Mathematics
https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 31414–31443.
DOI:10.3934/math.20241513
Received: 19 July 2024
Revised: 12 October 2024
Accepted: 25 October 2024
Published: 05 November 2024

Research article

Limited type subsets of locally convex spaces

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Abstract: Let $1 \le p \le q \le \infty$. Being motivated by the classical notions of limited, p-limited, and coarse p-limited subsets of a Banach space, we introduce and study (p, q)-limited subsets and their equicontinuous versions and coarse p-limited subsets of an arbitrary locally convex space E. Operator characterizations of these classes are given. We compare these classes with the classes of bounded, (pre)compact, weakly (pre)compact, and relatively weakly sequentially (pre)compact sets. If E is a Banach space, we show that the class of coarse 1-limited subsets of E coincides with the class of $(1, \infty)$-limited sets, and if $1 < p < \infty$, then the class of coarse p-limited sets in E coincides with the class of p-(V^*) sets of Pełczyński. We also generalize a known theorem of Grothendieck.

Keywords: (p, q)-limited set; coarse p-limited set; p-(V^*) set; p-convergent operator; p-barrelled space
Mathematics Subject Classification: 46A03, 46E10

1. Introduction

Let E be a locally convex space (lcs for short), and let E' denote the topological dual of E. For a bounded subset $A \subseteq E$ and a functional $\chi \in E'$, we put

$$\|\chi\|_A := \sup\{|\chi(x)| : x \in A \cup \{0\}\}.$$

Definition 1.1. *A bounded subset A of a Banach space E is called limited if each weak* null sequence $\{\chi_n\}_{n\in\omega}$ in E' converges to zero uniformly on A, that is* $\lim_{n\to\infty} \|\chi_n\|_A = 0$. *Denote by* $\mathsf{L}(E)$ *the family of all limited subsets of E.*

It is well-known that any compact subset of a Banach space E is limited; on the other hand, by the Josefson-Nissenzweig theorem, the closed unit ball B_E of E is limited if, and only if, E is finite-dimensional. Limited sets in Banach spaces were systematically studied by Bourgain and Diestel [5]; see also Schlumprecht [38]. Among other things, they proved the following result (all relevant definitions are given in Section 2).

Theorem 1.1 ([5]). *Let E be a Banach space. Then,*

(i) $\mathsf{L}(E)$ *is closed under taking subsets, finite sums, and absolutely convex hulls;*
(ii) *if E contains no copy of ℓ_1, then each $A \in \mathsf{L}(E)$ is relatively weakly compact;*
(iii) *every $A \in \mathsf{L}(E)$ is weakly sequentially precompact;*
(iv) *if E is separable or reflexive, then each $A \in \mathsf{L}(E)$ is relatively compact.*

Let E and H be locally convex spaces. Denote by $\mathcal{L}(E, H)$ the family of all operators (= continuous and linear maps) from E to H. If $p \in [1, \infty]$, a sequence $\{x_n\}_{n\in\omega}$ in E is called *weakly p-summable* if for every $\chi \in E'$ it follows that $(\langle\chi, x_n\rangle) \in \ell_p$ if $p \in [1, \infty)$, or $(\langle\chi, x_n\rangle) \in c_0$ if $p = \infty$. The family $\ell_p^w(E)$ (or $c_0^w(E)$ if $p = \infty$) of all weakly p-summable sequences in E is a vector space which admits a natural locally convex vector topology such that it is complete if so is E; for details, see Section 19.4 in [27] or Section 4 in [17]. Analogously, we say that a sequence $\{\chi_n\}_{n\in\omega}$ in E' is *weak* p-summable* if $(\langle\chi_n, x\rangle) \in \ell_p$ (or $(\langle\chi_n, x\rangle) \in c_0$ if $p = \infty$) for every $x \in E$. For the basic theory of locally convex spaces, see the classical books [26, 27, 32, 34].

Let $p \in [1, \infty]$, and let X and Y be Banach spaces. Generalizing the notion of completely continuous operators, Castillo and Sánchez defined in [6] an operator $T : X \to Y$ to be *p-convergent* if T sends weakly p-summable sequences of X into norm null-sequences of Y. The influential article of Castillo and Sánchez [6] inspired an intensive study of p-versions of numerous geometrical properties of Banach spaces. In particular, the following p-versions of limitedness were introduced by Karn and Sinha [28] and Galindo and Miranda [20].

Definition 1.2. *Let $p \in [1, \infty]$, and let X be a Banach space. A bounded subset A of X is called*

(i) *a p-limited set if*
$$\big(\sup_{a\in A} |\langle\chi_n, a\rangle|\big) \in \ell_p \quad \big(\text{or } \big(\sup_{a\in A} |\langle\chi_n, a\rangle|\big) \in c_0 \ \text{ if } p = \infty\big),$$
for every $(\chi_n) \in \ell_p^w(X^*)$ (or $(\chi_n) \in c_0^w(X^*)$ if $p = \infty$) ([28]);
(ii) *a coarse p-limited set if for every $T \in \mathcal{L}(X, \ell_p)$ (or $T \in \mathcal{L}(X, c_0)$ if $p = \infty$), the set $T(A)$ is relatively compact ([20]).*

Every p-limited subset of X is coarse p-limited, but the converse is not true in general; see [20]. It turns out that the family $\mathsf{L}_p(X)$ of all p-limited subsets of X and the family $\mathsf{CL}_p(X)$ of all coarse p-limited subsets of X have similar properties as $\mathsf{L}(X)$, described in Theorem 1.1; see [28] and [20], respectively.

Limited sets in Fréchet spaces were studied by Alonso [1]. The notion of a limited set in general locally convex spaces was introduced by Lindström and Schlumprecht in [30] and independently by Banakh and Gabriyelyan in [3]. Since limited sets in the sense of [30] are defined using equicontinuity, to distinguish both notions we called them in [3] by $\mathcal{E}$-limited sets. Recall that a subset B of E' is equicontinuous if there is a neighborhood U of zero in E such that B is contained in the polar U°.

Definition 1.3. *A subset A of a locally convex space E is called*

(i) *$\mathcal{E}$-limited if $\|\chi_n\|_A \to 0$ for every equicontinuous weak* null sequence $\{\chi_n\}_{n\in\omega}$ in E' ([30]);*
(ii) *limited if $\|\chi_n\|_A \to 0$ for every weak* null sequence $\{\chi_n\}_{n\in\omega}$ in E' ([3]).*

It is clear that if E is a c_0-barrelled space (i.e., all weak* null sequences are equicontinuous; and Banach space is c_0-barrelled), then A is limited if, and only if, it is $\mathcal{E}$-limited.

Definitions 1.1–1.3 and the notions of (p,q)-(V^*) subsets and (p,q)-(EV^*) subsets of a locally convex space E introduced and studied in [17] motivate the following notions.

Definition 1.4. *Let $p,q \in [1,\infty]$. A non-empty subset A of a separated topological vector space E (i.e., E' separates the points of E) is called*

(i) *a (p,q)-limited set (resp., (p,q)-$\mathcal{E}$-limited set) if*

$$\left(\|\chi_n\|_A\right) \in \ell_q \ \text{ if } q<\infty, \quad \text{or} \quad \|\chi_n\|_A \to 0 \ \text{ if } q=\infty,$$

for every (resp., equicontinuous) weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. We denote by $\mathsf{L}_{(p,q)}(E)$ and $\mathsf{EL}_{(p,q)}(E)$ the family of all (p,q)-limited subsets and all (p,q)-$\mathcal{E}$-limited subsets of E, respectively. (p,p)-limited sets and (∞,∞)-limited sets will be called simply p-limited sets and limited sets, respectively.

(ii) *a coarse p-limited set if for every $T \in \mathcal{L}(E,\ell_p)$ (or $T \in \mathcal{L}(E,c_0)$ if $p=\infty$), the set $T(A)$ is relatively compact. The family of all coarse p-limited sets is denoted by $\mathsf{CL}_p(E)$.*

The purpose of the article is to study (p,q)-limited subsets and coarse p-limited subsets of locally convex spaces in the spirit of Theorem 1.1 and the articles [28] and [20]. Now, we describe the content of the article.

In Section 2, we fix the main notions and some auxiliary results used in what follows.

In Section 3, we study the classes $\mathsf{L}_{(p,q)}(E)$ and $\mathsf{EL}_{(p,q)}(E)$. In Lemma 3.1, we generalize (i) of Theorem 1.1, and show that $\mathsf{L}_{(p,q)}(E) = \mathsf{EL}_{(p,q)}(E) = \{0\}$ if $q<p$. In Proposition 3.1, we characterize (p,q)-limited subsets and (p,q)-$\mathcal{E}$-limited subsets in products and direct sums of locally convex spaces. In Theorem 3.1 we give an operator characterization of (p,q)-limited and (p,q)-$\mathcal{E}$-limited subsets of the locally convex space E. The following diagram easily follows from Definition 1.4 (see also (vi) of Lemma 3.1)

$$\left.\begin{array}{r}\text{limited}\\ (p,q)\text{-limited}\end{array}\right\} \Longrightarrow (p,\infty)\text{-limited} \Longrightarrow (1,\infty)\text{-limited}.$$

This diagram motivates the study of (p,∞)-limited sets and $(1,\infty)$-limited sets. It is well-known that any (pre)compact subset of a Banach space is limited. In Proposition 3.2, we generalize this useful result by showing that each precompact subset of an lcs E is (p,∞)-$\mathcal{E}$-limited, and if, in addition, E is p-barrelled, then every precompact subset of E is (p,∞)-limited. Consequently, each precompact subset of a c_0-barrelled space is limited. In Theorem 3.3, we show that every precompact subset of $C_p(X)$ (= the space $C(X)$ of all continuous functions over a Tychonoff space X endowed with the pointwise topology) is (p,q)-limited if, and only if, X has no infinite functionally bounded subsets. As a corollary (see Example 3.1), we obtain that the metrizable space $C_p([0,\omega])$ has even compact subsets which are *not* limited. Being motivated by (iv) of Theorem 1.1, it is natural to consider the case when every (p,q)-limited set is precompact. This problem is solved in Theorem 3.4. In Proposition 3.3, we characterize $(1,\infty)$-limited subsets of barrelled locally convex spaces. In [23] (see also Theorem 3.11 of [25]), Grothendieck proved that if E is a Banach space, then a bounded subset B of $(E',\mu(E',E))$ is precompact if, and only if, it is limited. In Theorem 3.5, we generalize this result.

In Section 4, we study coarse p-limited subsets of locally convex spaces. Generalizing Proposition 2 of [20], we show in Lemma 4.1 that the family $\mathsf{CL}_p(E)$ of all coarse p-limited sets in E is closed under taking subsets, finite unions, closed absolutely convex hulls, and continuous linear images.

In Proposition 4.1 we show that every p-limited subset of E is coarse p-limited (this generalizes Proposition 1 of [20]), and under addition assumption we prove that even every (p, p)-(V^*) subset of E is coarse p-limited. A description of coarse p-limited subsets of direct products and direct sums is given in Proposition 4.2.

Let $1 \le p \le q \le \infty$. In Section 16 of [17], we naturally extend the notion of p-convergent operators between Banach spaces to the general case saying that a linear map $T : E \to L$ between locally convex spaces E and L is *(q, p)-convergent* if it sends weakly p-summable sequences in E to strongly q-summable sequences in L (so (∞, p)-convergent operators are exactly p-convergent operators). The notion of (q, p)-convergent operators is useful to solve the following general problem: Characterize those operators T which map *all bounded* sets into (p, q)-limited sets (or into coarse p-limited sets). If E and L are Banach spaces and $p = q$, a partial answer to this problem is given by Ghenciu; see Theorem 14 of [22]. In Section 5 we give a complete answer to this problem; see Theorem 5.1. The clauses (ii)–(iv) of Theorem 1.1 motivate the problem of finding conditions on a space E under which (p, q)-limited sets and coarse p-limited sets have additional topological properties. For p-limited subsets of Banach spaces, this problem was considered by Ghenciu; see Theorem 15 of [22]. In Theorem 5.2, we generalize Ghenciu's result. In Theorem 5.5, we characterize coarse 1-limited sets. As a consequence of the obtained results, we show in Corollary 5.5 that: (1) if $p = 1$, then the class of coarse 1-limited subsets of a Banach space E coincides with the class of $(1, \infty)$-limited sets, and (2) if $1 < p < \infty$, then the class of coarse p-limited sets in E coincides with the class of p-(V^*) sets. It should be mentioned that p-(V^*) sets in Banach spaces were defined and studied by Chen, Chávez-Domínguez, and Li in [7] and [29]. Using the idea of the proof of (iii) of Theorem 1.1, Galindo and Miranda proved in Proposition 3 of [20] that if $2 \le p < \infty$, then every coarse p-limited set is weakly sequentially precompact. In Theorem 5.6, we extend this result to locally convex spaces with the Rosenthal property.

The clause (iv) of Theorem 1.1 implies that each separable or reflexive Banach space has the Gelfand-Phillips property. By this reason generalizations of this clause will be given in the forthcoming article [18].

2. Preliminaries results

We start with some necessary definitions and notations used in the article. Set $\omega := \{0, 1, 2, \dots\}$. All topological spaces are assumed to be Tychonoff (= completely regular and T_1). The closure of a subset A of a topological space X is denoted by $\overline{A}$, $\overline{A}^X$, or $\mathrm{cl}_X(A)$. A function $f : X \to Y$ between topological spaces X and Y is called *sequentially continuous* if for any convergent sequence $\{x_n\}_{n\in\omega} \subseteq X$, the sequence $\{f(x_n)\}_{n\in\omega}$ converges in Y and $\lim_n f(x_n) = f(\lim_n x_n)$. A subset A of a topological space X is called *functionally bounded in X* if every $f \in C(X)$ is bounded on A.

All topological vector spaces are over the field $\mathbb{F}$ of real or complex numbers. The closed unit ball of the field $\mathbb{F}$ is denoted by $\mathbb{D}$.

Let E be a locally convex space. The span of a subset A of E and its closure are denoted by $E_A := \mathsf{span}(A)$ and $\overline{\mathsf{span}}(A)$, respectively; the absolutely convex hull of A and its closure are denoted by $\mathrm{acx}(A)$ and $\overline{\mathrm{acx}}(A)$, respectively. We denote by $\mathcal{N}_0(E)$ (resp., $\mathcal{N}_0^c(E)$) the family of all (resp., closed absolutely convex) neighborhoods of zero of E. The family of all bounded subsets of E is denoted by $\mathsf{Bo}(E)$. The value of $\chi \in E'$ on $x \in E$ is denoted by $\langle \chi, x \rangle$ or $\chi(x)$. A sequence $\{x_n\}_{n\in\omega}$ in E is said to be *Cauchy* if for every $U \in \mathcal{N}_0(E)$ there is $N \in \omega$ such that $x_n - x_m \in U$ for all $n, m \geq N$. It is easy to see that a sequence $\{x_n\}_{n\in\omega}$ in E is Cauchy if, and only if, $x_{n_k} - x_{n_{k+1}} \to 0$ for every (strictly) increasing sequence (n_k) in ω. If E is a normed space, B_E denotes the closed unit ball of E.

For an lcs E, we denote by E_w and E_β the space E endowed with the weak topology $\sigma(E, E')$ and with the strong topology $\beta(E, E')$, respectively. The topological dual space E' of E endowed with weak* topology $\sigma(E', E)$ or with the strong topology $\beta(E', E)$ is denoted by E'_{w^*} or E'_β, respectively. The closure of a subset A in the weak topology is denoted by $\overline{A}^{w}$ or $\overline{A}^{\sigma(E,E')}$, and $\overline{B}^{w^*}$ (or $\overline{B}^{\sigma(E',E)}$) denotes the closure of $B \subseteq E'$ in the weak* topology. The *polar* of a subset A of E is denoted by $A^\circ := \{\chi \in E' : \|\chi\|_A \leq 1\}$. A subset B of E' is *equicontinuous* if $B \subseteq U^\circ$ for some $U \in \mathcal{N}_0(E)$.

A subset A of a locally convex space E is called

- *precompact* if for every $U \in \mathcal{N}_0(E)$ there is a finite set $F \subseteq E$ such that $A \subseteq F + U$;
- *sequentially precompact* if every sequence in A has a Cauchy subsequence;
- *weakly* (*sequentially*) *compact* if A is (sequentially) compact in E_w;
- *relatively weakly compact* if its weak closure $\overline{A}^{\sigma(E,E')}$ is compact in E_w;
- *relatively weakly sequentially compact* if each sequence in A has a subsequence weakly converging to a point of E;
- *weakly sequentially precompact* if each sequence in A has a weakly Cauchy subsequence.

Note that each sequentially precompact subset of E is precompact, but the converse is not true in general; see Lemma 2.2 of [17].

In what follows, we shall actively use the following classical completeness type properties and weak barrelledness conditions. A locally convex space E is

- *quasi-complete* if each closed bounded subset of E is complete;
- *sequentially complete* if each Cauchy sequence in E converges;
- *locally complete* if the closed absolutely convex hull of a null sequence in E is compact;
- (*quasi*)*barrelled* if every $\sigma(E', E)$-bounded (resp., $\beta(E', E)$-bounded) subset of E' is equicontinuous;
- c_0-(*quasi*)*barrelled* if every $\sigma(E', E)$-null (resp., $\beta(E', E)$-null) sequence is equicontinuous.

It is well-known that $C_p(X)$ is quasibarrelled for every Tychonoff space X.

Recall that a locally convex space (E, τ) has the *Schur property* (resp., the *Glicksberg property*) if E and E_w have the same convergent sequences (resp., the same compact sets). If an lcs E has the Glicksberg property, then it has the Schur property. The converse is true for strict (*LF*)-spaces (in particular, for Banach spaces), but not in general; see Corollary 2.13 and Proposition 3.5 of [14]. We shall use the next two lemmas.

Lemma 2.1. *A locally convex space (E, τ) has the Schur property if, and only if, E and E_w have the same relatively sequentially compact sets.*

Proof. Assume that (E, τ) has the Schur property. If A is a relatively sequentially compact subset of E, then evidently A is relatively sequentially compact in E_w. Conversely, let A be a relatively sequentially compact subset of E_w. Take a sequence $S = \{a_n\}_{n\in\omega}$ in A. Then S has a subsequence $\{a_{n_k}\}_{k\in\omega}$ weakly converging to a point $x \in E$. By the Schur property $a_{n_k} \to x$ in τ. Hence A is a relatively sequentially compact subset of E. Thus, E and E_w have the same relatively sequentially compact sets.

Assume that E and E_w have the same relatively sequentially compact sets. To show that E has the Schur property, let $S = \{a_n\}_{n\in\omega}$ be a weakly null sequence. Then, S is relatively sequentially compact in E_w and hence, also in E. We show that $a_n \to 0$ also in E. Suppose for a contradiction that there is a $U \in \mathcal{N}_0(E)$ such that $a_n \notin U$ for each $n \in I$ for some infinite $I \subseteq \omega$. Since S is relatively sequentially compact in E, the sequence $S' = \{a_n\}_{n\in I}$ has a subsequence $\{b_n\}_{n\in\omega}$ converging to some point $b \in E$. As $\{b_n\}_{n\in\omega}$ is also weakly null, we have $b = 0$. However, since $\{b_n\}_{n\in\omega} \subseteq S'$, it follows that $b_n \notin U$ for every $n \in \omega$; so $b_n \not\to b$, a contradiction. □

Lemma 2.2. *Every weakly sequentially precompact subset A of a Schur space E is sequentially precompact. If, in addition, E is sequentially complete, then E_w is sequentially complete.*

Proof. Let $\{x_n\}_{n\in\omega}$ be a sequence in A. As A is weakly sequentially precompact, there is a subsequence $\{y_n\}_{n\in\omega}$ of $\{x_n\}_{n\in\omega}$ which is weakly Cauchy. Let (n_k) be a strictly increasing sequence in ω. Then, $\{y_{n_{k+1}} - y_{n_k}\}_{n\in\omega}$ is weakly null. By the Schur property of E, we obtain that $y_{n_{k+1}} - y_{n_k} \to 0$ in E. Thus, $\{y_n\}_{n\in\omega}$ is a Cauchy sequence in E, and hence, A is sequentially precompact.

Assume that E is, in addition, a sequentially complete space. Let $S = \{x_n\}_{n\in\omega}$ be a weakly Cauchy sequence. As we proved above, S is a Cauchy sequence in E. Since E is sequentially complete, there is $a \in E$ such that $x_n \to a$. Thus, x_n converges to a also in the weak topology. □

Two vector topologies τ and $\mathcal{T}$ on a vector space L are called *compatible* if $(L,\tau)' = (L,\mathcal{T})'$ algebraically. If (E,τ) is a locally convex space, then there is a finest locally convex vector topology $\mu(E,E')$ compatible with τ. The topology $\mu(E,E')$ is called the *Mackey topology*, and if $\tau = \mu(E,E')$, the space E is called a *Mackey space*. Set $E_\mu := (E,\mu(E,E'))$. It is well-known that any quasibarrelled space is Mackey; see Theorem 11.11.6 of [32].

Recall that an lcs E is called *semi-reflexive* if the canonical map $J_E : E \to E'' = (E'_\beta)'_\beta$ defined by $\langle J_E(x),\chi\rangle := \langle\chi,x\rangle$ ($\chi \in E'$) is an isomorphism; if in addition J_E is a topological isomorphism, the space E is called *reflexive*. Each reflexive space is barrelled; see Proposition 11.4.2 of [27].

We denote by $\bigoplus_{i\in I} E_i$ and $\prod_{i\in I} E_i$ the locally convex direct sum and the topological product of a nonempty family $\{E_i\}_{i\in I}$ of locally convex spaces, respectively. If $0 \neq \mathbf{x} = (x_i) \in \bigoplus_{i\in I} E_i$, then the set $\mathrm{supp}(\mathbf{x}) := \{i \in I : x_i \neq 0\}$ is called the *support* of $\mathbf{x}$. The *support* of a subset A, $\{0\} \subsetneq A$, of $\bigoplus_{i\in I} E_i$ is the set $\mathrm{supp}(A) := \bigcup_{a\in A} \mathrm{supp}(a)$. We shall also consider elements $\mathbf{x} = (x_i) \in \prod_{i\in I} E_i$ as functions on I and write $\mathbf{x}(i) := x_i$.

Below, we recall some of the basic classes of compact-type operators.

Definition 2.1. *Let E and L be locally convex spaces. An operator $T \in \mathcal{L}(E,L)$ is called compact (resp., sequentially compact, precompact, sequentially precompact, weakly compact, weakly sequentially compact, weakly sequentially precompact, bounded) if there is $U \in \mathcal{N}_0(E)$ such that $T(U)$ a relatively compact (relatively sequentially compact, precompact, sequentially precompact, relatively weakly compact, relatively weakly sequentially compact, weakly sequentially precompact or bounded) subset of E.*

Let $p \in [1,\infty]$. Then, p^* is defined to be the unique element of $[1,\infty]$ which satisfies $\frac{1}{p} + \frac{1}{p^*} = 1$. For $p \in [1,\infty)$, the space ℓ_{p^*} is the dual space of ℓ_p. We denote by $\{e_n\}_{n\in\omega}$ the canonical basis of ℓ_p, if $1 \le p < \infty$, or the canonical basis of c_0, if $p = \infty$. The canonical basis of ℓ_{p^*} is denoted by $\{e_n^*\}_{n\in\omega}$. Denote by ℓ_p^0 and c_0^0 the linear span of $\{e_n\}_{n\in\omega}$ in ℓ_p or c_0 endowed with the induced norm topology, respectively. We shall use repeatedly the following well-known description of relatively compact subsets of ℓ_p and c_0, see [8, p. 6].

Proposition 2.1. (i) *A bounded subset A of ℓ_p, $p \in [1,\infty)$, is relatively compact if, and only if,*

$$\lim_{m\to\infty} \sup\Big\{ \sum_{m\le n} |x_n|^p : x = (x_n) \in A\Big\} = 0.$$

(ii) *A bounded subset A of c_0 is relatively compact if, and only if,* $\lim_{n\to\infty} \sup\{|x_n| : x = (x_n) \in A\} = 0$.

One of the most important classes of locally convex spaces is the class of free locally convex spaces introduced by Markov in [31]. The *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map f from X to a locally convex space E gives rise to a unique continuous linear operator $\Psi_E(f) : L(X) \to E$ with $f = \Psi_E(f) \circ i$. The free locally convex space $L(X)$ always exists and is essentially unique, and X is the Hamel basis of $L(X)$. So, each nonzero $\chi \in L(X)$ has a unique decomposition $\chi = a_1 i(x_1) + \cdots + a_n i(x_n)$, where all a_k are nonzero and x_k are distinct. The set $\mathrm{supp}(\chi) := \{x_1, \dots, x_n\}$ is called the *support* of χ. In what follows, we shall identify $i(x)$ with x and consider $i(x)$ as the Dirac measure δ_x at the point $x \in X$. We also recall that $C_p(X)' = L(X)$ and $L(X)' = C(X)$. It is worth mentioning that $L(X)$ has the Glicksberg property for every Tychonoff space X, and if X is non-discrete, then $L(X)$ is not a Mackey space; see [15] and [12], respectively.

Let $p \in [1,\infty]$. A sequence $\{x_n\}_{n\in\omega}$ in a locally convex space E is called

- *weakly p-summable* if for every $\chi \in E'$, it follows:

$$(\langle \chi, x_n\rangle)_{n\in\omega} \in \ell_p \text{ if } p < \infty, \text{ and } (\langle \chi, x_n\rangle)_{n\in\omega} \in c_0 \text{ if } p = \infty;$$

- *weakly p-convergent to $x \in E$* if $\{x_n - x\}_{n\in\omega}$ is weakly p-summable;
- *weakly p-Cauchy* if for each pair of strictly increasing sequences $(k_n), (j_n) \subseteq \omega$, the sequence $(x_{k_n} - x_{j_n})_{n\in\omega}$ is weakly p-summable.

A sequence $\{\chi_n\}_{n\in\omega}$ in E' is called *weak* p-summable* (resp., *weak* p-convergent to $\chi \in E'$* or *weak* p-Cauchy*) if it is weakly p-summable (resp., weakly p-convergent to $\chi \in E'$ or weakly p-Cauchy) in E'_{w^*}.

The following weak barrelledness conditions introduced and studied in [17] will play a considerable role in the article. Let $p \in [1,\infty]$. A locally convex space E is called *p-barrelled* (resp., *p-quasibarrelled*) if every weakly p-summable sequence in E'_{w^*} (resp., in E'_β) is equicontinuous.

We shall consider also the following linear map introduced in [17]

$$S_p : \mathcal{L}(E, \ell_p) \to \ell_p^w(E'_{w^*}) \quad (\text{or } S_\infty : \mathcal{L}(E, c_0) \to c_0^w(E'_{w^*}) \text{ if } p = \infty)$$

defined by $S_p(T) := (T^*(e_n^*))_{n\in\omega}$.

The following class of subsets of an lcs E was introduced and studied in [17], and it generalizes the notion of p-(V^*) subsets of Banach spaces defined in [7].

Definition 2.2. *Let $p, q \in [1,\infty]$. A nonempty subset A of a locally convex space E is called a (p,q)-(V^*) set (resp., a (p,q)-(EV^*) set) if*

$$\Big(\sup_{a\in A} |\langle \chi_n, a\rangle|\Big) \in \ell_q \ \ \text{if } q < \infty, \quad \text{or} \quad \Big(\sup_{a\in A} |\langle \chi_n, a\rangle|\Big) \in c_0 \ \ \text{if } q = \infty,$$

for every (resp., equicontinuous) weakly p-summable sequence $\{\chi_n\}_{n\in\omega}$ *in* E'_β. (p,∞)-(V^*) *sets and* $(1,\infty)$-(V^*) *sets will be called simply p-*(V^*) *sets and* (V^*) *sets, respectively. Analogously,* (p,∞)-(EV^*) *sets and* $(1,\infty)$-(EV^*) *sets will be called p-*(EV^*) *sets and* (EV^*) *sets, respectively.*

The family of all (p,q)-(V^*) sets (resp. p-(V^*) sets, (p,q)-(EV^*) sets, (V^*) sets etc.) of an lcs E is denoted by $\mathsf{V}^*_{(p,q)}(E)$ (resp. $\mathsf{V}^*_p(E)$, $\mathsf{EV}^*_{(p,q)}(E)$, $\mathsf{V}^*(E)$ etc.).

Following [17], a nonempty subset B of E' is called a (p,q)-(V) *set* if

$$\Big(\sup_{\chi\in B}|\langle\chi,x_n\rangle|\Big)\in\ell_q \ \text{ if } q<\infty, \quad\text{or}\quad \Big(\sup_{\chi\in B}|\langle\chi,x_n\rangle|\Big)\in c_0 \ \text{ if } q=\infty,$$

for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E. (p,∞)-(V) sets and $(1,\infty)$-(V) sets will be called simply p-(V) *sets* and (V) *sets*, respectively.

Let $1\le p\le q\le\infty$, and let E and L be locally convex spaces. Following [17], a linear map $T:E\to L$ is called (q,p)-*convergent* if it sends weakly p-summable sequences in E to strongly q-summable sequences in L.

The following p-versions of weakly compact-type properties are defined in [17] generalizing the corresponding notions in the class of Banach spaces introduced in [6] and [22]. Let $p\in[1,\infty]$. A subset A of a locally convex space E is called

- (*relatively*) *weakly sequentially p-compact* if every sequence in A has a weakly p-convergent subsequence with limit in A (resp., in E);
- *weakly sequentially p-precompact* if every sequence from A has a weakly p-Cauchy subsequence.

A Tychonoff space X is called *Fréchet-Urysohn* if for any cluster point $a\in X$ of a subset $A\subseteq X$ there is a sequence $\{a_n\}_{n\in\omega}\subseteq A$ which converges to a. A Tychonoff space X is called an *angelic space* if (1) every relatively countably compact subset of X is relatively compact, and (2) any compact subspace of X is Fréchet-Urysohn. Note that any subspace of an angelic space is angelic, and a subset A of an angelic space X is compact if, and only if, it is countably compact if, and only if, A is sequentially compact; see Lemma 0.3 of [35].

Let $p\in[1,\infty]$. Following [16], a locally convex space E is called a *weakly sequentially p-angelic space* if the family of all relatively weakly sequentially p-compact sets in E coincides with the family of all relatively weakly compact subsets of E. The space E is a *weakly p-angelic space* if it is a weakly sequentially p-angelic space and each weakly compact subset of E is Fréchet-Urysohn.

3. Limited-type sets in locally convex spaces

Many of the results included in this section are similar to the corresponding results from [17] in which (p,q)-(V^*) sets are substituted by (p,q)-limited sets. In the next lemma we summarize some basic elementary properties of (p,q)-limited sets, cf. (i) of Theorem 1.1 (compare this lemma with Lemma 7.2 of [17]).

Lemma 3.1. *Let* $p,q\in[1,\infty]$, *and let* (E,τ) *be a locally convex space. Then,*

(i) *every* (p,q)*-limited set is* (p,q)*-*$\mathcal{E}$*-limited; the converse is true if* E *is a p-barrelled space;*
(ii) *every* (p,q)*-*$\mathcal{E}$*-limited set in* E *is bounded;*

(iii) *the family of all (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) sets in E is closed under taking subsets, finite unions and sums, and closed absolutely convex hulls;*

(iv) *the family of all (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) sets in E is closed under taking continuous linear images; in particular, if H is a subspace of E, then every (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in H is (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) in E;*

(v) *a subset A of E is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set if, and only if, every countable subset of A is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set;*

(vi) *if $p',q'\in[1,\infty]$ are such that $p'\le p$ and $q\le q'$, then every (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E is also (p',q')-limited (resp., (p',q')-$\mathcal{E}$-limited); in particular, any (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set is $(1,\infty)$-limited (resp., $(1,\infty)$-$\mathcal{E}$-limited);*

(vii) *the property of being a (p,q)-limited set depends only on the duality (E,E'), i.e., if $\mathcal{T}$ is a locally convex vector topology on E compatible with the topology τ of E, then the (p,q)-limited sets of $(E,\mathcal{T})$ are exactly the (p,q)-limited sets of (E,τ);*

(viii) *every (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E is a (p,q)-(V^*) (resp., (p,q)-(EV^*)) set; the converse is true for semi-reflexive spaces;*

(ix) *every (p,q)-limited subset of E'_β is a (p,q)-(V) set;*

(x) *if $q<p$ and A is a (p,q)-$\mathcal{E}$-limited subset of E, then $A=\{0\}$;*

(xi) *if $q\ge p$, then any finite subset of E is (p,q)-limited;*

(xii) *a bounded subset A of E is (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) if, and only if, for every sequence $\{x_n\}_{n\in\omega}$ in A and each (resp., equicontinuous) weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E', it follows that $(|\langle\chi_n,x_n\rangle|)\in\ell_q$ (or $\in c_0$ if $p=\infty$).*

Proof. (i) and (iii) are clear, and (viii) follows from Definitions 1.4 and 2.2 and the trivial fact that every (equicontinuous) weakly p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'_β is (resp., equicontinuous) weak* p-summable in E'. The clause (ii) follows from (viii) and (ii) of Lemma 7.2 of [17] (which states that every (p,q)-(EV^*) set is bounded). (vii) follows from the definition of (p,q)-limited sets, and (ix) follows from the easy fact that for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E, the sequence $\{J_E(x_n)\}_{n\in\omega}$ is weak* p-summable in E''.

(iv) Let $T:E\to L$ be an operator from E to an lcs L, and let A be a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E. Observe that the adjoint map $T^*:L'_{w^*}\to E'_{w^*}$ is continuous. Fix a (resp., equicontinuous) weak* p-summable sequence $S=\{\chi_n\}_{n\in\omega}$ in L'. It is easily seen (see Lemma 4.5 of [17]) that the sequence $\{T^*(\chi_n)\}$ is weak* p-summable in E'. If, in addition, the sequence S is equicontinuous, then its image $T^*(S)$ is equicontinuous as well.

Therefore,

$$\Big(\sup_{a\in A}|\langle\chi_n,T(a)\rangle|\Big)=\Big(\sup_{a\in A}|\langle T^*(\chi_n),a\rangle|\Big)\in\ell_q\ \ (\text{or}\ \ \in c_0\ \text{ if } q=\infty).$$

Thus, $T(A)$ is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in L.

The last assertion follows from the proved one applied to the identity embedding $T:H\to E$.

(v) The necessity follows from (iii). To prove the sufficiency suppose for a contradiction that A is not a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E. Then, there is a (resp., equicontinuous) weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E' such that

$$\Big(\sup_{a\in A}|\langle\chi_n,a\rangle|\Big)\notin\ell_q\ \text{ if } q<\infty,\quad\text{or}\quad\Big(\sup_{a\in A}|\langle\chi_n,a\rangle|\Big)\notin c_0\ \text{ if } q=\infty.$$

Assume that $q < \infty$ (the case $q = \infty$ can be considered analogously). For every $n \in \omega$, choose $a_n \in A$ such that $|\langle \chi_n, a_n\rangle| \geq \frac{1}{2} \cdot \sup_{a\in A} |\langle \chi_n, a\rangle|$. Then,

$$\sum_{n\in\omega} |\langle \chi_n, a_n\rangle|^q \geq \frac{1}{2^q} \sum_{n\in\omega} \big(\sup_{a\in A} |\langle \chi_n, a\rangle|\big)^q = \infty.$$

Thus, the countable subset $\{a_n\}_{n\in\omega}$ of A is not a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E, a contradiction.

(vi) Take any (resp., equicontinuous) weak* p'-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. Since $p' \leq p$, $\{\chi_n\}_{n\in\omega}$ is also (resp., equicontinuous) weak* p-summable and hence, $\big(\sup_{a\in A} |\langle \chi_n, a\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q = \infty$). It remains to note that $\ell_q \subseteq \ell_{q'}$ because $q \leq q'$.

(x) Let $q < p$ and A be a (p,q)-$\mathcal{E}$-limited subset of E. Then, by (viii), A is a (p,q)-(EV^*) set. Therefore, by Proposition 7.5 of [17], $A = \{0\}$.

(xi) By (iii), it suffices to show that $A = \{x\}$ is a (p,q)-limited set for every $x \in E$. Let $\{\chi_n\}_{n\in\omega}$ be a weak* p-summable sequence in E'_β. Then, $(\langle \chi_n, x\rangle) \in \ell_p$ (or $\in c_0$ if $p = \infty$). Since $p \leq q$ it follows that $\big(\sup_{x\in A} |\langle \chi_n, x\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q = \infty$). Thus, A is a (p,q)-limited set.

(xii) The necessity is clear. To prove the sufficiency, for every $n \in \omega$, choose $x_n \in A$ such that $|\langle \chi_n, x_n\rangle| \geq \frac{1}{2} \sup_{a\in A} |\langle \chi_n, a\rangle|$. By assumption, $(|\langle \chi_n, x_n\rangle|) \in \ell_q$ (or $\in c_0$ if $q = \infty$). Therefore, also $\big(\sup_{a\in A} |\langle \chi_n, a\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q = \infty$). Thus, A is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set. □

According to (viii) of Lemma 3.1, every (p,q)-limited set is a (p,q)-(V^*) set, but the converse is not true in general; see Corollary 3.2 below. It follows from (x) and (xi) that there is sense to consider only the case when $1 \leq p \leq q \leq \infty$.

Notation 3.1. *The family of all (p,q)-limited (resp., p-limited, limited, (p,q)-$\mathcal{E}$-limited, p-$\mathcal{E}$-limited, or $\mathcal{E}$-limited) sets of an lcs E is denoted by $\mathsf{L}_{(p,q)}(E)$ (resp., $\mathsf{L}_p(E)$, $\mathsf{L}(E)$, $\mathsf{EL}_{(p,q)}(E)$, $\mathsf{EL}_p(E)$, or $\mathsf{EL}(E)$).*

Below we characterize (p,q)-limited sets in products and direct sums.

Proposition 3.1. *Let $1 \leq p \leq q \leq \infty$, and let $\{E_i\}_{i\in I}$ be a nonempty family of locally convex spaces. Then,*

(i) *a subset K of $E = \prod_{i\in I} E_i$ is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set if, and only if, so are all its coordinate projections;*
(ii) *a subset K of $E = \bigoplus_{i\in I} E_i$ is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set if, and only if, so are all its coordinate projections and the support of K is finite.*

Proof. The necessity follows from (iv) of Lemma 3.1 because E_i is a direct summand of E and, for the case (ii), the well-known fact that any bounded subset of a locally convex direct sum has finite support.

To prove the sufficiency, let K be a subset of E such that each projection K_i of K is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E_i, and, for the case (ii), $K_i = \{0\}$ for all but finitely many indices $i \in I$. We distinguish between the cases (i) and (ii).

(i) Take an arbitrary (resp., equicontinuous) weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E', where $\chi_n = (\chi_{i,n})_{i\in I}$. By Lemma 4.18 of [17], the sequence $\{\chi_n\}_{n\in\omega}$ has finite support $F \subseteq I$ (i.e., $\chi_{i,n} = 0$ for all $n \in \omega$ and $i \in I\backslash F$), and for every $i \in F$, each sequence $\{\chi_{i,n}\}_{n\in\omega}$ is weak* p-summable in E'_i. If in addition

$\{\chi_n\}_{n\in\omega}$ is equicontinuous, then for every $i \in F$, the sequence $\{\chi_{i,n}\}_{n\in\omega} \subseteq E_i'$ is also equicontinuous (indeed, if $T_i : E_i \to E$ is the identity embedding, then $\{\chi_{i,n}\}_n = \{T_i^*(\chi_n)\}_n$ is equicontinuous). Then,

$$\sup_{x\in K} |\langle \chi_n, x\rangle| = \sup_{x\in K} \Big| \sum_{i\in F} \langle \chi_{i,n}, x(i)\rangle \Big| \le \sum_{i\in F} \sup_{x(i)\in K_i} |\langle \chi_{i,n}, x(i)\rangle|.$$

Since all K_i are (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) sets, we have $\big(\sup_{x(i)\in K_i} |\langle \chi_{i,n}, x(i)\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q=\infty$). Therefore, also $\big(\sup_{x\in K} |\langle \chi_n, x\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q=\infty$). Thus, K is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E.

(ii) Let $F \subseteq I$ be the finite support of K. Take an arbitrary (resp., equicontinuous) weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'_β, where $\chi_n = (\chi_{i,n})_{i\in I}$ with $\chi_{i,n} \in E_i'$. As in (i) above, if $\{\chi_n\}_{n\in\omega}$ is equicontinuous, then for every $i \in F$, the sequence $\{\chi_{i,n}\}_{n\in\omega} \subseteq E_i'$ is also equicontinuous. Then, by Lemma 4.18 of [17], for every $i \in F$, the sequence $\{\chi_{i,n}\}_{n\in\omega}$ is weak* p-summable in E_i' and hence,

$$\sup_{x\in K} |\langle \chi_n, x\rangle| = \sup_{x\in K} \Big| \sum_{i\in F} \langle \chi_{i,n}, x(i)\rangle \Big| \le \sum_{i\in F} \sup_{x(i)\in K_i} |\langle \chi_{i,n}, x(i)\rangle|.$$

Since all K_i are (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) sets, we have $\big(\sup_{x(i)\in K_i} |\langle \chi_{i,n}, x(i)\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q=\infty$). Therefore, also $\big(\sup_{x\in K} |\langle \chi_n, x\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q=\infty$). Thus, K is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set in E'. □

Let A be a bounded subset of a locally convex space E. We denote by $\ell_1^0(A)$ the subspace of the Banach space $\ell_1(A)$ consisting of all vectors with finite support. Then, by Proposition 16.10 of [17], the map $T_A : \ell_1^0(A) \to E$ defined by

$$T_A(\lambda_0 a_0 + \cdots + \lambda_n a_n) := \lambda_0 a_0 + \cdots + \lambda_n a_n \quad (n\in\omega,\ \lambda_0,\dots,\lambda_n \in \mathbb{F},\ a_0,\dots,a_n \in A) \tag{3.1}$$

is an operator. Now, we characterize (p,q)-limited sets (compare with Theorem 16.11 of [17]).

Theorem 3.1. *Let $1 \le p \le q \le \infty$, and let E be a (resp., p-barrelled) locally convex space. Then a bounded subset A of E is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set if, and only if, the adjoint operator $T_A^* : E'_{w^*} \to \ell_\infty(A)$ is (q,p)-convergent.*

Proof. Consider an operator $T_A : \ell_1^0(A) \to E$ defined in (3.1). Observe that for each $\chi \in E'$, the a-th coordinate $T_A^*(\chi)(a)$ of $T_A^*(\chi)$ is

$$T_A^*(\chi)(a) = \langle T_A^*(\chi), a\rangle = \langle \chi, T_A(a)\rangle = \langle \chi, a\rangle,$$

and hence,

$$\|T_A^*(\chi)\|_{\ell_\infty(A)} = \sup_{a\in A} |T_A^*(\chi)(a)| = \sup_{a\in A} |\langle \chi, a\rangle|. \tag{3.2}$$

Now, by definition, a subset A of E is a (p,q)-limited set if, and only if, $\big(\sup_{a\in A} |\langle \chi_n, a\rangle|\big) \in \ell_q$ (or $\in c_0$ if $q=\infty$) for every weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E', and hence, by (3.2), if, and only if, $\big(\|T_A^*(\chi_n)\|_{\ell_\infty(A)}\big) \in \ell_q$ (or $\in c_0$ if $q=\infty$) for every weakly p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'_{w^*}, i.e., T_A^* is a (q,p)-convergent linear map.

The case when E is p-barrelled follows from the fact that (p,q)-limited subsets of E are exactly (p,q)-$\mathcal{E}$-limited (see (i) of Lemma 3.1). □

We select the next theorem (compare with Theorem 16.8 of [17]).

Theorem 3.2. *Let $1 \leq p \leq q \leq \infty$, E be a locally convex space, and let T be an operator from a normed space L to E. Then, $T(B_L)$ is a (p,q)-limited subset of E if, and only if, $T^* : E'_{w^*} \to L'_\beta$ is (q,p)-convergent.*

Proof. Observe that for every $\chi \in E'$, we have

$$\|T^*(\chi)\|_{L'_\beta} = \sup_{y\in B_L} |\langle T^*(\chi), y\rangle| = \sup_{y\in B_L} |\langle \chi, T(y)\rangle|. \tag{3.3}$$

Let $\{\chi_n\}_{n\in\omega}$ be a weak* p-summable sequence in E'. Then, by (3.3), we have

$$\big(\|T^*(\chi_n)\|_{L'_\beta}\big)_{n\in\omega} = \big(\sup_{y\in B_L} |\langle \chi_n, T(y)\rangle|\big)_{n\in\omega}.$$

Now, the theorem follows from the definition of (p,q)-limited sets and the definition of (q,p)-convergent linear map. □

It is natural to find some classes of subsets which are (p,q)-limited. Below, under additional assumption on an lcs E, we show that any precompact subset A of E is (p,∞)-limited (compare with Proposition 7.6 of [17]).

Proposition 3.2. *Let $p \in [1,\infty]$, and let E be a locally convex space.*

(i) *Every precompact subset A of E is (p,∞)-$\mathcal{E}$-limited.*
(ii) *If E is p-barrelled, then every precompact subset A of E is (p,∞)-limited.*

Proof. Let $S = \{\chi_n\}_{n\in\omega}$ be a (resp., equicontinuous) weak* p-summable sequence in E'. If E is p-barrelled, then S is equicontinuous. Therefore, in both cases (i) and (ii) we can assume that S is equicontinuous. Hence, by Proposition 3.9.8 of [26], the weak* topology $\sigma(E',E)$ and the topology τ_{pc} of uniform convergence on precompact subsets of E coincide on S. Since S is weak* p-summable, it is a weak* null-sequence. Therefore, $\chi_n \to 0$ also in τ_{pc}. As A is precompact, we obtain $\sup_{x\in A} |\langle\chi_n, x\rangle| \to 0$. Thus, A is a (p,∞)-limited set (resp., a (p,∞)-$\mathcal{E}$-limited set). □

Since, by definition, ∞-barrelled spaces are exactly c_0-barrelled, setting $p = \infty$ in (ii) of Proposition 3.2, we obtain the next assertion.

Corollary 3.1. *If E is a c_0-barrelled space, then every precompact subset of E is limited.*

The condition in (ii) of Proposition 3.2 that E is p-barrelled is essential as the following theorem shows. Moreover, it may happen that a non-p-barrelled space contains even compact sets which are not limited; see Example 3.1 below.

Theorem 3.3. *Let $p \in [1,\infty]$, X be a Tychonoff space, and let $\mathcal{T}$ be a locally convex vector topology on $L(X)$ compatible with the duality $(L(X), C(X))$. Then, the following assertions are equivalent:*

(i) *the space $C_p(X)$ is p-barrelled;*
(ii) *every precompact (= bounded) subset of $C_p(X)$ is (p,∞)-limited;*
(iii) *X has no infinite functionally bounded subsets;*
(iv) *each bounded subset of $C_p(X)$ is (p,q)-limited for some (every) $p \leq q \leq \infty$;*

(v) $L_{\mathcal{T}}(X)$ *is quasi-complete;*
(vi) $L_{\mathcal{T}}(X)$ *is sequentially complete;*
(vii) $L_{\mathcal{T}}(X)$ *is locally complete.*

Proof. (i)⇒(ii) follows from (ii) of Proposition 3.2.

(ii)⇒(iii) Assume that every bounded subset of $C_p(X)$ is (p,∞)-limited, and suppose for a contradiction that X has an infinite functionally bounded subset A. Then, one can find a sequence $\{x_n\}_{n\in\omega}$ in A and a sequence $\{U_n\}_{n\in\omega}$ of open subsets of X such that $x_n \in U_n$ and $U_n \cap U_m = \emptyset$ for all distinct $n, m \in \omega$. Set

$$B := \Big\{ f \in C_p(X) : f(U_n) \subseteq [0, 2^{n+1}] \text{ for all } n \in \omega, \text{ and } f(X\setminus \bigcup_{n\in\omega} U_n) \subseteq \{0\} \Big\}.$$

Then, B is a bounded subset of $C_p(X)$, and hence, B is (p,∞)-limited. For every $n \in \omega$, set $\chi_n := \frac{1}{2^n}\delta_{x_n}$. Since A is functionally bounded, we obtain that the sequence $S = \{\chi_n\}_{n\in\omega}$ is weak* p-summable in the dual space $C_p(X)'$. For every $n \in \omega$, take a continuous function $g_n : X \to [0, 2^n]$ such that $g_n(X\setminus U_n) \subseteq \{0\}$ and $g_n(x_n) = 2^n$. It is clear that $g_n \in B$ for all $n \in \omega$. However, since

$$\sup_{f\in B} |\langle \chi_n, f\rangle| \geq |\langle \chi_n, g_n\rangle| = 1 \not\to 0,$$

we obtain that B is not (p,∞)-limited, a contradiction.

(iii)⇒(i) Assume that X has no infinite functionally bounded subsets. By the Buchwalter-Schmets theorem, the space $C_p(X)$ is barrelled and hence, it is p-barrelled.

(iii)⇒(iv) Fix $p \leq q \leq \infty$, and let B be a bounded subset of $C_p(X)$. Take an arbitrary weak* p-summable sequence $S = \{\chi_n\}_{n\in\omega}$ in $C_p(X)' = L(X)$. Since S is weak* bounded and the topology of the free lcs $L(X)$ is compatible with $\sigma(L(X), C_p(X))$, it follows that S is a bounded subset of $L(X)$. As all functionally bounded subsets of X are finite, Proposition 2.7 of [13] implies that S is finite-dimensional. By Lemma 4.6 of [17], there are linearly independent elements $\eta_1, \dots, \eta_s \in L(X)$ and sequences $(a_{1,n}), \dots, (a_{s,n}) \in \ell_p$ (or $\in c_0$ if $p = \infty$) such that

$$\chi_n = a_{1,n}\eta_1 + \cdots + a_{s,n}\eta_s \quad \text{for every } n \in \omega.$$

Now, since B is a bounded subset of $C_p(X)$, we obtain

$$\sup_{f\in B} |\langle \chi_n, f\rangle| \leq \sum_{i=1}^{s} |a_{i,n}| \cdot \sup_{f\in B} |\langle \eta_i, f\rangle|,$$

and hence, the inequality $p \leq q$ implies $\big(\sup_{f\in B} |\langle \chi_n, f\rangle|\big)_n \in \ell_q$ (or $\in c_0$ if $q = \infty$). Therefore, B is a (p,q)-limited set, as desired.

(iv)⇒(ii) Assume that each bounded subset of $C_p(X)$ is (p,q)-limited for some $p \leq q \leq \infty$. Then, by (vi) of Lemma 3.1, every bounded subset of $C_p(X)$ is (p,∞)-limited.

(iii)⇒(v) Since all functionally bounded subsets of X are finite, Proposition 2.7 of [13] implies that any bounded subset of $L_{\mathcal{T}}(X)$ is finite-dimensional. Thus, $L_{\mathcal{T}}(X)$ is quasi-complete.

The implication (v)⇒(vi) and (vi)⇒(vii) hold true for any lcs.

(vii)⇒(i) Assume that $L_{\mathcal{T}}(X)$ is locally complete. Since $(L(X), C(X))$ is a dual pair, and $L_{\mathcal{T}}(X)$ is locally complete, it follows that $L(X)_{w^*}$ is also locally complete. As $C_p(X)$ is quasibarrelled hence Mackey, Theorem 5.6 of [17], implies that $C_p(X)$ is p-barrelled. □

For numerous other equivalent conditions to (i)–(vii) of Theorem 3.3, see Theorem 3.5 of [2].

Corollary 3.2. *Let $1 \leq p \leq q \leq \infty$, and let X be a Tychonoff space which has infinite functionally bounded subsets. Then, $C_p(X)$ contains (p,q)-(V^*) sets which are not (p,q)-limited.*

Proof. By Corollary 7.11 of [17], for every Tychonoff space X, we have $\mathsf{V}^*_{(p,q)}(C_p(X)) = \mathsf{Bo}(C_p(X))$. Now, the assertion follows from Theorem 3.3 (iv). □

For a better understanding, it is convenient to have a concrete example of a compact subset which is not limited. Denote by $\mathbf{s} = [0, \omega]$ a convergent sequence.

Example 3.1. *There are compact subsets of $C_p(\mathbf{s})$ which are not limited.*

Proof. For every $n \in \omega$, let $f_n = 1_{\{n\}}$ be the characteristic function of the set $\{n\}$ and let $\chi_n := \delta_n - \delta_{n+1}$, where δ_x denoted the Dirac measure at the point x. Evidently, the sequence $S = \{f_n\}_{n\in\omega}$ is a null sequence in $C_p(\mathbf{s})$, and the sequence $\{\chi_n\}_{n\in\omega}$ is weak* null. Since $\sup\{|\langle\chi_n, f_i\rangle| : i \in \omega\} \geq |\langle\chi_n, f_n\rangle| = 1 \not\to 0$, it follows that S is not limited. □

Proposition 3.2 motivates the following inverse problem: *Characterize locally convex spaces whose (p,q)-limited subset (resp., (p,q)-$\mathcal{E}$-limited subset) of E are precompact*. We solve this problem in the next theorem (compare with Theorem 7.13 of [17]).

Theorem 3.4. *Let $1 \leq p \leq q \leq \infty$. For a locally convex space E, the following assertions are equivalent:*

(i) *every (p,q)-limited subset (resp., (p,q)-$\mathcal{E}$-limited subset) of E is precompact;*
(ii) *each operator $T : L \to E$ from an lcs L to E, which transforms bounded subsets of L to (p,q)-limited subsets (resp., (p,q)-$\mathcal{E}$-limited subset) of E, transforms bounded subsets of L to precompact subsets of E;*
(iii) *as in* (ii) *with a normed space L.*

If in addition E is locally complete, then (i)–(iii) *are equivalent to*

(iv) *as in* (ii) *with a Banach space L.*

Proof. (i)⇒(ii) Let $T : L \to E$ be an operator which transforms bounded subsets of an lcs L to (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) subsets of E. Let A be a bounded subset of L. Then $T(A)$ is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) subset of E, and hence, by (i), $T(A)$ is precompact. Thus, T transforms bounded subsets of L to precompact subsets of E.

(ii)⇒(iii) and (ii)⇒(iv) are trivial.

(iii)⇒(i) and (iv)⇒(i): Fix a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) subset A of E. By (iii) of Lemma 3.1, without loss of generality we can assume that $A = A^{\circ\circ}$. Consider the normed space E_A (if E is locally complete, then E_A is a Banach space), where the norm on E_A is defined by the gauge of A, and recall that the closed unit ball B_A of E_A is exactly A. By Propositions 3.2.2 and 5.1.6 of [34], the identity inclusion $T : E_A \to E$ is continuous and the set $T(B_A) = A$ is a (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) set. Since any bounded subset of E_A is contained in some aB_A, $a > 0$, Lemma 3.1 implies that T transforms bounded subsets of the normed (resp., Banach) space E_A to (p,q)-limited (resp., (p,q)-$\mathcal{E}$-limited) subsets of E. Therefore, by (iii) or (iv), the set $A = T(B)$ is precompact. □

By (vi) of Lemma 3.1, every (p,q)-limited set is $(1,\infty)$-limited. Therefore to characterize $(1,\infty)$-limited sets in locally convex spaces is an important problem. For barrelled spaces we solve this problem in Proposition 3.3 below (compare with Theorem 4.17 of [17]). Our proof is similar to the proof of Proposition 1.1 of [4], where it is obtained a characterization of (V^*) sets in Banach spaces. First, we prove the next lemma.

Lemma 3.2. *If a subset A of a locally convex space E is a $(1,\infty)$-limited set, then $T(A)$ is relatively compact for every operator $T : E \to \ell_1$.*

Proof. Suppose for a contradiction that $T(A)$ is not relatively compact in ℓ_1 for some operator $T : E \to \ell_1$. By (i) of Proposition 4.17 of [17], $T(x) = (\langle \chi_n, x\rangle)_{n\in\omega}$ for some equicontinuous weak* 1-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. Then, Proposition 2.1 implies that there are $\varepsilon > 0$, a sequence $r_0 < s_0 < r_1 < s_1 < \cdots$ in ω, and a sequence $\{a_j\}_{j\in\omega}$ in A such that

$$\sum_{n=r_j}^{s_j} |\langle \chi_n, a_j\rangle| > \varepsilon \quad \text{for every} \quad j \in \omega.$$

For every $j \in \omega$, by Lemma 6.3 of [36], there is a subset F_j of $[r_j, s_j]$ such that

$$\Big| \sum_{n\in F_j} \langle \chi_n, a_j\rangle \Big| > \tfrac{\varepsilon}{4}. \tag{3.4}$$

For every $j \in \omega$, set $\eta_j := \sum_{n\in F_j} \chi_n$. Then, the sequence $\{\eta_j\}_{j\in\omega}$ is weak* 1-summable in E'. By (3.4), we have

$$\sup_{a\in A} |\langle \eta_j, a\rangle| \geq |\langle \eta_j, a_j\rangle| > \tfrac{\varepsilon}{4}, \qquad \text{for every} \quad j \in \omega,$$

and hence, A is not a $(1,\infty)$-limited set, a contradiction. □

Proposition 3.3. *For a bounded subset A of a barrelled locally convex space E the following assertions are equivalent:*

(i) *A is a $(1,\infty)$-limited set;*
(ii) *$T(A)$ is relatively compact for every operator $T : E \to \ell_1$;*
(iii) *for any weak* 1-summable sequence $\{\chi_n\}_{n\in\omega}$ in E', it follows:*

$$\lim_{m\to\infty} \sup\Big\{ \sum_{m\leq n} |\langle \chi_n, x\rangle| : x \in A\Big\} = 0.$$

Proof. (i)⇒(ii) follows from Lemma 3.2.

(ii)⇒(iii) Let $\{\chi_n\}_{n\in\omega}$ be a weak* 1-summable sequence in E'. Then, by Proposition 4.19 of [17], there is an operator $T : E \to \ell_1$ such that $T(x) := (\langle \chi_n, x\rangle)$ for every $x \in E$. By (ii), the set $T(A)$ is relatively compact in ℓ_1. Therefore, by Proposition 2.1, we obtain

$$\lim_{m\to\infty} \sup\Big\{ \sum_{m\leq n} |y_n| : y = (y_n) \in T(A)\Big\} = 0.$$

It remains to note that if $y = (y_n) = T(x)$ for some $x \in A$, then $y_n = \langle \chi_n, x\rangle$ for all $n \in \omega$.

(iii) implies (i) since $\sup_{a\in A} |\langle \chi_m, a\rangle| \leq \sup\{ \sum_{m\leq n} |\langle \chi_n, x\rangle| : x \in A\} \to 0$ for every weak* 1-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. □

Corollary 3.3. *Weakly sequentially precompact subsets and precompact subsets of a barrelled locally convex space E are $(1,\infty)$-limited.*

Proof. Let A be a weakly sequentially precompact subset of E or a precompact subset of E. Then, for every operator $T : E \to \ell_1$, the image $T(A)$ is also weakly sequentially precompact or precompact in ℓ_1. By Lemma 2.2, $T(A)$ is (sequentially) precompact and hence, relatively compact in ℓ_1. Thus, by Proposition 3.3, A is a $(1,\infty)$-limited set. □

It is natural to characterize spaces for which *all* relatively weakly sequentially p-compact sets are (q,∞)-limited. For the case when E is a Banach space and $q = \infty$, the next proposition gives $(a) \Leftrightarrow (b)$ of Proposition 2.10 of [11]. The next assertion can be compared with Proposition 14.12 of [17].

Proposition 3.4. *Let $1 \le p \le q \le \infty$. For a locally convex space E, the following assertions are equivalent:*

(i) *all relatively weakly sequentially p-compact subsets of E are (q,∞)-limited (resp., (q,∞)-$\mathcal{E}$-limited);*
(ii) *every weakly p-summable sequence in E is (q,∞)-limited (resp., (q,∞)-$\mathcal{E}$-limited).*

Proof. (i)⇒(ii) is clear because every weakly p-summable sequence is relatively weakly sequentially p-compact.

(ii)⇒(i) Suppose for a contradiction that there is a relatively weakly sequentially p-compact subset A of E which is not (q,∞)-limited (resp., (q,∞)-$\mathcal{E}$-limited). Then, there are a weak* (resp., equicontinuous) q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E' and $\varepsilon > 0$ such that $\sup_{a\in A} |\langle\chi_n, a\rangle| \ge \varepsilon$ for every $n \in \omega$. For every $n \in \omega$, choose $a_n \in A$ such that

$$|\langle\chi_n, a_n\rangle| \ge \tfrac{\varepsilon}{2}. \tag{3.5}$$

Since A is relatively weakly sequentially p-compact, there are $a \in E$ and a subsequence $\{a_{n_k}\}_{k\in\omega}$ of $\{a_n\}_{n\in\omega}$ such that $\{a_{n_k} - a\}_{n\in\omega}$ is weakly p-summable. Taking into account that $\{\chi_{n_k}\}_{k\in\omega}$ is also weak* null, (ii) and (3.5) imply

$$\tfrac{\varepsilon}{2} \le |\langle\chi_{n_k}, a_{n_k}\rangle| \le |\langle\chi_{n_k}, a_{n_k} - a\rangle| + |\langle\chi_{n_k}, a\rangle| \le \sup_{i\in\omega} |\langle\chi_{n_k}, a_{n_i} - a\rangle| + |\langle\chi_{n_k}, a\rangle| \to 0,$$

a contradiction. □

If E is a Banach space and $p = \infty$, the next theorem was proved by Grothendieck in [23]. Note that Example 3.5 of [16] (which states that $\mathbb{F}^\omega$ is a weakly p-angelic space for every $p \in [1,\infty]$) shows that our result is indeed more general than the Grothendieck theorem.

Theorem 3.5. *Let $p \in [1,\infty]$, E be a weakly p-angelic and locally complete space, and let $H := (E', \mu(E',E))$. Then, a bounded subset B of H is precompact if, and only if, it is a (p,∞)-limited set.*

Proof. Assume that B is a $\mu(E',E)$-precompact subset of E'. Let $S = \{x_n\}_{n\in\omega}$ be a weak* p-summable sequence in $H' = E$. Then, S is weakly p-summable in E. Since E is locally complete, the set $K := \overline{\mathrm{acx}}(S)$ is weakly compact, and hence, by the Mackey-Arens theorem 8.7.3 of [32], K° is a neighborhood of zero in H. Consequently, $K = K^{\circ\circ}$ and hence, also S are $\mu(E',E)$-equicontinuous. The weak* p-summability of S implies that S is weak* null. Hence, by Proposition 3.9.8 of [26],

$x_n \to 0$ in the topology of uniform convergence on precompact subsets of H, in particular, $x_n \to 0$ uniformly on B. Since S was arbitrary, by definition this means that B is a (p,∞)-limited subset of H.

Conversely, assume that B is a (p,∞)-limited subset of H. Let $u = \mathsf{id}_E : E \to E$ be the identity map, $\mathfrak{S}$ be the family of all absolutely convex weakly compact subsets of E, and $\mathfrak{T}$ be the family of all (p,∞)-limited subsets of H. Then, the equivalence of (1) and (1') in Theorem 12 of [24, p. 91] can be formulated as follows: the (p,∞)-limited sets of H are precompact if, and only if, any set $K \in \mathfrak{S}$ is precompact for the topology $\mathcal{T}$ of uniform convergence on all (p,∞)-limited sets of H. Therefore to prove that B is precompact it suffices to show that any $K \in \mathfrak{S}$ is $\mathcal{T}$-precompact. To this end, fix a $K \in \mathfrak{S}$.

We claim that the topology $\mathcal{T}$ and the weak topology $\sigma(E,E')$ coincide on K. By (xi) of Lemma 3.1, we have $\sigma(E,E') \subseteq \mathcal{T}$. Therefore to prove the claim we have to show only that any $\mathcal{T}$-closed subset A of K is also weakly closed. Let $z \in \overline{A}^{\sigma(E,E')}$. Since E is a weakly p-angelic space and K is weakly compact, (i) of Lemma 3.6 of [16] implies that there is a sequence $\{x_n\}_{n\in\omega}$ in A which weakly p-converges to z, i.e., the sequence $\{x_n - z\}_{n\in\omega}$ is weakly p-summable. By the definition of (p,∞)-limited sets we obtain that $x_n \to z$ in the topology $\mathcal{T}$ and hence, $z \in A$. Thus, A is weakly closed. The claim is proved.

Since K is weakly compact, the claim implies that also K is compact for the topology $\mathcal{T}$. □

It is convenient to formulate Theorem 3.5 in a dual form.

Corollary 3.4. *Let $p \in [1,\infty]$, and let E be a Mackey space. If the space E'_{w^*} is weakly p-angelic and locally complete, then a bounded subset A of E is precompact if, and only if, it is a (p,∞)-limited set.*

Proof. Set $E_1 := E'_{w^*}$ (so E_1 carries its weak topology) and $H_1 := (E'_1, \mu(E'_1, E_1))$. Then, $E'_1 = E$ algebraically. Since, by the Mackey-Arens theorem, the polars of the weak* compact absolutely convex subsets of $E'_{w^*} = E_1$ define the Mackey topology on E and the Mackey topology $\mu(E'_1, E_1)$ on $E'_1 = E$, we obtain $\mu(E'_1, E_1) = \mu(E, E')$. As E is a Mackey space, it follows $H_1 = E$. Now, Theorem 3.5 applies. □

Theorem 5.6 of [17] (which states that a Mackey space E is p-barrelled if, and only if, E'_{w^*} is locally complete) and Corollary 3.4 imply the following.

Corollary 3.5. *Let $p \in [1,\infty]$, and let E be a Mackey p-barrelled space (for example, E is barrelled). If E'_{w^*} is a weakly p-angelic space, then a bounded subset A of E is precompact if, and only if, it is a (p,∞)-limited set.*

The case $p = \infty$ is of independent interest.

Corollary 3.6. *Let a locally convex space E satisfy one of the following conditions:*

(i) *E is a Mackey c_0-barrelled space such that E'_{w^*} is a weakly angelic space;*
(ii) *E is a reflexive space such that E'_β is a weakly angelic space;*
(iii) *E is a separable Mackey c_0-barrelled space.*

Then, a bounded subset A of E is precompact if, and only if, it is a limited set. Moreover, if in addition E is von Neumann complete, then a bounded subset A of E is relatively compact if, and only if, it is a limited set.

Proof. (i) Proposition 3.4 of [16] states that every weakly angelic space E is weakly ∞-angelic. Now Corollary 3.5 applies.

(ii) Recall that any reflexive space is barrelled; see Proposition 11.4.2 of [27]. Since E is semi-reflexive, E'_β is a weakly angelic space if, and only if, so is E'_{w^*}. Now (i) applies.

(iii) Since E is separable, the space E'_{w^*} admits a weaker metrizable locally convex topology $\mathcal{T}$. Therefore, $(E', \mathcal{T})$ and hence, also E'_{w^*} are even (weakly) angelic spaces. By Proposition 3.4 of [16], the space E'_{w^*} is weakly (sequentially) ∞-angelic. Now, (i) applies.

The last assertion follows from the fact that if, in addition, E is von Neumann complete, then any precompact subset of E is relatively compact. □

4. Coarse p-limited sets

Below we summarize some basic properties of coarse p-limited sets, cf. Proposition 2 of [20].

Lemma 4.1. *Let $p, q \in [1, \infty]$, and let (E, τ) be a locally convex space. Then,*

(i) *every coarse p-limited subset of E is bounded;*
(ii) *the family $\mathsf{CL}_p(E)$ of all coarse p-limited sets in E is closed under taking subsets, finite unions and sums, and closed absolutely convex hulls;*
(iii) *if L is a locally convex space and $T \in \mathcal{L}(E, L)$, and if $A \subseteq E$ is coarse p-limited, then $T(A)$ is a coarse p-limited subset of L;*
(iv) *a subset A of E is a coarse p-limited set if, and only if, every countable subset of A is a coarse p-limited set;*
(v) ([20, Remark 2]) *in general, even for Banach spaces, there is no inclusion relationships between the class of coarse p-limited sets and the class of coarse q-limited sets for $p \neq q$;*
(vi) *in general, the property of being a coarse p-limited set is not the property of the duality (E, E').*

Proof. The clauses (i)–(iii) are clear.

(iv) The necessity follows from (ii). To prove the sufficiency, we consider the case $1 \le p < \infty$ since the case $p = \infty$ can be considered analogously. Suppose for a contradiction that A is not a coarse p-limited set in E. Then there is an operator $T : E \to \ell_p$ such that $T(A)$ is not relatively compact in ℓ_p. Then, by Proposition 2.1, there is $\varepsilon > 0$ such that

$$\sup_{a\in A} \Big\{ \sum_{m\le n} |\langle T^*(e_n^*), a\rangle|^p \Big\} \ge \varepsilon \quad \text{for every } m \in \omega.$$

For every $m \in \omega$, choose $a_m \in A$ such that $\sum_{m\le n} |\langle T^*(e_n^*), a_m\rangle|^p > \frac{\varepsilon}{2}$. By assumption, the sequence $\{a_m\}_{m\in\omega}$ is a coarse p-limited set. Therefore, by Proposition 2.1 and the choice of a_m, we have

$$\tfrac{\varepsilon}{2} < \sup_{m\in\omega} \Big\{ \sum_{m\le n} |\langle T^*(e_n^*), a_m\rangle|^p \Big\} \to 0,$$

a contradiction.

(vi) Let $1 \le p < \infty$, and let $E = \ell_p$ (for $p = \infty$, one can consider $E = c_0$). Then, the unit ball B_E is not a coarse p-limited set in E (if $\mathsf{id} : E \to \ell_p$ is the identity map then $\mathsf{id}(B_E)$ is not relatively compact in ℓ_p). However, since every $T \in \mathcal{L}(E_w, \ell_p)$ is finite-dimensional by Lemma 17.18 of [17], it follows that B_E is a coarse p-limited set in E_w. □

If $1 \leq p < \infty$, Proposition 1 of [20] states that every p-limited subset of a Banach space is a coarse p-limited set. Below, we generalize this result.

Proposition 4.1. *Let E be a locally convex space. Then,*

(i) *if $p \in [1, \infty]$ and $S_p(\mathcal{L}(E, \ell_p)) \subseteq \ell_p^w(E'_\beta)$, then every (p, p)-(V^*) subset of E is a coarse p-limited set;*
(ii) *if $1 < p < \infty$, then every (p, p)-(V^*) subset A of E is a coarse p-limited set;*
(iii) *if $p = \infty$ and $S_\infty(\mathcal{L}(E, c_0)) = c_0^w(E'_\beta)$, then the class of ∞-(V^*) subsets of E coincides with the class of coarse ∞-limited subsets of E;*
(iv) *if $p \in [1, \infty]$, then every p-limited subset of E is coarse p-limited; in particular, every finite subset of E is coarse p-limited.*

Proof. (i) Let A be a (p, p)-(V^*) subset of E, and let $T : E \to \ell_p$ (or $T : E \to c_0$ if $p = \infty$) be an operator. For every $n \in \omega$, we set $\chi_n := T^*(e_n^*)$. Then, the inclusion $S_p(\mathcal{L}(E, \ell_p)) \subseteq \ell_p^w(E'_\beta)$ implies that the sequence $\{\chi_n\}_n$ is weakly p-summable in E'_β. Since A is a (p, p)-(V^*) set, it follows that

$$\big(\sup_{a \in A} |\langle \chi_n, a\rangle|\big) \in \ell_p \ \ (\text{or} \in c_0 \text{ if } p = \infty). \tag{4.1}$$

Assume that $p < \infty$. Then, (4.1) implies

$$\sup\left\{\sum_{n=m}^{\infty} |\langle e_n^*, T(a)\rangle|^p : a \in A\right\} \leq \sum_{n=m}^{\infty} \Big(\sup_{a \in A} |\langle e_n^*, T(a)\rangle|\Big)^p < \infty.$$

Therefore, by (i) of Proposition 2.1, $T(A)$ is relatively compact in ℓ_p. Thus, A is a coarse p-limited set.

If $p = \infty$, then (4.1) yields $\lim_{n\to\infty} \sup\{|\langle e_n^*, T(a)\rangle| : a \in A\} = 0$. Hence, by (ii) of Proposition 2.1, $T(A)$ is relatively compact in c_0. Thus, A is a coarse ∞-limited set.

(ii) Assume that $1 < p < \infty$. Then, by (iii) of Proposition 4.17 of [17], $S_p(\mathcal{L}(E, \ell_p)) \subseteq \ell_p^w(E'_\beta)$. Thus, by (i), A is a coarse p-limited set.

(iii) Taking account (i), we have to prove that every coarse ∞-limited subset A of E is an ∞-(V^*) set. To this end, let $\{\chi_n\}_n$ be a weakly ∞-summable sequence in E'_β. Then, the equality $S_\infty(\mathcal{L}(E, c_0)) = c_0^w(E'_\beta)$ implies that there is $T \in \mathcal{L}(E, c_0)$ such that $\chi_n = T^*(e_n^*)$ for every $n \in \omega$. Since A is a coarse ∞-limited set, we obtain that $T(A)$ is a relatively compact subset of c_0. Therefore, by (ii) of Proposition 2.1, we have

$$\sup\{|\langle \chi_n, a\rangle| : a \in A\} = \sup\{|\langle e_n^*, T(a)\rangle| : a \in A\} \to 0 \ \text{ as } n \to \infty,$$

which means that A is an ∞-(V^*) set.

(iv) Let A be a p-limited subset of E, and let $T : E \to \ell_p$ (or $T : E \to c_0$ if $p = \infty$) be an operator. For every $n \in \omega$, we set $\chi_n := T^*(e_n^*)$. Then, by Proposition 4.17(i) of [17], the sequence $\{\chi_n\}_n$ is weakly p-summable in E'_{w^*}. Now proceeding exactly as in (i), we obtain that A is a coarse p-limited subset of E.

For the last assertion it suffices to note that, by (xi) of Lemma 3.1, every finite subset of E is a p-limited set. □

Proposition 4.2. *Let $p \in [1, \infty]$, and let $\{E_i\}_{i \in I}$ be a nonempty family of locally convex spaces. Then,*

(i) *a subset K of $E = \prod_{i\in I} E_i$ is a coarse p-limited set if, and only if, so are all its coordinate projections;*
(ii) *a subset K of $E = \bigoplus_{i\in I} E_i$ is a coarse p-limited set if, and only if, so are all its coordinate projections and the support of K is finite.*

Proof. The necessity follows from (iii) of Lemma 4.1 since E_i is a direct summand of E and, for the case (ii), the well-known fact that any bounded subset of a direct locally convex sum has finite support.

To prove the sufficiency, let K be a subset of E such that each coordinate projection K_i of K is a coarse p-limited set in E_i, and, for the case (ii), $K_i = \{0\}$ for all but finitely many indices $i \in I$. By (ii) of Lemma 4.1, we can assume that $0 \in K$. We distinguish between the cases (i) and (ii).

(i) Let $T : E \to \ell_p$ (or $T : E \to c_0$ if $p = \infty$) be an operator. It is easy to show (see for example, Lemma 2.6 in [17]) that there is a finite subset F of I such that $\{0\}^F \times \prod_{i\in I\setminus F} E_i$ is in the kernel of T. Then, taking into account that $0 \in K$, we obtain $T(K) \subseteq \sum_{i\in F} T(K_i)$. Since, by assumption, all $T(K_i)$ are relatively compact in the Banach space ℓ_p (or in c_0) it follows that $\sum_{i\in F} T(K_i)$ (we identify K_i with $K_i \times \prod_{I\setminus\{i\}}\{0_i\}$) and hence, also $T(K)$ are relatively compact in ℓ_p (or c_0). Thus, K is a coarse p-limited set in E.

(ii) Let $F \subseteq I$ be the finite support of K. Then $T(K) \subseteq \sum_{i\in F} T(K_i)$. As above in (i), it follows that $T(K)$ is relatively compact in ℓ_p (or c_0). Thus, K is a coarse p-limited set in E. □

5. Limited type sets and p-convergent operators

Let $1 \le p \le q \le \infty$, and let E be a locally convex space. By Lemma 7.2 of [17] and Lemmas 3.1 and 4.1, the family $\mathsf{V}^*_{(p,q)}(E)$ of all (p,q)-(V^*) sets, the family $\mathsf{L}_{(p,q)}(E)$ of all (p,q)-limited sets and the family $\mathsf{CL}_p(E)$ of all coarse p-limited sets in E are saturated bornologies. Therefore, one can naturally define the following polar topologies on the dual space E'.

Definition 5.1. Let $1 \le p \le q \le \infty$, and let E be a locally convex space. Denote by $V^*_{(p,q)}(E', E)$ (resp., $EV^*_{(p,q)}(E', E)$, $L_{(p,q)}(E', E)$, $EL_{(p,q)}(E', E)$, and $CL_p(E', E)$) the polar topology on E' of uniform convergence on (p,q)-(V^*) (resp., (p,q)-(EV^*), (p,q)-limited, (p,q)-$\mathcal{E}$-limited, or coarse p-limited) subsets of E.

Since the families $\mathsf{V}^*_{(p,q)}(E)$ and $\mathsf{L}_{(p,q)}(E)$ depend only on the duality, the topologies $V^*_p(E', E)$ and $L_{(p,q)}(E', E)$ are topologies of the dual pair (E, E'). However, (vi) of Lemma 4.1 shows that the topology $CL_p(E', E)$ is not a topology of (E, E'). By this reason, in what follows we consider only the topologies $\mathsf{V}^*_{(p,q)}(E)$ and $\mathsf{L}_{(p,q)}(E)$.

For further references, we select the next simple lemma.

Lemma 5.1. *Let $1 \le p \le q \le \infty$, and let E be a locally convex space. Then,*

(i) $\sigma(E', E) \subseteq L_{(p,q)}(E', E) \subseteq V^*_{(p,q)}(E', E) \subseteq \beta(E', E)$;
(ii) $L_{(p,q)}(E', E) \subseteq \mu(E', E)$ *if, and only if, every (p,q)-limited set A in E is relatively weakly compact;*
(iii) $L_{(p,q)}(E', E) = \mu(E', E)$ *if, and only if, every (p,q)-limited set A in E is relatively weakly compact and every weakly compact absolutely convex subset of E is (p,q)-limited.*

Proof. (i) follows from (viii) and (xi) of Lemma 3.1.

(ii) and (iii) follow from the Mackey-Arens theorem and the fact that $\mathsf{L}_{(p,q)}(E)$ is a saturated bornology (see (iii) of Lemma 3.1). □

Remark 5.1. *The inclusion $L_{(p,q)}(E',E) \subseteq V^*_{(p,q)}(E',E)$ can be strict. Indeed, let X be a Tychonoff space containing an infinite functionally bounded subset. Then, by Corollary 3.2, the space $C_p(X)$ contains (p,q)-(V^*) sets which are not (p,q)-limited. This fact, the inclusion $L_{(p,q)}(E',E) \subseteq V^*_{(p,q)}(E',E)$ and the fact that $\mathsf{V}^*_{(p,q)}(C_p(X))$ and $\mathsf{L}^*_{(p,q)}(C_p(X))$ are saturated bornologies imply that $L_{(p,q)}(E',E) \subsetneq V^*_{(p,q)}(E',E)$.*

It is well-known that if $T \in \mathcal{L}(E,L)$, then, T^* is weak* and strongly continuous. The following assertion shows that T^* is also continuous with respect to the topology $L_{(p,q)}$.

Proposition 5.1. *Let $1 \leq p \leq q \leq \infty$, and let $T : E \to L$ be an operator between locally convex spaces E and L. Then,*

(i) *the adjoint map $T^* : (L', L_{(p,q)}(L',L)) \to (E', L_{(p,q)}(E',E))$ is continuous;*
(ii) *the adjoint map $T^* : L'_{w^*} \to (E', L_{(p,q)}(E',E))$ is p-convergent.*

Proof. (i) To show that T^* is continuous, let A° be a standard $L_{(p,q)}(E',E)$-neighborhood of zero, where A is a (p,q)-limited set in E. Then, by Lemma 3.1, $T(A)$ is a (p,q)-limited set in L. Then for every $\eta \in T(A)^\circ$ and each $a \in A$, we have $|\langle T^*(\eta), a\rangle| = |\langle \eta, T(a)\rangle| \leq 1$ and hence, $T^*(T(A)^\circ) \subseteq A^\circ$. Thus, T^* is continuous.

(ii) Since $L_{(p,q)}(E',E) \subseteq L_{(p,\infty)}(E',E)$ by (vi) of Lemma 3.1, it suffices to consider the case $q = \infty$. Let $\{\chi_n\}_{n\in\omega}$ be a weakly p-summable sequence in L'_{w^*}. To show that $T^*(\chi_n) \to 0$ in $(E', L_{(p,\infty)}(E',E))$, fix an arbitrary $B \in \mathsf{L}_{(p,\infty)}(E)$. By Lemma 3.1, $T(B)$ is a (p,∞)-limited set in L and hence,

$$\lim_{n\to\infty} \sup_{b\in B} \left|\langle T^*(\chi_n), b\rangle\right| = \lim_{n\to\infty} \sup_{b\in B} \left|\langle \chi_n, T(b)\rangle\right| = 0.$$

Therefore, $T^*(\chi_n) \in B^\circ$ for all sufficiently large $n \in \omega$. Since B was arbitrary this means that $T^*(\chi_n) \to 0$ in $L_{(p,\infty)}(E',E)$, as desired. □

Below we give a complete answer to the problem posed in the introduction for the (p,∞)-case (namely, characterize those operators T which map *all bounded* sets into (p,q)-limited sets or into coarse p-limited sets). We are interested in this special case because it is dually connected with p-convergent operators; see in particular Theorems 5.1 and 5.2 for the case $q = \infty$.

One can naturally also ask when the topology $L_{(p,q)}(E',E)$ in (ii) of Proposition 5.1 can be replaced by the strong topology $\beta(E',E)$. We answer this question in the next theorem. For $1 \leq p < \infty$, it generalizes a characterization of operators T between Banach spaces for which T^* is p-convergent; see Ghenciu [22]. Following Definition 13.8 of [17], if $p \in [1,\infty]$, a locally convex space E is called *weakly sequentially p-complete* if every weakly p-Cauchy sequence is weakly p-convergent.

Theorem 5.1. *Let $p \in [1,\infty]$, and let $T : E \to L$ be an operator between locally convex spaces E and L. Then, the following assertions are equivalent:*

(i) *for every $B \in \mathsf{Bo}(E)$, the image $T(B)$ is a (p,∞)-limited set in L;*
(ii) *$T^* : L'_{w^*} \to E'_\beta$ is p-convergent.*

If L'_{w^*} is sequentially complete and $T^* : L'_{w^*} \to (E'_\beta)_w$ is sequentially continuous, then (i) *and* (ii) *are equivalent to*

(iii) *$T^* \circ S$ is a sequentially precompact operator for any operator $S : \ell_{p^*} \to L'_{w^*}$ (or $S : c_0 \to L'_{w^*}$ if $p = 1$).*

If $1 < p < \infty$, L'_{w^*} *is sequentially complete and* $T^* : L'_{w^*} \to (E'_\beta)_w$ *is sequentially continuous, then* (i)–(iii) *are equivalent to the following:*

(iv) $T^* \circ S$ *is a sequentially compact operator for any operator* $S : \ell_{p^*} \to L'_{w^*}$.

If $p = 1$, E'_β *and* L'_{w^*} *are sequentially complete,* $T^* : L'_{w^*} \to (E'_\beta)_w$ *is sequentially continuous and* L'_{w^*} *is weakly sequentially* 1*-complete, then* (i)–(iii) *are equivalent to the following:*

(v) $T^* \circ S$ *is a sequentially compact operator for any operator* $S : c_0 \to L'_{w^*}$.

Proof. (i)⇒(ii) Let $\{\chi_n\}_{n\in\omega}$ be a weak* p-summable sequence in L'. To show that $T^*(\chi_n) \to 0$ in E'_β, fix an arbitrary $B \in \mathsf{Bo}(E)$. Since $T(B)$ is a (p,∞)-limited set in L, we have

$$\sup_{b\in B} \left|\langle T^*(\chi_n), b\rangle\right| = \sup_{b\in B} \left|\langle \chi_n, T(b)\rangle\right| \to 0 \ \text{ as } \ n \to \infty,$$

and hence, $T^*(\chi_n) \in B^\circ$ for all sufficiently large $n \in \omega$. Since B was arbitrary, this means that $T^*(\chi_n) \to 0$ in E'_β, as desired.

(ii)⇒(i) Let $B \in \mathsf{Bo}(E)$. To show that $T(B)$ is a (p,∞)-limited set in L, take any weakly p-summable sequence $\{\chi_n\}_{n\in\omega}$ in L'_{w^*}. For every $\varepsilon > 0$, the polar $\varepsilon B^\circ = \left(\frac{1}{\varepsilon}B\right)^\circ$ is a neighborhood of zero in E'_β. Since T^* is p-convergent, we have $T^*(\chi_n) \to 0$ in E'_β and hence, there is $N_\varepsilon \in \omega$ such that $T^*(\chi_n) \in \varepsilon B^\circ$ for all $n \geq N_\varepsilon$. Therefore,

$$\sup_{b\in B} \left|\langle \chi_n, T(b)\rangle\right| = \sup_{b\in B} \left|\langle T^*(\chi_n), b\rangle\right| \leq \varepsilon \ \text{ for all } \ n \geq N_\varepsilon.$$

As ε was arbitrary, it follows that $\sup_{b\in B} \left|\langle \chi_n, T(b)\rangle\right| \to 0$. Thus, $T(B)$ is a (p,∞)-limited set.

The equivalences (ii)⇔(iii), and (ii)⇔(iii)⇔(iv)⇔(v) immediately follow from Theorem 13.17 of [17] applied to $E_1 = L'_{w^*}$, $L_1 = E'_\beta$ and $T_1 = T^*$. □

Remark 5.2. *The condition on T to be such that* $T^* : L'_{w^*} \to E'_\beta$ *is continuous is sufficiently strong. It is satisfied if E is a feral space because* $E'_\beta = E'_{w^*}$ *and hence,* T^* *is automatically continuous by Theorem 8.10.5 of [32]. Recall that an lcs E is feral if every bounded subset of E is finite-dimensional.*

Theorem 5.1 applied to the identity map $T = \mathsf{id}_E : E \to E$ immediately implies the following characterization of spaces for which every bounded subset is a (p,∞)-limited set. Recall that a locally convex space E is called *Grothendieck* or has the *Grothendieck property* if the identity map $\mathsf{id}_{E'} : E'_{w^*} \to (E'_\beta)_w$ is sequentially continuous.

Corollary 5.1. *Let* $p \in [1,\infty]$, *and let E be a locally convex space. Then, the following conditions are equivalent:*

(i) *every bounded subset of E is a* (p,∞)*-limited set* (*i.e.,* $\mathsf{Bo}(E) = \mathsf{L}_{(p,\infty)}(E)$)*;*
(ii) *the identity map* $\mathsf{id}_{E'} : E'_{w^*} \to E'_\beta$ *is p-convergent.*

If E'_{w^*} *is sequentially complete and E has the Grothendieck property, then* (i) *and* (ii) *are equivalent to*

(iii) *any operator* $S : \ell_{p^*} \to E'_{w^*}$ (*or* $S : c_0 \to E'_{w^*}$ *if* $p = 1$) *is sequentially precompact.*

If $1 < p < \infty$, E'_{w^*} *is sequentially complete and E has the Grothendieck property, then* (i)–(iii) *are equivalent to the following:*

(iv) *any operator* $S : \ell_{p^*} \to E'_{w^*}$ *is sequentially compact.*

If $p = 1$, E'_{w^*} *is a sequentially complete, weakly sequentially* 1*-complete, Grothendieck space and* E'_β *is sequentially complete, then* (i)–(iii) *are equivalent to the following*

(v) *any operator* $S : c_0 \to E'_{w^*}$ *is sequentially compact.*

Applying Corollary 5.1 for $p = \infty$, we obtain the following assertion.

Corollary 5.2. *Let E be a locally convex space. Then, the following conditions are equivalent:*

(i) *every bounded subset of E is limited* (*i.e.,* $\mathsf{Bo}(E) = \mathsf{L}(E)$)*;*
(ii) *the identity map* $\mathsf{id}_{E'} : E'_{w^*} \to E'_\beta$ *is completely continuous* ($= \infty$*-convergent*)*; in particular, E is a Grothendieck space.*

Below we give a useful construction of operators from ℓ_1^0 into locally convex spaces whose adjoint is p-convergent.

Proposition 5.2. *Let* $\{x_n\}_{n\in\omega}$ *be a bounded sequence in a locally convex space* (E, τ), *and let* $T : \ell_1^0 \to E$ *be a linear map defined by*

$$T(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \dots, a_n \in \mathbb{F}).$$

Then, T is continuous. Moreover, if E is locally complete, then, T can be extended to a continuous operator from ℓ_1 *to E. In any case, if* $\{x_n\}_{n\in\omega}$ *is a* (p,∞)*-limited set, then,* $T^* : E'_{w^*} \to \ell_\infty$ *is p-convergent.*

Proof. The continuity of T and, in the case E is locally complete, the existence of the extension of T are proved in Proposition 14.9 of [17]. Assume now that $A = \{x_n\}_{n\in\omega}$ is a (p,∞)-limited set. To show that the adjoint linear map T^* is p-convergent, let $\{\chi_n\}_{n\in\omega}$ be a weak* p-summable sequence in E'. Since A is a (p,∞)-limited set, by definition, we have $\lim_{n\to\infty} \sup_{x\in A} |\langle\chi_n, x\rangle| = 0$. Therefore,

$$\|T^*(\chi_n)\|_{\ell_\infty} = \sup_{k\in\omega} \left|\langle T^*(\chi_n), e_k\rangle\right| = \sup_{k\in\omega} \left|\langle\chi_n, x_k\rangle\right| \le \sup_{x\in A} \left|\langle\chi_n, x\rangle\right| \to 0 \quad \text{as } n \to \infty.$$

Thus, $\{T^*(\chi_n)\}_{n\in\omega}$ is a null sequence in ℓ_∞, and hence, T^* is p-convergent. □

We know that (p,q)-limited sets are bounded. In the next theorem, we give an operator characterization of those spaces E in which the (p,q)-limited sets have stronger topological properties than just being bounded as, for example, being weakly sequentially (pre)compact.

Theorem 5.2. *Let* $1 \le p \le q \le \infty$, *and let E be a locally convex space. Then, the following assertions are equivalent:*

(i) *if L is a normed space and* $T : L \to E$ *is an operator such that* $T^* : E'_{w^*} \to L'_\beta$ *is* (q,p)*-convergent, then, T is weakly sequentially compact* (*resp., sequentially compact, weakly sequentially precompact, sequentially precompact, weakly sequentially p-compact, or weakly sequentially p-precompact*)*;*
(ii) *the same as* (i) *with* $L = \ell_1^0$*;*

(iii) *each* (p, q)*-limited subset of* E *is relatively weakly sequentially compact* (*resp., relatively sequentially compact, weakly sequentially precompact, sequentially precompact, relatively weakly sequentially* p*-compact, or weakly sequentially* p*-precompact*).

Moreover, if E *is locally complete, then* (i)-(iii) *are equivalent to*

(iv) *the same as* (i) *with* $L = \ell_1$.

Proof. (i)⇒(ii) and (i)⇒(iv) are clear.

(ii)⇒(iii) and (iv)⇒(iii): Let A be a (p, q)-limited subset of E. Fix an arbitrary sequence $S = \{x_n\}_{n\in\omega}$ in A, so S is a bounded subset of E. Therefore, by Proposition 5.2, the linear map $T : \ell_1^0 \to E$ (or $T : \ell_1 \to E$ if E is locally complete) defined by

$$T(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \ldots, a_n \in \mathbb{F})$$

is continuous. For every $n \in \omega$ and each $\chi \in E'$, we have $\langle T^*(\chi), e_n\rangle = \langle \chi, T(e_n)\rangle = \langle \chi, x_n\rangle$ and hence, $T^*(\chi) = (\langle \chi, x_n\rangle)_n \in \ell_\infty$. In particular, $\|T^*(\chi)\|_{\ell_\infty} = \sup_{n\in\omega} |\langle \chi, x_n\rangle|$.

Let now $\{\chi_n\}_{n\in\omega}$ be a weak* p-summable sequence in E'_{w^*}. Since A and hence, also S are (p, q)-limited sets, we obtain $(\|T^*(\chi_n)\|_{\ell_\infty}) = (\sup_{i\in\omega} |\langle \chi_n, x_i\rangle|) \in \ell_q$ (or $\in c_0$ if $q = \infty$). Therefore, T^* is (q, p)-convergent, and hence, by (ii) or (iv), the operator T belongs to the corresponding class described in (i). Therefore, $S = \{T(e_n)\}_{n\in\omega}$ has a weakly convergent (resp., convergent, weakly Cauchy, Cauchy, weakly p-convergent, or weakly p-Cauchy) subsequence, as desired.

(iii)⇒(i) Let $T : L \to E$ be an operator from a normed space such that $T^* : E'_{w^*} \to L'_\beta$ is a (q, p)-convergent operator. Then, by Theorem 3.2, $T(B_L)$ is a (p, q)-limited set and hence it is relatively weakly sequentially compact (resp., relatively sequentially compact, weakly sequentially precompact, sequentially precompact, relatively weakly sequentially p-compact, or weakly sequentially p-precompact). Thus, T belongs to the corresponding class described in (i). □

The definition of coarse p-limited sets allows us to reformulate Theorem 14.16 of [17] as follows.

Theorem 5.3. *Let* $1 < p < \infty$*, and let* E *be a quasibarrelled space such that* E'_β *is an* ℓ_∞-V_p*-barrelled space. Then, the class of* p-(V^*) *sets in* E *coincides with the class of coarse* p*-limited sets.*

In Theorem 5.3, the condition on E being a quasibarrelled space is essential as Example 5.1 below shows. First, we prove the next simple lemma.

Lemma 5.2. *Let* E *be a locally convex space such that* $E = E_w$*, and let* L *be a normed space. Then, every* $T \in \mathcal{L}(E, L)$ *is finite-dimensional.*

Proof. Observe that T can be extended to an operator $\bar{T}$ from a completion $\bar{E}$ of E to a completion $\bar{L}$ of L. As E carries its weak topology, we obtain $\bar{E} = \mathbb{F}^\kappa$ for some cardinal κ. Since $\bar{T}$ is continuous, there is a finite subset λ of κ such that $\bar{T}(\{0\}^\lambda \times \mathbb{F}^{\kappa\setminus\lambda})$ is contained in the unit ball $B_{\bar{L}}$ of $\bar{L}$. Taking into account that $B_{\bar{L}}$ contains no nontrivial linear subspaces we obtain that $\{0\}^\lambda \times \mathbb{F}^{\kappa\setminus\lambda}$ is contained in the kernel $\ker(\bar{T})$ of $\bar{T}$. Therefore, $\bar{T}[\mathbb{F}^\kappa] = \bar{T}[\mathbb{F}^\lambda]$ is finite-dimensional. Thus, also T is finite-dimensional. □

Example 5.1. *Let* $1 < q \le p < \infty$*. Then the space* $E := (\ell_q)_w$ *satisfies the following conditions:*

(i) *E is not quasibarrelled, but* $E'_\beta = \ell_{q^*}$ *is a Banach space;*

(ii) B_{ℓ_q} *is a coarse p-limited set in E which is not a p-*(V^*) *set.*

Proof. The clause (i) is clear, and Lemma 5.2 shows that B_{ℓ_q} is a coarse p-limited set in E. To show that B_{ℓ_q} is not a p-(V^*) set, let $\chi_n = e_n^*$ for every $n \in \omega$. Then, the sequence $\{\chi_n\}_{n\in\omega} \subseteq E'_\beta = \ell_{q^*}$ is weakly p-summable (indeed, if $(x_k) \in \ell_q = (\ell_{q^*})'$, then, $\big(\langle (x_k), \chi_n\rangle\big)_n = (x_n) \in \ell_q \subseteq \ell_p$). However, $\sup_{(x_k)\in B_{\ell_q}} |\langle \chi_n, (x_k)\rangle| = 1 \not\to 0$. Thus, B_{ℓ_q} is not a p-(V^*) set. □

Below for an important case which includes all strict (LF)-spaces, we characterize coarse 1-limited sets. To begin, we recall some definitions and results. Following [14], a sequence $A = \{a_n\}_{n\in\omega}$ in an lcs E is said to be *equivalent to the standard unit basis* $\{e_n : n \in \omega\}$ *of* ℓ_1 if there exists a linear topological isomorphism R from $\overline{\mathsf{span}}(A)$ onto a subspace of ℓ_1 such that $R(a_n) = e_n$ for every $n \in \omega$ (we do not assume that the closure $\overline{\mathsf{span}}(A)$ of the $\mathsf{span}(A)$ of A is complete or that R is onto). We shall say also that A is an ℓ_1*-sequence*. Following [19], a locally convex space E is said to have the *Rosenthal property* if every bounded sequence in E has a subsequence which either (1) is Cauchy in the weak topology, or (2) is equivalent to the unit basis of ℓ_1. The following remarkable extension of the celebrated Rosenthal ℓ_1-theorem was proved by Ruess [37].

Theorem 5.4. *Every locally complete locally convex space E whose separable bounded sets are metrizable has the Rosenthal property.*

Note that for every ℓ_1-sequence $A = \{a_n\}_{n\in\omega}$ in E in Theorem 5.4, a topological isomorphism R from $\overline{\mathsf{span}}(A)$ to ℓ_1 is *onto*. Observe also that strict (LF)-spaces satisfy Theorem 5.4 (recall that an lcs E is a *strict* (LF)*-space* if E is the strict inductive limit of a sequence $\{(E_n, \tau_n)\}_{n\in\omega}$ of Fréchet spaces; for more details, we refer the reader to Section 4.5 of [27]).

Theorem 5.5. *Let E be a locally complete space whose separable bounded sets are metrizable. Then for a bounded subset A of E, the following assertions are equivalent:*

(i) *A is a coarse* 1*-limited set;*
(ii) *A does not contain an* ℓ_1*-sequence* $\{x_n\}_{n\in\omega}$ *such that the closed span* $\overline{\mathsf{span}}\{x_n\}_{n\in\omega}$ *is complemented in E.*

If, in addition, E is barrelled, then, (i) *and* (ii) *are equivalent to*

(iii) *A is a* $(1,\infty)$*-limited set.*

Proof. (i)⇒(ii) Assume that E has only the Rosenthal property, and suppose for a contradiction that there is an ℓ_1-sequence $\{x_n\}_{n\in\omega}$ in A such that $L := \overline{\mathsf{span}}\{x_n : n \in \omega\}$ is complemented in E. Let S be a projection from E onto L, and let R be a linear homeomorphism of L onto a subspace of ℓ_1 such that $R(x_n) = e_n$ for every $n \in \omega$. Then $T := R \circ S : E \to \ell_1$ is an operator such that $T(A)$ contains $\{e_n : n \in \omega\}$. Therefore, $T(A)$ is not relatively compact in ℓ_1. Thus, A is not coarse 1-limited, a contradiction.

(ii)⇒(i) Suppose for a contradiction that A is not a coarse 1-limited set. Then, there is $T \in \mathcal{L}(E, \ell_1)$ such that $T(A)$ is not relatively compact in ℓ_1. By Theorem 1.4 of [33], there is a sequence $\{x_n\}_{n\in\omega}$ in A such that the sequence $S_0 = \{T(x_n)\}_{n\in\omega}$ is equivalent to the standard unit basis $\{e_n\}_{n\in\omega}$ of ℓ_1 and such that the subspace $H_0 := \overline{\mathsf{span}}(S_0)$ is a complemented subspace of ℓ_1. Let $R_0 : H_0 \to \ell_1$ be a linear topological isomorphism such that $R_0(T(x_n)) = e_n$ for every $n \in \omega$. Since a continuous image of a weakly Cauchy sequence is weakly Cauchy, the sequence $\{x_n\}_{n\in\omega}$ has no weakly Cauchy subsequences, and hence, by the Rosenthal property of E (see Theorem 5.4), there is a subsequence $\{x_{n_k}\}_{k\in\omega}$ of $\{x_n\}_{n\in\omega}$ which is equivalent to $\{e_k\}_{k\in\omega}$. Let $R : \overline{\mathsf{span}}\{x_{n_k}\}_{k\in\omega} \to \ell_1$ be a linear topological isomorphism such that $R(x_{n_k}) = e_k$ for every $k \in \omega$. Observe that the subspace $H_1 := \overline{\mathsf{span}}\{T(x_{n_k})\}_{k\in\omega}$ of H_0 satisfies the following two conditions:

(1) H_1 is complemented in H_0 (since R_0 is a topological isomorphism, $R_0(T(x_{n_k})) = e_{n_k}$, and $\overline{\mathsf{span}}\{e_{n_k}\}_{k\in\omega}$ is complemented in ℓ_1), and hence, H_1 is complemented also in ℓ_1, and
(2) H_1 is topologically isomorphic to ℓ_1 (since R_0 is a topological isomorphism and $\overline{\mathsf{span}}\{e_{n_k}\}_{k\in\omega}$ is topologically isomorphic to ℓ_1).

Let $Q : \ell_1 \to H_1$ be a projection (so $Q(T(x_{n_k})) = T(x_{n_k})$ for every $k \in \omega$), and let $R_1 : H_1 \to \ell_1$ be a linear topological isomorphism such that $R_1(T(x_{n_k})) = e_k$ for every $k \in \omega$. Since

$$R^{-1} \circ R_1 \circ Q \circ T(x_{n_k}) = R^{-1} \circ R_1(T(x_{n_k})) = R^{-1}(e_k) = x_{n_k} \quad \text{for every } k \in \omega,$$

it follows that $R^{-1} \circ R_1 \circ Q \circ T$ is a continuous projection from E onto $\overline{\mathsf{span}}\{x_{n_k}\}_{k\in\omega}$ and $\{x_{n_k}\}_{k\in\omega}$ is equivalent to $\{e_k\}_{k\in\omega}$. However, this contradicts (ii).

(i)⇔(iii) immediately follows from Proposition 3.3. □

Corollary 5.3. *Let E be a strict (LF)-space which does not contain an isomorphic copy of ℓ_1 which is complemented in E. Then every bounded subset A of E is a coarse 1-limited and a $(1,\infty)$-limited set.*

By the classical Pitt theorem [10, 4.49], all operators $\mathcal{L}(\ell_p, \ell_1)$ $(1 < p < \infty)$ and $\mathcal{L}(c_0, \ell_1)$ are compact. Below, we generalize this result.

Corollary 5.4. *If E is a Banach space containing no an isomorphic copy of ℓ_1 which is complemented in E, then the class of all bounded subsets of E coincides with the class of all coarse 1-limited sets. Consequently, every $T \in \mathcal{L}(E, \ell_1)$ is compact.*

The condition of being a barrelled space in (iii) of Theorem 5.5 is essential as the following example shows.

Example 5.2. *Let $E = (c_0)_p$ be the Banach space c_0 endowed with the pointwise topology induced from $\mathbb{F}^\omega$, and let*

$$B = \{(x_n)_{n\in\omega} \in E : |x_n| \le (n+1)^2 \text{ for every } n \in \omega\}.$$

Then, B is a coarse 1-limited set in E which is not $(1,\infty)$-limited.

Proof. It is clear that B is a bounded subset of E. Therefore, by (ii) of Example 5.4 of [17], B is a coarse p-limited set for every $p \in [1,\infty]$. To show that B is not $(1,\infty)$-limited, consider the sequence $\{\chi_n\}_{n\in\omega} = \{\frac{e_n^*}{(n+1)^2}\}_{n\in\omega}$ in E'. In the proof of (i) of Example 5.4 of [17], we showed that $\{\chi_n\}_{n\in\omega}$ is a weak* 1-summable sequence in E'. Since

$$\sup_{(x_k)\in B} |\langle \chi_n, (x_k)\rangle| = 1 \quad \text{for every } n \in \omega,$$

it follows that B is not a $(1,\infty)$-limited set in E. □

Corollary 5.5. *Let E be a Banach space. Then,*

(i) *if* $p = 1$*, then the class of coarse* 1*-limited sets in E coincides with the class of* $(1, \infty)$*-limited sets;*
(ii) *if* $1 < p < \infty$*, then the class of coarse p-limited sets in E coincides with the class of* p-(V^*) *sets.*

Proof. (i) follows from Theorem 5.5, and (ii) follows from Theorem 5.3. □

To generalize (iii) of Theorem 1.1 and its extension given in Proposition 3 of [20], first we prove the next lemma. Recall that an lcs X is called *injective* if for every subspace H of a locally convex space E, each operator $T : H \to X$ can be extended to an operator $\bar{T} : E \to X$.

Lemma 5.3. *Every injective Banach space is also an injective locally convex space.*

Proof. Let X be an injective Banach space, H be a subspace of a locally convex space E, and let $T : H \to X$ be an operator. It is well-known (see, for example, Exercise 5.27 of [10]) that there are a set Γ and a closed subspace Y of $\ell_\infty(\Gamma)$ such that $X \oplus Y = \ell_\infty(\Gamma)$. Denote by $\pi_X : \ell_\infty(\Gamma) \to X$ the canonical projection, and let $I_X : X \to \ell_\infty(\Gamma)$, $I_X(x) := (x, 0)$, be the canonical embedding. Since, by Proposition 7.4.5 of [27], $\ell_\infty(\Gamma)$ is an injective locally convex space, the operator $I_X \circ T : H \to \ell_\infty(\Gamma)$ can be extended to an operator $\overline{I_X \circ T} : E \to \ell_\infty(\Gamma)$. Set $\bar{T} := \pi_X \circ (\overline{I_X \circ T})$. Then $\bar{T}$ is an operator from E to X such that

$$\bar{T}(h) = \pi_X \circ (I_X \circ T)(h) = \pi_X((T(h), 0)) = T(h) \quad \text{for each } h \in H.$$

Thus, $\bar{T}$ extends T and hence, X is an injective locally convex space. □

Theorem 5.6. *Let* $2 \le p \le \infty$*, and let E be a locally convex space with the Rosenthal property. Then, every coarse p-limited subset of E is weakly sequentially precompact.*

Proof. We consider only the case $2 \le p < \infty$ since the case $p = \infty$ can be considered analogously replacing ℓ_p by c_0. Suppose for a contradiction that there is a coarse p-limited subset A of E which is not weakly sequentially precompact. So, there is a sequence $S = \{x_n\}_{n\in\omega}$ in A that does not have a weakly Cauchy subsequence. By the Rosenthal property of E and passing to a subsequence if needed, we can assume that S is an ℓ_1-sequence. Set $H := \overline{\mathsf{span}}(S)$ and let $P : H \to \ell_1$ be a topological isomorphism of H onto a subspace of ℓ_1 such that $P(x_n) = e_n$ for every $n \in \omega$. Let $J : \ell_1 \to \ell_p$, $I_1 : \ell_1 \to \ell_2$, and $I_2 : \ell_2 \to \ell_p$ be the natural inclusions, so $J = I_2 \circ I_1$. By the Grothendieck Theorem 1.13 of [9], the operator I_1 is 1-summing. By the Ideal Property 2.4 of [9], J is also 1-summing, and hence, by the Inclusion Property 2.8 of [9], the operator J is 2-summing. By the discussion after Corollary 2.16 of [9], the operator J has a factorization

$$J : \ \ell_1 \xrightarrow{R} L_\infty(\mu) \xrightarrow{J_2^\infty} L_2(\mu) \xrightarrow{Q} \ell_p \ ,$$

where μ is a regular probability measure on some compact space K and $J_2^\infty : L_\infty(\mu) \to L_2(\mu)$ is the natural inclusion. By Theorem 4.14 of [9], the Banach space $L_\infty(\mu)$ is injective. Therefore, by Lemma 5.3, $L_\infty(\mu)$ is an injective locally convex space. In particular, the operator $R \circ P : H \to L_\infty(\mu)$ can be extended to an operator $T_\infty : E \to L_\infty(\mu)$. Set $T := Q \circ J_2^\infty \circ T_\infty$. Then, T is an operator from E to ℓ_p such that

$$T(x_n) = Q \circ J_2^\infty \circ R \circ P(x_n) = J \circ P(x_n) = e_n \quad \text{for every } n \in \omega.$$

Since A and hence, also S are coarse p-limited sets, (iii) of Lemma 4.1 implies that the canonical basis $\{e_n\}_{n\in\omega}$ of ℓ_p is also a coarse p-limited set. Therefore, $\mathsf{id}_{\ell_p} \circ T(S) = \{e_n\}_{n\in\omega}$ is a relatively compact subset of ℓ_p, a contradiction. □

It is noticed in [20, p. 944] that, in general, Theorem 5.6 is not true for $p = 1$, even for Banach spaces (in fact, the closed unit ball of $C([0, 1])$ is a coarse 1-limited set which is not weakly sequentially precompact).

Corollary 5.6. (i) *If $2 \leq p \leq \infty$ and E is a locally convex space with the Rosenthal property, then every p-limited subset of E is weakly sequentially precompact.*

(ii) *If E is a Banach space and $1 \leq p < \infty$, then every p-limited subset of E is relatively weakly (sequentially) compact.*

Proof. (i) Since p-limited sets are coarse p-limited by Proposition 4.1(iv), the assertion follows from Theorem 5.6.

(ii) immediately follows from Theorem 17.19 of [17] (which states that every (p, p)-(V^*) subset of E is relatively weakly compact) and (viii) of Lemma 3.1. □

Concerning the case $p = \infty$ in (ii) of Corollary 5.6, we note that if a Banach space E does not contain an isomorphic copy of ℓ_1, then every limited subset of E is relatively weakly (sequentially) compact; for the proof, see [5] (an alternative proof is given in Theorem 1.9 of [21]).

Remark 5.3. By Corollary 5.6, each limited subset of a Banach space is weakly sequentially precompact. It turns out that for non-Banach spaces, this very useful assertion is not true in general. Indeed, by Example 7.12 of [17], the product $\mathbb{R}^{\mathfrak{c}}$ contains a uniformly bounded sequence $S = \{f_n\}_{n\in\omega}$, which is a (p, q)-(V^*) set for all $1 \leq p \leq q \leq \infty$ but is not (weakly) sequentially precompact. Since $\mathbb{R}^{\mathfrak{c}}$ is reflexive, by (viii) of Lemma 3.1, S is also a (p, q)-limited set.

6. Conclusions

In our work we generalize the classical notions of limited, p-limited, and coarse p-limited subsets of a Banach space by introducing (p, q)-limited subsets and their equicontinuous versions and coarse p-limited subsets of an arbitrary locally convex space E, where $1 \leq p \leq q \leq \infty$. We give operator characterizations of these classes and compare them with the classes of bounded, (pre)compact, weakly (pre)compact, and relatively weakly sequentially (pre)compact sets. We also generalize numerous results from Banach space theory to the general theory of locally convex spaces, in particular, a generalization of a known theorem of Grothendieck is given. The obtained results motivates the study of limited type sets in non-Banach spaces (such as strict (LF)-space) which are of high importance to applications, and allow us to introduce and study some natural generalizations of the Gelfand-Phillips property in non-Banach cases.

Conflict of interest

The author declares no conflicts of interest in this article.

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