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Research article

An analysis of fractional integral calculus and inequalities by means of coordinated center-radius order relations

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Abstract: Interval-valued maps adjust integral inequalities using different types of ordering relations, including inclusion and center-radius, both of which behave differently. Our purpose was to develop various novel bounds and refinements for weighted Hermite-Hadamard inequalities as well as their product form by employing new types of fractional integral operators under a cr-order relation. Mostly authors have used inclusion order to adjust inequalities in interval maps, but they have some flaws, specifically they lack the property of comparability between intervals. However, we show that under cr-order, it satisfies all relational properties of intervals, including reflexivity, antisymmetry, transitivity, and comparability and preserves integrals as well. Furthermore, we provide numerous interesting remarks, corollaries, and examples in order to demonstrate the accuracy of our findings.

Keywords: weighted Hermite-Hadamard; symmetric mappings; fractional calculus; coordinated cr-order
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1. Introduction

Fractional calculus represents a significant extension of classical calculus, enabling the analysis of phenomena that cannot be adequately described by integer-order derivatives and integrals. Over the centuries, mathematicians like Euler, Laplace, Riemann, and Liouville contributed to its development. However, it was not until the 20th century that fractional calculus started gaining significant attention and applications. Fractional calculus often provides more accurate and flexible models for real-world phenomena. It can capture memory effects, non-locality, and complex dynamics that traditional integer-order calculus might miss. It has numerous applications in various fields of science and engineering, including: It is applied in the analysis of electromagnetic fields and waves, where fractional derivatives help model complex behaviors. Fractional calculus is utilized in control systems to model and analyze dynamical systems, enhancing the design of controllers for systems with memory effects. In materials science, fractional calculus models the behavior of viscoelastic materials, which exhibit both viscous and elastic characteristics, allowing for a more accurate representation of their stress-strain relationships. For some other applications in various domains, check [1–4] and the references therein.

Mathematical inequalities provide a foundational framework for understanding the behavior of functions under integration, leading to significant applications in both theoretical and applied mathematics. Convex integral inequalities are a powerful tool in mathematical analysis, providing relationships between integrals of convex functions and their values at specific points. They find applications in various fields, including probability theory, information theory, and optimization (see [5–8]). These inequalities are crucial for numerical methods, especially for estimating the error bounds in numerical integration techniques like the trapezoidal rule, Simpson's rule, and others. Several notable integral inequalities have been documented in the literature, including Hermite-Hadamard [9], Newton [10], Simpson [11], Bullen [12], and others.

Through the use of the different classes of generalized convex mappings, authors have interpreted the double Hermite and Hadamard (H-H) inequality in various ways. The double inequality was proposed by Hermite (1822–1901) and Hadamard (1865–1963). In addition to their contributions to number theory, nonlinear analysis, and complex analysis, Hermite and Hadamard have made significant contributions to other fields as well. To learn more about their contributions, see [13]. This inequality is a significant discovery in convex analysis and is widely used in various fields of applied analysis, particularly optimality analysis. Let us describe it as below.

Suppose that $\psi : \Omega \subseteq \mathsf{R} \to \mathsf{R}$ is a convex mapping defined over the interval Ω of real numbers, as well as $\omega_1, \omega_2 \in \Omega$ together with $\omega_1 \neq \omega_2$. Then one has (see [14]):

$$\psi\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \psi(\theta)\, \mathrm{d}\theta \leq \frac{\psi(\omega_1) + \psi(\omega_2)}{2}. \tag{1.1}$$

This inequality is applied in geometric contexts to determine correlations between a function's value at the midpoint of an interval and its average over that interval. In information theory, the inequality has been used to set boundaries and estimations, especially in relation to quantum integral inequalities and quantum calculus. An effective tool for examining a variety of economic phenomena involving convex functions, such as asset pricing and optimization as well as income distribution and production, is due to Hermite-Hadamard inequality (see [15]).

The bidimensional convex function is primarily used to prove that all convex mappings are convex over their coordinates. Additionally, there are bidimensional convex functions that are not convex (see, for instance, [16]). Hermite-Hadamard type inequality of the following kind was established for convex co-ordinated mappings on R^2, that is:

Let a function $\psi : [\omega_1, \omega_2] \times [\omega_3, \omega_4] \subseteq \mathsf{R}^2 \to \mathsf{R}$ is convex across its coordinate plane. Then, we have the following double inequalities (see [17]):

$$
\begin{aligned}
&4\psi\Big(\frac{\omega_1+\omega_2}{2}, \frac{\omega_3+\omega_4}{2}\Big)\\
\leq&\Big[\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\psi\Big(\mathtt{x}, \frac{\omega_3+\omega_4}{2}\Big)\,d\mathtt{x} + \frac{1}{\omega_4-\omega_3}\int_{\omega_3}^{\omega_4}\psi\Big(\frac{\omega_1+\omega_2}{2}, \mathtt{y}\Big)\,d\mathtt{y}\Big]\\
\leq&\frac{1}{(\omega_2-\omega_1)(\omega_4-\omega_3)}\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\psi(\mathtt{x},\mathtt{y})\,d\mathtt{y}\,d\mathtt{x}\\
\leq&\Big[\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}[\psi(\mathtt{x},\omega_3)+\psi(\mathtt{x},\omega_4)]\,d\mathtt{x} + \frac{1}{\omega_4-\omega_3}\int_{\omega_3}^{\omega_4}[\psi(\omega_1,\mathtt{y})+\psi(\omega_2,\mathtt{y})]\,d\mathtt{y}\Big]\\
\leq&\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}
\tag{1.2}
$$

First, the authors in [18] employed inclusion order and interval-valued functions for two-dimensional double inequality and proposed a number of innovative variations of inequalities in terms of inclusions. They further show that when the interval is wrapped, these results generalize many earlier discoveries. In [19], the authors provide several new refinements and reversals for bidimensional double inequality using $\mathtt{s}$-convex mappings on the rectangular plane. Furthermore, this inequality has been investigated using different types of convex mappings as well as integral operators. For example, in [20], the authors used different types of convex, continuous, and differentiable mappings and found various bounds of double inequalities. In [21], the authors define two variable logarithmic convex mappings and develop Hermite-Hadamard inequality with applications. In [22], the authors define two-dimensional preinvex type mappings and develop various types of Hermite-Hadamard inequalities. In [23], the authors exploited the q-jackson quantum double integral on the plane to established the double inequality with its fascinating applications in quantum calculus. Kalsoom et al. [24] used quantum integrals to established the Fejer and Pachapatte type inequalities utilizing two different forms of invex mappings. In [25], authors introduced a new type of fractional integral operators with singular kernels to develop Hermite-Hadamard inequality by employing several types of integral identities. In the realm of interval maps, Shi et al. [26] created multiple new bounds for double Hermite-Hadamard inequality using two types of generalized convex mappings. Zareen et al. [27] exploited generalized double fractional integrals and generated numerous novel Hermite-Hadamard type inequalities with coordinated convex mappings. Saeed et al. [28] employed generalized double fractional integrals to establish numerous novel Hermite-Hadamard type inequalities over convex set relevant to fuzzy-number-valued settings using coordinated convex mappings. In [29], the authors refined the Hadamard inequality for coordinated convex functions and explored their applications. Afzal et al. [30] exploited coordinated convex mappings to produce Hermite-Hadamard, Pachpatte, and Fejer type integral inequalities using innovative fractional integral operators via fully interval-order relations. For some further recent advancements for coordinated convex mappings, we refer to [31–34] and the references therein.

Since this article mainly deals with a cr-interval order relations, we should recollect recent advancements focusing on center-radius order relations using some other form of convex mapping. The concept of cr-order was introduced for the first time by the authors in [35] in 2014. This relation is more compatible than other order relations and has several additional properties that other interval order relations do not have. Based on their work, authors in [36] defined a new type of convex mapping for convex optimizing problems in the realm of cr-order. Liu et al. [37,38] first created discrete versions of Jensen and Hermite-Hadamard inequalities utilizing two distinct kinds of generalized convex mappings employing cr-order as a result of these discoveries. Khan and Saad [39] created several novel bounds for various kinds of double inequalities by utilizing superquadratic functions in the relam of fractional frame of reference via cr-order relation. Fahad et al. [40] used geometric and arithmetic-cr-convex functions to explore characteristics and several applications related to entropy and means. Using the concepts of cr-order and various classes of convex and Godunova-Levin functions, the authors of [41–44] generated a number of novel inequalities connected to these findings with applications. For more recent developments on comparable results using different types of related convex classes, see [45–49] and the references therein.

"Order relations" and "convex mappings" are the basic concepts for adjusting inequalities within interval set-valued mappings. Despite that, authors have extensively used the interval partial order relation "$\subseteq_{\mathtt{p}}$" to develop different types of inequalities and analyze that this type of order relation though it is not very suitable for adjusting inequalities in interval maps because there are some inequalities that are not adjusted under the same assumptions. For instance, please see reference [50] in which several major results are not developed in interval maps. To overcome this limitation, the authors established a new sort of order relation known as cr-order, whose definition is now standard. This is considered a natural generalization of all recently developed order relations such as inclusion, left-right, pseudo, up and down, and various others. Furthermore, the cr-order is full order, which means it possesses all relational features between intervals, including reflexivity, anti-symmetry, transitivity, and comparability, but the inclusion partial order relation lacks comparability between two intervals.

Inspired by well built appropriate literature, particularly these works [18, 24, 39–41], we derived a novel and improved form of inequalities employing cr-interval order. This paper is organized into five sections, beginning with an introduction and fundamental discussion of the subject connected to preliminary. In Section 3, we first show that double integral preserves cr-order, and then we show that the newly developed mappings after we apply cr-order, including midpoint and center, are both convex in nature. Next, we developed different variants of double inequalities that generalize various previous findings. Finally, in Section 4, we provide a precise conclusion and some future prospects.

2. Preliminaries

In this section, we discuss some basic concepts related to fractional and interval calculus. Further some key concepts are not thoroughly discussed here, thus we refer to [39].

- $\mathsf{R}_{\mathtt{i}}$: intervals in R;
- $\underline{\psi} = \overline{\psi}$: interval maps become dysfunctional;
- $\subseteq$: inclusion interval order;
- $\preceq_{\mathtt{cr}}$: cr-interval order;
- $\leq$: basic order;

- *ivfs*: interval-valued functions.

2.1. Interval operations

Let R be the one-dimensional Euclidean space, and consider $\mathsf{R_c}$ the family of all non-empty compact convex subsets of R, that is

$$\mathsf{R_c} = \{[\omega_1, \omega_2] : \omega_1, \omega_2 \in \mathsf{R} \text{ and } \omega_1 \leq \omega_2\}.$$

To define the Hausdorff metric in $\mathsf{R_c}$, use this formula:

$$\mathtt{D}(\omega_1, \omega_2) = \max\{\mathtt{d}(\omega_1, \omega_2), \mathtt{d}(\omega_2, \omega_1)\}, \tag{2.1}$$

where $\mathtt{d}(\omega_1, \omega_2) = \sup_{\nu\in\omega_1} \mathtt{d}(\nu, \omega_2)$, and $\mathtt{d}(\nu, \omega_2) = \min_{\mu\in\omega_2} \mathtt{d}(\nu, \mu) = \min_{\mu\in\omega_2} |\nu - \mu|$.

Remark 2.1. *The Hausdorff metric described in (2.1) is alternatively represented in the following manner:*

$$\mathtt{H}([\underline{\omega_1}, \overline{\omega_1}], [\underline{\omega_2}, \overline{\omega_2}]) = \max\{|\underline{\omega_1} - \underline{\omega_2}|, |\overline{\omega_1} - \overline{\omega_2}|\}.$$

In interval space, we call this the Moore metric.

For instance, if $\mathtt{H}_1 = [\underline{\omega_1}, \overline{\omega_1}]$ and $\mathtt{H}_2 = [\underline{\omega_2}, \overline{\omega_2}]$ are two closed intervals, then the Minkowski sum, scalar multiplication, and difference are defined as follows:

$$\mathtt{H}_1 + \mathtt{H}_2 = \{\omega_1 + \omega_2 \mid \omega_1 \in \mathtt{H}_1, \omega_2 \in \mathtt{H}_2\} \text{ and} \Gamma\mathtt{H}_1 = \{\Gamma\omega_1 \mid \omega_1 \in \mathtt{H}_1\}$$

and

$$\mathtt{H}_1 - \mathtt{H}_2 = [\underline{\omega_1} - \overline{\omega_2}, \overline{\omega_1} - \underline{\omega_2}],$$

with the product

$$\mathtt{H}_1 \cdot \mathtt{H}_2 = [\min\{\underline{\omega_1\omega_2}, \underline{\omega_1}\overline{\omega_2}, \overline{\omega_1}\underline{\omega_2}, \overline{\omega_1\omega_2}\}, \sup\{\underline{\omega_1\omega_2}, \underline{\omega_1}\overline{\omega_2}, \overline{\omega_1}\underline{\omega_2}, \overline{\omega_1\omega_2}\}],$$

and the division

$$\frac{\mathtt{H}_1}{\mathtt{H}_2} = \left[\min\left\{\frac{\underline{\omega_1}}{\underline{\omega_2}}, \frac{\underline{\omega_1}}{\overline{\omega_2}}, \frac{\overline{\omega_1}}{\underline{\omega_2}}, \frac{\overline{\omega_1}}{\overline{\omega_2}}\right\}, \max\left\{\frac{\underline{\omega_1}}{\underline{\omega_2}}, \frac{\underline{\omega_1}}{\overline{\omega_2}}, \frac{\overline{\omega_1}}{\underline{\omega_2}}, \frac{\overline{\omega_1}}{\overline{\omega_2}}\right\}\right],$$

where $0 \notin \mathtt{H}_2$.

The order relation that permeates our primary findings is outlined by Bhunia and Samanta [51]; it is commonly referred to as center-radius order.

Definition 2.1. *[41] For any two intervals the center-radius order relation is defined as* $\mathtt{H}_1 = [\underline{\omega_1}, \overline{\omega_2}] = \langle\omega_c, \omega_r\rangle = \left\langle\frac{\underline{\omega_1}+\overline{\omega_2}}{2}, \frac{\overline{\omega_2}-\underline{\omega_1}}{2}\right\rangle$, $\mathtt{H}_2 = [\underline{\Omega_1}, \overline{\Omega_2}] = \langle\Omega_c, \Omega_r\rangle = \left\langle\frac{\underline{\Omega_1}+\overline{\Omega_2}}{2}, \frac{\overline{\Omega_2}-\underline{\Omega_1}}{2}\right\rangle$, *where*

$$\mathtt{H}_1 \preceq_{\mathtt{cr}} \mathtt{H}_2 \iff \begin{cases} \omega_c < \Omega_c, & \text{if} \omega_c \neq \Omega_c; \\ \omega_r \leq \Omega_r, & \text{if} \omega_r = \Omega_r. \end{cases}$$

The relation $\preceq_{\mathtt{cr}}$ satisfies the following relational properties for any three intervals $\mathtt{H}_1 = [\underline{\omega_1}, \overline{\omega_2}] = \langle\omega_c, \omega_r\rangle$, $\mathtt{H}_2 = [\underline{\Omega_1}, \overline{\Omega_2}] = \langle\Omega_c, \Omega_r\rangle$ *and* $\mathtt{H}_3 = [\underline{\eta_1}, \overline{\eta_2}] = \langle\eta_c, \eta_r\rangle$: *Reflexivity:* $\mathtt{H}_1 \preceq_{\mathtt{cr}} \mathtt{H}_1$. *Anti-symmetry:* $\mathtt{H}_1 \preceq_{\mathtt{cr}} \mathtt{H}_2$ *and* $\mathtt{H}_2 \preceq_{\mathtt{cr}} \mathtt{H}_1$. *Transitivity:* $\mathtt{H}_1 \preceq_{\mathtt{cr}} \mathtt{H}_2$ *and* $\mathtt{H}_2 \preceq_{\mathtt{cr}} \mathtt{H}_3$, *then* $\mathtt{H}_1 \preceq_{\mathtt{cr}} \mathtt{H}_3$. *Comparability:* $\mathtt{H}_2 \preceq_{\mathtt{cr}} \mathtt{H}_3$ *or* $\mathtt{H}_3 \preceq_{\mathtt{cr}} \mathtt{H}_2$.

Theorem 2.1. *[42] Let $\psi : [\omega_1, \omega_2] \to \mathsf{R_c}$ be an ivfs represented by $\psi(\omega) = [\underline{\psi}(\omega), \overline{\psi}(\omega)]$. $\psi \in \mathsf{IR}_{([\omega_1,\omega_2])}$, iff $\underline{\psi}(\omega), \overline{\psi}(\omega) \in \mathsf{R}_{([\omega_1,\omega_2])}$ and*

$$(\mathrm{IR}) \int_{\omega_1}^{\omega_2} \psi(\omega)\, \mathrm{d}\omega = \Big[(\mathsf{R}) \int_{\omega_1}^{\omega_2} \underline{\psi}(\omega)\, \mathrm{d}\omega, (\mathsf{R}) \int_{\omega_1}^{\omega_2} \overline{\psi}(\omega)\, \mathrm{d}\omega \Big].$$

Theorem 2.2. *[41] Let $\psi, \chi : [\omega_1, \omega_2] \to \mathsf{R_c}$ be an ivfs defined by $\chi = [\underline{\chi}, \overline{\chi}], \psi = [\underline{\psi}, \overline{\psi}]$. If $\psi(\omega) \preceq_{\mathtt{cr}} \chi(\omega)$ for all $\omega \in [\omega_1, \omega_2]$, then*

$$\int_{\omega_1}^{\omega_2} \psi(\omega) \mathrm{d}\omega \preceq_{\mathtt{cr}} \int_{\omega_1}^{\omega_2} \chi(\omega) \mathrm{d}\omega.$$

Theorem 2.3. *[18] Let $\Delta = [\omega_1, \omega_2] \times [\omega_3, \omega_4]$. If $\psi : \Delta \to \mathsf{R_c}$ is $\mathtt{UD}$-integrable over Δ, then we have*

$$(\mathtt{UD}) \iint_{\Delta} \psi(\omega, \Omega)\, \mathtt{dA} = (\mathrm{IR}) \int_{\omega_1}^{\omega_2} (\mathrm{IR}) \int_{\omega_3}^{\omega_4} \psi(\omega, \Omega)\, \mathrm{d}\omega\, \mathrm{d}\Omega.$$

Using Zhao et al. [18] concept of interval-valued double integrals, we provide the following definitions for Hadamard and Katugampola integrals as follows:

Definition 2.2. *Let $\psi \in \mathrm{IR}_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$. For bidimensional interval-valued functions, the Hadamard integrals of order $\theta_1, \theta_2 > 0$ with $\omega_1, \omega_3 \geq 0$ are represented as*

$$\mathfrak{H}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\mathtt{x},\mathtt{y}) := \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\omega_1}^{\mathtt{x}} \int_{\omega_3}^{\mathtt{y}} \left(\ln \frac{\mathtt{x}}{\mathtt{t}}\right)^{\theta_1-1} \left(\ln \frac{\mathtt{y}}{\mathtt{s}}\right)^{\theta_2-1} \frac{\psi(\mathtt{t},\mathtt{s})}{\mathtt{ts}} \mathtt{dtds},\ \mathtt{x} > \omega_1,\ \mathtt{y} > \omega_3,$$

$$\mathfrak{H}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\mathtt{x},\mathtt{y}) := \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\omega_1}^{\mathtt{x}} \int_{\mathtt{y}}^{\omega_4} \left(\ln \frac{\mathtt{x}}{\mathtt{t}}\right)^{\theta_1-1} \left(\ln \frac{s}{y}\right)^{\theta_2-1} \frac{\psi(\mathtt{t},\mathtt{s})}{\mathtt{ts}} \mathtt{dtds},\ \mathtt{x} > \omega_1,\ \mathtt{y} < \omega_4,$$

$$\mathfrak{H}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\mathtt{x},\mathtt{y}) := \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\mathtt{x}}^{\omega_2} \int_{\omega_3}^{\mathtt{y}} \left(\ln \frac{\mathtt{t}}{\mathtt{x}}\right)^{\theta_1-1} \left(\ln \frac{\mathtt{y}}{\mathtt{s}}\right)^{\theta_2-1} \frac{\psi(\mathtt{t},\mathtt{s})}{\mathtt{ts}} \mathtt{dtds},\ \mathtt{x} < \omega_2,\ \mathtt{y} > \omega_3,$$

and

$$\mathfrak{H}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\mathtt{x},\mathtt{y}) := \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\mathtt{x}}^{\omega_2} \int_{\mathtt{y}}^{\omega_4} \left(\ln \frac{\mathtt{t}}{\mathtt{x}}\right)^{\theta_1-1} \left(\ln \frac{\mathtt{y}}{\mathtt{s}}\right)^{\theta_2-1} \frac{\psi(\mathtt{t},\mathtt{s})}{\mathtt{ts}} \mathtt{dtds},\ \mathtt{x} < \omega_2,\ \mathtt{y} < \omega_4.$$

Definition 2.3. *Let $\psi \in \mathrm{IR}_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$. For bidimensional interval-valued functions, the Katugampola fractional integrals are defined as*

$$\eta,\sigma\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\mathtt{x},\mathtt{y}) := \frac{\eta^{1-\theta_1}\sigma^{1-\theta_2}}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\omega_1}^{\mathtt{x}} \int_{\omega_3}^{\mathtt{y}} \frac{\mathtt{t}^{\eta-1}}{[\mathtt{x}^\eta - \mathtt{t}^\eta]^{1-\theta_1}} \frac{\mathtt{s}^{\sigma-1}}{[y^\sigma - \mathtt{s}^\sigma]^{1-\theta_2}} \psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x} > \omega_1,\ \mathtt{y} > \omega_3,$$

$$^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\mathtt{x},\mathtt{y}) := \frac{\eta^{1-\theta_1}\sigma^{1-\theta_2}}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\omega_1}^{\mathtt{x}} \int_{\mathtt{y}}^{\omega_4} \frac{\mathtt{t}^{\eta-1}}{[\mathtt{x}^\eta - \mathtt{t}^\eta]^{1-\theta_1}} \frac{\mathtt{s}^{\sigma-1}}{[\mathtt{s}^\sigma - y^\sigma]^{1-\theta_2}} \psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x} > \omega_1,\ \mathtt{y} < \omega_4,$$

$$\eta,\sigma\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\mathtt{x},\mathtt{y}) := \frac{\eta^{1-\theta_1}\sigma^{1-\theta_2}}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\mathtt{x}}^{\mathtt{y}} \int_{\omega_3}^{\mathtt{y}} \frac{\mathtt{t}^{\eta-1}}{[\mathtt{t}^\eta - \mathtt{x}^\eta]^{1-\theta_1}} \frac{\mathtt{s}^{\sigma-1})}{[y^\sigma - \mathtt{s}^\sigma]^{1-\theta_2}} \psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x} < \omega_2,\ \mathtt{y} > \omega_3,$$

and

$$^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\mathtt{x},\mathtt{y}) := \frac{\eta^{1-\theta_1}\sigma^{1-\theta_2}}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR}) \int_{\mathtt{x}}^{\omega_2} \int_{\mathtt{t}}^{\omega_4} \frac{\mathtt{t}^{\eta-1}}{[\mathtt{t}^\eta - \mathtt{x}^\eta]^{1-\theta_1}} \frac{\mathtt{s}^{\sigma-1}}{[\mathtt{s}^\sigma - y^\sigma]^{1-\theta_2}} \psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x} < \omega_2,\ \mathtt{y} < \omega_4.$$

We have now established a new type of double fractional integral operators under cr-order that generalize various existing operators by specifying different sorts of functions.

Definition 2.4. *Let $\zeta : [\omega_1, \omega_2] \to \mathsf{R}$ be a increasing and positive monotone function on $(\omega_1, \omega_2]$, having a continuous derivative on (ω_1, ω_2), and let $\kappa : [\omega_3, \omega_4] \to \mathsf{R}$ be a increasing and positive monotone function on $(\omega_3, \omega_4]$, having a continuous derivative on (ω_3, ω_4). Let $\psi \in \mathrm{IR}_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$. The fractional integral operators for interval-valued functions of two variables are defined by*

$$\begin{aligned}
&\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\mathtt{x},\mathtt{y})\\
:=&\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_1}^{\mathtt{x}}\int_{\omega_3}^{\mathtt{y}}\frac{\zeta'(\mathtt{t})}{[\zeta(\mathtt{x})-\zeta(\mathtt{t})]^{1-\theta_1}}\frac{\kappa'(\mathtt{s})}{[\kappa(\mathtt{y})-\kappa(\mathtt{s})]^{1-\theta_2}}\psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x}>\omega_1,\ \mathtt{y}>\omega_3,\\
&\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\mathtt{x},\mathtt{y})\\
:=&\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_1}^{\mathtt{x}}\int_{\mathtt{y}}^{\omega_4}\frac{\zeta'(\mathtt{t})}{[\zeta(\mathtt{x})-\zeta(\mathtt{t})]^{1-\theta_1}}\frac{\kappa'(\mathtt{s})}{[\kappa(\mathtt{s})-\kappa(\mathtt{y})]^{1-\theta_2}}\psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x}>\omega_1,\ \mathtt{y}<\omega_4,\\
&\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\mathtt{x},\mathtt{y})\\
:=&\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\mathtt{x}}^{\omega_2}\int_{\omega_3}^{\mathtt{y}}\frac{\zeta'(\mathtt{t})}{[\zeta(\mathtt{t})-\zeta(\mathtt{x})]^{1-\theta_1}}\frac{\kappa'(\mathtt{s})}{[\kappa(\mathtt{y})-\kappa(\mathtt{s})]^{1-\theta_2}}\psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x}<\omega_2,\ \mathtt{y}>\omega_3,
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\psi(\mathtt{x},\mathtt{y})\\
:=&\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\mathtt{x}}^{\omega_2}\int_{\mathtt{y}}^{\omega_4}\frac{\zeta'(\mathtt{t})}{[\zeta(\mathtt{t})-\zeta(\mathtt{x})]^{1-\theta_1}}\frac{\kappa'(\mathtt{s})}{[\kappa(\mathtt{s})-\kappa(\mathtt{y})]^{1-\theta_2}}\psi(\mathtt{t},\mathtt{s})\mathtt{dtds},\ \mathtt{x}<\omega_2,\ \mathtt{y}<\omega_4
\end{aligned}$$

for $\theta_1, \theta_2 > 0$.

Similar to the preceding definitions, we can provide the following integrals:

$$\begin{aligned}
\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi\left(\mathtt{x},\frac{\omega_3+\omega_4}{2}\right)&:=\frac{1}{\Gamma(\theta_1)}(\mathrm{IR})\int_{\omega_1}^{\mathtt{x}}\frac{\zeta'(\mathtt{t})}{[\zeta(\mathtt{x})-\zeta(\mathtt{t})]^{1-\theta_1}}\psi\left(t,\frac{\omega_3+\omega_4}{2}\right)\mathtt{dt},\ \mathtt{x}>\omega_1,\\
\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi\left(\mathtt{x},\frac{\omega_3+\omega_4}{2}\right)&:=\frac{1}{\Gamma(\theta_1)}(\mathrm{IR})\int_{\mathtt{x}}^{\omega_2}\frac{\zeta'(\mathtt{t})}{[\zeta(\mathtt{t})-\zeta(\mathtt{x})]^{1-\theta_1}}\psi\left(t,\frac{\omega_3+\omega_4}{2}\right)\mathtt{dt},\ \mathtt{x}<\omega_2,\\
\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\mathtt{y}\right)&:=\frac{1}{\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_3}^{\mathtt{y}}\frac{\kappa'(\mathtt{t})}{[\kappa(\mathtt{y})-\kappa(\mathtt{s})]^{1-\theta_2}}\psi\left(\frac{\omega_1+\omega_2}{2},\mathtt{s}\right)\mathtt{ds},\ \mathtt{y}>\omega_3,
\end{aligned}$$

and

$$\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\mathtt{y}\right):=\frac{1}{\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_3}^{\mathtt{y}}\frac{\kappa'(\mathtt{t})}{[\kappa(\mathtt{s})-\kappa(\mathtt{y})]^{1-\theta_2}}\psi\left(\frac{\omega_1+\omega_2}{2},\mathtt{s}\right)\mathtt{ds},\ \mathtt{y}<\omega_4.$$

Next, according to the definition provided by the authors [17, 18], we discuss the bidimensional convexity via classical and interval order relation.

Definition 2.5. *[17] Let $\psi : \Omega = [\omega_1, \omega_2] \times [\omega_3, \omega_4] \subseteq \mathsf{R}^2 \to \mathsf{R}^+$ be a two-variable convex function if*

$$\begin{aligned}
&\psi(\mathtt{s}_1\omega_1+(1-\mathtt{s}_1)\omega_2,\mathtt{r}_1\omega_3+(1-\mathtt{r}_1)\omega_4)\\
\leq\ &\mathtt{r}_1\mathtt{s}_1\psi(\omega_1,\omega_3)+\mathtt{s}_1(1-\mathtt{r}_1)\psi(\omega_1,\omega_4)+\mathtt{r}_1(1-\mathtt{s}_1)\psi(\omega_2,\omega_3)+(1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)
\end{aligned}$$

holds true for every $(\omega_1, \omega_2), (\omega_3, \omega_4) \in \Omega$ along with $\mathtt{r}_1, \mathtt{s}_1 \in [0, 1]$.

Definition 2.6. *[18] Let* $\psi : \Omega = [\omega_1, \omega_2] \times [\omega_3, \omega_4] \subseteq \mathsf{R}^2 \to \mathsf{R_i}^+$ *be a two-variable set-valued convex mapping defined as* $\psi = [\underline{\psi}(\omega, \Omega), \overline{\psi}(\omega, \Omega)]$ *with* $0 \leq \omega_1 < \omega_2, 0 \leq \omega_3 < \omega_4$*, then we have*

$$\begin{aligned}&\psi(\mathtt{s}_1\omega_1 + (1-\mathtt{s}_1)\omega_2, \mathtt{r}_1\omega_3 + (1-\mathtt{r}_1)\omega_4)\\ &\supseteq \mathtt{r}_1\mathtt{s}_1\psi(\omega_1,\omega_3) + \mathtt{s}_1(1-\mathtt{r}_1)\psi(\omega_1,\omega_4) + \mathtt{r}_1(1-\mathtt{s}_1)\psi(\omega_2,\omega_3) + (1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)\end{aligned}$$

holds true for every $(\omega_1, \omega_2), (\omega_3, \omega_4) \in \Omega$ *along with* $\mathtt{r}_1, \mathtt{s}_1 \in [0, 1]$.

Definition 2.7. *[52] Let* $\psi : \Omega = [\omega_1, \omega_2] \times [\omega_3, \omega_4] \subseteq \mathsf{R}^2 \to \mathsf{R_i}^+$ *be a two-variable set-valued strongly convex mapping defined as* $\psi = [\underline{\psi}(\omega, \Omega), \overline{\psi}(\omega, \Omega)]$ *with* $0 \leq \omega_1 < \omega_2, 0 \leq \omega_3 < \omega_4$ *and* κ_1, κ_2 *are positive real numbers, then we have*

$$\begin{aligned}&\psi(\mathtt{s}_1\omega_1 + (1-\mathtt{s}_1)\omega_2, \mathtt{r}_1\omega_3 + (1-\mathtt{r}_1)\omega_4)\\ &\supseteq \mathtt{r}_1\mathtt{s}_1\psi(\omega_1,\omega_3) + \mathtt{s}_1(1-\mathtt{r}_1)\psi(\omega_1,\omega_4) + \mathtt{r}_1(1-\mathtt{s}_1)\psi(\omega_2,\omega_3) + (1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)\\ &\quad - \kappa_1\mathtt{s}_1(1-\mathtt{s}_1)(\omega_1-\omega_2)^2 - 2\kappa_2\mathtt{r}_1(1-\mathtt{r}_1)(\omega_3-\omega_4)^2\end{aligned}$$

holds true for every $(\omega_1, \omega_2), (\omega_3, \omega_4) \in \Omega$ *along with* $\mathtt{r}_1, \mathtt{s}_1 \in [0, 1]$.

Definition 2.8. *[46] Let* $\psi : \Omega = [\omega_1, \omega_2] \to \mathsf{R_i}^+$ *be a ivfs defined as* $\psi = [\underline{\psi}(\omega), \overline{\psi}(\omega)]$ *with* $0 \leq \omega_1 < \omega_2$*. Then* ψ *is* $\mathtt{cr}$*-convex if*

$$\psi(\mathtt{r}_1\omega_1 + (1-\mathtt{r}_1)\omega_2) \preceq_{\mathtt{cr}} \mathtt{r}_1\psi(\omega_1) + (1-\mathtt{r}_1)\psi(\omega_2)$$

holds true for every $(\omega_1, \omega_2) \in \Omega$ *along with* $\mathtt{r}_1 \in [0, 1]$.

Motivated by the aforementioned definitions, we now effectively extend Definition 2.8 into R^2 by the use of cr-order.

Definition 2.9. *Let* $\psi : \Omega = [\omega_1, \omega_2] \times [\omega_3, \omega_4] \subseteq \mathsf{R}^2 \to \mathsf{R_i}^+$ *be a two-variable set-valued* $\mathtt{cr}$ *convex mapping defined as* $\psi = [\underline{\psi}(\omega, \Omega), \overline{\psi}(\omega, \Omega)]$ *with* $0 \leq \omega_1 < \omega_2, 0 \leq \omega_3 < \omega_4$*, then we have*

$$\begin{aligned}&\psi(\mathtt{s}_1\omega_1 + (1-\mathtt{s}_1)\omega_2, \mathtt{r}_1\omega_3 + (1-\mathtt{r}_1)\omega_4)\\ &\preceq_{\mathtt{cr}} \mathtt{r}_1\mathtt{s}_1\psi(\omega_1,\omega_3) + \mathtt{s}_1(1-\mathtt{r}_1)\psi(\omega_1,\omega_4) + \mathtt{r}_1(1-\mathtt{s}_1)\psi(\omega_2,\omega_3) + (1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)\end{aligned}$$

holds true for every $(\omega_1, \omega_2), (\omega_3, \omega_4) \in \Omega$ *along with* $\mathtt{r}_1, \mathtt{s}_1 \in [0, 1]$.

Remark 2.2. *(1) If* $\underline{\psi} \neq \overline{\psi}$*, we have Definition 2, as stated by Zhao et al. in [18] and Definition 6, as stated by Khan et al. in [53].*

(2) If $\underline{\psi} = \overline{\psi}$*, we have Definition 2.1, as stated by Silvestru Sever in [17].*

Example 2.1. Let $\psi : [\omega_1, \omega_2] \times [\omega_3, \omega_4] \subseteq \mathsf{R}^2 \to \mathsf{R_i}^+$ be a two variable set-valued function defined as (see Figure 1)

$$\begin{aligned}\psi =&[-\omega^2 - \Omega^2 - 10e^{5\omega\Omega+1} - 7e^{4\omega\Omega} + 7, 2\omega^2 + 2\Omega^2 + 12e^{5\omega\Omega+1} + 10e^{4\omega\Omega} + 9],\\ &(\omega, \Omega) \in [-1, 1] \times [-1, 1].\end{aligned} \tag{2.2}$$

Then (see Figure 2),

$$\psi_{\mathtt{c}} = \frac{\omega^2 + \Omega^2 + 2e^{5\omega\Omega+1} + 3e^{4\omega\Omega} + 16}{2} \quad \text{and} \quad \psi_{\mathtt{r}} = \frac{3\omega^2 + 3\Omega^2 + 22e^{5\omega\Omega+1} + 17e^{4\omega\Omega} + 2}{2}. \tag{2.3}$$

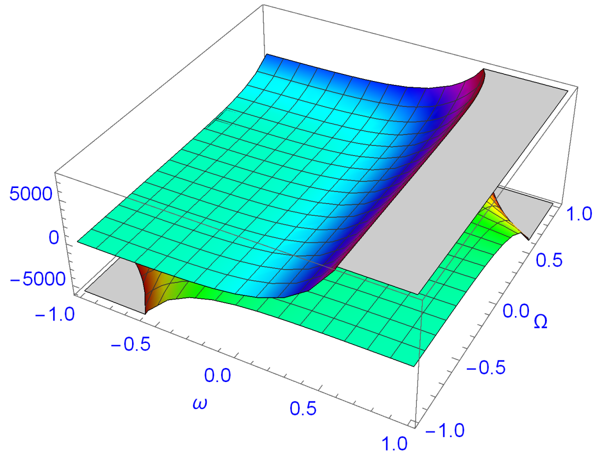

Figure 1. pictorial view of the cr set-valued function ψ.

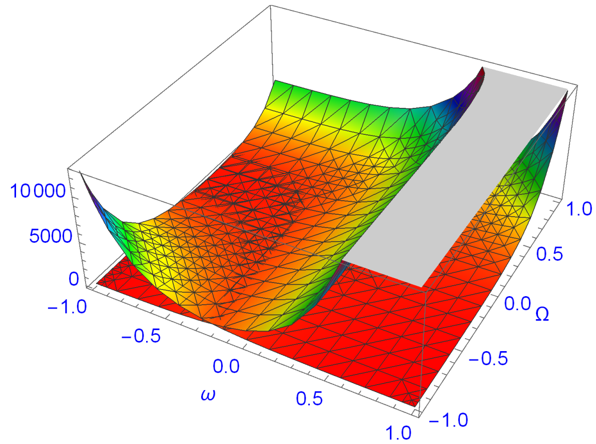

Figure 2. pictorial view of the cr set-valued functions ψ_c and ψ_r.

3. The major results

In this section, we first show that double integral preserves cr-order, and then we show that the newly developed mappings after we apply cr-order, including midpoint and center, are both convex in nature. Finally, we developed different variants of double inequalities that generalize various previous findings.

Theorem 3.1. *Let* $\psi, \phi : [\omega_1, \omega_2] \times [\omega_3, \omega_4] \to \mathsf{R}_c$ *given by* $\phi = [\underline{\phi}(\omega, \Omega), \overline{\phi}(\omega, \Omega)]$, *and* $\psi = [\underline{\psi}(\omega, \Omega), \overline{\psi}(\omega, \Omega)]$. *If* $\psi, \phi \in IR_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$, *and* $\psi(\omega, \Omega) \preceq_{cr} \phi(\omega, \Omega)$ *for all* $\omega, \Omega \in [\omega_1, \omega_2] \times [\omega_3, \omega_4] \to \mathsf{R}_c$, *then*

$$\iint_\Delta \psi(\omega, \Omega)\, dA = \int_{\omega_1}^{\omega_2} \int_{\omega_3}^{\omega_4} \psi(\omega, \Omega)\, d\omega\, d\Omega \preceq_{cr} \int_{\omega_1}^{\omega_2} \int_{\omega_3}^{\omega_4} \phi(\omega, \Omega)\, d\omega\, d\Omega.$$

Proof. Since $\psi(\omega, \Omega) \preceq_{cr} \phi(\omega, \Omega)$ for all $\omega, \Omega \in [\omega_1, \omega_2] \times [\omega_3, \omega_4] \to \mathsf{R}_c$, then we have

$$\begin{cases} \psi_c(\omega, \Omega) \preceq_{cr} \phi_c(\omega, \Omega), & \text{if} \quad \psi_c(\omega, \Omega) \neq \phi_c(\omega, \Omega), \\ \psi_r(\omega, \Omega) \preceq_{cr} \phi_r(\omega, \Omega), & \text{if} \quad \psi_r(\omega, \Omega) \neq \phi_r(\omega, \Omega). \end{cases}$$

Since $\psi,\phi \in IR_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$, by Theorem 2.1, we have $\underline{\psi}(\omega,\Omega),\overline{\psi}(\omega,\Omega),\underline{\phi}(\omega,\Omega),\overline{\phi}(\omega,\Omega) \in IR_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$. When $\psi_c(\omega,\Omega)\neq\phi_c(\omega,\Omega), \forall\omega,\Omega\in[\omega_1,\omega_2]\times[\omega_3,\omega_4]$, then

$$\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}(\underline{\psi}(\omega,\Omega)+\overline{\psi}(\omega,\Omega))\mathrm{d}\omega\mathrm{d}\Omega \leq \int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}(\underline{\phi}(\omega,\Omega)+\overline{\phi}(\omega,\Omega))\mathrm{d}\omega\mathrm{d}\Omega.$$

That is

$$\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\psi(\omega,\Omega)\,\mathrm{d}\omega\,\mathrm{d}\Omega \preceq_{\mathtt{cr}} \int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\phi(\omega,\Omega)\,\mathrm{d}\omega\,\mathrm{d}\Omega.$$

When $\psi_c(\omega,\Omega)=\phi_c(\omega,\Omega), \forall\omega,\Omega\in[\omega_1,\omega_2]\times[\omega_3,\omega_4]$, then

$$\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}(\overline{\psi}(\omega,\Omega)-\underline{\psi}(\omega,\Omega))\mathrm{d}\omega\mathrm{d}\Omega \leq \int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}(\overline{\phi}(\omega,\Omega)-\underline{\phi}(\omega,\Omega))\mathrm{d}\omega\mathrm{d}\Omega.$$

That is

$$\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\psi(\omega,\Omega)\,\mathrm{d}\omega\,\mathrm{d}\Omega \preceq_{\mathtt{cr}} \int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\phi(\omega,\Omega)\,\mathrm{d}\omega\,\mathrm{d}\Omega.$$

This completes the proof. □

Now, we have defined a coordinated strongly convex mapping utilizing the center-radius ordering relation.

Definition 3.1. *Let* $\psi:\Omega=[\omega_1,\omega_2]\times[\omega_3,\omega_4]\subseteq\mathsf{R}^2\to\mathsf{R_i}^+$ *be a two-variable cr set-valued strongly convex mapping defined as* $\psi=[\underline{\psi}(\omega,\Omega),\overline{\psi}(\omega,\Omega)]$ *with* $0\leq\omega_1<\omega_2, 0\leq\omega_3<\omega_4$ *and* κ_1,κ_2 *are positive real numbers, then we have*

$$\begin{aligned}&\psi(\mathtt{s}_1\omega_1+(1-\mathtt{s}_1)\omega_2,\mathtt{r}_1\omega_3+(1-\mathtt{r}_1)\omega_4)\\ \preceq_{\mathtt{cr}}\ &\mathtt{r}_1\mathtt{s}_1\psi(\omega_1,\omega_3)+\mathtt{s}_1(1-\mathtt{r}_1)\psi(\omega_1,\omega_4)+\mathtt{r}_1(1-\mathtt{s}_1)\psi(\omega_2,\omega_3)\\ &+(1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)-\kappa_1\mathtt{s}_1(1-\mathtt{s}_1)(\omega_1-\omega_2)^2-2\kappa_2\mathtt{r}_1(1-\mathtt{r}_1)(\omega_3-\omega_4)^2\end{aligned}$$

holds true for every $(\omega_1,\omega_2),(\omega_3,\omega_4)\in\Omega$ *along with* $\mathtt{r}_1,\mathtt{s}_1\in[0,1]$.

Proposition 3.1. *Let* $\psi:\Omega=[\omega_1,\omega_2]\times[\omega_3,\omega_4]\subseteq\mathsf{R}^2\to\mathsf{R_i}^+$ *be a two-variable set-valued function represented as* $\psi=[\underline{\psi}(\omega,\Omega),\overline{\psi}(\omega,\Omega)]$ *with* $0\leq\omega_1<\omega_2,\ 0\leq\omega_3<\omega_4$ *and* κ_1,κ_2. *Then* ψ *is set-valued* $\mathtt{cr}$*-strongly convex iff* $\psi_{\mathtt{c}}$ *and* $\psi_{\mathtt{r}}$ *are strongly convex functions.*

Proof. As ψ_c and ψ_r are strongly coordinated convex in nature, then for all $(\omega_1,\omega_2),(\omega_3,\omega_4)\in[0,1]\times[0,1]$, we have

$$\begin{aligned}&\psi_{\mathtt{c}}(\mathtt{s}_1\omega_1+(1-\mathtt{s}_1)\omega_2,\mathtt{r}_1\omega_3+(1-\mathtt{r}_1)\omega_4)\\ \preceq_{\mathtt{cr}}\ &\mathtt{r}_1\mathtt{s}_1\psi_{\mathtt{c}}(\omega_1,\omega_3)+\mathtt{s}_1(1-\mathtt{r}_1)\psi_{\mathtt{c}}(\omega_1,\omega_4)+\mathtt{r}_1(1-\mathtt{s}_1)\psi_{\mathtt{c}}(\omega_2,\omega_3)\\ &+(1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)-\kappa_1\mathtt{s}_1(1-\mathtt{s}_1)(\omega_1-\omega_2)^2-2\kappa_2\mathtt{r}_1(1-\mathtt{r}_1)(\omega_3-\omega_4)^2,\end{aligned}$$

and

$$\begin{aligned}&\psi_{\mathtt{r}}(\mathtt{s}_1\omega_1+(1-\mathtt{s}_1)\omega_2,\mathtt{r}_1\omega_3+(1-\mathtt{r}_1)\omega_4)\\ \preceq_{\mathtt{cr}}\ &\mathtt{r}_1\mathtt{s}_1\psi_{\mathtt{r}}(\omega_1,\omega_3)+\mathtt{s}_1(1-\mathtt{r}_1)\psi_{\mathtt{c}}(\omega_1,\omega_4)+\mathtt{r}_1(1-\mathtt{s}_1)\psi_{\mathtt{r}}(\omega_2,\omega_3)\\ &+(1-\mathtt{s}_1)(1-\mathtt{r}_1)\psi(\omega_2,\omega_4)-\kappa_1\mathtt{s}_1(1-\mathtt{s}_1)(\omega_1-\omega_2)^2-2\kappa_2\mathtt{r}_1(1-\mathtt{r}_1)(\omega_3-\omega_4)^2.\end{aligned}$$

Now, if

$$\begin{aligned}&\psi_{c}(s_1\omega_1+(1-s_1)\omega_2, r_1\omega_3+(1-r_1)\omega_4)\\ &\neq r_1s_1\psi_{c}(\omega_1,\omega_3)+s_1(1-r_1)\psi_{c}(\omega_1,\omega_4)+r_1(1-s_1)\psi_{c}(\omega_2,\omega_3)\\ &\quad+(1-s_1)(1-r_1)\psi(\omega_2,\omega_4)-\kappa_1s_1(1-s_1)(\omega_1-\omega_2)^2-2\kappa_2r_1(1-r_1)(\omega_3-\omega_4)^2.\end{aligned}$$

This implies

$$\begin{aligned}&\psi_{c}(s_1\omega_1+(1-s_1)\omega_2, r_1\omega_3+(1-r_1)\omega_4)\\ &\preceq_{cr} r_1s_1\psi_{c}(\omega_1,\omega_3)+s_1(1-r_1)\psi_{c}(\omega_1,\omega_4)+r_1(1-s_1)\psi_{c}(\omega_2,\omega_3)\\ &\quad+(1-s_1)(1-r_1)\psi(\omega_2,\omega_4)-\kappa_1s_1(1-s_1)(\omega_1-\omega_2)^2-2\kappa_2r_1(1-r_1)(\omega_3-\omega_4)^2.\end{aligned}$$

Otherwise, we have

$$\begin{aligned}&\psi_{r}(s_1\omega_1+(1-s_1)\omega_2, r_1\omega_3+(1-r_1)\omega_4)\\ &\leq r_1s_1\psi_{r}(\omega_1,\omega_3)+s_1(1-r_1)\psi_{c}(\omega_1,\omega_4)+r_1(1-s_1)\psi_{r}(\omega_2,\omega_3)\\ &\quad+(1-s_1)(1-r_1)\psi(\omega_2,\omega_4)-\kappa_1s_1(1-s_1)(\omega_1-\omega_2)^2-2\kappa_2r_1(1-r_1)(\omega_3-\omega_4)^2.\end{aligned}$$

This implies

$$\begin{aligned}&\psi_{r}(s_1\omega_1+(1-s_1)\omega_2, r_1\omega_3+(1-r_1)\omega_4)\\ &\preceq_{cr} r_1s_1\psi_{r}(\omega_1,\omega_3)+s_1(1-r_1)\psi_{c}(\omega_1,\omega_4)+r_1(1-s_1)\psi_{r}(\omega_2,\omega_3)\\ &\quad+(1-s_1)(1-r_1)\psi(\omega_2,\omega_4)-\kappa_1s_1(1-s_1)(\omega_1-\omega_2)^2-2\kappa_2r_1(1-r_1)(\omega_3-\omega_4)^2.\end{aligned}$$

This can be summed up as follows using the Definition 3.1 and the previously mentioned results:

$$\begin{aligned}&\psi(s_1\omega_1+(1-s_1)\omega_2, r_1\omega_3+(1-r_1)\omega_4)\\ &\preceq_{cr} r_1s_1\psi(\omega_1,\omega_3)+s_1(1-r_1)\psi(\omega_1,\omega_4)+r_1(1-s_1)\psi(\omega_2,\omega_3)\\ &\quad+(1-s_1)(1-r_1)\psi(\omega_2,\omega_4)-\kappa_1s_1(1-s_1)(\omega_1-\omega_2)^2-2\kappa_2r_1(1-r_1)(\omega_3-\omega_4)^2.\end{aligned}$$

This concludes the proof. □

3.1. Hermite-Hadamard type inequalities using coordinated center-radius order relations

In this section, we employ coordinated center and radius order relations to create various new bounds for double inequality. Let $\psi \in IR_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$. First, we specify the functions that will be utilized frequently:

$$\begin{aligned}\psi_1(\omega,\Omega)&=\psi(\omega_1+\omega_2-\omega,\Omega),\\ \psi_2(\omega,\Omega)&=\psi(\omega,\omega_3+\omega_4-\Omega),\\ \psi_3(\omega,\Omega)&=\psi(\omega_1+\omega_2-\omega,\omega_3+\omega_4-\Omega),\\ \mathcal{G}(\omega,\Omega)&=\psi(\omega,\Omega)+\psi_2(\omega,\Omega),\\ \mathcal{H}(\omega,\Omega)&=\psi(\omega,\Omega)+\psi_1(\omega,\Omega),\\ \mathcal{K}(\omega,\Omega)&=\psi_1(\omega,\Omega)+\psi_3(\omega,\Omega),\end{aligned}$$

$$\begin{aligned}\mathcal{L}(\omega,\Omega) &= \psi_2(\omega,\Omega)+\psi_3(\omega,\Omega),\\ \mathcal{F}(\omega,\Omega) &= \psi_1(\omega,\Omega)+\psi_2(\omega,\Omega)+\psi_3(\omega,\Omega)+\psi(\omega,\Omega)\\ &= \frac{\mathcal{G}(\omega,\Omega)+\mathcal{H}(\omega,\Omega)+\mathcal{K}(\omega,\Omega)+\mathcal{L}(\omega,\Omega)}{2}.\end{aligned}$$

Theorem 3.2. *Let $\zeta : [\omega_1,\omega_2] \to \mathsf{R}$ be a increasing and positive monotone function on $(\omega_1,\omega_2]$, having a continuous derivative on (ω_1,ω_2), and let $\kappa : [\omega_3,\omega_4] \to \mathsf{R}$ be a increasing and positive monotone function on $(\omega_3,\omega_4]$, having a continuous derivative on (ω_3,ω_4). Let $\psi \in \mathrm{IR}_{([\omega_1,\omega_2]\times[\omega_3,\omega_4])}$, then for $\theta_1,\theta_2 > 0$ the following Hermite-Hadamard-type relation hold:*

$$\begin{aligned}&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\ \preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\ &\times\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\psi(\omega_1,\omega_3)\right]\\ \preceq_{\mathrm{cr}}\ &\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).\end{aligned}\tag{3.1}$$

Proof. As ψ represents a set-valued coordinated cr-convex mapping on Δ, we have

$$\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\preceq_{\mathrm{cr}}\frac{\psi(\kappa_1,\kappa_3)+\psi(\kappa_1,\kappa_4)+\psi(\kappa_3,\kappa_1)+\psi(\kappa_2,\kappa_4)}{4},\tag{3.2}$$

for $(\kappa_1,\kappa_3),(\kappa_2,\kappa_4)\in\Delta$. Now, for $\mathtt{t},\mathtt{s}\in[0,1]$, let $\kappa_1=\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,\kappa_2=(1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,\kappa_3=\omega_3\mathtt{s}+(1-\mathtt{s})\omega_4$, and $\kappa_4=(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4$. Then, we have

$$\begin{aligned}&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\ \preceq_{\mathrm{cr}}\ &\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)+\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)\\ &+\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)+\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4).\end{aligned}\tag{3.3}$$

Multiply the above relation by $\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}$, then integrate the resulting relation with respect to $\mathtt{t},\mathtt{s}$ over $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)(\mathrm{IR})\int_0^1\int_0^1\left[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\right.\\ &\left.\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\right]\mathtt{dtds}\\ \preceq_{\mathrm{cr}}\ &\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\left[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\right.\\ &\left.\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)\right]\mathtt{dtds}\\ &+\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\left[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\right.\\ &\left.\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)\right]\mathtt{dtds}\end{aligned}$$

$$+\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\left[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\right.$$
$$\left.\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)\right]\mathtt{dtds}$$
$$+\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\left[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\right.$$
$$\left.\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)\right]\mathtt{dtds}.$$

Using basic computation, we have

$$(\mathrm{IR})\int_0^1\int_0^1\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\mathtt{dtds}$$
$$=\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\theta_1\theta_2(\omega_2-\omega_1)(\omega_4-\omega_3)}.$$

Making use of the variable change $\tau=(1-\mathtt{t})\omega_1+\mathtt{t}\omega_2$ and $\eta=(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4$, we obtain

$$\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)$$
$$\preceq_{\mathtt{cr}}\frac{1}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\frac{\zeta'(\tau)}{[\zeta(\omega_2)-\zeta(\tau)]^{1-\theta_1}}\frac{\kappa'(\eta)}{[\kappa(\omega_4)-\kappa(\eta)]^{1-\theta_2}}\psi(\omega_1+\omega_2-\tau,\omega_3+\omega_4-\eta)d\eta d\tau$$
$$+\frac{1}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\frac{\zeta'(\tau)}{[\zeta(\omega_2)-\zeta(\tau)]^{1-\theta_1}}\frac{\kappa'(\eta)}{[\kappa(\omega_4)-\kappa(\eta)]^{1-\theta_2}}\psi(\omega_1+\omega_2-\tau,\eta)d\eta d\tau$$
$$+\frac{1}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\frac{\zeta'(\tau)}{[\zeta(\omega_2)-\zeta(\tau)]^{1-\theta_1}}\frac{\kappa'(\eta)}{[\kappa(\omega_4)-\kappa(\eta)]^{1-\theta_2}}\psi(\tau,\omega_3+\omega_4-\eta)d\eta d\tau$$
$$+\frac{1}{4\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_{\omega_1}^{\omega_4}\int_{\omega_3}^{\omega_4}\frac{\zeta'(\tau)}{[\zeta(\omega_2)-\zeta(\tau)]^{1-\theta_1}}\frac{\kappa'(\eta)}{[\kappa(\omega_4)-\kappa(\eta)]^{1-\theta_2}}\psi(\tau,\eta)d\eta d\tau$$
$$=\frac{1}{4}\left[\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi_3(\omega_2.\omega_4)+\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+:\zeta,\kappa}\psi_1(\omega_2.\omega_4)+\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi_2(\omega_2.\omega_4)+\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)\right]$$
$$=\frac{1}{4}\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4).\tag{3.4}$$

That is, we have

$$\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\preceq_{\mathtt{cr}}\frac{1}{4}\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4).\tag{3.5}$$

In a similar manner, multiplying (3.3) on both sides by

$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)-\kappa(\omega_3)]^{1-\theta_2}},$$

and by integrating the above relation across $[0,1]\times[0,1]$, we get

$$\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\preceq_{\mathtt{cr}}\frac{1}{4}\mathtt{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3).\tag{3.6}$$

Moreover, multiplying both sides of (3.3) by

$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)-\zeta(\omega_1)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}},$$

and

$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)-\zeta(\omega_1)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)-\kappa(\omega_3)]^{1-\theta_2}},$$

then incorporating the aforementioned relation with reference to $\mathtt{t},\mathtt{s}$ across $[0,1]\times[0,1]$, we get

$$\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\preceq_{\mathtt{cr}}\frac{1}{4}\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4), \tag{3.7}$$

and

$$\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\preceq_{\mathtt{cr}}\frac{1}{4}\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\psi(\omega_1,\omega_3), \tag{3.8}$$

respectively.

Summing the relations (3.5) to (3.8), we have

$$\begin{aligned}&\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\ \preceq_{\mathtt{cr}}\ &\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{16[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\ &\times\Big[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\psi(\omega_1,\omega_3)\Big]. \end{aligned}\tag{3.9}$$

This concludes the demonstration of the first part of relation in (3.1). To prove the second part in (3.1), again taking into account bidimensional convex mappings, we have

$$\begin{aligned}&\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)+\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)\\ &+\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)+\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)\\ \preceq_{\mathtt{cr}}\ &\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).\end{aligned}\tag{3.10}$$

Multiplying both sides of (3.10) by

$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}},$$

then incorporating the aforementioned relation with reference to $\mathtt{t},\mathtt{s}$ across $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\Bigg[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\\ &\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)\Bigg]\mathtt{dtds}\\ &+\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\Bigg[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\end{aligned}$$

$$\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\psi(\mathtt{t}\omega_1+(1-\mathtt{t})\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)\Bigg]\mathtt{dtds}$$
$$+\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}(\mathrm{IR})\int_0^1\int_0^1\Bigg[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}$$
$$\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,\mathtt{s}\omega_3+(1-\mathtt{s})\omega_4)]\mathtt{dtds}}$$
$$+\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\int_0^1\int_0^1\Bigg[\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}$$
$$\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}\psi((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2,(1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]\mathtt{dtds}}$$
$$2\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}[\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4)]$$
$$\times(\mathrm{IR})\int_0^1\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}}\mathtt{dtds}.$$

Then, we get

$$\begin{aligned}&\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi_3(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi_1(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi_2(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)\\&\preceq_{\mathtt{cr}}[\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4)]\frac{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)},\end{aligned}\tag{3.11}$$

that is,

$$\begin{aligned}&\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)\\&\preceq_{\mathtt{cr}}\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).\end{aligned}\tag{3.12}$$

Similarly, multiplying both sides of (3.10) by

$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta(\omega_2)-\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)-\kappa(\omega_3)]^{1-\theta_2}},$$
$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)-\zeta(\omega_1)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa(\omega_4)-\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)]^{1-\theta_2}},$$

and

$$\frac{(\omega_2-\omega_1)(\omega_4-\omega_3)}{\Gamma(\theta_1)\Gamma(\theta_2)}\frac{\zeta'((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)}{[\zeta((1-\mathtt{t})\omega_1+\mathtt{t}\omega_2)-\zeta(\omega_1)]^{1-\theta_1}}\frac{\kappa'((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)}{[\kappa((1-\mathtt{s})\omega_3+\mathtt{s}\omega_4)-\kappa(\omega_3)]^{1-\theta_2}},$$

and then incorporating the aforementioned relation with reference to $\mathtt{t},\mathtt{s}$ across $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)\\&\preceq_{\mathtt{cr}}\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4),\end{aligned}$$

$$\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}} \mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)$$
$$\preceq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4), \tag{3.13}$$

and

$$\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}} \mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3-;\zeta,\kappa}\psi(\omega_1,\omega_3)$$
$$\preceq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4). \tag{3.14}$$

By adding the relations (3.12) to (3.14), we have

$$\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}$$
$$\times\Big[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta\cdot\kappa}\psi(\omega_1,\omega_3)\Big]$$
$$\preceq_{\mathrm{cr}} 4[\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4)]. \tag{3.15}$$

Dividing both sides of relation (3.15) by 16 yields the second relation in (3.1). This concludes the proof.

Remark 3.1. *The relationship for set-valued Riemann-Liouville integrals is as follows if we take into account $\zeta(\omega)=\omega$ and $\kappa(\Omega)=\Omega$ with $\underline{\psi}\neq\overline{\psi}$ in Theorem 3.2:*

$$4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)$$
$$\supseteq\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{(\omega_2-\omega_1)^{\theta_1}(\omega_4-\omega_3)^{\theta_2}}$$
$$\Big[\mathcal{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\Big]$$
$$\supseteq\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4),$$

which was proved by the authors in [54].

Remark 3.2. *The relationship for Riemann integrals is as follows if we take into account $\zeta(\omega)=\omega$ and $\kappa(\Omega)=\Omega$, $\theta_1=\theta_2=1$ with $\underline{\psi}\neq\overline{\psi}$ in Theorem 3.2:*

$$4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)$$
$$\supseteq\frac{4}{(\omega_2-\omega_1)(\omega_4-\omega_3)}\int_{\omega_1}^{\omega_2}\int_{\omega_3}^{\omega_4}\psi(\omega,\Omega)\mathrm{d}\Omega\mathrm{d}\omega$$
$$\supseteq\psi(\omega_1,\omega_3)+\psi(\omega_2,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_4),$$

which was proved by the authors in [18].

Remark 3.3. *The relationship for Riemann-Liouville integrals is as follows if we take into account $\zeta(\omega)=\omega$ and $\kappa(\Omega)=\Omega$ with $\underline{\psi}=\overline{\psi}$ in Theorem 3.2:*

$$\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\leq \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{(\omega_2-\omega_1)^{\theta_1}(\omega_4-\omega_3)^{\theta_2}}\\
&\quad\left[\mathcal{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\right]\\
&\leq \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4),
\end{aligned}$$

which was proved by the authors in [55].

Remark 3.4. *The relationship for fractional integrals is as follows if we take into account $\underline{\psi}=\overline{\psi}$ in Theorem 3.2:*

$$\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\leq \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\
&\quad\times\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\psi(\omega_1,\omega_3)\right]\\
&\leq \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}$$

which was proved by the authors in [34].

Remark 3.5. *The relationship for Riemann integrals is as follows if we take into account $\zeta(\omega)=\omega$ and $\kappa(\Omega)=\Omega$, $\theta_1=\theta_2=1$ with $\underline{\psi}\neq\overline{\psi}$ in Theorem 3.2, which was proved by the authors in [53].*

Corollary 3.1. *The relationship for Hadamard fractional integrals is as follows if we take into account $\zeta(\omega)=\ln\omega$ and $\kappa(\Omega)=\ln\Omega$ in Theorem 3.2:*

$$\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\leq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4\left[\ln\omega_2-\ln\omega_1\right]^{\theta_1}\left[\ln\omega_4-\ln\omega_3\right]^{\theta_2}}\\
&\quad\left[\mathfrak{H}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+\mathfrak{H}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+\mathfrak{H}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+\mathfrak{H}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\right]\\
&\leq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}$$

Corollary 3.2. *The relationship for Katugampola fractional integrals is as follows if we take into account $\zeta(\omega)=\frac{\omega^{\mathrm{p}}}{\eta}$ and $\kappa(\Omega)=\frac{\Omega^{\sigma}}{\sigma},\eta,\sigma>0$ in Theorem 3.2:*

$$\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\leq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)\eta^{\theta_1}\sigma^{\theta_2}}{4\left[{\omega_2}^{\eta}-{\omega_1}^{\eta}\right]^{\theta_1}\left[{\omega_4}^{\sigma}-{\omega_3}^{\sigma}\right]^{\theta_2}}\\
&\quad\left[{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\right]\\
&\leq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}$$

Example 3.1. Let $\psi(\omega,\Omega)=\left[24e^{\omega}+24e^{\Omega}+4e^{\Omega\omega}+144, 216e^{\omega}+216e^{\Omega}+36e^{\Omega\omega}+1296\right]$, $[\omega_1,\omega_2]=[0,1]$, $[\omega_3,\omega_4]=[0,1]$, $\theta_1=\theta_2=1$, $\zeta(\omega)=\ln\omega$ and $\kappa(\Omega)=\ln\Omega$, then we have

$$\begin{aligned}
&\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\approx[88.4109, 351.0176],\\
&\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{16[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\
&\times\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\psi(\omega_1,\omega_3)\right]\\
\approx{}&[117.4421, 557.1243],\\
&\frac{\psi(\omega_2,\omega_3)+\psi(\omega_1,\omega_3)+\psi(\omega_2,\omega_4)+\psi(\omega_1,\omega_4)}{4}\approx[123.5321, 741.1931].
\end{aligned}$$

Thus,

$$[88.4109, 351.0176]\preceq_{cr}[117.4421, 557.1243]\preceq_{cr}[123.5321, 741.1931].$$

As a result, Theorem 3.2 holds true.

Theorem 3.3. *Considering the same hypotheses that were considered in Theorem 3.2, we get the following center-radius order relationships:*

$$\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
\preceq_{cr}{}&\frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right]\\
&+\frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right]\\
\preceq_{cr}{}&\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\
&\times\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta\cdot\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta\cdot\kappa}\psi(\omega_1,\omega_3)\right]\\
\preceq_{cr}{}&\frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}(\omega_1,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}(\omega_1,\omega_4)\right]\\
&+\frac{\Gamma(\theta_2+1)}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}(\omega_1,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}(\omega_2.\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}(\omega_1,\omega_3)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}(\omega_2,\omega_3)\right]\\
\preceq_{cr}{}&\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}\tag{3.16}$$

Proof. As ψ is a bidimesional convex on Δ, if we have the mapping $\sigma^1_{\mathrm{x}}:[\omega_3,\omega_4]\to\mathsf{R}$, $\sigma^1_{\mathrm{x}}(\mathrm{y})=\psi(\mathrm{x},\mathrm{y})$, then $\sigma^1_{\mathrm{x}}(\mathrm{y})$ is convex $\forall x\in[\omega_1,\omega_2]$ and $\mathcal{H}^1_{\mathrm{x}}(\mathrm{y})=\sigma^1_{\mathrm{x}}(\mathrm{y})+\widetilde{\sigma^1_{\mathrm{x}}}(\mathrm{y})=\psi(\mathrm{x},\mathrm{y})+\psi_2(\mathrm{x},\mathrm{y})=\mathcal{G}(\mathrm{x},\mathrm{y})$, then for the convex mapping $\sigma^1_{\mathrm{x}}(\mathrm{y})$, we have

$$\begin{aligned}
&\sigma^1_{\mathrm{x}}\left(\frac{\omega_3+\omega_4}{2}\right)\\
\preceq_{cr}{}&\frac{\Gamma(\theta_2+1)}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{H}^1_{\mathrm{x}}(\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{H}^1_{\mathrm{x}}(\omega_3)\right]\\
\preceq_{cr}{}&\frac{\sigma^1_{\mathrm{x}}(\omega_3)+\sigma^1_{\mathrm{x}}(\omega_4)}{2},
\end{aligned}$$

that is,

$$
\begin{aligned}
&\psi\left(\mathtt{x}, \frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{cr} \frac{\theta_2}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\
&\left[(IR)\int_{\omega_3}^{\omega_4} \frac{\kappa'(\mathtt{y})}{[\kappa(\omega_4)-\kappa(\mathtt{y})]^{1-\theta_2}} C(\mathtt{x},\mathtt{y})d\mathtt{y} + (IR)\int_{\omega_3}^{\omega_4} \frac{\kappa'(\mathtt{y})}{[\kappa(\mathtt{y})-\kappa(\omega_3)]^{1-\theta_2}} C(\mathtt{x},\mathtt{y})d\mathtt{y}\right]\\
&\preceq_{cr} \frac{\psi(\mathtt{x},\omega_3)+\psi(\mathtt{x},\omega_4)}{2}.
\end{aligned} \tag{3.17}
$$

Multiplying the relation (3.17) by

$$
\frac{\theta_1}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}} \frac{\zeta'(\mathtt{x})}{[\zeta(\omega_2)-\zeta(\mathtt{x})]^{1-\theta_1}},
$$

and

$$
\frac{\theta_1}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}} \frac{\zeta'(\mathtt{x})}{[\zeta(\mathtt{x})-\zeta(\omega_1)]^{1-\theta_1}}.
$$

Next, we obtain by integrating the given results from ω_1 to ω_2 with regard to $\mathtt{x}$.

$$
\begin{aligned}
&\frac{\Gamma(\theta_1+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}} \mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi\left(\omega_2, \frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}} \left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa} C(\omega_2,\omega_4) + \mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa} \mathcal{G}(\omega_2,\omega_3)\right]\\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}} \left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi(\omega_2,\omega_3) + \mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi(\omega_2,\omega_4)\right],
\end{aligned} \tag{3.18}
$$

and

$$
\begin{aligned}
&\frac{\Gamma(\theta_1+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}} \mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi\left(\omega_1, \frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}} \left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa} \mathcal{G}(\omega_1,\omega_4) + \mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa} \mathcal{G}(\omega_1,\omega_3)\right]\\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}} \left[\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi(\omega_1,\omega_3) + \mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi(\omega_1,\omega_4)\right],
\end{aligned} \tag{3.19}
$$

respectively.

Also, if we define another mapping $\sigma^2_{\mathtt{x}} : [\omega_3, \omega_4] \to \mathsf{R}, \sigma^2_{\mathtt{x}}(\mathtt{y}) = \psi_1(\mathtt{x},\mathtt{y})$, then $\sigma^2_{\mathtt{x}}(\mathtt{y})$ is convex $\forall x \in [\omega_1, \omega_2]$ and $\mathcal{H}^2_{\mathtt{x}}(\mathtt{y}) = \psi_1(\mathtt{x},\mathtt{y}) + \psi_3(\mathtt{x},\mathtt{y}) = \mathcal{K}(\mathtt{x},\mathtt{y})$, then for the convex function $\sigma^2_{\mathtt{x}}(\mathtt{y})$, we have

$$
\begin{aligned}
&\sigma^2_{\mathtt{x}}\left(\frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{cr} \frac{\Gamma(\theta_2+1)}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}} \left[\mathsf{J}_{\omega_3+;\kappa}\mathcal{H}^2_{\mathtt{x}}(\omega_4) + \mathsf{J}_{\omega_4-;\kappa}\mathcal{H}^2_{\mathtt{x}}(\omega_3)\right]\\
&\preceq_{cr} \frac{\sigma^2_{\mathtt{x}}(\omega_3)+\sigma^2_{\mathtt{x}}(\omega_4)}{2},
\end{aligned}
$$

that is,

$$\begin{aligned}
&\psi_1\left(\mathtt{x}, \frac{\omega_3+\omega_4}{2}\right) \\
&\preceq_{cr} \frac{\theta_2}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}} \\
&\left[(IR)\int_{\omega_3}^{\omega_4} \frac{\kappa'(\mathtt{y})}{[\kappa(\omega_4)-w(\mathtt{y})]^{1-\theta_2}}\mathcal{K}(\mathtt{x},\mathtt{y})\mathtt{y} + (IR)\int_{\omega_3}^{\omega_4} \frac{\kappa'(\mathtt{y})}{[w(\mathtt{y})-\kappa(\omega_3)]^{1-\theta_2}}\mathcal{K}(\mathtt{x},\mathtt{y})d\mathtt{y}\right] \\
&\preceq_{cr} \frac{\psi_1(\mathtt{x},\omega_3)+\psi_1(\mathtt{x},\omega_4)}{2}.
\end{aligned} \tag{3.20}$$

Similarly, multiplying the relation (3.20) by

$$\frac{\theta_1}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\frac{\zeta'(\mathtt{x})}{[\zeta(\omega_2)-\zeta(\mathtt{x})]^{1-\theta_1}},$$

and

$$\frac{\theta_1}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\frac{\zeta'(\mathtt{x})}{[\zeta(\mathtt{x})-\zeta(\omega_1)]^{1-\theta_1}}.$$

Therefore, from ω_1 to ω_2, integrating the acquired output with reference to $\mathtt{x}$, we obtain

$$\begin{aligned}
&\frac{\Gamma(\theta_1+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi_1\left(\omega_2, \frac{\omega_3+\omega_4}{2}\right) \\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,,\omega_3+;\zeta,\kappa}\mathcal{K}(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\mathcal{K}(\omega_2,\omega_3)\right] \\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi_1(\omega_2,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi_1(\omega_2,\omega_4)\right],
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\frac{\Gamma(\theta_1+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi_1\left(\omega_1, \frac{\omega_3+\omega_4}{2}\right) \\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\mathcal{K}(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\mathcal{K}(\omega_1,\omega_3)\right] \\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi_1(\omega_1,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\psi_1(\omega_1,\omega_4)\right],
\end{aligned} \tag{3.22}$$

respectively.

Moreover, if we have mapping ${\sigma_{\mathtt{y}}}^1 : [\omega_1,\omega_2] \to \mathsf{R}, {\sigma_{\mathtt{y}}}^1(\mathtt{x}) = \psi(\mathtt{x},\mathtt{y})$, then ${\sigma_{\mathtt{y}}}^1(\mathtt{x})$ is convex for all $\forall \mathtt{y} \in [\omega_3,\omega_4]$ and $\mathcal{H}^1_y(\mathtt{x}) = {\sigma_{\mathtt{y}}}^1(\mathtt{x}) + \widetilde{{\sigma_{\mathtt{y}}}^1}(\mathtt{x}) = \psi(\mathtt{x},\mathtt{y}) + \psi_1(\mathtt{x},\mathtt{y}) = \mathcal{H}(\mathtt{x},\mathtt{y})$, then for the convex function ${\sigma_{\mathtt{y}}}^1(\mathtt{x})$, we have

$$\begin{aligned}
&{\sigma_{\mathtt{y}}}^1\left(\frac{\omega_1+\omega_2}{2}\right) \\
&\preceq_{cr} \frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}^1_y(\omega_2)+\mathsf{J}^{\theta_1}_{\omega_2-;\kappa}\mathcal{H}^1_y(\omega_1)\right] \\
&\preceq_{cr} \frac{{\sigma_{\mathtt{y}}}^1(\omega_1)+{\sigma_{\mathtt{y}}}^1(\omega_2)}{2},
\end{aligned}$$

that is,

$$\begin{aligned}
&\psi\left(\frac{\omega_1+\omega_2}{2},\mathrm{y}\right)\\
\preceq_{\mathrm{cr}}\ &\frac{\theta_1}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\\
&\left[(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\omega_1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\mathcal{H}(\mathrm{x},\mathrm{y})d\mathrm{x}+(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathrm{x})}{[\zeta(\mathrm{x})-\zeta(\omega_1)]^{1-\theta_1}}\mathcal{H}(\mathrm{x},\mathrm{y})d\mathrm{x}\right]\\
\preceq_{\mathrm{cr}}\ &\frac{\psi(\omega_1,\mathrm{y})+\psi(\omega_2,\mathrm{y})}{2}.
\end{aligned}\tag{3.23}$$

Multiplying the relation (3.23) by

$$\frac{\theta_2}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\frac{\kappa'(\mathrm{y})}{[\kappa(\omega_4)-w(\mathrm{y})]^{1-\theta_2}},\tag{3.24}$$

and

$$\frac{\theta_2}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\frac{\kappa'(\mathrm{y})}{[w(\mathrm{y})-\kappa(\omega_3)]^{1-\theta_2}}.\tag{3.25}$$

Next, by integrating the known findings from ω_3 to ω_4 with respect to y, we derive the subsequent relation:

$$\begin{aligned}
&\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathrm{J}^{\theta_2}_{\omega_3+;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_1,\theta_2}_{\omega_1+,,\omega_3+;\zeta,\kappa}\mathcal{H}(\omega_2.\omega_4)+\mathrm{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\mathcal{H}(\omega_1,\omega_4)\right]\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_2}_{\omega_3+;\kappa}\psi(\omega_1,\omega_4)+\mathrm{J}^{\theta_2}_{\omega_3+;\kappa}\psi(\omega_2,\omega_4)\right],
\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}
&\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathrm{J}^{\theta_2}_{\omega_4-;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\mathcal{H}(\omega_2,\omega_3)+\mathrm{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta,\kappa}\mathcal{H}(\omega_1,\omega_3)\right]\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_2}_{\omega_4-;\kappa}\psi(\omega_1,\omega_3)+\mathrm{J}^{\theta_2}_{\omega_4-;\kappa}\psi(\omega_2,\omega_3)\right],
\end{aligned}\tag{3.27}$$

respectively.

Furthermore, if we have mapping ${\sigma_{\mathrm{y}}}^2:[\omega_1,\omega_2]\to\mathrm{R},{\sigma_{\mathrm{y}}}^2(\mathrm{x})=\psi_2(\mathrm{x},\mathrm{y})$, then ${\sigma_{\mathrm{y}}}^2(\mathrm{x})$ is convex for all $\mathrm{y}\in[\omega_3,\omega_4]$ and $\mathcal{H}^2_{\mathrm{y}}(\mathrm{x})={\sigma_{\mathrm{y}}}^2(\mathrm{x})+\widetilde{{\sigma_{\mathrm{y}}}^2}(\mathrm{x})=\psi_2(\mathrm{x},\mathrm{y})+\psi_3(\mathrm{x},\mathrm{y})=\mathcal{L}(\mathrm{x},\mathrm{y})$, the for the convex mapping ${\sigma_{\mathrm{y}}}^2(\mathrm{x})$, we have

$${\sigma_{\mathrm{y}}}^2\left(\frac{\omega_1+\omega_2}{2}\right)\preceq_{\mathrm{cr}}\frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathrm{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}^2_{\mathrm{y}}(\omega_2)+\mathrm{J}^{\theta_1}_{\omega_2-;\kappa}\mathcal{H}^2_{\mathrm{y}}(\omega_1)\right]\preceq_{\mathrm{cr}}\frac{{\sigma_{\mathrm{y}}}^2(\omega_1)+{\sigma_{\mathrm{y}}}^2(\omega_2)}{2},$$

that is,

$$\begin{aligned}
&\psi_2\left(\frac{\omega_1+\omega_2}{2},\mathrm{y}\right)\\
&\preceq_{\mathrm{cr}} \frac{\theta_1}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\\
&\left[(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathrm{x})}{[\zeta(\omega_2)-\zeta(\mathrm{x})]^{1-\theta_1}}\mathcal{L}(\mathrm{x},\mathrm{y})d\mathrm{x}+(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathrm{x})}{[\zeta(\mathrm{x})-\zeta(\omega_1)]^{1-\theta_1}}\mathcal{L}(\mathrm{x},\mathrm{y})d\mathrm{x}\right]\\
&\preceq_{\mathrm{cr}} \frac{\psi_2(\omega_1,\mathrm{y})+\psi_2(\omega_2,\mathrm{y})}{2}.
\end{aligned} \tag{3.28}$$

Multiplying the relations (3.28) by

$$\frac{\theta_2}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\frac{\kappa'(\mathrm{y})}{[\kappa(\omega_4)-\kappa(\mathrm{y})]^{1-\theta_2}},$$

and

$$\frac{\theta_2}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\frac{\kappa'(\mathrm{y})}{[\kappa(\mathrm{y})-\kappa(\omega_3)]^{1-\theta_2}},$$

subsequently integrating the established findings from ω_3 to ω_4 with respect to y, we derive the following relations:

$$\begin{aligned}
&\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathrm{J}^{\theta_2}_{\omega_3+;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\mathcal{H}(\omega_2.\omega_4)+\mathrm{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\mathcal{H}(\omega_1,\omega_4)\right]\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_2}_{\omega_3+;\kappa}\psi(\omega_1,\omega_4)+\mathrm{J}^{\theta_2}_{\omega_3+;\kappa}\psi(\omega_2,\omega_4)\right],
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathrm{J}^{\theta_2}_{\omega_4-;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\mathcal{H}(\omega_2,\omega_3)+\mathrm{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta\cdot\kappa}\mathcal{H}(\omega_1,\omega_3)\right]\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathrm{J}^{\theta_2}_{\omega_4-;\kappa}\psi(\omega_1,\omega_3)+\mathrm{J}^{\theta_2}_{\omega_4-;\kappa}\psi(\omega_2,\omega_3)\right],
\end{aligned} \tag{3.30}$$

respectively.

Again, if we define the mapping $\sigma_{\mathrm{y}}{}^2:[\omega_1,\omega_2]\to \mathsf{R},\sigma_{\mathrm{y}}{}^2(\mathrm{x})=\psi_2(\mathrm{x},\mathrm{y})$, then $\sigma_{\mathrm{y}}{}^2(\mathrm{x})$ is convex $\forall\mathrm{y}\in[\omega_3,\omega_4]$ and $\mathcal{H}^2_{\mathrm{y}}(\mathrm{x})=\sigma_{\mathrm{y}}{}^2(\mathrm{x})+\widetilde{\sigma_{\mathrm{y}}{}^2}(\mathrm{x})=\psi_2(\mathrm{x},\mathrm{y})+\psi_3(\mathrm{x},\mathrm{y})=\mathcal{L}(\mathrm{x},\mathrm{y})$, then for the convex mapping $\sigma_{\mathrm{y}}{}^2(\mathrm{x})$, we have

$$\begin{aligned}
\sigma_{\mathrm{y}}{}^2\left(\frac{\omega_1+\omega_2}{2}\right)&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathrm{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}^2_{\mathrm{y}}(\omega_2)+\mathrm{J}^{\theta_1}_{\omega_2-;\kappa}\mathcal{H}^2_{\mathrm{y}}(\omega_1)\right]\\
&\preceq_{\mathrm{cr}} \frac{\sigma_{\mathrm{y}}{}^2(\omega_1)+\sigma_{\mathrm{y}}{}^2(\omega_2)}{2},
\end{aligned}$$

that is,

$$\begin{aligned}&\psi_2\left(\frac{\omega_1+\omega_2}{2},\mathrm{y}\right)\\ &\preceq_{\mathrm{cr}}\ \frac{\theta_1}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\\ &\left[(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathrm{x})}{[\zeta(\omega_2)-\zeta(\mathrm{x})]^{1-\theta_1}}\mathcal{L}(\mathrm{x},\mathrm{y})d\mathrm{x}+(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathrm{x})}{[\zeta(\mathrm{x})-\zeta(\omega_1)]^{1-\theta_1}}\mathcal{L}(\mathrm{x},\mathrm{y})d\mathrm{x}\right]\\ &\preceq_{\mathrm{cr}}\ \frac{\psi_2(\omega_1,\mathrm{y})+\psi_2(\omega_2,\mathrm{y})}{2}.\end{aligned} \tag{3.31}$$

Similarly, multiplying the relations (3.31) by

$$\frac{\theta_2}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\frac{\kappa'(\mathrm{y})}{[\kappa(\omega_4)-\kappa(\mathrm{y})]^{1-\theta_2}},$$

and

$$\frac{\theta_2}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\frac{\kappa'(\mathrm{y})}{[\kappa(\mathrm{y})-\kappa(\omega_3)]^{1-\theta_2}},$$

upon integrating the acquired outcomes concerning y between ω_3 and ω_4, we derive the subsequent relationships:

$$\begin{aligned}&\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi_2\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)\\ &\preceq_{\mathrm{cr}}\ \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\mathcal{L}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\mathcal{L}(\omega_1,\omega_4)\right]\\ &\preceq_{\mathrm{cr}}\ \frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi_2(\omega_1,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi_2(\omega_2.\omega_4)\right],\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi_2\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\\ &\preceq_{\mathrm{cr}}\ \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\mathcal{L}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-;\zeta\cdot\kappa}\mathcal{L}(\omega_1,\omega_3)\right]\\ &\preceq_{\mathrm{cr}}\ \frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi_2(\omega_1,\omega_3)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi_2(\omega_2,\omega_3)\right],\end{aligned} \tag{3.33}$$

respectively.

Summing the relations (3.18), (3.19), (3.21), (3.22), (3.26), (3.28)–(3.30) and (3.33), we have the following result:

$$\begin{aligned}&\frac{\Gamma(\theta_1+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+\zeta}\psi\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathsf{J}^{\theta_1}_{\omega_2-\zeta}\psi\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right.\\ &\left.+\mathsf{J}^{\theta_1}_{\omega_1+\zeta}\psi_1\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathsf{J}^{\theta_1}_{\omega_2-\zeta}\psi_1\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right]\\ &+\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi_2\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi_2\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\Bigg]\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\
&\times\Big[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+:\zeta,\kappa}\mathcal{G}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-:\zeta,\kappa}\mathcal{G}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\mathcal{G}(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-:\zeta,\kappa}\mathcal{G}(\omega_1,\omega_3)\\
&+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\mathcal{K}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\mathcal{K}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+:\zeta,\kappa}\mathcal{K}(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-:\zeta,\kappa}\mathcal{K}(\omega_1,\omega_3)\\
&+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\mathcal{H}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;\zeta,\kappa}\mathcal{H}(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-:\zeta,\kappa}\mathcal{H}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-:\zeta,\kappa}\mathcal{H}(\omega_1,\omega_3)\\
&+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;\zeta,\kappa}\mathcal{L}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+:\zeta,\kappa}\mathcal{L}(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-:\zeta,\kappa}\mathcal{L}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-:\zeta,\kappa}\mathcal{L}(\omega_1,\omega_3)\Big]\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\Big[\mathsf{J}^{\theta_1}_{\omega_1+\zeta}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_1+\zeta}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1}_{\omega_2-\zeta}\psi(\omega_1,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_2-\zeta}\psi(\omega_1,\omega_4)\\
&+\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\psi_1(\omega_2,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_1+\zeta}\psi_1(\omega_2.\omega_4)+\mathsf{J}^{\theta_1}_{\omega_2-\zeta}\psi_1(\omega_1,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_2-\zeta}\psi_1(\omega_1,\omega_4)\Big]\\
&+\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\Big[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi(\omega_1,\omega_3)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi(\omega_2,\omega_3)\\
&+\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi_2(\omega_1,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\psi_2(\omega_2.\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi_2(\omega_1,\omega_3)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\psi_2(\omega_2,\omega_3)\Big].
\end{aligned}
$$

That is, we have

$$
\begin{aligned}
&\frac{\Gamma(\theta_1+1)}{[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right]\\
&+\frac{\Gamma(\theta_2+1)}{[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathsf{J}^{\theta_2}_{\omega_4-;,\kappa}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right]\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\\
&\times\Big[\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+;,\kappa}\psi(\omega_2,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-;\zeta,\kappa}\psi(\omega_2,\omega_3)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+;,\kappa}\psi(\omega_1,\omega_4)+\mathsf{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4+;,\kappa}\psi(\omega_1,\omega_3)\Big]\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\theta_1+1)}{2[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\Big[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}(\omega_1,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}(\omega_1,\omega_4)\Big]\\
&+\frac{\Gamma(\theta_2+1)}{2[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\Big[\mathsf{J}^{\theta_2}_{\omega_3+;,\kappa}\mathcal{G}(\omega_1,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_3+;,\kappa}\mathcal{G}(\omega_2.\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}(\omega_1,\omega_3)+\mathsf{J}^{\theta_2}_{\omega_4-\kappa;,}\mathcal{G}(\omega_2,\omega_3)\Big].
\end{aligned}
$$

This concludes the verification of relations second and third in (3.16). On the other hand, for symmetric function, we have the following relation:

$$
\begin{aligned}
\psi\left(\frac{\omega_1+\omega_2}{2}\right)\preceq_{\mathrm{cr}}\ &\frac{\theta_1}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathtt{x})}{[\zeta(\omega_2)-\zeta(\mathtt{x})]^{\theta_1}}[\psi(\mathtt{x})+\psi(\omega_1+\omega_2-\mathtt{x})]d\mathtt{x}\right.\\
&\left.+\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathtt{x})}{[\zeta(\mathtt{x})-\zeta(\omega_1)]^{\theta_1}}[\psi(\mathtt{x})+\psi(\omega_1+\omega_2-\mathtt{x})]d\mathtt{x}\right].
\end{aligned}
\tag{3.34}
$$

As ψ is set-valued bidimensional convex on Δ, by using relation (3.34), we obtain

$$
\begin{aligned}
&\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
\preceq_{\mathrm{cr}}\ &\frac{\theta_1}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathtt{x})}{[\zeta(\omega_2)-\zeta(\mathtt{x})]^{\theta_1}}\left[\psi\left(\mathtt{x},\frac{\omega_3+\omega_4}{2}\right)+\psi\left(\omega_1+\omega_2-\mathtt{x},\frac{\omega_3+\omega_4}{2}\right)\right]d\mathtt{x}\right.
\end{aligned}
$$

$$+(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathtt{x})}{[\zeta(\mathtt{x})-\zeta(\omega_1)]^{\theta_1}}\left[\psi\left(\mathtt{x},\frac{\omega_3+\omega_4}{2}\right)+\psi\left(\omega_1+\omega_2-\mathtt{x},\frac{\omega_3+\omega_4}{2}\right)\right]\mathtt{dx}\Bigg]$$
$$=\frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right],\tag{3.35}$$

and similarly, we have

$$\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)$$
$$\preceq_{\mathrm{cr}}\frac{\theta_2}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\Bigg[(\mathrm{IR})\int_{\omega_3}^{\omega_4}\frac{\kappa'(\mathtt{y})}{[\kappa(\omega_4)-\kappa(\mathtt{y})]^{\theta_1}}\left[\psi\left(\frac{\omega_1+\omega_2}{2},\mathtt{y}\right)+\psi\left(\frac{\omega_1+\omega_2}{2},\omega_3+\omega_4-\mathtt{y}\right)\right]\mathtt{dy}$$
$$+(\mathrm{IR})\int_{\omega_3}^{\omega_4}\frac{\kappa'(\mathtt{y})}{[\kappa(\mathtt{y})-\kappa(\omega_3)]^{\theta_1}}\left[\psi\left(\frac{\omega_1+\omega_2}{2},\mathtt{y}\right)+\psi\left(\frac{\omega_1+\omega_2}{2},\omega_3+\omega_4-\mathtt{y}\right)\right]\mathtt{dy}\Bigg]$$
$$=\frac{\Gamma(\theta_2+1)}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right].\tag{3.36}$$

Combining the relations (3.35) and (3.36), we obtain the first relation in (3.16). For the second relation, we following double relation:

$$\frac{\theta_1}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\Bigg[(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathtt{x})}{[\zeta(\omega_2)-\zeta(\mathtt{x})]^{\theta_1}}[\psi(\mathtt{x})+\psi(\omega_1+\omega_2-\mathtt{x})]\mathtt{dx}$$
$$+(\mathrm{IR})\int_{\omega_1}^{\omega_2}\frac{\zeta'(\mathtt{x})}{[\zeta(\mathtt{x})-\zeta(\omega_1)]^{\theta_1}}[\psi(\mathtt{x})+\psi(\omega_1+\omega_2-\mathtt{x})]\mathtt{dx}\Bigg]$$
$$\preceq_{\mathrm{cr}}\frac{\psi(\omega_1)+\psi(\omega_2)}{2}.\tag{3.37}$$

By using relation (3.37), we obtain the following relations:

$$\frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}(\omega_2,\omega_3)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}(\omega_1,\omega_3)\right]\preceq_{\mathrm{cr}}\frac{\psi(\omega_1,\omega_3)+\psi(\omega_2,\omega_3)}{2},$$
$$\frac{\Gamma(\theta_1+1)}{4[\zeta(\omega_2)-\zeta(\omega_1)]^{\theta_1}}\left[\mathsf{J}^{\theta_1}_{\omega_1+;\zeta}\mathcal{H}(\omega_2.\omega_4)+\mathsf{J}^{\theta_1}_{\omega_2-;\zeta}\mathcal{H}(\omega_1,\omega_4)\right]\preceq_{\mathrm{cr}}\frac{\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_4)}{2},$$
$$\frac{\Gamma(\theta_2+1)}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}(\omega_1,\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}(\omega_1,\omega_3)\right]\preceq_{\mathrm{cr}}\frac{\psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)}{2},\tag{3.38}$$

and

$$\frac{\Gamma(\theta_2+1)}{4[\kappa(\omega_4)-\kappa(\omega_3)]^{\theta_2}}\left[\mathsf{J}^{\theta_2}_{\omega_3+;\kappa}\mathcal{G}(\omega_2.\omega_4)+\mathsf{J}^{\theta_2}_{\omega_4-;\kappa}\mathcal{G}(\omega_2,\omega_3)\right]\preceq_{\mathrm{cr}}\frac{\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4)}{2}.\tag{3.39}$$

Combining the relations (3.38) to (3.39), we obtain the last relation in (3.16). This completes the proof. □

Remark 3.6. *If we consider $\zeta(\mathtt{t})=\mathtt{t}$ and $\kappa(\mathtt{s})=\mathtt{s}$ in Theorem 3.3, then we obtain the following relationship for Riemann-Liouville integrals:*

$$
\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)}{(\omega_2-\omega_1)^{\theta_1}}\left[\mathcal{J}^{\theta_1}_{\omega_1+}\psi\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathcal{J}^{\theta_1}_{\omega_2-}\psi\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right]\\
&\quad+\frac{\Gamma(\theta_2+1)}{(\omega_4-\omega_3)^{\theta_2}}\left[\mathcal{J}^{\theta_2}_{\omega_3+}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathcal{J}^{\theta_2}_{\omega_4-}\psi\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right]\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4(\omega_2-\omega_1)^{\theta_1}(\omega_4-\omega_3)^{\theta_2}}\\
&\quad\left[\mathcal{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+\mathcal{J}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\right]\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)}{2(\omega_2-\omega_1)^{\theta_1}}\left[\mathcal{J}^{\theta_1}_{\omega_1+}\psi(\omega_2,\omega_3)+\mathcal{J}^{\theta_1}_{\omega_1+}\psi(\omega_2,\omega_4)+\mathcal{J}^{\theta_1}_{\omega_2-}\psi(\omega_1,\omega_3)+\mathcal{J}^{\theta_1}_{\omega_2-}\psi(\omega_1,\omega_4)\right]\\
&\quad+\frac{\Gamma(\theta_2+1)}{2(\omega_4-\omega_3)^{\theta_2}}\left[\mathcal{J}^{\theta_2}_{\omega_3+}\psi(\omega_1,\omega_4)+\mathcal{J}^{\theta_2}_{\omega_3+}\psi(\omega_2,\omega_4)+\mathcal{J}^{\theta_2}_{\omega_4-}\psi(\omega_1,\omega_3)+\mathcal{J}^{\theta_2}_{\omega_4-}\psi(\omega_2,\omega_3)\right]\\
&\preceq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}
$$

Corollary 3.3. *If we consider $\zeta(\mathtt{t})=\mathtt{t}$ and $\kappa(\mathtt{s})=\mathtt{s}$ in Theorem 3.3, then we obtain the following relationship for Hadamard integrals:*

$$
\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)}{2\left[\ln\omega_2-\ln\omega_1\right]^{\theta_1}}\left[\mathfrak{G}^{\theta_1}_{\omega_1+}\mathcal{H}\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+\mathfrak{G}^{\theta_1}_{\omega_2-}\mathcal{H}\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right]\\
&\quad+\frac{\Gamma(\theta_2+1)}{2\left[\ln\omega_4-\ln\omega_3\right]^{\theta_2}}\left[\mathfrak{G}^{\theta_2}_{\omega_3+}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+\mathfrak{G}^{\theta_2}_{\omega_4-}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right]\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}{4\left[\ln\omega_2-\ln\omega_1\right]^{\theta_1}\left[\ln\omega_4-\ln\omega_3\right]^{\theta_2}}\\
&\quad\left[\mathfrak{G}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+\mathfrak{G}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+\mathfrak{G}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+\mathfrak{G}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\right]\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)}{4\left[\ln\omega_2-\ln\omega_1\right]^{\theta_1}}\left[\mathfrak{T}^{\theta_1}_{\omega_1+}\mathcal{H}(\omega_2,\omega_3)+\mathfrak{G}^{\theta_1}_{\omega_1+}\mathcal{H}(\omega_2.\omega_4)+\mathfrak{G}^{\theta_1}_{\omega_2-}\mathcal{H}(\omega_1,\omega_3)+\mathfrak{G}^{\theta_1}_{\omega_2-}\mathcal{H}(\omega_1,\omega_4)\right]\\
&\quad+\frac{\Gamma(\theta_2+1)}{4\left[\ln\omega_4-\ln\omega_3\right]^{\theta_2}}\left[\mathfrak{G}^{\theta_2}_{\omega_3+}\mathcal{G}(\omega_1,\omega_4)+\mathfrak{G}^{\theta_2}_{\omega_3+}\mathcal{G}(\omega_2.\omega_4)+\mathfrak{G}^{\theta_2}_{\omega_4-}\mathcal{G}(\omega_1,\omega_3)+\mathfrak{G}^{\theta_2}_{\omega_4-}\mathcal{G}(\omega_2,\omega_3)\right]\\
&\preceq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}
$$

Corollary 3.4. *If we consider $\zeta(\mathtt{t})=\frac{\mathtt{t}^\eta}{\eta}$ and $\kappa(\mathtt{s})=\frac{\mathtt{s}^\sigma}{\sigma}$ in Theorem 3.3, then we obtain the following relationship for Katugampola integrals:*

$$
\begin{aligned}
&4\psi\left(\frac{\omega_1+\omega_2}{2},\frac{\omega_3+\omega_4}{2}\right)\\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\eta^{\theta_1}}{2\left[\omega_2{}^\eta-\omega_1{}^\eta\right]^{\theta_1}}\left[{}^{\theta_1}_{\omega_1+}\mathcal{H}\left(\omega_2,\frac{\omega_3+\omega_4}{2}\right)+{}^\eta\mathrm{I}^{\theta_1}_{\omega_2-}\mathcal{H}\left(\omega_1,\frac{\omega_3+\omega_4}{2}\right)\right]\\
&\quad+\frac{\Gamma(\theta_2+1)\sigma^{\theta_2}}{2\left[\omega_4{}^\sigma-\omega_3{}^\sigma\right]^{\theta_2}}\left[{}^\sigma\mathrm{I}^{\theta_2}_{\omega_3+}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_4\right)+{}^\sigma\mathrm{I}^{\theta_2}_{\omega_4-}\mathcal{G}\left(\frac{\omega_1+\omega_2}{2},\omega_3\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\Gamma(\theta_2+1)\eta^{\theta_1}\sigma^{\theta_2}}{4\left[\omega_2{}^{\eta}-\omega_1{}^{\eta}\right]^{\theta_1}\left[\omega_4{}^{\sigma}-\omega_3{}^{\sigma}\right]^{\theta_2}} \\
&\quad \left[{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_3+}\psi(\omega_2,\omega_4)+{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_1+,\omega_4-}\psi(\omega_2,\omega_3)+{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_2-,\omega_3+}\psi(\omega_1,\omega_4)+{}^{\eta,\sigma}\mathsf{I}^{\theta_1,\theta_2}_{\omega_2-,\omega_4-}\psi(\omega_1,\omega_3)\right] \\
&\preceq_{\mathrm{cr}} \frac{\Gamma(\theta_1+1)\eta^{\theta_1}\sigma^{\theta_2}}{4\left[b^{\eta}-a^{\eta}\right]^{\theta_1}}\left[{}^{\eta}\mathsf{I}^{\theta_1}_{\omega_1+}\mathcal{H}(\omega_2,\omega_3)+{}^{\eta}\mathsf{I}^{\theta_1}_{\omega_1+}\mathcal{H}(\omega_2.\omega_4)+{}^{\eta}\mathsf{I}^{\theta_1}_{\omega_2-}\mathcal{H}(\omega_1,\omega_3)+{}^{\eta}\mathsf{I}^{\theta_1}_{\omega_2-}\mathcal{H}(\omega_1,\omega_4)\right] \\
&\quad +\frac{\Gamma(\theta_2+1)\sigma^{\theta_2}}{4\left[\omega_4{}^{\sigma}-\omega_3{}^{\sigma}\right]^{\theta_2}}\left[{}^{\sigma}\mathsf{I}^{\theta_2}_{\omega_3+}\mathcal{G}(\omega_1,\omega_4)+{}^{\sigma}\mathsf{I}^{\theta_2}_{\omega_3+}\mathcal{G}(\omega_2.\omega_4)+{}^{\sigma}\mathsf{I}^{\theta_2}_{\omega_4-}\mathcal{G}(\omega_1,\omega_3)+{}^{\sigma}\mathsf{I}^{\theta_2}_{\omega_4-}\mathcal{G}(\omega_2,\omega_3)\right] \\
&\preceq_{\mathrm{cr}} \psi(\omega_1,\omega_3)+\psi(\omega_1,\omega_4)+\psi(\omega_2,\omega_3)+\psi(\omega_2,\omega_4).
\end{aligned}
$$

4. Discussion and conclusions with future remarks

Coordinated convex functions apply the concept of convexity to functions defined as a product of intervals, allowing for more detailed analysis in multivariable settings. The conclusion of these functions focuses on their integral inequalities and applications in mathematical analysis. These functions enable the application of mathematical inequalities that are essential for analyzing and ensuring the stability, optimality, and control of complex systems. In this paper, we develop various novel bounds and refinements for weighted Hermite-Hadamard inequalities as well as their product form by employing new types of fractional integral operators under a cr-order relation. Additionally, we demonstrate that by means of coordinated center-radius order relations for these integral operators, various new findings can be obtained for Katugampola and Hadamard integrals operators. Furthermore, we show that this type of order relation preserves the integral structure. In addition, as a distinct characteristic from pictorial view, we show that this order is convex in nature, whereas inclusion order is non-convex. A number of interesting examples are presented in support of the major findings. It will be interesting in the future if readers take motivation from these findings and construct results using Itô's lemma as well as quantum calculas, multiplicative calculas, fuzzy order relations, and various other fractional integrals.

Additionally, in a very recent study, Afzal et al. [56, 57] developed results using a new approach; more specifically, they used tensor Hilbert spaces and variable exponent spaces, which is an extremely new approach to Hermite-Hadamard inequality and its related results. We also suggest readers extend these results in the sense of these spaces that utilize a variety of convex mappings under norm structures and modular function spaces that have not been initiated yet for coordinated convexity of any kind.

Author's contributions

W. Afzal: Conceptualization, Data curation, Writing-original draft, Funding acquisition, Investigation, Visualization; M. Abbas: Conceptualization, Formal analysis, Writing-original draft, Supervision, Validation, Writing-review & editing; J. Ro: Investigation, Project administration, Visualization; K. H. Hakami: Data curation, Formal analysis, Methodology; H. Zogan: Methodology, Project administration, Software. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no competing interests.

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