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Research article

Kink phenomena of the time-space fractional Oskolkov equation

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Abstract: In this study, we applied the Riccati-Bernoulli sub-ODE method and Bäcklund transformation to analyze the time-space fractional Oskolkov equation for kink solutions by matching the coefficients and optimal series parameters. The time-space fractional Oskolkov equation is used to analyze the behavior of solitons for different applications such as fluid dynamics and viscoelastic flow. The kink solutions derived have important consequences for stability analysis and interaction dynamic in these systems, and these are useful in controlling the physical behaviour of systems described by this equation. Such effects are illustrated by 2D and 3D plots, showing that the proposed model can handle both fractional and integer-order solitons with different but equally efficient outcomes. This research contributes to a valuable analytical method that can determine and manage processes in diversified systems based on fractional differential equations. This work provides a basis for subsequent analysis in other branches of science and technology in which the fractional Oskolkov model is used.

Keywords: fractional Oskolkov equation; Bäcklund transformation; non-linear differential equations; exact solutions
Mathematics Subject Classification: 32W50, 34A25, 83C15

1. Introduction

Over the last few decades, nonlinear partial differential equations (NLPDEs) have emerged as a major field of interest in the study of mathematical sciences. Since the natural world is highly complex, the interconnection between its two components is fascinating; many authors believe that, while studying the nonlinear science true triumphs of the human mind, one gets the greatest opportunity to understand the basics of specific physical laws. Countless physical behaviors and scientific disciplines including engineering, climatology, applied mathematics, biologic, and chemical reactions are described through the help of NLPDEs [1, 2]. In this context of evaluated systems to comprehend these evaluated systems, solving NLPDEs and finding the numerical as well as the analytical solutions proves to be of significant importance. Numerous researchers have developed various methods to obtain solitary wave solutions for the non-linear PDEs [3, 4]. These include the lie group method [5], $exp(-\phi(x))$-expansion method [6], extended direct algebraic method [7], variational iteration method [8], unified method [9], (G'/G^2)-expansion method [10], sine-Gordon expansion method [11], tanh-coth method [12], homotopy analysis method [13], auxiliary ordinary differential equation (ODE) method [14], and tanh function method [15].

Fractional derivatives are widely used in mathematics, with the focus being put on the non-integer order derivatives [16–18]. These derivatives are crucial when the system under investigation is characterized by power-law processes and memory. In recent years, a large number of scientists have introduced fractional derivatives to analyze the characteristics of the stability of solitons in various fields [19–21]. Many researches have been devoted to seeking solitary wave solutions of diverse nonlinear PDEs and these have become an essential part of the development of our thoughts on these systems [22–24]. In order to increase the precision of the model, several forms of the fractional have been built. For example, fractional derivatives have been used in signal systems [25], plasma physics models [26], fractional epidemiologic models [27] and different models involving fractional derivatives. In this context, the Riccati-Bernoulli sub-ODE method with Bäcklund transformation [28–30] was used on the truncated time-space fractional Oskolkov equation. In this method, one can succeed in the systematic simplification of complicated fractional PDEs into ODEs and obtain explicit exact solutions. This method is very useful in the presence of nonlinear techniques and is very flexible with respect of the type of equation used. Also, it yields precise solutions, and such solutions are good predictors that give additional understanding of the behavior of the system in question relative to numerical or strictly approximate approaches. It is crucial to substitute the conventional Oskolkov equation with the equation of its fractional analogue because the description of phenomena involving some memory and non-locality, which are quite common in numerous applications, is impossible with assistance from the standard approach. The fractional differential parameter (α) provides an opportunity to approximate the intricacy of diffusion and wave propagation that the simple Oskolkov equation cannot consider. This is quite suitable in areas like fluid dynamics and plasma physics since the fractional model is much more detailed in the description of such processes as viscoelastic flow and fractal dispersion. These phenomena can be described more thoroughly using the fractional Oskolkov equation, which provides the researchers and engineers a more accurate approach to investigate complex systems which demonstrate the fractional-order behavior. Hence to ensure that the phenomena under consideration correspond to real-life occurrences, this model was examined in its fractional form [31–33]:

$$\frac{\partial f}{\partial t} - \beta \frac{\partial}{\partial t}\left(\frac{\partial^2 f}{\partial x^2}\right) - \gamma \frac{\partial^2 f}{\partial x^2} + f\left(\frac{\partial f}{\partial x}\right) = 0. \tag{1.1}$$

This equation is helpful in the calculation of dimensions and geometry of thin-walled pressure vessels like tanks and reactors and is used extensively across the field of chemical and mechanical engineering. It is particularly useful in the design of pressure vessels for high pressure and high-temperature severe service environments. The Oskolkov equation is in fact a major model for viscoelastic, non-Newtonian fluids, capable of approximating major manifestations of the flow behavior such as shear-thinning and viscoelasticity [34]. Hence, incorporating fractional derivatives, the fractional Oskolkov equation also covers such behaviors as memory effects and non-local interactions inherent in non-Newtonian media. This extension expands the applicability of this method to various real-life situations such as polymer solutions and biological fluids. Also, the Oskolkov equation is used in estimating mechanical properties like stress and strain of pressure vessels. Still, many authors studied soliton solutions of similar models by employing significance approaches. For instance, different approaches were used to investigate solitonic wave solutions of the Oskolkov equation. The MSE scheme was used in [35], and another approach was used by the authors of [36]. The modified $exp-(-\phi(\zeta))$-expansion function was given in [37] and the (Φ, Ψ) expansion method was also studied in [38]. In addition the Sine-Gordon expansion method was used in [39]. Altogether these investigations have helped in put into place various solitonic wave solutions. Current trends of study also involve optical solitons [40], wave propagation in plasma [41], and nonparaxial solitons [42], including both fractional as well as traditional approaches to vibrations of cross kink solitons [43].

To construct solitonic wave solutions, several approaches have been employed in previous studies to examine the soliton solutions of models similar to the present one. These investigations span various applications, which include optical solitons, wave propagation in plasma, and discrete as well as fractional approaches. Nevertheless, in this work, we provide a new approach to this problem by using the Riccati-Bernoulli sub-ODE method in combination with the Bäcklund transformation to study the fractional version of the Oskolkov equation. This method offers fresh perspectives on the behavior of solitonic waves and describes some exciting kink characteristics in the fractional Oskolkov model. In this context, the presentation of our analysis concerns the way that fractional parameters affect these solitonic solutions and uses 2D as well as 3D plots to visualize the dependence of these parameters. This contribution stretches the area of application of the fractional Oskolkov equation and provides deeper arguments to further investigate soliton solutions in fractional models.

Further, the operator integrating α-derivatives of powers agrees exactly to the idea of conformable fractional derivatives [44] for $W(\phi)$ of order $\alpha \in (0, 1)$ and for $t > 0$ is defined as:

$$D_{\phi}^{\alpha} W(\phi) = \lim_{i \to 0} \frac{W(i(\phi)^{i-\alpha} - W(\phi))}{i}, \quad 0 < \alpha \leq 1, \tag{1.2}$$

$$\begin{cases} D_{\phi}^{\alpha} \phi^{p} = p\phi^{p-\alpha}, \\ D_{\phi}^{\alpha} \left(p_1 \eta(\phi) \pm p_2 t(\phi)\right) = p_1 D_{\phi}^{\alpha}(\eta(\phi)) \pm m_2 D_{\phi}^{\alpha}(t(\phi)), \\ D_{\phi}^{\alpha} \left[f \circ g\right] = \phi^{1-\alpha} g\left(\phi\right) D_{\phi}^{\alpha} f\left(g(\phi)\right). \end{cases} \tag{1.3}$$

For smooth functions, the derivative simplifies to $D_{\alpha} f(t) = t^{1-\alpha} \frac{df(t)}{dt}$. One of the major benefits as to the proposed conformable derivative is that it does generalize the classical derivative in a manner as elementary as the concept itself. Indeed, when $\alpha = 1$, it returns the standard derivative, thus making a transition from fractional derivation to classical derivation rather smooth.

Unlike Caputo and Riemann-Liouville derivatives, the conformable derivative does not use complex integral formulations, which makes it more easy to utilize when finding the derivative of differential equations, yet the conformable derivative keeps a lot of characteristics in fractional calculus. In our analyses, the defined conformable fractional derivative is incorporated to present the time-space fractional Oskolkov equation to describe the fluid flow and other processes with better accuracy as compared to the conventional integer order. This derivative will help us incorporate memory and hereditary properties in the system, which is very important in fractional models.

Section 2 gives a brief overview of the method that has been used here, which will be explained in detail in Section 3 by providing the solution of the new fractional Oskolkov system. In Section 4, results and discussion are provided and some graphical illustrations are given. Finally, Section 5 provides the conclusion of our work.

2. Algorithm

Here, for the clear understanding of the working procedure of the mentioned process, let us explain it broadly.

Step 1. Consider nonlinear PDEs in the following form:

$$P_1\left(R_1, D_t^{\alpha}(R_1), D_{q_1}^{\alpha}(R_1), D_{q_2}^{\alpha}(R_1), R_1 D_{q_1}^{\alpha}(R_1), \ldots\right) = 0, \qquad 0 < \alpha \leq 1, \tag{2.1}$$

where $R_1 = R(t, q_1, q_2, q_3, \ldots, q_k)$ is a function of $(t, q_1, q_2, q_3, \ldots, q_k)$ and its partial derivatives.

Step 2. This transformation changes Eq (2.1) into a nonlinear ODE of the following form:

$$Q_1\left(F, F'(\phi), F''(\phi), FF'(\phi), \ldots\right) = 0. \tag{2.2}$$

Step 3. Let us suppose that Eq (2.2) has the following solution:

$$G(\phi) = \sum_{j=-n}^{n} s_j g(\phi)^j, \tag{2.3}$$

where s_j are constants and $g(\phi)$ is obtained from the Bäcklund transformation,

$$g(\phi) = \frac{-\zeta p_2 + p_1 Z(\phi)}{p_1 + p_2 Z(\phi)}.$$

Here, (ζ), (p_1), and (p_2) are constants such that $p_2 \neq 0$ and $Z(\phi)$ is the solution of the following ODE:

$$\frac{dZ}{d\phi} = \zeta + Z(\phi)^2. \tag{2.4}$$

The Ricatti Eq (2.4) possess the following general solutions [45]:

$$Z(\phi) = \begin{cases} -\sqrt{-\zeta}\tanh(\sqrt{-\zeta}\phi), & \text{as } \zeta < 0, \\ -\sqrt{-\zeta}\coth(\sqrt{-\zeta}\phi), & \text{as } \zeta < 0, \end{cases}$$
$$Z(\phi) = -\frac{1}{\psi}, \quad \text{as } \zeta = 0, \tag{2.5}$$
$$Z(\phi) = \begin{cases} \sqrt{\zeta}\tan(\sqrt{\zeta}\phi), & \text{as } \zeta > 0, \\ -\sqrt{\zeta}\cot(\sqrt{\zeta}\phi), & \text{as } \zeta > 0. \end{cases}$$

Step 4. Solving for the homogeneous balance of the largest nonlinear term and the highest-order derivative in Eq (2.2) gives the positive integer (n) as presented in Eq (2.3). First, it has to be noted that the balance number of a processor can be calculated as follows [46]:

$$D\left[\frac{d^m f}{d\psi^m}\right] = n + m, \quad D\left[f^J \frac{d^m f}{d\psi^m}\right]^w = nJ + w(m+n). \tag{2.6}$$

Step 5. Next, we replace the same function with the help of Eq (2.3) into Eq (2.2) or into the expression which appears after integration of Eq (2.2), and collect all terms containing $g(\phi)$. The coefficients of the polynomial are then set equal to zero and a system of algebraic equations in (s_i) and other parameters are obtained.

Step 6. These equations are solved using the Maple computational tool and, in the end, the solitary wave solutions are provided for Eq (2.1).

3. Problem execution

By applying the considered model with the Riccati-Bernoulli sub-ODE, we get the wave solutions. The (1 + 1)-dimensional fractional Oskolkov equation in its fractional form is given by

$$D_t^\alpha(f) - \beta D_t^\alpha\left(D_x^{2\alpha}(f)\right) - \gamma\left(D_x^{2\alpha}(f)\right) + f\left(D_x^\alpha(f)\right) = 0. \tag{3.1}$$

Equation (3.1) encompasses the temporal evolution of viscoelastic fluids where the material memory and the refractive index both play pertinent roles. The term $D_t^\alpha(f)$ represents the time fractionality, expressing the fact that the current state of the fluid depends on its prior behavior. The second term, therefore, includes fractional temporal and spatial derivatives and is used to capture dispersive effects that depend on the spatial distribution. The dissipation term $-\left(D_x^{2\alpha}(f)\right)$ is used to express the generalized diffusion processes and the nonlinear interface $f\left(D_x^\alpha(f)\right)$ depicts how solitons and wave patterns interact and stay preserved while travelling. This equation is very important in modeling this behavior of fluids where conventional models fail to capture the behavior well. Here, β and γ are constants and $f(x,t)$ represents an unknown wave front. Therefore, $f(x,t) = F(\psi)$ and $\psi = \frac{\lambda x^\alpha}{\alpha} - \omega\frac{t^\alpha}{\alpha}$ are used to change Eq (3.1). It is converted into the following ordinary differential system:

$$2\lambda^2\omega\beta\frac{d^2F}{d\psi^2} - 2\gamma\lambda^2\frac{dF}{d\psi} - 2\omega F + \lambda F^2 = 0. \tag{3.2}$$

We use the proposed approach that takes advantage of properties that exist inherently in the system balancing equations to reduce and solve for wave structures. In this way, integrating specific terms, it is possible to extract the individual characteristics of the primary components of wave phenomena. By substituting Eqs (2.3) and (2.4) into Eq (3.2) and then collecting the coefficients of $Z(\phi)$, we derive the following system of equations:

$$\begin{aligned}
&-12\,\lambda^2\omega\beta\,s_{-2}{p_2}^8\zeta^2-\lambda\,{s_{-2}}^2{p_2}^8=0,\\
&4\,\lambda^2\omega\beta\,s_{-1}{p_2}^8\zeta^3+2\,\lambda\,s_{-2}{p_2}^8s_{-1}\zeta+4\,\lambda^2\gamma\,s_{-2}{p_2}^8\zeta^2=0,\\
&-16\,\lambda^2\omega\beta\,s_{-2}{p_2}^8\zeta^3+2\,\omega\,s_{-2}{p_2}^8\zeta^2-\lambda\,{s_{-1}}^2{p_2}^8\zeta^2-2\,\lambda\,s_{-2}{p_2}^8s_0\zeta^2-2\,\lambda^2\gamma\,s_{-1}{p_2}^8\zeta^3=0,\\
&-2\,\omega\,s_{-1}{p_2}^8\zeta^3+4\,\lambda^2\gamma\,s_{-2}{p_2}^8\zeta^3+4\,\lambda^2\omega\beta\,s_{-1}{p_2}^8\zeta^4+2\,\lambda\,s_{-1}{p_2}^8s_0\zeta^3+2\,\lambda\,s_{-2}{p_2}^8s_1\zeta^3=0,\\
&-2\,\lambda\,s_{-1}{p_2}^8s_1\zeta^4-4\,\lambda^2\omega\beta\,s_2{p_2}^8\zeta^6-2\,\lambda^2\gamma\,s_{-1}{p_2}^8\zeta^4+2\,\lambda^2\gamma\,s_1{p_2}^8\zeta^5\\
&-2\,\lambda\,s_{-2}{p_2}^8s_2\zeta^4+2\,\omega\,s_0{p_2}^8\zeta^4-\lambda\,{s_0}^2{p_2}^8\zeta^4-4\,\lambda^2\omega\beta\,s_{-2}{p_2}^8\zeta^4=0,\\
&-2\,\omega\,s_1\zeta^5{p_2}^8+2\,\lambda\,s_0s_1\zeta^5{p_2}^8-4\,\lambda^2\gamma\,s_2{p_2}^8\zeta^6+4\,\lambda^2\omega\beta\,s_1{p_2}^8\zeta^6+2\,\lambda\,s_{-1}{p_2}^8s_2\zeta^5=0,\\
&2\,\omega\,s_2\zeta^6{p_2}^8-\lambda\,{s_1}^2\zeta^6{p_2}^8-16\,\lambda^2\omega\beta\,s_2{p_2}^8\zeta^7-2\,\lambda\,s_0s_2\zeta^6{p_2}^8+2\,\lambda^2\gamma\,s_1{p_2}^8\zeta^6=0,\\
&4\,\lambda^2\omega\beta\,s_1{p_2}^8\zeta^7+2\,\lambda\,s_1\zeta^7{p_2}^8s_2-4\,\lambda^2\gamma\,s_2{p_2}^8\zeta^7=0,\\
&-\lambda\,{s_2}^2\zeta^8{p_2}^8-12\,\lambda^2\omega\beta\,s_2{p_2}^8\zeta^8=0.
\end{aligned}\tag{3.3}$$

This give us the algebraic equations by setting $Z(\phi)=0$. The solutions of this system of algebraic equations obtained from Maple are:

Set 1.

$$\begin{aligned}
&s_0=3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}},\,s_1=2\,\sqrt{6}\,\sqrt{\beta}\omega,\,s_{-1}=0,\,s_{-2}=0,\\
&s_2=-10\,\frac{\sqrt{6}\beta^{3/2}\omega^2}{\gamma},\zeta=-\frac{1}{100}\,\frac{\gamma^2}{\omega^2\beta^2},\lambda=5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega}{\gamma},\omega=\omega.
\end{aligned}\tag{3.4}$$

Set 2.

$$\begin{aligned}
&s_0=1/4\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}},\,s_1=2\,\sqrt{6}\,\sqrt{\beta}\omega,\,s_{-1}=\frac{1}{200}\,\frac{\sqrt{6}\gamma^2}{\beta^{3/2}\omega},\,s_{-2}=-\frac{1}{16000}\,\frac{\sqrt{6}\gamma^3}{\beta^{5/2}\omega^2},\\
&s_2=-10\,\frac{\sqrt{6}\beta^{3/2}\omega^2}{\gamma},\zeta=-\frac{1}{400}\,\frac{\gamma^2}{\omega^2\beta^2},\lambda=5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega}{\gamma},\omega=\omega.
\end{aligned}\tag{3.5}$$

Solution Group 1. For Set 1 ($\zeta<0$) provided that $\omega=\frac{-1}{10}$, we obtain the following set of solutions for Eq (3.1), where in this case

$$\zeta=-\frac{1}{100}\,\frac{\gamma^2}{\omega^2\beta^2},\psi=5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega\,x^{\alpha}}{\gamma\,\alpha}-\frac{\omega\,t^{\alpha}}{\alpha},$$

$$\begin{aligned}
f_1(x,t)=&\,3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}}+2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2p_2}{\omega^2\beta^2}-p_1\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)\left(p_1-p_2\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^{-1}\\
&-10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2p_2}{\omega^2\beta^2}-p_1\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^2\gamma^{-1}\left(p_1-p_2\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^{-2}
\end{aligned}\tag{3.6}$$

or

$$f_2(x,t) = 3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)\left(p_1 - p_2\,\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^{-1}$$
$$-\,10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^2\gamma^{-1}\left(p_1 - p_2\,\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^{-2}. \tag{3.7}$$

Solution Group 2. For Set 1 ($\zeta > 0$) provided that $\omega = \frac{-1}{10}$, we obtain the following set of solutions for Eq (3.1):

$$f_3(x,t) = 3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\,\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)\left(p_1 + p_2\,\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^{-1}$$
$$-\,10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\,\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^2\gamma^{-1}\left(p_1 + p_2\,\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^{-2} \tag{3.8}$$

or

$$f_4(x,t) = 3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)\left(p_1 - p_2\,\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^{-1}$$
$$-\,10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^2\gamma^{-1}\left(p_1 - p_2\,\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^{-2}. \tag{3.9}$$

Solution Group 3. For Set 1 ($\zeta = 0$), we obtain the following set of solutions for Eq (3.1):

$$f_5(x,t) = 3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)\left(p_1 - \frac{p_2}{\psi}\right)^{-1}$$
$$-\,10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)^2\gamma^{-1}\left(p_1 - \frac{p_2}{\psi}\right)^{-2}. \tag{3.10}$$

Solution Group 4. For Set 2 ($\zeta < 0$) provided that $\omega = \frac{-1}{20}$, we obtain the following set of solutions for Eq (3.1), where in this case

$$\zeta = -\frac{1}{400}\,\frac{\gamma^2}{\omega^2\beta^2},\ \psi = 5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega\,x^\alpha}{\gamma\,\alpha} - \frac{\omega\,t^\alpha}{\alpha},$$

$$f_6(x,t) = s_{-2}\left(p_1 - p_2\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^2\left(\frac{1}{400}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^{-2}$$
$$+\,s_{-1}\left(p_1 - p_2\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)\left(\frac{1}{400}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^{-1}$$
$$+\,s_0 + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{400}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)\left(p_1 - p_2\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^{-1}$$
$$+\,s_2\omega^2\beta\left(\frac{1}{400}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^2\left(p_1 - p_2\,\sqrt{-\zeta}\tanh\left(\sqrt{-\zeta}\psi\right)\right)^{-2} \tag{3.11}$$

or

$$
\begin{aligned}
f_7(x,t) =\ & s_{-2}\left(p_1 - p_2\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^2\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^{-2} \\
& + s_{-1}\left(p_1 - p_2\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^{-1} \\
& + s_0 + 2\sqrt{6}\sqrt{\beta}\omega\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)\left(p_1 - p_2\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^{-1} \\
& + s_2\omega^2\beta\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^2\left(p_1 - p_2\sqrt{-\zeta}\coth\left(\sqrt{-\zeta}\psi\right)\right)^{-2}.
\end{aligned}
\tag{3.12}
$$

Solution Group 5. For Set 2 ($\zeta > 0$) provided that $\omega = \frac{-1}{20}$, we obtain the following set of solutions for Eq (3.1):

$$
\begin{aligned}
f_8(x,t) =\ & s_{-2}\left(p_1 + p_2\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^2\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^{-2} \\
& + s_{-1}\left(p_1 + p_2\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^{-1} \\
& + s_0 + 2\sqrt{6}\sqrt{\beta}\omega\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)\left(p_1 + p_2\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^{-1} \\
& + s_2\omega^2\beta\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^2\left(p_1 + p_2\sqrt{\zeta}\tan\left(\sqrt{\zeta}\psi\right)\right)^{-2}
\end{aligned}
\tag{3.13}
$$

or

$$
\begin{aligned}
f_9(x,t) =\ & s_{-2}\left(p_1 - p_2\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^2\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^{-2} \\
& + s_{-1}\left(p_1 - p_2\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^{-1} \\
& + s_0 + 2\sqrt{6}\sqrt{\beta}\omega\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)\left(p_1 - p_2\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^{-1} \\
& + s_2\omega^2\beta\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^2\left(p_1 - p_2\sqrt{\zeta}\cot\left(\sqrt{\zeta}\psi\right)\right)^{-2}.
\end{aligned}
\tag{3.14}
$$

Solution Group 6. For Set 2 ($\zeta = 0$), we obtain the following set of solutions for Eq (3.1):

$$
\begin{aligned}
f_10(x,t) =\ & s_{-2}\left(p_1 - \frac{p_2}{\psi}\right)^2\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)^{-2} + s_{-1}\left(p_1 - \frac{p_2}{\psi}\right)\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)^{-1} \\
& + s_0 + 2\sqrt{6}\sqrt{\beta}\omega\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)\left(p_1 - \frac{p_2}{\psi}\right)^{-1} \\
& + s_2\omega^2\beta\left(\frac{1}{400}\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)^2\left(p_1 - \frac{p_2}{\psi}\right)^{-2}.
\end{aligned}
\tag{3.15}
$$

4. Results and discussion

In this work, we have used the Riccati-Bernoulli sub-ODE approach in conjunction with the Bäcklund transformation to obtain explicit solitonic solutions of the fractional Oskolkov equation. Although finding these solutions was possible due to the use of computational tools such as Maple, the emphasis is much more than just writing down these solutions. The solutions offer essential information on the nonlinear dynamics determined by the fractional differential operator (α), especially for the description of solitonic phenomena in multiscale systems. Thus, we build a strong theoretical background by the detailed analysis of the fractional order and the proper consideration of the results for the fluid dynamics and the wave propagation. This framework also aided in improving knowledge about the fractional Oskolkov equation as well as providing assessment of the concepts in fractional calculus and real-world applications of fractional phenomena. The novelty of this work is to connect these solutions to the physical systems and, thus, to the field of applied mathematics and theory of nonlinear waves. In the next section, we discuss the spatial visualization of wave solutions attained through the fractional Oskolkov equation. The identified solution types, trigonometric, hyperbolic, and rational, are shown in the following figures in 3D and 2D views Figures 1–4. This equation turns out to be extremely useful in arriving at the dimensions and geometry of thin-walled pressure vessels like tanks and reactors and hence forms a part of the standard tools used by chemical and mechanical engineers. In particular, it contributes to such vessels' design, which are destined to operate at elevated pressure and temperature, typical for severe service conditions [47]. Moreover, to predict the mechanical properties, stresses, and strains of those pressure vessels, one makes use of the Oskolkov equation. Table 1 has been created to presents side by side comparison between solutions achieved in this study and those obtained using modified Kudryashov method in other study. This table presents the difference and advantage of the current approach and shows how our work deviates from the previous method used in similar issue.

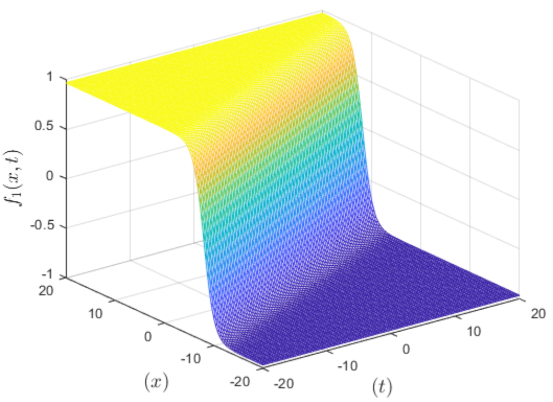
The solution for an anti-kink wave is depicted in Figure 1. Thus, one may claim that as the value of the fractional order parameter (α) increases, the wave propagation features change dramatically. More particularly, the amplitude of the solution increases manifold in the medium in which the wave propagates.

The graphical solutions presented in Figure 1 can be discussed distinguishing some characteristics that can be significant considering wave propagation, especially in areas of fluid dynamics. Such large amplitude solutions indicate large coupling within the medium which is again important for quantifying the wave phenomena in different systems. This understanding proves very useful in areas of engineering, particularly in the estimation and design of thin-walled pressure vessels. In chemical and mechanical engineering, these designs have to be optimized with significant awareness of wave behavior at different pressures and temperatures. For instance, the solutions proffered out of the time-space fractional Oskolkov equation are instrumental in estimating stress, strain, and mechanical characteristics of materials toward enhancing the design effectiveness and reliability of engineered structures.

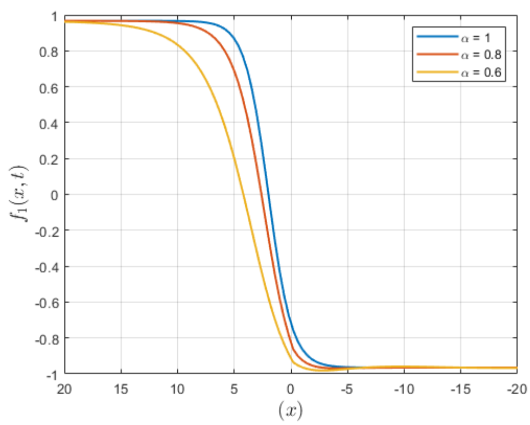
Figure 2 plots the fuzziness diagnostic of the soliton solution kink type provided in this paper with the deterioration of the fractional-order derivative parameter (α). The deviation, as a result, with an application of the fractional order parameter with reduced values aligns to Figure 1 below. In the physical sense, this implies that smaller fractional orders for the reduced order model can have a beneficial effect on the dynamic characteristics of the system. In particular, the increase in the amplitudes for $\alpha = 0.99$ and $\alpha = 0.98$ compared to $\alpha = 1$ hints at the possibility that the interior force within the nuclear matter could be boosted by minimal fractions. Such enhancement can, therefore, result in deeper kink formations, implying improved angular density, and hence, sharper and profound state transition during the fission and fusion process inherent in nuclear reactions.

In detail, Figure 3 illustrates the dependence of the amplitude of a kink-type solitary solution on the fractional operator parameter (α). When varying the fractional parameter, a noticeable decline of the amplitude is observed in the bottom zones, whereas the amplitude in the top zones does not get altered. Such selectiveness indicates that (α) affects the solution energy distribution in a differential manner. At a physical level, it could be the manifestation, within this model, of the selective impact of the fractional parameter on the energy of the system. The trough in the lower energy state fluctuates more due to the memory effect and the strength of the nuclei matter. On the other hand, the crests which are a location of the higher energy states are left unscathed hence there is a sense of nuclear transitions in the fission and fusion at these locations to be stable and uninterfered with. This understanding could have crucial implications for the analysis of energy interactions across behaviors, especially in structural and dynamic nuclear physical systems and engineering where wave properties impact the structure stability and energy changes.

It is clearly seen from Figure 4 that the effect of an increase in the fractional order parameter (α) is uniform and is manifested through the decrease in the amplitude of the kink-type solitary solution at all points. From this plot, one can infer that with the decrease in memory effects and interaction strengths in nuclear matters, the total strength of the kink-type solution decreases. From a physical perspective, it shows that lesser fractional weights negatively affect the solution by providing a weaker bend. Also, notable behavior is essential for enhancing thin-wall pressure vessels utilized in high-pressure and high-temperature susceptible applications. It is useful in predicting mechanical properties such as stress and strain by applying the Oskolkov equation, thereby guaranteeing high performance and reliability, especially in operations involving fluids and nuclear- related processes.

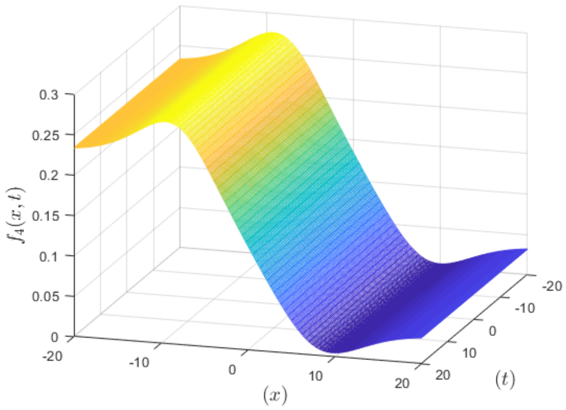


(a) 3D graphical representation for the integer order derivative parameter.

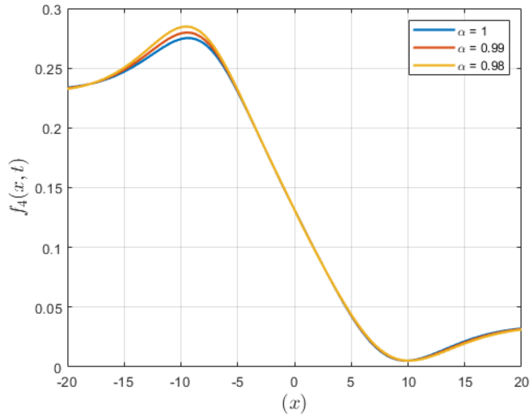


(b) 2D graphical representation for the fractional order derivative parameter α.

Figure 1. Different variation analysis of the derivative parameter (α) of the solution $f_1(x,t)$.

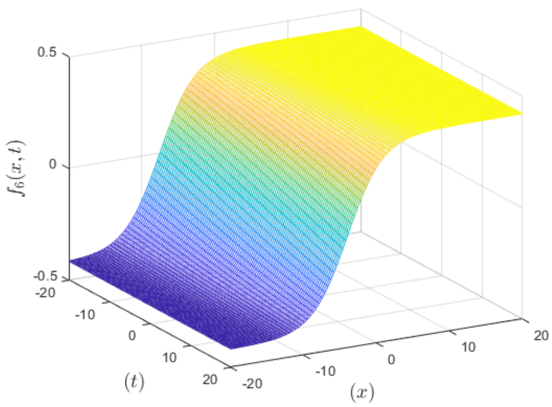


(a) 3D graphical representation for the integer order derivative parameter.

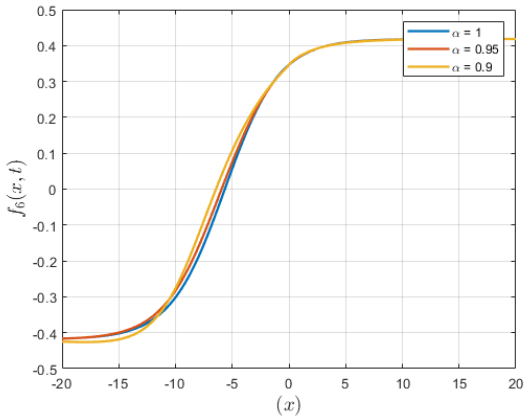


(b) 2D graphical representation for the fractional order derivative parameter α.

Figure 2. Different variation analysis of the derivative parameter (α) of the solution $f_4(x, t)$.

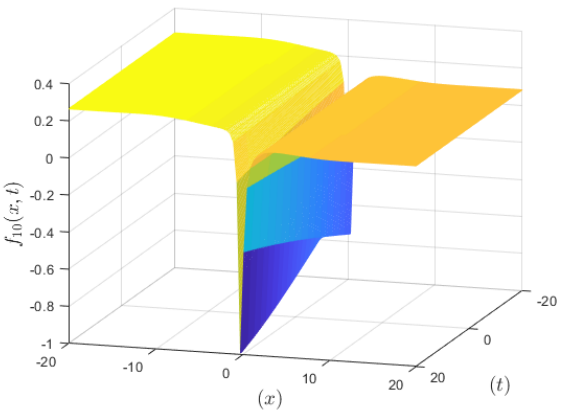


(a) 3D graphical representation for the integer order derivative parameter.

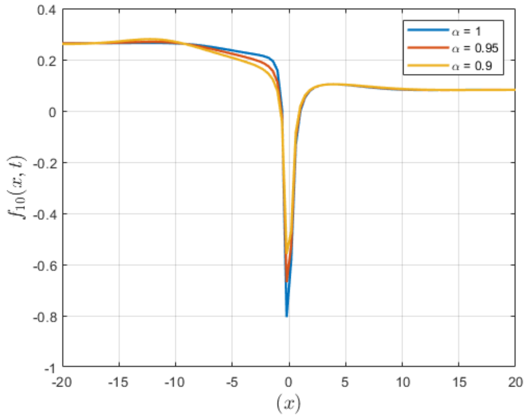


(b) 2D graphical representation for the fractional order derivative parameter α.

Figure 3. Different variation analysis of the derivative parameter (α) of the solution $f_6(x, t)$.



(a) 3D graphical representation for the integer order derivative parameter.



(b) 2D graphical representation for the fractional order derivative parameter α.

Figure 4. Different variation analysis of the derivative parameter (α) of the solution $f_{10}(x, t)$.

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Table 1. Comparison of the Riccati-Bernaoulli sub-ODE along with Bäcklund transformation with the modified Kudryashov method [47].

	Present method	Modified Kudryashov method
Case I: $\zeta < 0$,	$f(x,t) = \frac{3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\tanh(\,\sqrt{-\zeta}\psi)\right)}{(p_1 - p_2\,\sqrt{-\zeta}\tanh(\,\sqrt{-\zeta}\psi))}$	$M(x,t) = \frac{12\alpha}{\sqrt[5]{6\beta}}\frac{\mathscr{N}}{\mathscr{N} + \ell(\mathscr{N} - \mathcal{J})\cdot m^{\pm\frac{1}{\mathrm{kn}(m)\sqrt{6\beta}}}}\left\{\pm 2\left(-\frac{\mathscr{N}}{\mathscr{N} + \ell(\mathscr{N} - \mathcal{J})\cdot m^{\pm\frac{1}{k(m)\sqrt{6\beta}}\cdot\xi}}\right)\right\}$
$\psi = 5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega\,x^{\alpha}}{\gamma\,\alpha} - \frac{\omega\,t^{\alpha}}{\alpha}$	$-\frac{10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - p_1\,\sqrt{-\zeta}\tanh(\,\sqrt{-\zeta}\psi)\right)^2}{\gamma\left(p_1 - p_2\,\sqrt{-\zeta}\tanh(\,\sqrt{-\zeta}\psi)\right)^{-2}}$	for $\xi = kx \mp \frac{6k\alpha\Gamma(\sigma+1)}{5\delta\sqrt{6\beta}}\tau^{\delta}$
Case II: $\zeta > 0$,	$f(x,t) = \frac{3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\,\sqrt{\zeta}\tan(\,\sqrt{\zeta}\psi)\right)}{(p_1 + p_2\,\sqrt{\zeta}\tan(\,\sqrt{\zeta}\psi))}$	$M(x,t) = \frac{12\alpha}{\sqrt[5]{6\beta}}\left\{\pm 1 \mp \frac{\mathscr{N}^2}{\left\{\mathscr{N} + \ell(\mathscr{N} - \mathcal{J})m^{\pm\frac{1}{k\ln(m)\sqrt{6\beta}}\xi}\right\}^2}\right\}$
$\psi = 5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega\,x^{\alpha}}{\gamma\,\alpha} - \frac{\omega\,t^{\alpha}}{\alpha}$	$-\frac{10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} + p_1\,\sqrt{\zeta}\tan(\,\sqrt{\zeta}\psi)\right)^2}{\gamma\left(p_1 + p_2\,\sqrt{\zeta}\tan(\,\sqrt{\zeta}\psi)\right)^{-2}}$	for $\xi = kx \mp \frac{6k\alpha\Gamma(\sigma+1)}{5\delta\sqrt{6\beta}}\tau^{\delta}$
Case III: $\zeta = 0$,	$f_5(x,t) = \frac{3/10\,\frac{\sqrt{6}\gamma}{\sqrt{\beta}} + 2\,\sqrt{6}\,\sqrt{\beta}\omega\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)}{\left(p_1 - \frac{p_2}{\psi}\right)}$	$M(x,t) = \frac{12i\alpha}{\sqrt[5]{6\beta}}\left\{\pm 1 \mp \frac{2\mathscr{N}}{\mathscr{N} + \ell(\mathscr{N} - \mathscr{J})m^{\pm\frac{i}{k\ln(m)\sqrt{6\beta}}\xi}}\right\}\left\{\mp\left(\frac{\mathscr{N}}{\mathscr{N} + \ell(\mathscr{N} - \mathscr{J})m^{\pm\frac{i}{k\ln(m)\sqrt{6\beta}}\xi}}\right)^2\right\}$
$\psi = 5/6\,\frac{\sqrt{6}\,\sqrt{\beta}\omega\,x^{\alpha}}{\gamma\,\alpha} - \frac{\omega\,t^{\alpha}}{\alpha}$	$-\frac{10\,\sqrt{6}\beta^{3/2}\omega^2\left(\frac{1}{100}\,\frac{\gamma^2 p_2}{\omega^2\beta^2} - \frac{p_1}{\psi}\right)^2}{\gamma\left(p_1 - \frac{p_2}{\psi}\right)^2}$	for $\xi = kx \mp \frac{6ik\alpha\Gamma(\sigma+1)}{5\delta\sqrt{6\beta}}\tau^{\delta}$

5. Conclusions

Here we considered the spatial nature and dynamic properties of the solutions obtained from the fractional Oskolkov equation, and especially the solitons of the kink type. It is shown that by applying systematic Bäcklund transformation and the sub-ODE of the given Riccati-Bernoulli equation, the types of different solutions derived and discussed include trigonometric, hyperbolic, and rational forms.

As shown in our results depicted in Figures 1–4, a considerable difference in the amplitude and propagation of solutions is distinguished when the value of the fractional order parameter is changed. Reduction of the fractional order also improves the dynamic characteristics of the system that increases its strength within the nuclear matter and, therefore, sharpens the kink formations. These ideas have great relevance to the generation and modeling of thin-walled pressure vessels in chemical and mechanical engineering. Owing to the Oskolkov equation, it is possible to precisely forecast and regulate mechanical characteristics, including stress and strain, to increase the efficiency within high-pressure and high-temperature conditions. The given approach based on the Bäcklund transformation with the use of the Riccati-Bernoulli sub-ODE method is a solid platform for further detailed examination of emerging wave solutions in the context of fractional systems.

The study is beneficial for developing the quantitative descriptions on the realistic wave systems with potential applications in hydrodynamics, plasma physics, and nonlinear optics. Applying conformable fractional derivatives, the work provides an enriched insight into the behavior of dynamic waves and the significance of the use of fractional-order models for a better description of the nonlocal phenomena and memory of the physical processes. It is noteworthy that the aforementioned analysis and the two proposed methods, namely the Riccati-Bernoulli sub-ODE method and Bäcklund transformation, provide efficient ways of obtaining the exact solutions for the time-space fractional Oskolkov equation. For instance, the solutions that have been found are limited by the range of fractional orders, and how these solutions might behave when the orders transcend beyond these stated limits has not been explained. Also, it must be noted that the model is still simplified and deals with the concepts of an idealized environment, and the effects of turbulence, the higher-order effects, or the interactions between multiple solitons have not been fully considered. Also it was found that using two and three dimensionals plots can help us understand the solitonic behavior of the analytically computed solutions but the experimental confirmation of these theoretical findings has been left for another future work. These limitations present the scope for coming up with additional research to enhance this approach. Future work could include looking at the theoretical and experimental results of this work and making comparisons with other nonlinear models and an in-depth analysis of boundary layers. This will further strengthen the usage as well as the relialibility of the models presented here.

Author contributions

Conceptualization, M.M.A; Data curation, H.Y; Formal analysis, A.M.M; Resources, M.M.A; Investigation, H.Y; Project administration, A.M.M; Validation, M.M.A.; Software, H.Y; Validation, A.M.M; Visualization, M.M.A.; Validation, H.Y.; Visualization, M.M.A.; Resources, A.M.M.; Project administration, M.M.A.; Writingreview & editing, A.M.M; Funding, H.Y. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. S. Noor, W. Albalawi, R. Shah, M. M. Al-Sawalha, S. M. E. Ismaeel, Mathematical frameworks for investigating fractional nonlinear coupled Korteweg-de Vries and Burger's equations, *Front. Phys.*, **12** (2024), 1374452. https://doi.org/10.3389/fphy.2024.1374452
2. S. Noor, W. Albalawi, R. Shah, M. M. Al-Sawalha, S. M. E. Ismaeel, S. A. El-Tantawy, On the approximations to fractional nonlinear damped Burger's-type equations that arise in fluids and plasmas using Aboodh residual power series and Aboodh transform iteration methods, *Front. Phys.*, **12** (2024), 1374481. https://doi.org/10.3389/fphy.2024.1374481
3. H. Yasmin, A. S. Alshehry, A. H. Ganie, A. M. Mahnashi, R. Shah, Perturbed Gerdjikov-Ivanov equation: soliton solutions via Backlund transformation, *Optik*, **298** (2024), 171576. https://doi.org/10.1016/j.ijleo.2023.171576
4. S. A. El-Tantawy, H. A. Alyousef, R. T. Matoog, R. Shah, On the optical soliton solutions to the fractional complex structured $(1+1)$-dimensional perturbed Gerdjikov-Ivanov equation, *Phys. Scr.*, **99** (2024), 035249. https://doi.org/10.1088/1402-4896/ad241b
5. H. Jafari, N. Kadkhoda, D. Baleanu, Fractional Lie group method of the time-fractional Boussinesq equation, *Nonlinear Dyn.*, **81** (2015), 1569–1574. https://doi.org/10.1007/s11071-015-2091-4
6. N. Raza, M. R. Aslam, H. Rezazadeh, Analytical study of resonant optical solitons with variable coefficients in Kerr and non-Kerr law mediaa, *Opt. Quant. Electron.*, **51** (2019), 59. https://doi.org/10.1007/s11082-019-1773-4
7. W. Gao, H. Rezazadeh, Z. Pinar, H. M. Baskonus, S. Sarwar, G. Yel, Novel explicit solutions for the nonlinear Zoomeron equation by using newly extended direct algebraic technique, *Opt. Quant. Electron.*, **52** (2020), 52. https://doi.org/10.1007/s11082-019-2162-8
8. H. I. Abdel-Gawad, M. S. Osman, On the variational approach for analyzing the stability of solutions of evolution equations, *Kyungpook Math. J.*, **53** (2013), 661–680. https://doi.org/10.5666/KMJ.2013.53.4.680

9. H. I. Abdel-Gawad, M. Osman, On shallow water waves in a medium with time-dependent dispersion and nonlinearity coefficients, *J. Adv. Res.*, **6** (2014), 593–599. https://doi.org/10.1016/j.jare.2014.02.004

10. M. N. Ali, M. S. Osman, S. M. Husnine, On the analytical solutions of conformable time-fractional extended ZakharovCKuznetsov equation through (G'/G^2)-expansion method and the modified Kudryashov method, *SeMA J.*, **76** (2019), 15–25. https://doi.org/10.1007/s40324-018-0152-6

11. K. K. Ali, M. S. Osman, M. Abdel-Aty, New optical solitary wave solutions of Fokas-Lenells equation in optical fiber via Sine-Gordon expansion method, *Alex. Eng. J.*, **59** (2020), 1191–1196. https://doi.org/10.1016/j.aej.2020.01.037

12. A. Kumar, R. D. Pankaj, Tanh-coth scheme for traveling wave solutions for nonlinear wave interaction model, *J. Egypt. Math. Soc.*, **23** (2015), 282–285. https://doi.org/10.1016/j.joems.2014.05.002

13. D. R. G. Domairry, A. Mohsenzadeh, M. Famouri, The application of homotopy analysis method to solve nonlinear differential equation governing Jeffery-Hamel flow, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 85–95. https://doi.org/10.1016/j.cnsns.2007.07.009

14. Z. Y. Zhang, Exact traveling wave solutions of the perturbed Klein-Gordon equation with quadratic nonlinearity in (1 + 1)-dimension, part I: without local inductance and dissipation effect, *Turk. J. Phys.*, **37** (2013), 259–267. https://doi.org/10.3906/fiz-1205-13

15. K. A. Abro, A. Atangana, Dual fractional modeling of rate type fluid through non-local differentiation, *Numer. Methods Partial Differ. Equ.*, **38** (2020), 390–405. https://doi.org/10.1002/num.22633

16. S. Alshammari, K. Moaddy, R. Shah, M. Alshammari, Z. Alsheekhhussain, M. M. Al-sawalha, et al., Analysis of solitary wave solutions in the fractional-order Kundu-Eckhaus system, *Sci. Rep.*, **14** (2024), 3688. https://doi.org/10.1038/s41598-024-53330-7

17. S. A. El-Tantawy, R. T. Matoog, R. Shah, A. W. Alrowaily, S. M. E. Ismaeel, On the shock wave approximation to fractional generalized Burger-Fisher equations using the residual power series transform method, *Phys. Fluids*, **36** (2024), 023105. https://doi.org/10.1063/5.0187127

18. Z. Alsheekhhussain, K. Moaddy, R. Shah, S. Alshammari, M. Alshammari, M. M. Al-Sawalha, et al., Extension of the optimal auxiliary function method to solve the system of a fractional-order Whitham-Broer-Kaup equation, *Fractal Fract.*, **8** (2023), 1. https://doi.org/10.3390/fractalfract8010001

19. M. Alqhtani, K. M. Saad, R. Shah, W. M. Hamanah, Discovering novel soliton solutions for $(3+1)$-modified fractional Zakharov-Kuznetsov equation in electrical engineering through an analytical approach, *Opt. Quant. Electron.*, **55** (2023), 1149. https://doi.org/10.1007/s11082-023-05407-2

20. S. Noor, B. M. Alotaibi, R. Shah, S. M. E. Ismaeel, S. A. El-Tantawy, On the solitary waves and nonlinear oscillations to the fractional Schrödinger-KdV equation in the framework of the Caputo operator, *Symmetry*, **15** (2023), 1616. https://doi.org/10.3390/sym15081616

21. H. Yasmin, A. A. Alderremy, R. Shah, A. Hamid Ganie, S. Aly, Iterative solution of the fractional Wu-Zhang equation under Caputo derivative operator, *Front. Phys.*, **12** (2024), 1333990. https://doi.org/10.3389/fphy.2024.1333990

22. Y. Kai, S. Chen, K. Zhang, Z. Yin, Exact solutions and dynamic properties of a nonlinear fourth-order time-fractional partial differential equation, *Waves Random Complex Media*, 2022, 1–12. https://doi.org/10.1080/17455030.2022.2044541

23. Y. Kai, Z. Yin, Linear structure and soliton molecules of Sharma-Tasso-Olver-Burgers equation, *Phys. Lett. A*, **452** (2022), 128430. https://doi.org/10.1016/j.physleta.2022.128430

24. C. Zhu, M. Al-Dossari, S. Rezapour, B. Gunay, On the exact soliton solutions and different wave structures to the (2 + 1) dimensional Chaffee-Infante equation, *Results Phys.*, **57** (2024), 107431. https://doi.org/10.1016/j.rinp.2024.107431

25. M. D. Ortigueira, J. A. T. Machado, Fractional calculus applications in signals and systems, *Signal Process.*, **86** (2006), 2503–2504. https://doi.org/10.1016/j.sigpro.2006.02.001

26. M. A. Abdou, An analytical method for space-time fractional nonlinear differential equations arising in plasma physics, *J. Ocean Eng. Sci.*, **2** (2017), 288–292. https://doi.org/10.1016/j.joes.2017.09.002

27. C. Baishya, S. J. Achar, P. Veeresha, D. G. Prakasha, Dynamics of a fractional epidemiological model with disease infection in both the populations, *Chaos*, **31** (2021), 043130. https://doi.org/10.1063/5.0028905

28. M. A. E. Abdelrahman, M. A. Sohaly, Solitary waves for the modified Korteweg-de Vries equation in deterministic case and random case, *J. Phys. Math.*, **8** (2017), 1–6. https://doi.org/10.4172/2090-0902.1000214

29. S. Meng, F. Meng, H. Chi, H. Chen, A. Pang, A robust observer based on the nonlinear descriptor systems application to estimate the state of charge of lithium-ion batteries, *J. Franklin Inst.*, **360** (2023), 11397–11413. https://doi.org/10.1016/j.jfranklin.2023.08.037

30. M. A. E. Abdelrahman, M. A. Sohaly, Solitary waves for the nonlinear Schrödinger problem with the probability distribution function in the stochastic input case, *Eur. Phys. J. Plus*, **132** (2017), 339. https://doi.org/10.1140/epjp/i2017-11607-5

31. S. Kumar, B. Mohan, R. Kumar, Lump, soliton, and interaction solutions to a generalized two-mode higher-order nonlinear evolution equation in plasma physics, *Nonlinear Dyn.*, **110** (2022), 693–704. https://doi.org/10.1007/s11071-022-07647-5

32. G. P. Agrawal, Nonlinear fiber optics, In: P. L. Christiansen, M. P. Sørensen, A. C. Scott, *Nonlinear science at the dawn of the 21st century*, Lecture Notes in Physics, Springer, **542** (2000), 195–211. https://doi.org/10.1007/3-540-46629-0_9

33. B. Ghanbari, New analytical solutions for the Oskolkov-type equations in fluid dynamics via a modified methodology, *Results Phys.*, **28** (2021), 104610. https://doi.org/10.1016/j.rinp.2021.104610

34. E. S. Baranovskii, The Navier-Stokes-Voigt equations with position-dependent slip boundary conditions, *Z. Angew. Math. Phys.*, **74** (2023), 6. https://doi.org/10.1007/s00033-022-01881-y

35. M. M. Roshid, H. O. Roshid, Exact and explicit traveling wave solutions to two nonlinear evolution equations which describe incompressible viscoelastic Kelvin-Voigt fluid, *Heliyon*, **4** (2018), e00756. https://doi.org/10.1016/j.heliyon.2018.e00756

36. L. Yu, Y. Lei, Y. Ma, M. Liu, J. Zheng, D. Dan, et al., A comprehensive review of fluorescence correlation spectroscopy, *Front. Phys.*, **9** (2021), 644450. https://doi.org/10.3389/fphy.2021.644450
37. O. A. Ilhan, H. Bulut, T. A. Sulaiman, H. M. Baskonus, Dynamic of solitary wave solutions in some nonlinear pseudoparabolic models and Dodd-Bullough-Mikhailov equation, *Indian J. Phys.*, **92** (2018), 999–1007. https://doi.org/10.1007/s12648-018-1187-3
38. M. N. Alam, M. S. Uddin, C. Tunç, Soliton wave solutions of the Oskolkov equation arising in incompressible visco-elastic Kelvin-Voigt fluid, *Abstr. Appl. Anal.*, **5** (2021), 334–342.
39. N. Kheaomaingam, S. Phibanchon, S. Chimchinda, Sine-Gordon expansion method for the kink soliton to Oskolkov equation, *J. Phys.: Conf. Ser.*, **2431** (2023), 012097. https://doi.org/10.1088/1742-6596/2431/1/012097
40. R. Li, Z. A. B. Sinnah, Z. M. Shatouri, J. Manafian, M. F. Aghdaei, A. Kadi, Different forms of optical soliton solutions to the Kudryashov's quintuple self-phase modulation with dual-form of generalized nonlocal nonlinearity, *Results Phys.*, **46** (2023), 106293. https://doi.org/10.1016/j.rinp.2023.106293
41. X. Liu, B. Abd Alreda, J. Manafian, B. Eslami, M. F. Aghdaei, M. Abotaleb, et al., Computational modeling of wave propagation in plasma physics over the Gilson-Pickering equation, *Results Phys.*, **50** (2023), 106579. https://doi.org/10.1016/j.rinp.2023.106579
42. Y. Qian, J. Manafian, M. Asiri, K. H. Mahmoud, A. I. Alanssari, A. S. Alsubaie, Nonparaxial solitons and the dynamics of solitary waves for the coupled nonlinear Helmholtz systems, *Opt. Quant. Electron.*, **55** (2023), 1022. https://doi.org/10.1007/s11082-023-05232-7
43. R. Li, J. Manafian, H. A. Lafta, H. A. Kareem, K. F. Uktamov, M. Abotaleb, The nonlinear vibration and dispersive wave systems with cross-kink and solitary wave solutions, *Int. J. Geom. Methods Mod. Phys.*, **19** (2022), 2250151. https://doi.org/10.1142/S0219887822501511
44. X. F. Yang, Z. C. Deng, Y. Wei, A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application, *Adv. Differ. Equ.*, **2015** (2015), 117. https://doi.org/10.1186/s13662-015-0452-4
45. D. Lu, Q. Shi, New Jacobi elliptic functions solutions for the combined KdV-mKdV equation, *Int. J. Nonlinear Sci.*, **10** (2010), 320–325.
46. H. Yao, D. Pugliese, M. Lancry, Y. Dai, Ultrafast laser direct writing nanogratings and their engineering in transparent materials, *Laser Photonics Rev.*, **18** (2024), 2300891. https://doi.org/10.1002/lpor.202300891
47. H. O. Roshid, M. Roshid, A. Abdeljabbar, M. Begum, H. Basher, Abundant dynamical solitary waves through Kelvin-Voigt fluid via the truncated M-fractional Oskolkov model, *Results Phys.*, **55** (2023), 107128. https://doi.org/10.1016/j.rinp.2023.107128

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