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Research article

A novel method for Mannheim curves in the Galilean 3−space G_3

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Abstract: This research presents a novel method for Mannheim curves in three-dimensional Galilean space G_3. Using this method, the necessary and sufficient conditions, along with the established results, must be satisfied for a curve in G_3 to qualify as a Mannheim curve. Furthermore, relevant examples and graphs are provided to demonstrate how Mannheim curves and their partners can correspond to Salkowski and anti-Salkowski curves. Finally, in G_3, the Mannheim partner curves are described.

Keywords: Mannheim curves; slant helices; Salkowski curves; Curves of Anti-Salkowski; Galilean Space
Mathematics Subject Classification: 51A05, 53A35

1. Introduction

In the Galilean 3-space G_3, Ogreenmis et al. [11] proposed that if both curvatures, κ and τ are positive constants along the curve β, then the curve β is considered a circular helix with respect to the Frenet frame $\mathbf{t}(s), \mathbf{n}(s)$, and $\mathbf{b}(s)$.

Furthermore, concerning the Frenet frame, a curve β is characterized as a general helix if the ratio $\frac{\kappa}{\tau}$ remains constant, and the converse is also true.

Additionally, the concept of a slant helix was introduced by Karacan and Tuncer [10]. If a constant vector field W exists in G_3 and the function $\langle W, n(s)\rangle_{G_3}$ is constant, then a curve β is called a slant helix. Moreover, if one of the two functions,

$$\pm \frac{\kappa^2}{\tau^3} \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$$

is constant everywhere and τ does not vanish, then the curve is identified as a slant helix in G_3, and the converse is also true.

Bertrand explored curves in Euclidean 3-space whose principal normal is the principal normal of another curve in the classical differential geometry of curves. He demonstrated that a linear relationship with constant coefficients between the first and second curvatures of the original curve is a necessary and sufficient condition for the existence of such a second curve [3], i.e., κ and τ satisfy the equation $c_1\kappa + c_2\tau = 1$, where c_1 and c_2 belong to $\mathbb{R}$. A related curve is called a Mannheim curve, in which the binormal vector field of one curve is the principal normal vector field of another. Liu and Wang [17] examined Mannheim partner curves in both Minkowski and Euclidean 3-space. Since these works, many studies on Mannheim curves in Euclidean 3-space, Minkowski 3-space, dual 3-space, and Galilean spaces have been published [2, 4, 7–9, 12–15].

Let $\gamma(s)$ be an admissible Mannheim curve, and take $\widetilde{\gamma}(\widetilde{s})$ as the admissible Mannheim companion curve of $\gamma(s)$ through the Galilean 3−space G_3. In scholarly works, $\widetilde{\gamma}(\widetilde{s})$ is expressed as

$$\widetilde{\gamma}(\widetilde{s}) = \widetilde{\gamma}\big(g(s)\big) = \gamma(s) + \lambda(s)\,\mathbf{n}(s), \tag{1.1}$$

in which case the function $g(s)$ is differentiable, and $\mathbf{n}(s)$ is the major normal line of $\gamma(s)$ [16]. In relation (1.1), the vector $\overrightarrow{\widetilde{\gamma}\,\gamma}$ does not have to be linearly dependent with the $\mathbf{n}(s)$ vector.

Hence, the Mannheim mate curve, $\widetilde{\gamma}$, is assumed to be produced by

$$\widetilde{\gamma}(\widetilde{s}) = \widetilde{\gamma}\big(g(s)\big) = \gamma(s) + \lambda_1(s)\,\mathbf{t}(s) + \lambda_2(s)\,\mathbf{n}(s) + \lambda_3(s)\,\mathbf{b}(s), \tag{1.2}$$

where the Frenet frame of $\gamma(s)$ is represented by $\mathbf{t}(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$. During this situation, if we accept $\lambda_1(s) = \lambda_3(s) = 0$, we receive the instance that is referenced inside the written works. Therefore, in our article we took $\lambda_1 \neq 0$, and $\lambda_3 \neq 0$ to provide the Mannheim generalized curve in G_3.

Throughout this work, we establish the circumstances that must be satisfied for a G_3 curve to qualify as a Mannheim curve and give instances that correspond to this new way of defining Mannheim curves. Finally, we describe Mannheim mate curves within G_3.

2. Galilean 3−space curves

We will provide a few definitions in this section that will be used in our paper. For more fundamental concepts, see [1, 5, 6].

Galilean space in three dimensions, G_3, is defined as the Cayley-Klein space, where the characteristic protective metric has the signature $(0, 0, +, +)$. In Galilean space, the absolute is represented by a triple (V, E, J). V indicates the ideal plane, E defines a line in V, and J represents an elliptic point of involution $(0 : 0 : r_2 : r_3) \to (0 : 0 : r_3 : -r_2)$.

A plane is called Euclidean if it contains E, otherwise it is called isotropic. A vector $r = (r_1, r_2, r_3)$ is described as not being isotropic if $r_1 \neq 0$. The form of all unit non-isotropic vectors is $r = (1, r_2, r_3)$. Regarding the isotropic vectors, r_1 is equal to zero.

Allow $\overrightarrow{\eta} = (\eta_1, \eta_2, \eta_3)$ and $\overrightarrow{\xi} = (\xi_1, \xi_2, \xi_3)$ to be two vectors in Galilean 3−space G_3. In G_3, a dot product is described as

$$< \overrightarrow{\eta}, \overrightarrow{\xi} >_{G_3} = \begin{cases} \eta_1\xi_1 & \text{if } \eta_1 \neq 0 \ \text{ or } \ \xi_1 \neq 0; \\ \eta_2\xi_2 + \eta_3\xi_3 & \text{if } \eta_1 = 0 \ \text{ and } \ \xi_1 = 0. \end{cases}$$

The vector $\overrightarrow{\zeta} = \big(\zeta_1, \zeta_2, \zeta_3\big)$, in G_3, has the following norm: $\| \overrightarrow{\zeta} \| = \sqrt{< \overrightarrow{\zeta}, \overrightarrow{\zeta} >}$.
The Galilean vector product can be defined as

$$\overrightarrow{\eta} \times \overrightarrow{\xi} = \begin{vmatrix} 0 & e_2 & e_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix}.$$

In a coordinate form, assume that $\beta : I \to G_3$ is a curve in Galilean space G_3 given by

$$\beta(t) = \Big(u(t),\ v(t),\ w(t)\Big),$$

such that $u(t)$, $v(t)$, and $w(t) \in C^3$, $t \in I$. In the case of $u'(t) \neq 0$, then β is known as an admissible curve.

Assuming that the curve β is admissible in G_3, and that the parameter s is the arc length, which is derived by

$$\beta(s) = \Big(s, v(s), w(s)\Big),$$

$\kappa(s)$, and $\tau(s)$, the first and second curvature functions, respectively, are given by

$$\kappa(s) = \| \beta^{''}(s) \| = \sqrt{v''^2(s) + w''^2(s)},$$

and

$$\tau(s) = \frac{det\Big(\beta^{'}(s),\ \beta^{''}(s),\ \beta^{'''}(s)\Big)}{\kappa^2(s)},$$

respectively. The Frenet frame associated with $\beta(s)$ is written as

$$\begin{aligned} \mathbf{t}(s) &= \beta^{'}(s) = \Big(1, v^{'}(s), w^{'}(s)\Big), \\ \mathbf{n}(s) &= \frac{\beta^{''}(s)}{\kappa(s)} = \frac{1}{\kappa(s)}\Big(0, v^{''}(s), w^{''}(s)\Big), \\ \mathbf{b}(s) &= \frac{1}{\kappa(s)}\Big(0, -w^{''}(s), v^{''}(s)\Big), \end{aligned}$$

so that the velocity vector, principal normal vector, and binormal vector of the curve β are represented, respectively, by the symbols $\mathbf{t}(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$.

Regarding $\beta(s)$, the Frenet equations are written as

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix}.$$

3. A new method for Mannheim curves within Galilean space G_3

Definition 3.1. *An admissible curve $\gamma : I \subset \mathbb{R} \to G_3$ with non-vanishing curvatures is claimed to be a Mannheim curve if there exists a curve $\widetilde{\gamma} : \widetilde{I} \subset \mathbb{R} \to G_3$ in such a way that the principle normal vector field of $\gamma(s)$ coincides with the bi-normal vector field of $\widetilde{\gamma}(\widetilde{s})$ at $s \in I$ and $\widetilde{s} \in \widetilde{I}$. $\gamma(s)$ will be referred to as a Mannheim curve, $\widetilde{\gamma}(\widetilde{s})$ as a Mannheim partner, and the pair $(\gamma, \widetilde{\gamma})$ as a pair of Mannheim's.*

Let $\gamma : I \to G_3$, $I \subset \mathbb{R}$ be an admissible Mannheim curve in G_3 with the Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ and non-vanishing curvatures $\kappa(s)$ and $\tau(s)$. Furthermore, assuming the Frenet frame $\{\widetilde{\mathbf{t}}, \widetilde{\mathbf{n}}, \widetilde{\mathbf{b}}\}$, let $\widetilde{\gamma} : \widetilde{I} \to G_3$, $\widetilde{I} \subset \mathbb{R}$ be an admissible Mannheim partner curve for γ using non-vanishing curvatures $\widetilde{\kappa}(s)$ and $\widetilde{\tau}(s)$.

Then, $\widetilde{\gamma}$ can be formulated as

$$\widetilde{\gamma}(\widetilde{s}) = \widetilde{\gamma}\Big(g(s)\Big) = \gamma(s) + \lambda_1(s)\,\mathbf{t}(s) + \lambda_2(s)\,\mathbf{n}(s) + \lambda_3(s)\,\mathbf{b}(s),$$

where the functions $\lambda_1(s)$, $\lambda_2(s)$, and $\lambda_3(s)$ exhibit differentiability on I.

Theorem 3.1. *Assume that $\gamma : I \to G_3$, $I \subset \mathbb{R}$ is an admissible Mannheim curve within G_3 whose curvatures $\kappa(s)$, $\tau(s)$ do not vanish. If there are differentiable functions λ_1, λ_2, and λ_3 that satisfy the given conditions:*

$$\lambda_2' + \kappa\lambda_1 = \tau\lambda_3, \ \Big(1 + \lambda_1'\Big)\kappa = \Big(\lambda_3' + \tau\lambda_2\Big)\tau, \ \lambda_3' + \tau\lambda_2 \neq 0. \tag{3.1}$$

Hence, curve γ is a Mannheim curve with $\widetilde{\gamma}$ as its Mannheim partner; in addition, the converse is also true.

Proof. Consider that γ is an admissible Mannheim curve with non-zero curvatures κ, τ, using arc length s as the parameter, and $\widetilde{\gamma}$ represents the Mannheim partner of γ, where the parameter is the arc length $\widetilde{s}$. Then, the curve $\widetilde{\gamma}$ can be represented as

$$\widetilde{\gamma}(\widetilde{s}) = \widetilde{\gamma}\Big(g(s)\Big) = \gamma(s) + \lambda_1(s)\,\mathbf{t}(s) + \lambda_2(s)\,\mathbf{n}(s) + \lambda_3(s)\,\mathbf{b}(s), \tag{3.2}$$

for $s \in I$, such that $\lambda_1(s)$, $\lambda_2(s)$, and $\lambda_3(s)$ are differentiable functions on I. By using s to differentiate Eq (3.2), we arrive at

$$g'(s)\,\widetilde{t}(\widetilde{s}) = \Big(1 + \lambda_1'(s)\Big)\,\mathbf{t}(s) + \Big(\kappa(s)\lambda_1(s) - \tau(s)\lambda_3(s) + \lambda_2'(s)\Big)\,\mathbf{n}(s) + \Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)\,\mathbf{b}(s). \tag{3.3}$$

Taking the scalar product of Eq (3.3) with $\mathbf{n}(s)$, we get

$$\kappa(s)\lambda_1(s) - \tau(s)\lambda_3(s) + \lambda_2'(s) = 0. \tag{3.4}$$

Substituting from (3.4) into (3.3), implies

$$g'(s)\,\widetilde{t}(\widetilde{s}) = \Big(1 + \lambda_1'(s)\Big)\,\mathbf{t}(s) + \Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)\,\mathbf{b}(s). \tag{3.5}$$

Taking the scalar product of Eq (3.5) with itself yields

$$\Big(g'(s)\Big)^2 = \Big(1 + \lambda_1'(s)\Big)^2 + \Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)^2. \tag{3.6}$$

If we take

$$\alpha(s) = \frac{1 + \lambda_1'(s)}{g'(s)}, \ \ and \ \ \theta(s) = \frac{\tau(s)\lambda_2(s) + \lambda_3'(s)}{g'(s)},$$

we have

$$\widetilde{t}(\widetilde{s}) = \alpha(s)\,\mathbf{t}(s) + \theta(s)\,\mathbf{b}(s). \tag{3.7}$$

When we differentiate Eq (3.7) in relation to s, we obtain

$$\widetilde{\kappa}(\widetilde{s})\,\widetilde{n}(\widetilde{s})\,g'(s) = \alpha'(s)\,\mathbf{t}(s) + \Big(\kappa(s)\,\alpha(s) - \tau(s)\,\theta(s)\Big)\,\mathbf{n}(s) + \theta'(s)\,\mathbf{b}(s). \tag{3.8}$$

Using the scalar product of Eq (3.8) and $\mathbf{n}(s)$, we get $\kappa(s)\,\alpha(s) - \tau(s)\,\theta(s) = 0$, and therefore

$$\Big(1 + \lambda_1'(s)\Big)\kappa(s) = \Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)\tau(s), \tag{3.9}$$

such that $\tau(s)\lambda_2(s) + \lambda_3'(s) \neq 0$.

In contrast, consider γ to be a curve with the non-vanishing curvatures κ and τ and conditions (3.1) hold for differentiable functions λ_1, λ_2, and λ_3.

Then, we can create an additional curve, $\widetilde{\gamma}$, as follows

$$\widetilde{\gamma}(\widetilde{s}) = \widetilde{\gamma}\Big(g(s)\Big) = \gamma(s) + \lambda_1(s)\,\mathbf{t}(s) + \lambda_2(s)\,\mathbf{n}(s) + \lambda_3(s)\,\mathbf{b}(s). \tag{3.10}$$

Differentiating Eq (3.10) with respect to s yields

$$g'(s)\,\widetilde{t}(\widetilde{s}) = \Big(1 + \lambda_1'(s)\Big)\,\mathbf{t}(s) + \Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)\,\mathbf{b}(s), \tag{3.11}$$

which gives that

$$g'(s) = \Big(< \widetilde{t}(s), \widetilde{t}(s) >_{G_3} \Big)^{\frac{1}{2}} = \frac{q_1}{\kappa}\Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)\sqrt{\kappa^2 + \tau^2}\,, \tag{3.12}$$

where $q_1 = sgn\Big(\tau(s)\lambda_2(s) + \lambda_3'(s)\Big)$.

Substituting into Eq (3.11), we have

$$\widetilde{t}(\widetilde{s}) = \frac{q_1}{\sqrt{\kappa^2 + \tau^2}}\Big(\tau(s)\,\mathbf{t}(s) + \kappa(s)\,\mathbf{b}(s)\Big), \tag{3.13}$$

and hence $< \widetilde{t}(\widetilde{s}), \widetilde{t}(\widetilde{s}) >_{G_3} = 1$.

Taking

$$\alpha_1(s) = \frac{q_1\tau(s)}{\sqrt{\kappa^2 + \tau^2}}, \;\; and \;\; \alpha_2(s) = \frac{q_1\kappa(s)}{\sqrt{\kappa^2 + \tau^2}},$$

then

$$\widetilde{t}(\widetilde{s}) = \alpha_1(s)\,\mathbf{t}(s) + \alpha_2(s)\,\mathbf{b}(s). \tag{3.14}$$

Equation (3.14) can be differentiated with respect to s to yield

$$\frac{d\widetilde{t}}{d\widetilde{s}} = \frac{1}{g'(s)}\Big(\alpha_1'\,\mathbf{t}(s) + \alpha_2'\,\mathbf{b}(s)\Big),$$

and then

$$\widetilde{\kappa} = \sqrt{< \frac{d\widetilde{t}}{d\widetilde{s}}, \frac{d\widetilde{t}}{d\widetilde{s}} >} = \frac{\sqrt{\alpha_1'^2(s) + \alpha_2'^2(s)}}{g'(s)} = \frac{q_2\Big(\kappa\tau' - \tau\kappa'\Big)}{g'(s)\Big(\kappa^2 + \tau^2\Big)} = \frac{q_2\,\kappa^2\Big(\frac{\tau}{\kappa}\Big)'}{g'(s)\Big(\kappa^2 + \tau^2\Big)}, \tag{3.15}$$

such that $q_2 = sgn\Big(\kappa\tau' - \tau\kappa'\Big)$. Then, we may determine $\widetilde{n}(\widetilde{s})$ as follows:

$$\widetilde{n}(\widetilde{s}) = \frac{q_1}{q_2\,\sqrt{\kappa^2 + \tau^2}}\Big(\kappa\,\mathbf{t}(s) - \tau\,\mathbf{b}(s)\Big), \tag{3.16}$$

and $<\widetilde{n}(\widetilde{s}),\widetilde{n}(\widetilde{s})>_{G_3}=1$. Also, $\widetilde{b}(\widetilde{s})$ can be defined as

$$\widetilde{b}(\widetilde{s})=\widetilde{t}(\widetilde{s})\times\widetilde{n}(\widetilde{s})=\frac{1}{q_2}\,\mathbf{n}(s), \tag{3.17}$$

and $<\widetilde{b}(\widetilde{s}),\widetilde{b}(\widetilde{s})>_{G_3}=1$. Finally, we get

$$\widetilde{\tau}(\widetilde{s})=-<\widetilde{b}'(\widetilde{s}),\widetilde{n}(\widetilde{s})>=\frac{q_1\tau^2}{\sqrt{\kappa^2+\tau^2}}\neq 0. \tag{3.18}$$

Therefore, $\widetilde{\gamma}$ is a Mannheim partner curve of γ. Then γ is a Mannheim curve. □

In Theorem (3.1), if we set $\lambda_1(s)=\lambda_3(s)=0$, using the Mannheim mate curve $\widetilde{\gamma}$, which is expressed as

$$\widetilde{\gamma}(s)=\gamma(s)+\lambda_2(s)\,\mathbf{n}(s),$$

we may derive the criteria of classical Mannheim curves seen in the literature.

Corollary 3.2. *Assume that the admissible curve $\gamma: I\to G_3$, $I\subset\mathbb{R}$ has non-vanishing curvatures κ and τ. After that, the curve γ is a Mannheim curve, and its Mannheim partner $\widetilde{\gamma}$ is defined as*

$$\widetilde{\gamma}(s)=\gamma(s)+\lambda_2(s)\,\mathbf{n}(s),$$

if $\lambda_2(s)$ exists and it fulfills $\kappa=\lambda_2\tau^2$, and the opposite is again valid.

Corollary 3.3. *Let $\gamma(s)$ constitute an admissible Mannheim curve through G_3 using s as the parameter for arc length, and let $\widetilde{\gamma}(\widetilde{s})$ be an admissible Mannheim partner curve of γ. Assume $\gamma(s)$ is a generalized helix, then $\widetilde{\gamma}(\widetilde{s})$ represents a straight line.*

Proof. Assume $\gamma: I\to G_3$, $I\subset\mathbb{R}$ is an admissible Mannheim general helix that has non-vanishing curvatures κ and τ. Following that, the ratio $\frac{\tau}{\kappa}$ is constant, which gives from Eq (3.15) that $\widetilde{\kappa}=0$. Then, $\widetilde{\gamma}(\widetilde{s})$ defines a line. □

Corollary 3.4. *Take $\gamma: I\subset\mathbb{R}\to G_3$ to be an admissible Mannheim curve with non-vanishing curvatures κ and τ, and let curve $\widetilde{\gamma}$ be the Mannheim partner curve of γ having $\widetilde{\kappa}$ and $\widetilde{\tau}$ non-zero curvatures. Thus, given that $g'(s)$ has a non-zero value, $\widetilde{\gamma}$ indicates a general helix if the slant helix is represented by γ. In addition, the opposite also holds.*

Proof. Consider $\gamma: I\subset\mathbb{R}\to G_3$ as an admissible Mannheim curve with κ; and τ non-vanishing curvatures and let $\widetilde{\gamma}$ be the Mannheim partner curve of γ having non-zero curvatures $\widetilde{\kappa}$ and $\widetilde{\tau}$. And, following that, from Eqs (3.15) and, (3.18), we obtain

$$\frac{\widetilde{\kappa}}{\widetilde{\tau}}=\frac{q_2\,\kappa^2\left(\frac{\tau}{\kappa}\right)'}{q_1 g'(s)\tau^3\sqrt{\left(\frac{\tau}{\kappa}\right)^2+1}}.$$

If $g'(s)$ is a non-zero constant, then $\widetilde{\gamma}$ can only be considered a general helix in the event that γ is a slant helix. □

In the next part, we introduce special cases for the Mannheim curve in G_3 and its Mannheim partner curve.

Case 3.5. *Assuming that $\gamma : I \to G_3$, $I \subset \mathbb{R}$ is an admissible Mannheim curve with non-zero curvatures $\kappa(s)$ and $\tau(s)$, then the requirements of Theorem (3.1) have been met. Suppose that $\lambda_2 = \lambda \in \mathbb{R}$. Next, we are able to acquire*

$$\kappa(s)\lambda_1(s) = \tau(s)\lambda_3(s) \quad and \quad \Big(1 + \lambda_1'(s)\Big)\kappa(s) = \Big(\lambda_3'(s) + \lambda\tau(s)\Big)\tau(s),$$

which gives that

$$\lambda_3(s) = \frac{\lambda\tau^2(s) - \kappa(s)}{\kappa(s)\left(\frac{\tau}{\kappa}\right)'} \quad and \quad \lambda_1(s) = \frac{\tau(s)\Big[\lambda\tau^2(s) - \kappa(s)\Big]}{\kappa^2(s)\left(\frac{\tau}{\kappa}\right)'}.$$

Therefore, we get the Mannheim partner curve $\widetilde{\gamma}$ as

$$\widetilde{\gamma}(\widetilde{s}) = \gamma(s) + \frac{\tau(s)\Big[\lambda\tau^2(s) - \kappa(s)\Big]}{\kappa^2(s)\left(\frac{\tau}{\kappa}\right)'}\,\mathbf{t}(s) + \lambda\,\mathbf{n}(s) + \frac{\lambda\tau^2(s) - \kappa(s)}{\kappa(s)\left(\frac{\tau}{\kappa}\right)'}\,\mathbf{b}(s).$$

The following illustration displays the Mannheim curve as an example of the Salkowski curve.

Example 3.6. *Consider the Salkowski curve in G_3 given by*

$$\gamma(s) = \left(s,\ \frac{1}{4}\Big(3 - 4s)cos(2\sqrt{s}) + 6\sqrt{s}\ sin(2\sqrt{s})\Big),\ \frac{1}{4}\Big((3 - 4s)sin(2\sqrt{s}) - 6\sqrt{s}\ cos(2\sqrt{s})\Big)\right),$$

along with curvatures $\kappa(s) = 1$ and $\tau(s) = \frac{1}{\sqrt{s}}$ and the Frenet frame

$$\begin{aligned}
\mathbf{t}(s) &= \left(1,\ \sqrt{s}sin(2\sqrt{s}) + \frac{1}{2}\Big(cos(2\sqrt{s})\Big),\ -\sqrt{s}cos(2\sqrt{s}) + \frac{1}{2}\Big(sin(2\sqrt{s})\Big)\right),\\
\mathbf{n}(s) &= \Big(0,\ cos(2\sqrt{s}),\ sin(2\sqrt{s})\Big),\\
\mathbf{b}(s) &= \Big(0,\ -sin(2\sqrt{s}),\ cos(2\sqrt{s})\Big).
\end{aligned}$$

If we set $\lambda = 0$ in case (3.5), the Mannheim partner curve $\widetilde{\gamma}(\widetilde{s})$ may be derived as

$$\widetilde{\gamma}(\widetilde{s}) = \left(3s,\ \frac{3}{4}\Big(cos(2\sqrt{s}) + 2\sqrt{s}sin(2\sqrt{s})\Big),\ \frac{3}{4}\Big(sin(2\sqrt{s}) - 2\sqrt{s}cos(2\sqrt{s})\Big)\right), \tag{3.19}$$

along with curvatures

$$\widetilde{\kappa} = \frac{\sqrt{s(s+1)}}{6\ s\ (s+1)^2},\ and\, \widetilde{\tau} = \frac{\sqrt{s(s+1)}}{s\ (s+1)},$$

and it is possible to acquire the Frenet frame as follows

$$\begin{aligned}
\widetilde{t}(\widetilde{s}) &= \left(\frac{1}{\sqrt{s+1}},\ \frac{1}{2\sqrt{s+1}}cos(2\sqrt{s}),\ \frac{1}{2\sqrt{s+1}}sin(2\sqrt{s})\right),\\
\widetilde{n}(\widetilde{s}) &= \left(\frac{-\sqrt{s}}{\sqrt{s+1}},\ -\sqrt{s+1}\ sin(2\sqrt{s}) - \frac{\sqrt{s}}{2\sqrt{s+1}}cos(2\sqrt{s}),\ \sqrt{s+1}cos(2\sqrt{s}) - \frac{\sqrt{s}}{2\sqrt{s+1}}sin(2\sqrt{s})\right),
\end{aligned}$$

$$\widetilde{b}(\widetilde{s}) = \Big(0,\ -cos(2\sqrt{s}),\ -sin(2\sqrt{s})\Big).$$

We can easily obtain $n(s) = -\widetilde{b}(\widetilde{s})$*, implying that the Mannheim curve* $\gamma(s)$ *possesses a Mannheim partner curve* $\widetilde{\gamma}(\widetilde{s})$*. And,* $\widetilde{\gamma}(\widetilde{s})$ *is not a general helix (Figure 1).*

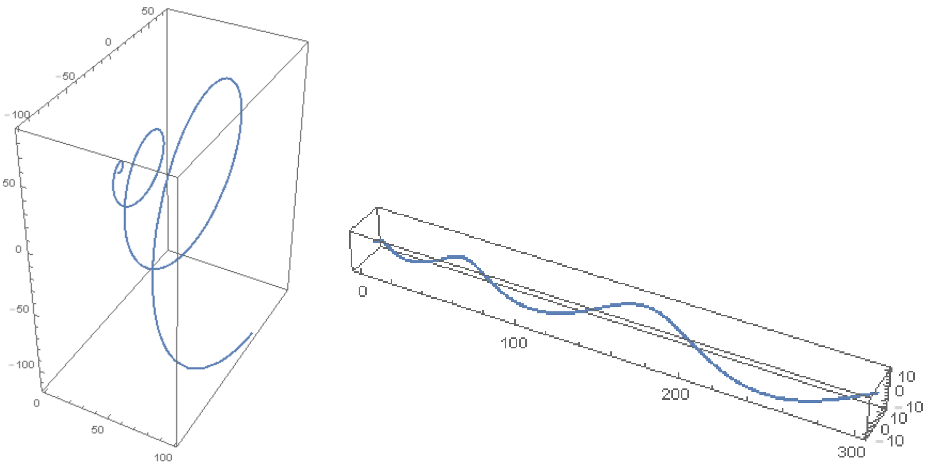

Figure 1. Curves $\gamma(s)$ and $\widetilde{\gamma}(\widetilde{s})$ diagram of Example 3.6.

In the next part, we present a Mannheim Curve as an illustration of the anti-Salkowski curve inside G_3.

Example 3.7. *Define* $\gamma(s)$ *to be an anti-Salkowski curve in* G_3*, which is given by*

$$\gamma(s) = \Big(s,\ \frac{16}{289}\Big[8\ sin(s)\ sinh\Big(\frac{s}{4}\Big) - 15\ cos(s)\ cosh\Big(\frac{s}{4}\Big)\Big], -\frac{16}{289}\Big[8\ cos(s)\ sinh\Big(\frac{s}{4}\Big) + 15\ sin(s)\ cosh\Big(\frac{s}{4}\Big)\Big]\Big)$$

with curvatures $\kappa(s) = cosh\big(\frac{s}{4}\big)$ *and* $\tau(s) = 1$*, and the Frenet frame in the manner described below:*

$$\mathbf{t}(s) = \Big(1,\ \frac{16}{17}\Big[sin(s)\ cosh\Big(\frac{s}{4}\Big) + \frac{1}{4}\ cos(s)\ sinh\Big(\frac{s}{4}\Big)\Big],\ -\frac{16}{17}\Big[cos(s)\ cosh\Big(\frac{s}{4}\Big) - \frac{1}{4}\ sin(s)\ sinh\Big(\frac{s}{4}\Big)\Big]\Big),$$

$$\mathbf{n}(s) = \Big(0,\ cos(s),\ sin(s)\Big),$$

$$\mathbf{b}(s) = \Big(0,\ -sin(s),\ cos(s)\Big).$$

Taking $\lambda = 0$ *in case (3.5), the Mannheim partner curve* $\widetilde{\gamma}(\widetilde{s})$ *is then obtained in the following manner:*

$$\widetilde{\gamma}(\widetilde{s}) = \Big(s + 4\ coth\Big(\frac{s}{4}\Big),\ \frac{128}{289}\ sin(s)\ sinh\Big(\frac{s}{4}\Big) + \frac{32}{289}\ cos(s)\ cosh\Big(\frac{s}{4}\Big) - \frac{4}{17}\ sin(s)\ cosh\Big(\frac{s}{4}\Big)\ coth\Big(\frac{s}{4}\Big),$$
$$-\frac{128}{289}\ cos(s)\ sinh\Big(\frac{s}{4}\Big) + \frac{32}{289}\ sin(s)\ cosh\Big(\frac{s}{4}\Big) + \frac{4}{17}\ cos(s)\ cosh\Big(\frac{s}{4}\Big)\ coth\Big(\frac{s}{4}\Big)\Big),$$

with curvatures $\widetilde{\kappa} = \dfrac{-sgn\Big(\frac{1}{4} sinh\big(\frac{s}{4}\big)\Big)\ tanh\big(\frac{s}{4}\big)}{4\Big(1-csech^2\big(\frac{s}{4}\big)\Big)\Big(1+cosh^2\big(\frac{s}{4}\big)\Big)^{\frac{3}{2}}}$ *and* $\widetilde{\tau} = \dfrac{sgn\Big(1-cosech^2\big(\frac{s}{4}\big)\Big)}{\sqrt{1+cosh^2\big(\frac{s}{4}\big)}}$*, and the Frenet frame obtained as*

$$\widetilde{t}(\widetilde{s}) = \frac{sgn\Big(1 - cosech^2\big(\frac{s}{4}\big)\Big)}{\sqrt{1 + cosh^2\big(\frac{s}{4}\big)}}\Big(1,\ \frac{1}{17}\Big[-sin(s)cosh\Big(\frac{s}{4}\Big) + 4cos(s)sinh\Big(\frac{s}{4}\Big)\Big],\ \frac{1}{17}\Big[cos(s)cosh\Big(\frac{s}{4}\Big) + 4sin(s)sinh\Big(\frac{s}{4}\Big)\Big]\Big),$$

$$\widetilde{n}(\widetilde{s}) = \frac{sgn\left(1 - cosech^2(\frac{s}{4})\right)}{sgn\left(\frac{1}{4}sinh(\frac{s}{4})\right)\sqrt{1 + cosh^2(\frac{s}{4})}}\left(cosh\Big(\frac{s}{4}\Big), cosh\Big(\frac{s}{4}\Big)\Big[\frac{16}{17}\, sin(s)cosh\Big(\frac{s}{4}\Big) + \frac{4}{17}\, cos(s)sinh\Big(\frac{s}{4}\Big)\Big] + sin(s),\right.$$
$$\left. cosh\Big(\frac{s}{4}\Big)\Big[-\frac{16}{17}\, cos(s)cosh\Big(\frac{s}{4}\Big) + \frac{4}{17}\, sin(s)sinh\Big(\frac{s}{4}\Big)\Big] - cos(s)\right),$$

$$\widetilde{b}(\widetilde{s}) = \frac{1}{sgn\left(\frac{1}{4}sinh\left(\frac{s}{4}\right)\right)}\Big(0, cos(s), sin(s)\Big).$$

It is simple to obtain $n(s) = sgn\left(\frac{1}{4}sinh(\frac{s}{4})\right)\widetilde{b}(\widetilde{s})$, *indicating that* $\gamma(s)$ *is a Mannheim curve and that* $\widetilde{\gamma}(\widetilde{s})$ *is its Mannheim partner curve. Again,* $\widetilde{\gamma}(\widetilde{s})$ *is not a general helix (Figure 2).*

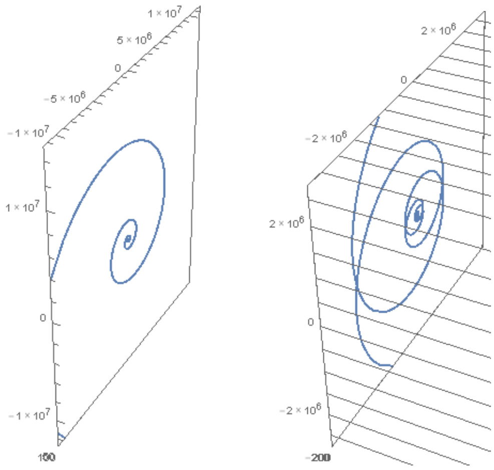

Figure 2. Diagram of curves $\gamma(s)$ and $\widetilde{\gamma}(\widetilde{s})$ in Example 3.7.

Example 3.8. *Consider the curve in* G_3 *defined by*
$\gamma(s) = \left(s, -\sqrt{\pi}\ s\ FresnelS\left(\frac{s}{\sqrt{\pi}}\right) - 2cos\left(\frac{s^2}{2}\right), \sqrt{\pi}\ s\ FresnelC\left(\frac{s}{\sqrt{\pi}}\right) - 2sin\left(\frac{s^2}{2}\right)\right)$, *with the curvatures* $\kappa(s) = s^2$ *and* $\tau(s) = s$, *where* $FresnelS(x) = \int sin\left(\frac{\pi x^2}{2}\right)dx$ *and* $FresnelC(x) = \int cos\left(\frac{\pi x^2}{2}\right)dx$, *and the Frenet frame is obtained as*

$$\begin{aligned}
\mathbf{t}(s) &= \left(1,\ s\ sin\Big(\frac{s^2}{2}\Big) - \int sin\Big(\frac{s^2}{2}\Big)ds,\ -s\ cos\Big(\frac{s^2}{2}\Big) + \int cos\Big(\frac{s^2}{2}\Big)ds\right),\\
\mathbf{n}(s) &= \left(0,\ cos\Big(\frac{s^2}{2}\Big),\ sin\Big(\frac{s^2}{2}\Big)\right),\\
\mathbf{b}(s) &= \left(0,\ -sin\Big(\frac{s^2}{2}\Big),\ cos\Big(\frac{s^2}{2}\Big)\right).
\end{aligned}$$

Taking $\lambda = 0$ *in case (3.5), we acquire the Mannheim partner curve* $\widetilde{\gamma}(\widetilde{s})$ *according to*

$$\widetilde{\gamma}(\widetilde{s}) = \left(2s,\ -2s\int sin\Big(\frac{s^2}{2}\Big)ds - 2cos\Big(\frac{s^2}{2}\Big),\ 2s\int cos\Big(\frac{s^2}{2}\Big)ds - 2sin\Big(\frac{s^2}{2}\Big)\right),$$

with curvatures $\widetilde{\kappa} = \frac{1}{2\left(s^2+1\right)^{\frac{3}{2}}}$ *and* $\widetilde{\tau} = \frac{s}{\sqrt{s^2+1}}$. *The Frenet frame is calculated as*

$$\begin{aligned}
\widetilde{t}(\widetilde{s}) &= \Big(\frac{1}{\sqrt{s^2+1}},\ \frac{-1}{\sqrt{s^2+1}}\int sin\big(\frac{s^2}{2}\big)ds,\ \frac{1}{\sqrt{s^2+1}}\int cos\big(\frac{s^2}{2}\big)ds\Big),\\
\widetilde{n}(\widetilde{s}) &= \Big(-\frac{s}{\sqrt{s^2+1}},\ -\sqrt{s^2+1}\ sin\big(\frac{s^2}{2}\big)+\frac{s}{\sqrt{s^2+1}}\int sin\big(\frac{s^2}{2}\big)ds,\ \sqrt{s^2+1}\ cos\big(\frac{s^2}{2}\big)-\frac{s}{\sqrt{s^2+1}}\int cos\big(\frac{s^2}{2}\big)ds\Big),\\
\widetilde{b}(\widetilde{s}) &= \Big(0,\ -cos\big(\frac{s^2}{2}\big),\ -sin\big(\frac{s^2}{2}\big)\Big).
\end{aligned}$$

Then, we can deduce that $n(s) = -\widetilde{b}(\widetilde{s})$, which means that $\gamma(s)$ is a Mannheim curve, along with $\widetilde{\gamma}(\widetilde{s})$ serving as its Mannheim partner curve. Also, $\widetilde{\gamma}(\widetilde{s})$ is not a general helix (Figure 3).

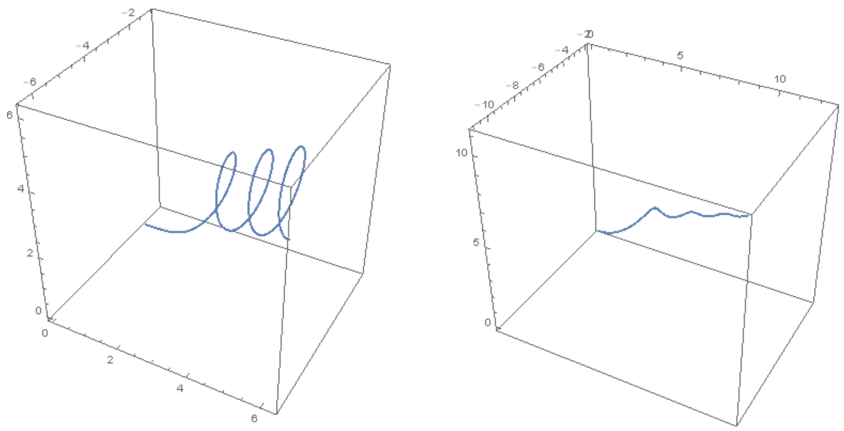

Figure 3. Curves $\gamma(s)$ and $\widetilde{\gamma}(\widetilde{s})$ of Example 3.8.

Theorem 3.9. *Let $\widetilde{\gamma} : \widetilde{I} \to G_3$, $\widetilde{I} \subset \mathbb{R}$ be an admissible curve utilizing the $\{\widetilde{t}, \widetilde{n}, \widetilde{b}\}$ Frenet frame as well as the non-zero curvatures $\widetilde{\kappa}$ and $\widetilde{\tau}$. For each given Mannheim curve, if its Mannheim partner curve is represented by $\widetilde{\gamma}$, then there exist differentiable functions μ_1, μ_2, μ_3, and δ satisfying the following two cases:*
(1) If $\dot{B} \neq 0$, then we have

$$\widetilde{\tau}\mu_2(s) + \dot{\mu}_3(\widetilde{s}) = 0,\ \frac{1+\dot{\mu}_1(\widetilde{s})}{\dot{f}} \neq 0,\ \dot{\delta} = -\widetilde{\kappa},\ \dot{\mu}_2(\widetilde{s}) + \widetilde{\kappa}\mu_1(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s}) = \delta\Big(1+\dot{\mu}_1(\widetilde{s})\Big). \tag{3.20}$$

(2) If $\dot{B} = 0$, then we have

$$\widetilde{\tau}\mu_2(\widetilde{s}) + \dot{\mu}_3(\widetilde{s}) = 0,\ \dot{\mu}_2(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s}) + \widetilde{\kappa}\Big(-\widetilde{s} + c_o\Big) = d\dot{f}, \tag{3.21}$$

where c_o and d are non-zero constants and $B = \frac{\dot{\mu}_2(\widetilde{s})+\widetilde{\kappa}\mu_1(\widetilde{s})-\widetilde{\tau}\mu_3(\widetilde{s})}{\dot{f}}$. In this case, "." signifies the derivative in relation to $\widetilde{s}$.

Proof. Suppose that γ is an admissible Mannheim curve with non-zero κ and τ curvatures, specified by the arc length s, and the Mannheim partner curve of curve γ is represented by curve $\widetilde{\gamma}$, where $\widetilde{\gamma}$ is parameterized by the arc length $\widetilde{s}$. Thus, we may define the curve γ as

$$\gamma(s) = \gamma\Big(f(\widetilde{s})\Big) = \widetilde{\gamma}(\widetilde{s}) + \mu_1(\widetilde{s})\,\widetilde{t}(\widetilde{s}) + \mu_2(\widetilde{s})\,\widetilde{n}(\widetilde{s}) + \mu_3(\widetilde{s})\,\widetilde{b}(\widetilde{s}), \tag{3.22}$$

with every $\widetilde{s} \in \widetilde{I}$, such that $\mu_1(\widetilde{s})$, $\mu_2(\widetilde{s})$, and $\mu_3(\widetilde{s})$ are differentiable functions on $\widetilde{I}$.

When we differentiate Eq (3.22) considering $\widetilde{s}$, we arrive at

$$\mathbf{t}(s)\dot{f} = \Big(1 + \dot{\mu}_1(\widetilde{s})\Big)\widetilde{t}(\widetilde{s}) + \Big(\dot{\mu}_2(\widetilde{s}) + \widetilde{\kappa}\mu_1(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s})\Big)\widetilde{n}(\widetilde{s}) + \Big(\widetilde{\tau}\mu_2(\widetilde{s}) + \dot{\mu}_3(\widetilde{s})\Big)\widetilde{b}(\widetilde{s}). \tag{3.23}$$

By taking the dot product of (3.23) with $\widetilde{b}(\widetilde{s})$, we get

$$\widetilde{\tau}\mu_2(\widetilde{s}) + \dot{\mu}_3(\widetilde{s}) = 0. \tag{3.24}$$

Inserting (3.24) in (3.23), we have

$$\mathbf{t}(s)\dot{f} = \Big(1 + \dot{\mu}_1(\widetilde{s})\Big)\widetilde{t}(\widetilde{s}) + \Big(\dot{\mu}_2(\widetilde{s}) + \widetilde{\kappa}\mu_1(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s})\Big)\widetilde{n}(\widetilde{s}). \tag{3.25}$$

By taking the scalar product of (3.25) with itself, we deduce

$$(\dot{f}\,)^2 = \Big(1 + \dot{\mu}_1(\widetilde{s})\Big)^2 + \Big(\dot{\mu}_2(\widetilde{s}) + \widetilde{\kappa}\mu_1(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s})\Big)^2. \tag{3.26}$$

If we set

$$A = \frac{1 + \dot{\mu}_1(\widetilde{s})}{\dot{f}}, \quad and \quad B = \frac{\dot{\mu}_2(\widetilde{s}) + \widetilde{\kappa}\mu_1(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s})}{\dot{f}}, \tag{3.27}$$

we obtain

$$\mathbf{t}(s) = A\,\widetilde{t}(\widetilde{s}) + B\,\widetilde{n}(\widetilde{s}). \tag{3.28}$$

Differentiating (3.28) with respect to $\widetilde{s}$, we obtain

$$\dot{f}\,\kappa\, n(s) = \dot{A}\,\widetilde{t}(\widetilde{s}) + \Big(A\widetilde{\kappa} + \dot{B}\Big)\,\widetilde{n}(\widetilde{s}) + \widetilde{\tau}B\,\widetilde{b}(\widetilde{s}). \tag{3.29}$$

Since the principle normal vector $\mathbf{n}(s)$ of the curve $\gamma(s)$ and the bi-normal vector $\widetilde{b}(\widetilde{s})$ of its Mannheim Partner curve is linearly dependent, we reach

$$\widetilde{\kappa} = \frac{-\dot{B}}{A}.$$

We have two cases:

- If $\dot{B} \neq 0$, then $A \neq 0$ and $B \neq 0$, which implies

  $$\frac{1 + \dot{\mu}_1(\widetilde{s})}{\dot{f}} \neq 0, \quad and \quad \dot{\mu}_2(\widetilde{s}) + \widetilde{\kappa}\mu_1(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s}) = \delta\Big(1 + \dot{\mu}_1(\widetilde{s})\Big),$$

  where $\delta = \frac{B}{A}$.

  Again, from (3.29), utilizing $\widetilde{b}(\widetilde{s})$ to obtain the scalar product, we acquire $\dot{A} = 0$ and $A\widetilde{\kappa} + \dot{B} = 0$, which gives

  $$\dot{\delta} = -\widetilde{\kappa}.$$

- If $\dot{B} = 0$, then $A = 0$, which implies $\dot{\mu}_1 = -1$ and $\mu_1 = -\widetilde{s} + c_o$, where c_o is constant. Also, we can deduce that B is non-zero constant (d, say). Now, from Eq (3.27), we have

  $$\dot{\mu}_2(\widetilde{s}) - \widetilde{\tau}\mu_3(\widetilde{s}) + \widetilde{\kappa}\Big(- \widetilde{s} + c_o\Big) = d\dot{f}$$

□

If we set $\mu_1(\widetilde{s}) = \mu_2(\widetilde{s}) = 0$ within Theorem (3.9), we arrive at the requirements of traditional Mannheim partner curves found in the literature by using

$$\gamma(s) = \gamma\Big(f(\widetilde{s})\Big) = \widetilde{\gamma}(\widetilde{s}) + \mu_3(\widetilde{s})\widetilde{b}(\widetilde{s}).$$

Also, we get

$$\dot{\mu}_3(\widetilde{s}) = 0, \ \dot{\delta} = -\widetilde{\kappa}, \ -\widetilde{\tau}\mu_3(\widetilde{s}) = \delta.$$

So, we have

$$\widetilde{\tau} = \frac{-1}{\mathbf{a}} \int \widetilde{\kappa} d\widetilde{s},$$

for some non-zero real number **a**. Therefore, this leads us to the next corollary.

Corollary 3.10. *Suppose that $\widetilde{\gamma} : \widetilde{I} \to G_3$, $\widetilde{I} \subset \mathbb{R}$ is an admissible curve with non-zero curvatures $\widetilde{\kappa}$ and $\widetilde{\tau}$, and with the Frenet frame $\{\widetilde{t}, \widetilde{n}, \widetilde{b}\}$. If $\widetilde{\gamma}$ is a Mannheim partner curve of a certain Mannheim curve given by*

$$\gamma(s) = \gamma\Big(f(\widetilde{s})\Big) = \widetilde{\gamma}(\widetilde{s}) + \mu_3(s)\widetilde{b}(\widetilde{s}),$$

then a non-zero real number ***a*** *exists such that*

$$\widetilde{\tau} = \frac{-1}{\boldsymbol{a}} \int \widetilde{\kappa} d\widetilde{s}.$$

4. Conclusions

In this study, a novel method for Mannheim curves in three-dimensional Galilean space was presented. The necessary and sufficient conditions for a curve to be a Mannheim curve were obtained. Finally, some examples were introduced.

Author contributions

Mervat Elzawy: conceptualization, methodology, software, reviewing and editing; Safaa Mosa: data curation, writing-original draft preparation, visualization, investigation, reviewing and editing.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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