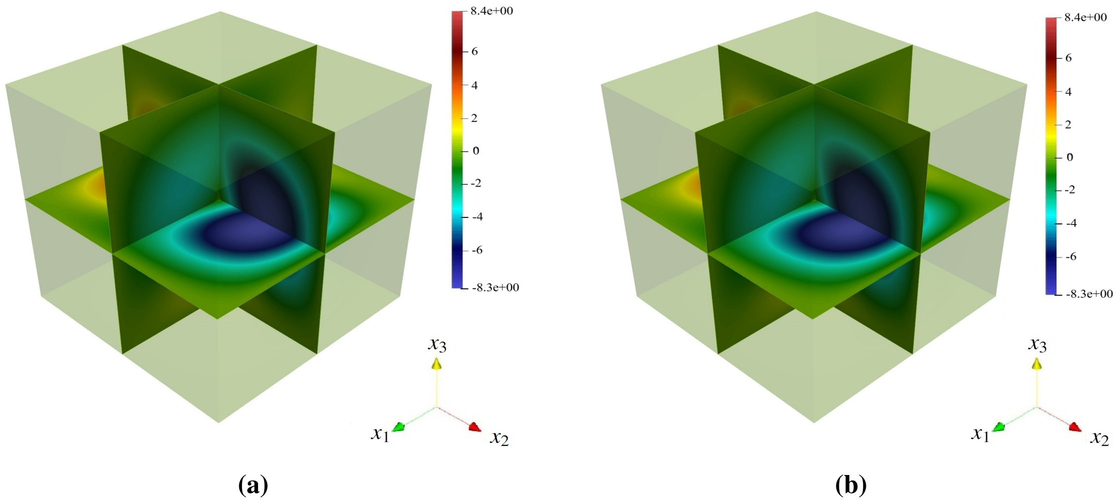

Figure 3. (a) The TGCNMFE solution of ϖ at $t = 1$. (b) The SGCNMFE solution of ϖ at $t = 1$.

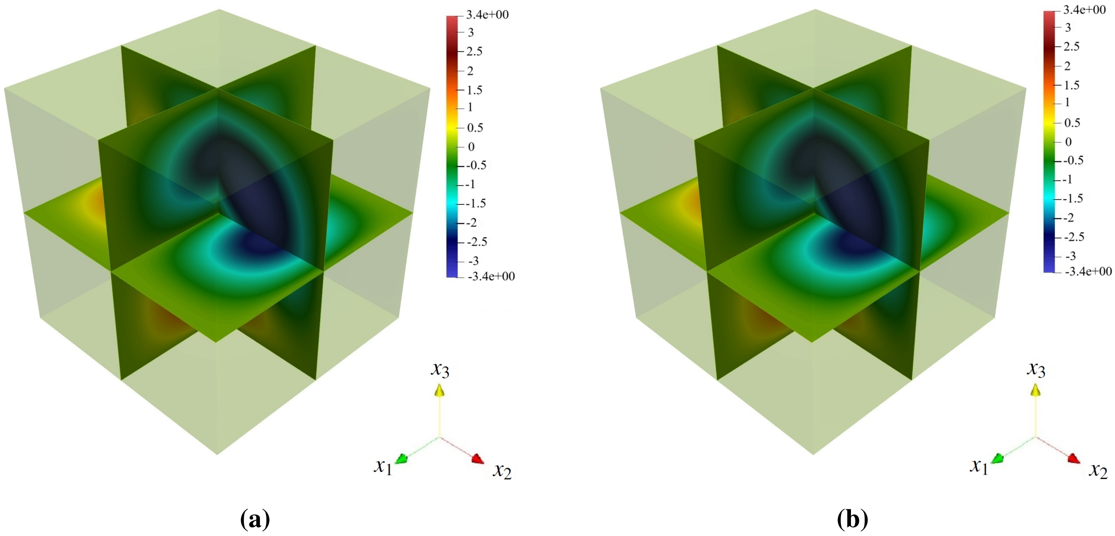

Figure 4. (a) The TGCNMFE solution of φ at $t = 1$. (b) The SGCNMFE solution of φ at $t = 1$.

By comparing the each pair of graphs in Figures 3 and 4, it can also be easy to see that the TGCNMFE solutions are almost same as the SGCNMFE solutions at $t = 1$.

To truly showcase the benefits of the TGCNMFE format, we also record the CPU running time for finding the TGCNMFE and SGCNMFE solutions and their errors when $t = 0.2, 0.4, 0.6, 0.8$, and 1.0, where errors are also respectively estimated by $\|\varpi_h^k - \varpi_h^{k-1}\|_0 + \|\varphi_h^k - \varphi_h^{k-1}\|_0$ and $\|\hat{\varpi}_h^k - \hat{\varpi}_h^{k-1}\|_0 + \|\hat{\varphi}_h^k - \hat{\varphi}_h^{k-1}\|_0$, shown in Table 2.

Table 2. The errors of the SGCNMFE and TGCNMMFE solutions and CPU running-time at $t = 0.2, 0.4, 0.6, 0.8$, and 1.0.

t	SGCNMFE solutions errors	TGCNMFE solutions errors	SGCNMFE method CPU running-time	TGCNMFE method CPU running-time
0.2	2.5781×10^{-6}	3.1563×10^{-6}	422.562 s	211.731 s
0.4	2.6436×10^{-6}	3.2253×10^{-6}	424.826 s	212.813 s
0.6	2.7841×10^{-6}	3.3265×10^{-6}	426.716 s	213.832 s
0.8	2.8862×10^{-6}	3.4453×10^{-6}	428.261 s	214.764 s
1.0	2.9631×10^{-6}	3.5626×10^{-6}	429.142 s	215.635 s

The data in Table 2 also show that when $t = 0.2$, 0.4, 0.6, 0.8, and 1.0, the numerical errors of the SGCNMFE and TGCNMFE solutions coincide with the theoretical errors $O(10^{-6})$, but the CPU running time for finding SGCNMFE solutions is also almost twice as long as that for finding SGCNMFE solutions. It is further shown that the TGCNMFE method is indeed better than the SGCNMFE method, and the TGCNMFE method is indeed feasible and effective to solve the NFOSG equation.

5. Conclusions and prospect

Above, we have proposed a new NFOSG equation, a new TSDMCN scheme, and a new TGCNMFE method for the NFOSG equation, and have strictly proved the existence, stability, and error estimates of the TSDMCN and TGCNMFE solutions theoretically. We have also employed the two sets of numerical experiments to confirm the correctness of the obtained theoretical results and showed the superiorities for the TGCNMFE method. The TSDMCN scheme and the TGCNMFE method for the NFOSG equation are first proposed in this paper. They are completely different from the existing methods in [10–12]. Therefore, they are original and indeed fire-new.

Although we only study the TGCNMFE method for the NFOSG equation, the method of this paper can be extended to the more complex nonlinear PDEs, for example, the nonlinear Cahn-Hilliard equation and the Schrödinger equation in [19, 20], in addition to the actual engineering nonlinear problems. Therefore, the TGCNMFE method has a wide range of applications.

Although the TGCNMFE method here can greatly simplify calculation, save CPU operating-time, and improve computational efficiency, when it is applied to settling practical engineering computations, it usually contains many (often more than tens of millions) unknowns and needs to take a long-time to obtain results on a computer. Thus, after a long computer operating time, owing to the accumulation of computing errors, the obtained TGCNMFE solutions could deviate from right solutions, or could even generate floating-point overflow, resulting in erroneous calculation results. Hence, in future study, we will use a proper orthogonal decomposition method (see [5, 21, 22]) to lessen the unknowns of the TGCNMFE method and develop some new reduced-dimensionality methods for the NFOSG equation.

Author contributions

Yanjie Zhou: conceptualization, investigation, methodology, validation, writing-original draft, formal analysis; Xianxiang Leng: conceptualization, methodology, formal analysis, writing-review and editing; Yuejie Li: validation, visualization, writing-review and editing; Qiuxiang Deng:

inspection, writing-review and editing; Zhendong Luo: conceptualization, investigation, methodology, formal analysis, writing-original draft, writing-review, editing and proofreading. All authors have read and agreed to the published version of the manuscript.

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Conflicts of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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