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Research article

Some convergence results in modular spaces with application to a system of integral equations

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Abstract: The paper aimed to achieve three primary objectives. First, it introduced significant common fixed point results in the context of newly proposed partial modular $b-$metric spaces, thus contributing to the advancement of this field. Second, it presented unique results using a direct approach that did not depend on the strong continuity of the mapping, thereby offering a valuable perspective. Finally, it applied previously established convergence techniques to determine a common solution for a system of Fredholm integral equations, demonstrating the practical implications of the theoretical findings.

Keywords: fixed point; partial modular $b-$metric; Proinov type contraction; Fredholm integral equation
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1. Introduction and Preliminaries

The field of metric fixed point theory, denoted as $(\mathcal{FP})$, holds significant relevance for the mathematical research community and scholarly discourse. The foundational work of S. Banach, particularly the introduction of the Banach contraction principle in his doctoral dissertation [1], stands as a seminal contribution within this domain. This significant result has not only been the foundation for several developments in fixed point theory, but it has also been the impetus for a wide variety of generalizations and novel adaptations to the idea of contraction maps [2].

Fixed point theory, a cornerstone of contemporary mathematical sciences, is characterized by its dynamic evolution and the vibrancy of its research community. This field, rooted in rich foundational principles and methodological innovations, extends its influence far beyond its initial mathematical confines, offering broad applications across a multitude of disciplines. The versatility and utility of fixed point theory's methodological approaches render it an indispensable tool for tackling complex problem-solving tasks within myriad mathematical contexts. Its capacity to bridge theoretical and practical aspects across diverse areas underscores its pivotal role in advancing both the understanding and application of mathematical principles; see [3–6].

Moreover, the study of metric spaces, symbolized as ($\mathcal{MS}$), occupies a central position in the realm of mathematical analysis and its wide-ranging applications, as evidenced by references such as [7–10]. This concept has undergone substantial refinement and expansion, with scholars broadening the scope of $\mathcal{MS}$ to encompass more abstract spaces, thus enhancing its applicability across diverse domains. Notably, the introduction of $b-\mathcal{MS}$, initially proposed by Bakhtin [11], represents a significant extension in this direction. Bakhtin's definition of the $b-$metric function has garnered widespread acceptance and has been subject to extensive development by researchers such as Czerwik [12, 13], thereby enriching the theoretical landscape of metric fixed point theory. A $b-$metric function differs from a typical metric function by relaxing the triangle inequality to a more general form, as seen below:

$$\mathscr{b}(\lambda, \hbar) \leq \rho\left[\mathscr{b}(\lambda, z) + \mathscr{b}(z, \hbar)\right].$$

In this definition, the function $\mathscr{b} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is regarded a $b-$metric on the set $\mathfrak{X}$, where ρ is a positive real number (≥ 1) and the pair $(\mathfrak{X}, \mathscr{b})$ defines a $b-\mathcal{MS}$. Upon the condition $\rho = 1$, the conceptualizations of a $b-$metric and the canonical metric converge, thereby suggesting that the $b-$metric framework serves as an expansion of the conventional $\mathcal{MS}$. On the other hand, unlike its canonical counterpart, the $b-$metric formulation does not always display continuity, even though the canonical metric does. This augmentation facilitates the exploration of $\mathcal{MS}$ within realms of increased complexity or abstraction.

In the seminal work of Matthews, a groundbreaking concept of partial $\mathcal{MS}$ was introduced, which built upon and extended the foundational principles of denotational semantics within computer languages [14]. This innovative framework diverges from the conventional understanding of $\mathcal{MS}$ through its utilization of a partial metric that allows for nonzero self-distances, thereby broadening the scope for mathematical analysis and practical application. It is important to note that while traditional $\mathcal{MS}$ configurations can be considered as special cases of partial $\mathcal{MS}$ (wherein self-distances are uniformly zero), the incorporation of nonzero self-distances significantly enriches the versatility and adaptability of this conceptual model across a diverse spectrum of computational and theoretical domains [15, 16]. This nuanced extension offers a more comprehensive and flexible approach, paving the way for enhanced computational and theoretical explorations in various domains.

In their pivotal work published in 2014, Mustafa et al. [17] proposed a pioneering advancement in the $\mathcal{MS}$ framework by introducing the concept of partial $b-$metrics. This novel distance function not only incorporates the fundamental principles of partial metrics and $b-$metrics but also extends the existing theoretical framework. Moreover, the authors went on to establish a robust analogue to the Banach contraction principle within these spaces, which represents a significant theoretical development and has the potential to enrich the field of study further.

Definition 1.1. [17] A partial $b-$metric on a nonempty set $\mathfrak{X}$ is a mapping $p_{\mathscr{b}} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ such that for all $\lambda, \hbar, z \in \mathfrak{X}$, which fulfills the subsequent circumstances:

$\left(p_{\mathscr{b}_1}\right)$ $p_{\mathscr{b}}(\lambda, \lambda) = p_{\mathscr{b}}(\hbar, \hbar) = p_{\mathscr{b}}(\lambda, \hbar) \Leftrightarrow \lambda = \hbar$,

$\left(p_{\mathscr{b}_2}\right)$ $p_{\mathscr{b}}(\lambda, \lambda) \leq p_{\mathscr{b}}(\lambda, \hbar)$,

$\left(p_{\mathscr{b}_3}\right)$ $p_{\mathscr{b}}(\lambda, \hbar) = p_{\mathscr{b}}(\hbar, \lambda)$,

$\left(p_{\mathfrak{b}_4}\right)$ $p_{\mathfrak{b}}(ƛ,\hbar) \leq \rho\left[p_{\mathfrak{b}}(ƛ,z)+p_{\mathfrak{b}}(z,\hbar)-p_{\mathfrak{b}}(z,z)\right]+\left(\frac{1-\rho}{2}\right)\left(p_{\mathfrak{b}}(ƛ,ƛ)-p_{\mathfrak{b}}(\hbar,\hbar)\right)$.

A partial $b-$metric is a pair $\left(\mathfrak{X}, p_{\mathfrak{b}}\right)$ such that $\mathfrak{X}$ is a nonempty set and $p_{\mathfrak{b}}$ is a partial $b-$metric on $\mathfrak{X}$. The number $\rho \geq 1$ is called the coefficient of $\left(\mathfrak{X}, p_{\mathfrak{b}}\right)$.

In the inspiring work by Shukla [18], a pivotal modification was proposed to the triangle property inherent to partial $b-$metric spaces. This modification was meticulously designed to ensure that every partial $b-$metric space is associated with a corresponding $b-$metric space. Through this innovative approach, Shukla not only established a comprehensive convergence criterion but also delineated a set of operational guidelines within the framework of partial $b-\mathcal{MS}$. This breakthrough significantly enhances our comprehension of metric spaces and extends the utility of $b-$metrics across various domains in mathematical analysis and adjacent fields (see [19]). The proposed convergence criterion and operational guidelines offer a sophisticated framework for examining partial $b-$metric spaces, thereby facilitating further advancements in this intricate area of mathematical research. Definition 1.1 has been modified in [18] by considering the following condition instead of $\left(p_{\mathfrak{b}_4}\right)$: $\left(p_{\mathfrak{b}'_4}\right)$ for all $ƛ,\hbar, z \in \mathfrak{X}$:

$$p_{\mathfrak{b}}(ƛ,\hbar) \leq \rho\left[p_{\mathfrak{b}}(ƛ,z)+p_{\mathfrak{b}}(z,\hbar)\right]-p_{\mathfrak{b}}(z,z).$$

As $\rho \geq 1$, from $\left(p_{\mathfrak{b}_4}\right)$ we have

$$p_{\mathfrak{b}}(ƛ,\hbar) \leq \rho\left[p_{\mathfrak{b}}(ƛ,z)+p_{\mathfrak{b}}(z,\hbar)-p_{\mathfrak{b}}(z,z)\right] \leq \rho\left[p_{\mathfrak{b}}(ƛ,z)+p_{\mathfrak{b}}(z,\hbar)\right]-p_{\mathfrak{b}}(z,z).$$

Remark 1. If $ƛ,\hbar \in \mathfrak{X}$ and $p_{\mathfrak{b}}(ƛ,\hbar)=0$, then $ƛ=\hbar$, but the converse may not be true. The notion of partial $b-$metric and partial metric coincide in the case of $\rho=1$. Moreover, a partial $b-$metric on $\mathfrak{X}$ is neither a partial metric nor a $b-$metric. As far as we understand, a partial $b-\mathcal{MS}$ includes the set of a $b-\mathcal{MS}$ and partial $\mathcal{MS}$.

In 2006, Chistyakov [20] pioneered the introduction of the concept of a modular metric on an arbitrary set. This innovative metric represents a significant departure from classical metrics, offering a novel framework for quantifying distances between elements within a set that boasts greater flexibility and versatility. Furthermore, Chistyakov embarked on the formulation of the corresponding modular space, presenting a paradigm that encompasses a more extensive array of structures in contrast to the conventional mathematical structure $\mathcal{MS}$. Building on this foundational premise, Chistyakov, in subsequent research conducted in 2010, made substantial strides in the advancement of the theory of modular $\mathcal{MS}$. This research phase was principally centered on exploring spaces constituted by such modular metrics. The contributions made through this line of inquiry have significantly facilitated a profound comprehension of modular metric spaces, alongside fostering their application in various domains.

In 2018, Ege and Alaca [7] introduced the notion of modular $b-\mathcal{MS}$ as follows:

Definition 1.2. [7] Let $\mathfrak{X} \neq \emptyset$. A function $\omega:(0,\infty)\times\mathfrak{X}\times\mathfrak{X} \rightarrow [0,\infty]$, defined by $\omega(\lambda, ƛ,\hbar)=\omega_{\lambda}(ƛ,\hbar)$, is called a modular $b-$metric on $\mathfrak{X}$ if it satisfies the following statements for all $ƛ,\hbar, z \in \mathfrak{X}$, $\lambda, \mu>0$:

(ω_1) $\omega_{\lambda}(ƛ,\hbar)=0$ for all $\lambda>0 \Leftrightarrow ƛ=\hbar$,
(ω_2) $\omega_{\lambda}(ƛ,\hbar)=\omega_{\lambda}(\hbar,ƛ)$ for all $\lambda>0$,

(ω_3) there exists $\rho \geq 1$ such that

$$\omega_{\lambda+\mu}(\lambda, \hbar) \leq \rho\left[\omega_\lambda(\lambda, z) + \omega_\mu(z, \hbar)\right].$$

Modular $\mathcal{MS}$ can be achieved from modular $b - \mathcal{MS}$ in the case of $\rho = 1$. Also, the set

$$\mathfrak{X}^*_\omega(\lambda_0) = \{\lambda \in \mathfrak{X} : \exists\lambda > 0 \text{ such that } \omega_\lambda(\lambda, \lambda_0) < \infty\}$$

is mentioned as modular $b - \mathcal{MS}$ (around λ_0). For further synthesis, we refer the reader to [21–24].

In 2010, the concept of partial modular $\mathcal{MS}$ was introduced by Hosseinzadeh and Parvaneh [25] as a combination of partial $\mathcal{MS}$ and modular $\mathcal{MS}$.

In 2023, Kesik et al. [26] made a significant contribution to the field of topology by proposing the concept of the partial modular $b-$metric function. This novel concept represents a synthesis of the principles underlying partiality, modularity, and the $b-$metric framework. By doing so, they have not only introduced a new perspective but have also delineated several results that explicate the topological properties intrinsic to this innovative space. This development marks a notable advancement in the understanding and application of topological structures, providing a foundation for further explorations and applications within the domain.

Definition 1.3. [26] Let $\mathfrak{X}$ be a non-void set and $\rho \geq 1$ be a real number. A mapping $\varpi^{p_b} : (0, \infty) \times \mathfrak{X} \times \mathfrak{X} \to [0, \infty]$ is called a partial modular $b-$metric (briefly $\mathcal{PM}_b\mathcal{M}$) on $\mathfrak{X}$ if the following conditions hold for all $\lambda, \hbar, z \in \mathfrak{X}$,

$\left(\varpi_1^{p_b}\right)$ $\varpi_\lambda^{p_b}(\lambda, \lambda) = \varpi_\mu^{p_b}(\lambda, \lambda)$ and $\varpi_\lambda^{p_b}(\lambda, \lambda) = \varpi_\lambda^{p_b}(\hbar, \hbar) = \varpi_\lambda^{p_b}(\lambda, \hbar) \Leftrightarrow \lambda = \hbar$,
$\left(\varpi_2^{p_b}\right)$ $\varpi_\lambda^{p_b}(\lambda, \lambda) \leq \varpi_\lambda^{p_b}(\lambda, \hbar)$, for all $\lambda > 0$,
$\left(\varpi_3^{p_b}\right)$ $\varpi_\lambda^{p_b}(\lambda, \hbar) = \varpi_\lambda^{p_b}(\hbar, \lambda)$, for all $\lambda > 0$,
$\left(\varpi_4^{p_b}\right)$ $\varpi_{\lambda+\mu}^{p_b}(\lambda, \hbar) \leq \rho\left[\varpi_\lambda^{p_b}(\lambda, z) + \varpi_\mu^{p_b}(z, \hbar)\right] - \varpi_\lambda^{p_b}(z, z)$, for all $\lambda, \mu > 0$.

Then, $\left(\mathfrak{X}, \varpi_\lambda^{p_b}\right) = \mathfrak{X}_{\varpi^{p_b}}$ is called a partial modular $b - \mathcal{MS}$ which indicates $\mathcal{PM}_b\mathcal{MS}$.

Definition 1.4. [26] Let ϖ^{p_b} be a $\mathcal{PM}_b\mathcal{M}$ on a set $\mathfrak{X}$. For given $\lambda_0 \in \mathfrak{X}$, we define

- $\mathfrak{X}_{\varpi^{p_b}}(\lambda_0) = \left\{\lambda \in \mathfrak{X} : \lim_{\lambda\to\infty} \varpi_\lambda^{p_b}(\lambda_0, \lambda) = c\right\}$, for some $c \geq 0$ and
- $\mathfrak{X}^*_{\varpi^{p_b}}(\lambda_0) = \left\{\lambda \in \mathfrak{X} : \exists\ \lambda = \lambda(\lambda) > 0, \varpi_\lambda^{p_b}(\lambda_0, \lambda) < \infty\right\}$.

Then, two sets $\mathfrak{X}_{\varpi^{p_b}}$ and $\mathfrak{X}^*_{\varpi^{p_b}}$ are called $\mathcal{PM}_b\mathcal{MS}$ centered at λ_0.

It is clear that a partial modular $\mathcal{MS}$ and $\mathcal{PM}_b\mathcal{MS}$ coincide in the case of $\rho = 1$, and every modular $b - \mathcal{MS}$ is a $\mathcal{PM}_b\mathcal{MS}$ with the same coefficient and zero self-distance. However, the converse of these facts need not hold in general.

Because a $\mathcal{PM}_b\mathcal{M}$ is a partial modular when $\rho = 1$, the $\mathcal{PM}_b\mathcal{MS}$ class is more significant than that of partial modular $\mathcal{MS}$.

Now, we derive different examples, which evidently hold the conditions of this newly enunciated generalized $\mathcal{MS}$.

Example 1.5. Let $\mathfrak{X} = \mathbb{R}$ and $\varpi^{p_b} : (0, \infty) \times \mathfrak{X} \times \mathfrak{X} \to [0, \infty]$ be defined by, for all $\lambda, \hbar \in \mathfrak{X}$,

$$\varpi_\lambda^{p_b}(\lambda, \hbar) = e^{-\lambda}|\lambda - \hbar|^2 + |\lambda| + |\hbar|, \quad \forall\lambda > 0.$$

Then, ϖ^{p_b} is a $\mathcal{PM}_b\mathcal{M}$ on $\mathfrak{X}$ with the coefficient $\rho = 2$.

Example 1.6. Let $\mathfrak{X} = \mathbb{R}$ and $\varpi^{p_b} : (0, \infty) \times \mathfrak{X} \times \mathfrak{X} \to [0, \infty]$ be defined by for all $\lambda, \hbar \in \mathfrak{X}$

$$\varpi_\lambda^{p_b}(\lambda, \hbar) = \frac{|\lambda - \hbar|^2}{\lambda + |\lambda - \hbar|^2}, \quad \forall \lambda > 0.$$

Then, ϖ^{p_b} is a $\mathcal{PM}_b\mathcal{M}$ on $\mathfrak{X}$ with the coefficient $\rho = 2$.

To get acquainted with different notions and concepts within the structure of $\mathcal{PM}_b\mathcal{MS}$, such as completeness, convergence, etc., we refer to [26].

Lemma 1.7. [26] *Let ϖ^{p_b} be a $\mathcal{PM}_b\mathcal{M}$ on a nonempty set $\mathfrak{X}$. Define*

$$\omega_\lambda(\lambda, \hbar) = 2\varpi_\lambda^{p_b}(\lambda, \hbar) - \varpi_\lambda^{p_b}(\lambda, \lambda) - \varpi_\lambda^{p_b}(\hbar, \hbar). \tag{1.1}$$

Then, ω is a modular $b-$metric on $\mathfrak{X}$.

Lemma 1.8. [26] *Let ϖ^{p_b} be a $\mathcal{PM}_b\mathcal{M}$ on $\mathfrak{X}$ and $\{\lambda_\varrho\}_{\rho\in\mathbb{N}}$ be a sequence in $\mathfrak{X}^*_{\varpi^{p_b}}$. Then:*

(*i*) $\{\lambda_\varrho\}_{n\in\mathbb{N}}$ *is a $\varpi^{p_b}-$Cauchy sequence in the $\mathcal{PM}_b\mathcal{MS}$ $\mathfrak{X}^*_{\varpi^{p_b}}$ $\Leftrightarrow$ it is an $\omega-$Cauchy sequence in modular $b-\mathcal{MS}$ $\mathfrak{X}^*_\omega$ induced by $\mathcal{PM}_b\mathcal{MS}$ ϖ^{p_b}.*

(*ii*) *A $\mathcal{PM}_b\mathcal{MS}$ $\mathfrak{X}^*_{\varpi^{p_b}}$ is $\varpi^{p_b}-$complete $\Leftrightarrow$ the modular $b-\mathcal{MS}$ $\mathfrak{X}^*_\omega$ induced by $\mathcal{PM}_b\mathcal{MS}$ ϖ^{p_b} is $\omega-$complete. Furthermore,*

$$\lim_{n\to\infty} \omega_\lambda(\lambda_\varrho, \lambda) = 0 \Leftrightarrow \lim_{n\to\infty} \left[2\varpi_\lambda^{p_b}(\lambda_\varrho, \lambda) - \varpi_\lambda^{p_b}(\lambda_\varrho, \lambda_\varrho) - \varpi_\lambda^{p_b}(\lambda, \lambda)\right] = 0.$$

(*iii*) $\{\lambda_\varrho\}_{\rho\in\mathbb{N}}$ *is called $\varpi^{p_b}-$convergent to $\lambda^* \in \mathfrak{X}^*_{\varpi^{p_b}}$ $\Leftrightarrow$ $\lim_{n\to\infty} \varpi_\lambda^{p_b}(\lambda_\varrho, \lambda^*) = \lim_{n,m\to\infty} \varpi_\lambda^{p_b}(\lambda_\varrho, \lambda_m) = \varpi_\lambda^{p_b}(\lambda^*, \lambda^*), \forall \lambda > 0$, as $n \to \infty$.*

During the subsequent analysis, we employed auxiliary functions to get a broader range of outcomes in the field of fixed point theory. Proinov [27] recently presented a new fixed point theorem by adding auxiliary functions. This theorem has led to the discovery of several significant findings.

Theorem 1.9. [27] *Let $\mathcal{G} : \mathfrak{X} \to \mathfrak{X}$ be a self-map on a complete $\mathcal{MS}$ $(\mathfrak{X}, d)$. Presume that $\forall\ \lambda, \hbar \in \mathfrak{X}$, $d(\mathcal{G}\lambda, \mathcal{G}\hbar) > 0$, and the following condition*

$$\daleth(d(\mathcal{G}\lambda, \mathcal{G}\hbar)) \le \Gamma(d(\lambda, \hbar))$$

is met, where $\daleth, \Gamma : (0, \infty) \to \mathbb{R}$ are two functions that fulfill the below axioms:

($\wp_1$) *$\daleth$ is nondecreasing,*
($\wp_2$) *$\Gamma(\ell) < \daleth(\ell)$ for all $\ell > 0$,*
($\wp_3$) *$\limsup_{\ell\to\ell_0+} \Gamma(\ell) < \daleth(\ell_0+)$ for any $\ell_0 > 0$.*

Then, $\mathcal{G}$ is called a Proinov type contraction and admits a unique fixed point ($\mathcal{UFP}$).

Because of its diverse applications, several $\mathcal{FP}$ results, including the Proinov type contraction, may be found in the literature; see, for instance, the noteworthy articles [28–31].

On the other hand, in 2009, Suzuki [32] proved the below theorem and, subsequently, it was mentioned as a Suzuki type contraction.

Theorem 1.10. [32] *Let $\mathcal{G} : \mathfrak{X} \to \mathfrak{X}$ be a self-map on a compact $\mathcal{MS}$ $(\mathfrak{X}, d)$. If the expression*

$$\frac{1}{2} d(\mathcal{A}, \mathcal{G}\mathcal{A}) < d(\mathcal{A}, \hbar) \Rightarrow d(\mathcal{G}\mathcal{A}, \mathcal{G}\hbar) < d(\mathcal{A}, \hbar)$$

is satisfied for all distinct $\mathcal{A}, \hbar \in \mathfrak{X}$, then $\mathcal{G}$ owns a $\mathcal{UFP}$.

Motivated by the diverse applications of Proinov type $\mathcal{FP}$ results and keeping in view the applicability and adaptability of $\mathcal{PM}_b\mathcal{MS}$ in various computational and theoretical contexts, in this article, we articulate Suzuki-type contraction and Proinov-type contraction in the realm of $\mathcal{PM}_b\mathcal{MS}$. We provide an illustrative example to uphold our results with an application to a system of Fredholm integral equations.

2. Common fixed point results

This section is devoted to enunciating some novel common $\mathcal{FP}$ in the realm of partial-modular $\mathcal{MS}$. In order to demonstrate the subsequent $\mathcal{FP}$ outcomes in the sequel, two requirements must be met:

(Ξ_1) $\varpi_\lambda^{p_b}(\mathcal{A}, \mathcal{G}\mathcal{A}) < \infty$ for all $\mathcal{A} > 0$ where $\mathcal{A} \in \mathfrak{X}^*_{\varpi^{p_b}}$.
(Ξ_2) $\varpi_\lambda^{p_b}(\mathcal{A}, \hbar) < \infty$ for all $\mathcal{A} > 0$ where $\mathcal{A}, \hbar \in \mathfrak{X}^*_{\varpi^{p_b}}$.

Now, we establish some common $\mathcal{FP}$ theorems considering Suzuki contraction and Proinov type contraction in the context of $\mathcal{PM}_b\mathcal{MS}$.

Theorem 2.1. *Let $\mathfrak{X}^*_{\varpi^{p_b}}$ be a ϖ^{p_b}–complete $\mathcal{PM}_b\mathcal{MS}$ with $\rho \geq 1$ and $\mathcal{G}, \mathcal{R} : \mathfrak{X}^*_{\varpi^{p_b}} \to \mathfrak{X}^*_{\varpi^{p_b}}$ be self-maps. If the underneath axioms are contended:*

(*i*) *For all $\mathcal{A}, \hbar \in \mathfrak{X}^*_{\varpi^{p_b}}$ and all $\lambda > 0$ with $\varpi_\lambda^{p_b}(\mathcal{G}\mathcal{A}, \mathcal{R}\mathcal{G}\hbar) > 0$ such that*

$$\frac{1}{2\rho} \min\left\{\varpi_\lambda^{p_b}(\mathcal{A}, \mathcal{G}\mathcal{A}), \varpi_\lambda^{p_b}(\mathcal{G}\hbar, \mathcal{R}\mathcal{G}\hbar)\right\} \leq \varpi_\lambda^{p_b}(\mathcal{A}, \mathcal{G}\hbar)$$

implies

$$\daleth\left(\rho^3 \varpi_\lambda^{p_b}(\mathcal{G}\mathcal{A}, \mathcal{R}\mathcal{G}\hbar)\right) \leq \Gamma\left(\chi\left(\varpi_\lambda^{p_b}(\mathcal{A}, \mathcal{G}\hbar)\right) \max\left\{\begin{array}{l} \varpi_\lambda^{p_b}(\mathcal{A}, \mathcal{G}\hbar), \varpi_\lambda^{p_b}(\mathcal{A}, \mathcal{G}\mathcal{A}), \\ \varpi_\lambda^{p_b}(\mathcal{G}\hbar, \mathcal{R}\mathcal{G}\hbar), \\ \frac{\varpi_{2\lambda}^{p_b}(\mathcal{A}, \mathcal{R}\mathcal{G}\hbar) + \varpi_{2\lambda}^{p_b}(\mathcal{G}\hbar, \mathcal{G}\mathcal{A})}{2\rho} \end{array}\right\}\right), \tag{2.1}$$

where $\chi : \bar{P} \to \mathbb{R}^+$ is upper semicontinuous on $\bar{P} := \left\{\varpi_\lambda^{p_b}(\mathcal{A}, \hbar) : \mathcal{A}, \hbar \in \mathfrak{X}^*_{\varpi^{p_b}}\right\}$, $\chi(t) < t$ for each $t \in \bar{P}$, and the functions $\daleth, \Gamma : (0, \infty) \to \mathbb{R}$ are fulfill the following circumstances:

(c_1) *$\daleth$ is lower semicontinuous and nondecreasing;*
(c_2) *$\Gamma(\ell) < \daleth(\ell)$ for all $\ell > 0$;*
(c_3) *$\limsup_{\ell \to \ell_0+} \Gamma(\ell) < \daleth(\ell_0+)$ for any $\ell_0 > 0$.*

(*ii*) *The mapping $\mathcal{G}$ is continuous.*

So, $\mathcal{G}$ and $\mathcal{R}$ own a common $\mathfrak{FP}$ provided that the (Ξ_1) *is met. Furthermore, by* (Ξ_2)*, $\mathcal{G}$ and $\mathcal{R}$ possess a common $\mathfrak{UFP}$.*

Proof. Let $\lambda_0 \in \mathfrak{X}^*_{\varpi^{p_b}}$. Then, there exists $\lambda_1 \in \mathfrak{X}^*_{\varpi^{p_b}}$ such that $\lambda_1 = \mathcal{G}\lambda_0$. Likewise, there exists $\lambda_2 \in \mathfrak{X}^*_{\varpi^{p_b}}$ such that $\lambda_2 = \mathcal{R}\lambda_1$. By proceeding in this line, we constitute a sequence $\left\{\lambda_\varrho\right\}_{\varrho\in\mathbb{N}}$ in $\mathfrak{X}^*_{\varpi^{p_b}}$ featured

$$\lambda_{2\varrho+1} = \mathcal{G}\lambda_{2\varrho} \quad \text{and} \quad \lambda_{2\varrho+2} = \mathcal{R}\lambda_{2\varrho+1}.$$

Presume that $\varpi^{p_b}_\lambda\left(\lambda_\varrho, \lambda_{\varrho+1}\right) = 0$ for some $\varrho \in \mathbb{N}$ and for all $\lambda > 0$. Without loss of generality, if we consider $\varrho = 2k$ for some $k \in \mathbb{N}$, then we achieve $\varpi^{p_b}_\lambda(\lambda_{2k}, \lambda_{2k+1}) = 0$ for all $\lambda > 0$. So, assume that $\varpi^{p_b}_\lambda(\lambda_{2k+1}, \lambda_{2k+2}) > 0$, and we have

$$\frac{1}{2\rho}\min\left\{\varpi^{p_b}_\lambda(\lambda_{2k}, \mathcal{G}\lambda_{2k}), \varpi^{p_b}_\lambda(\mathcal{G}\lambda_{2k}, \mathcal{R}\mathcal{G}\lambda_{2k})\right\} \le \varpi^{p_b}_\lambda(\lambda_{2k}, \mathcal{G}\lambda_{2k}),$$

which implies, by (2.1), that

$$\begin{aligned}\daleth\left(\rho^3\varpi^{p_b}_\lambda(\mathcal{G}\lambda_{2k}, \mathcal{R}\mathcal{G}\lambda_{2k})\right) \le \Gamma\Big(\chi\left(\varpi^{p_b}_\lambda(\lambda_{2k}, \mathcal{G}\lambda_{2k})\right)\max\Big\{&\varpi^{p_b}_\lambda(\lambda_{2k}, \mathcal{G}\lambda_{2k}),\\ &\varpi^{p_b}_\lambda(\lambda_{2k}, \mathcal{G}\lambda_{2k}), \varpi^{p_b}_\lambda(\mathcal{G}\lambda_{2k}, \mathcal{R}\mathcal{G}\lambda_{2k}), \tfrac{\varpi^{p_b}_{2\lambda}(\lambda_{2k}, \mathcal{R}\mathcal{G}\lambda_{2k}) + \varpi^{p_b}_{2\lambda}(\mathcal{G}\lambda_{2k}, \mathcal{G}\lambda_{2k})}{2\rho}\Big\}\Big).\end{aligned}$$

Let $\eta_k = \varpi^{p_b}_\lambda(\lambda_k, \lambda_{k+1})$. Thereby, the above inequality becomes

$$\daleth\left(\rho^3\eta_{2k+1}\right) \le \Gamma\left(\chi(\eta_{2k})\max\left\{\eta_{2k}, \eta_{2k+1}, \frac{\varpi^{p_b}_{2\lambda}(\lambda_{2k}, \lambda_{2k+2}) + \varpi^{p_b}_{2\lambda}(\lambda_{2k+1}, \lambda_{2k+1})}{2\rho}\right\}\right).$$

Utilizing the fact that $\varpi^{p_b}_{2\lambda}(\lambda_{2k}, \lambda_{2k+2}) \le \rho(\eta_{2k} + \eta_{2k+1})$ and since $\eta_{2k} = \varpi^{p_b}_\lambda(\lambda_{2k}, \lambda_{2k+1}) = 0$, we achieve

$$\max\left\{0, \eta_{2k+1}, \frac{\eta_{2k+1}}{2}\right\} = \eta_{2k+1}.$$

Hence, by using (c_2), we conclude that

$$\daleth\left(\rho^3\eta_{2k+1}\right) \le \Gamma(\chi(0)\eta_{2k+1}) < \daleth(\chi(0)\eta_{2k+1}).$$

Considering the property of (c_2), we get

$$\eta_{2k+1} \le \rho^3\eta_{2k+1} < \chi(0)\eta_{2k+1},$$

which causes a contradictory situation because of $\chi(0) < 1$.

Consequently, we procure $\eta_{2k+1} = 0$, i.e., $\lambda_{2k+1} = \lambda_{2k+2}$. Thus, $\lambda_{2k} = \lambda_{2k+1} = \lambda_{2k+2}$ and $\lambda_{2k} = \mathcal{G}\lambda_{2k} = \mathcal{R}\lambda_{2k}$ are met and this results in λ_{2k} being a common $\mathfrak{FP}$ of $\mathcal{G}$ and $\mathcal{R}$. Henceforth, we also assume that $\lambda_\varrho \ne \lambda_{\varrho+1}$. Hence, taking into consideration the above fact, by (2.1) and (c_2), we arrive at

$$\begin{aligned}\daleth\left(\rho^3\eta_{2\varrho+1}\right) &\le \Gamma\left(\chi\left(\eta_{2\varrho}\right)\max\left\{\eta_{2\varrho}, \eta_{2\varrho}, \eta_{2\varrho+1}, \tfrac{\eta_{2\varrho}+\eta_{2\varrho+1}}{2}\right\}\right)\\ &= \Gamma\left(\chi\left(\eta_{2\varrho}\right)\max\left\{\eta_{2\varrho}, \eta_{2\varrho+1}\right\}\right)\\ &< \daleth\left(\chi\left(\eta_{2\varrho}\right)\max\left\{\eta_{2\varrho}, \eta_{2\varrho+1}\right\}\right).\end{aligned}$$

Also, taking the properties of $\daleth$ into account, the above inequality turns into

$$\rho^3 \eta_{2\varrho+1} < \chi\left(\eta_{2\varrho}\right) \max\left\{\eta_{2\varrho}, \eta_{2\varrho+1}\right\}. \tag{2.2}$$

If $\max\left\{\eta_{2\varrho}, \eta_{2\varrho+1}\right\} = \eta_{2\varrho+1}$, then (2.2) becomes

$$\eta_{2\varrho+1} \leq \rho^3 \eta_{2\varrho+1} < \chi\left(\eta_{2\varrho}\right) \eta_{2\varrho+1} < \eta_{2\varrho+1},$$

and this causes a contradiction. Then, $\max\left\{\eta_{2\varrho}, \eta_{2\varrho+1}\right\}$ must be equal to $\eta_{2\varrho}$. Hence, from (2.2), we achieve

$$\daleth\left(\eta_{2\varrho+1}\right) \leq \daleth\left(\rho^3 \eta_{2\varrho+1}\right) \leq \Gamma\left(\chi\left(\eta_{2\varrho}\right) \eta_{2\varrho}\right) < \daleth\left(\chi\left(\eta_{2\varrho}\right) \eta_{2\varrho}\right), \tag{2.3}$$

for all $\varrho \in \mathbb{N}$. Again, by considering the property (c_1), the inequality (2.3) becomes

$$\eta_{2\varrho+1} < \chi\left(\eta_{2\varrho}\right) \eta_{2\varrho} < \eta_{2\varrho}.$$

Similarly, one can conclude that $\eta_{2\varrho} < \eta_{2\varrho-1}$. Thereby, we guarantee that $\left\{\eta_\varrho\right\}_{\varrho\in\mathbb{N}} = \left\{\varpi_\lambda^{p_b}\left(\lambda_\varrho, \lambda_{\varrho+1}\right)\right\}_{\varrho\in\mathbb{N}}$ is a nonincreasing sequence of nonnegative real numbers. Also, a similar consequence can be obtained when k is an odd number. Then, there exists $p \geq 0$ such that $\lim_{\varrho\to\infty} \eta_\varrho = p$. Assume, on the contrary, we aim to demonstrate that $p > 0$. Then, by (2.3), we have

$$\daleth(p) = \lim_{\varrho\to\infty} \daleth\left(\eta_{2\varrho+1}\right) \leq \limsup_{\varrho\to\infty} \Gamma\left(\chi\left(\eta_{2\varrho}\right) \eta_{2\varrho}\right) < \limsup_{\ell\to p} \daleth\left(\chi(\ell)\,\ell\right) \leq \limsup_{\ell\to p} \daleth(\ell),$$

such that this contradicts with the assumption (c_3). Then, we notice that our assumption is false, that is, for all $\lambda > 0$,

$$\lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_\varrho, \lambda_{\varrho+1}\right) = 0. \tag{2.4}$$

By the second condition of Definition 1.3, we derive that

$$\lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_\varrho, \lambda_\varrho\right) \leq \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_\varrho, \lambda_{\varrho+1}\right) = 0.$$

Thus, taking Lemma 1.7 into account, for all $\varrho, m \geq 1$, we obtain

$$\lim_{\varrho\to\infty} \omega_\lambda\left(\lambda_m, \lambda_\varrho\right) = 2 \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_m, \lambda_\varrho\right) - \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_m, \lambda_m\right) - \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_\varrho, \lambda_\varrho\right),$$

such that

$$\lim_{\varrho\to\infty} \omega_\lambda\left(\lambda_m, \lambda_\varrho\right) = 2 \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\lambda_m, \lambda_\varrho\right). \tag{2.5}$$

In the next step, we show that $\left\{\lambda_\varrho\right\}_{\varrho\in\mathbb{N}}$ is a $\varpi^{p_b}-$ Cauchy sequence in $\mathfrak{X}^*_{\varpi^{p_b}}$. For this, it is necessary to prove that $\left\{\lambda_\varrho\right\}_{\varrho\in\mathbb{N}}$ is a $\omega-$ Cauchy sequence in $\mathfrak{X}^*_\omega$ (see Lemma 1.11). Suppose, on the contrary, that $\left\{\lambda_\varrho\right\}_{\varrho\in\mathbb{N}}$ is not a $\omega-$ Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two sequences $\left\{\lambda_{2m_q}\right\}$ and $\left\{\lambda_{2\varrho_q}\right\}$ that can be constructed of positive integers satisfying $m_q > \varrho_q > q$ such that

$$\omega_{4\lambda}\left(\lambda_{2\varrho_q}, \lambda_{2m_q}\right) \geq \varepsilon \tag{2.6}$$

for all $\lambda > 0$, which yields that $\omega_{2\lambda}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q}\right) \geq \varepsilon$. Also, let m_q be the smallest index satisfying the above condition such that

$$\omega_{\lambda}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q-2}\right) < \varepsilon. \tag{2.7}$$

Then, by using (2.4) and (2.6), we get

$$\varepsilon \leq \omega_{4\lambda}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q}\right) \leq \rho\omega_{2\lambda}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2\varrho_q+1}\right) + \rho^2\omega_{\lambda}\left(\lambdabar_{2\varrho_q+1}, \lambdabar_{2m_q+2}\right)$$
$$+\rho^3\omega_{\lambda/2}\left(\lambdabar_{2m_q+2}, \lambdabar_{2m_q+1}\right) + \rho^3\omega_{\lambda/2}\left(\lambdabar_{2m_q+1}, \lambdabar_{2m_q}\right),$$

such that

$$\limsup_{q\to\infty}\omega_{\lambda}\left(\lambdabar_{2\varrho_q+1}, \lambdabar_{2m_q+2}\right) \geq \frac{\varepsilon}{\rho^2}. \tag{2.8}$$

Likewise, we have

$$\omega_{\lambda}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q+1}\right) \leq \rho\omega_{\lambda/2}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q-2}\right) + \rho^2\omega_{\lambda/4}\left(\lambdabar_{2m_q-2}, \lambdabar_{2m_q-1}\right)$$
$$+\rho^3\omega_{\lambda/8}\left(\lambdabar_{2m_q-1}, \lambdabar_{2m_q}\right) + \rho^3\omega_{\lambda/8}\left(\lambdabar_{2m_q}, \lambdabar_{2m_q+1}\right),$$

such that

$$\limsup_{q\to\infty}\omega_{\lambda}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q+1}\right) \leq \rho\varepsilon. \tag{2.9}$$

Similarly, considering the property of triangular inequality, we obtain

$$\omega_{\lambda}\left(\lambdabar_{\varrho_q}, \lambdabar_{m_q+2}\right) \leq \rho\omega_{\frac{\lambda}{2}}\left(\lambdabar_{\varrho_q}, \lambdabar_{m_q+1}\right) + \rho\omega_{\frac{\lambda}{2}}\left(\lambdabar_{m_q+1}, \lambdabar_{m_q+2}\right),$$

and

$$\omega_{\lambda}\left(\lambdabar_{m_q+1}, \lambdabar_{\varrho_q+1}\right) \leq \rho\omega_{\frac{\lambda}{2}}\left(\lambdabar_{m_q+1}, \lambdabar_{m_q}\right) + \rho\omega_{\frac{\lambda}{2}}\left(\lambdabar_{m_q}, \lambdabar_{\varrho_q+1}\right).$$

By means of (2.4) and (2.9), we conclude that

$$\limsup_{q\to\infty}\omega_{\lambda}\left(\lambdabar_{m_q}, \lambdabar_{\varrho_q+2}\right) = \limsup_{q\to\infty}\omega_{\lambda}\left(\lambdabar_{m_q+1}, \lambdabar_{\varrho_q+1}\right) \leq \rho^2\varepsilon. \tag{2.10}$$

On the other hand, by using (2.5), if we apply it to (2.8)–(2.10), we attain the following:

$$\limsup_{q\to\infty}\varpi_{\lambda}^{p_b}\left(\lambdabar_{2\varrho_q+1}, \lambdabar_{2m_q+2}\right) \geq \frac{\varepsilon}{2\rho^2}, \tag{2.11}$$

$$\limsup_{q\to\infty}\varpi_{\lambda}^{p_b}\left(\lambdabar_{2\varrho_q}, \lambdabar_{2m_q+1}\right) \leq \frac{\rho\varepsilon}{2}, \tag{2.12}$$

$$\limsup_{q\to\infty}\varpi_{\lambda}^{p_b}\left(\lambdabar_{m_q}, \lambdabar_{\varrho_q+2}\right) = \limsup_{q\to\infty}\varpi_{\lambda}^{p_b}\left(\lambdabar_{m_q+1}, \lambdabar_{\varrho_q+1}\right) \leq \frac{\rho^2\varepsilon}{2}. \tag{2.13}$$

For a sufficiently large $q \in \mathbb{N}$, if $m_q > \varrho_q > q$, we infer

$$\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}\left(\lambdabar_{2\varrho_q}, \mathcal{G}\lambdabar_{2\varrho_q}\right), \varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambdabar_{2m_q}, \mathcal{R}\mathcal{G}\lambdabar_{2m_q}\right)\right\} \leq \varpi_{\lambda}^{p_b}\left(\lambdabar_{2\varrho_q}, \mathcal{G}\lambdabar_{2m_q}\right). \tag{2.14}$$

Given the fact that $\varrho_q > m_q$ and the sequence $\left\{\varpi_\lambda^{p_b}\left(\lambda_\varrho, \lambda_{\varrho+1}\right)\right\}_{\varrho\geq 1}$ is nondecreasing, we acquire

$$\varpi_\lambda^{p_b}\left(\mathcal{G}\lambda_{2m_q}, \mathcal{R}\mathcal{G}\lambda_{2m_q}\right) = \varpi_\lambda^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2m_q+2}\right) \leq \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q+1}, \lambda_{2\varrho_q+2}\right) \leq \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2\varrho_q+1}\right)$$

$$= \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \mathcal{G}\lambda_{2\varrho_q}\right).$$

Hence,

$$\tfrac{1}{2\rho}\min\left\{\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \mathcal{G}\lambda_{2\varrho_q}\right), \varpi_\lambda^{p_b}\left(\mathcal{G}\lambda_{2m_q}, \mathcal{R}\mathcal{G}\lambda_{2m_q}\right)\right\} = \tfrac{1}{2\rho}\varpi_\lambda^{p_b}\left(\mathcal{G}\lambda_{2m_q}, \mathcal{R}\mathcal{G}\lambda_{2m_q}\right)$$

$$= \tfrac{1}{2\rho}\varpi_\lambda^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2m_q+2}\right).$$

According to (2.4), there exists $q_1 \in \mathbb{N}$ such that for any $q > q_1$,

$$\varpi_\lambda^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2m_q+2}\right) < \frac{\varepsilon}{2\rho}.$$

Also, there exists $q_2 \in \mathbb{N}$ such that for any $q > q_2$,

$$\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2\varrho_q+1}\right) < \frac{\varepsilon}{2\rho}.$$

Therefore, for any $q > \max\{q_1, q_2\}$ and $m_q > \varrho_q > q$, we have

$$\varepsilon \leq \varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q}\right) \leq \rho\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+1}\right) + \rho\varpi_\lambda^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2m_q+2}\right)$$

$$\leq \rho\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+1}\right) + \rho\tfrac{\varepsilon}{2\rho}.$$

So, one concludes that

$$\frac{\varepsilon}{2\rho} \leq \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+1}\right).$$

Thus, we deduce that for any $q > \max\{q_1, q_2\}$ and $\varrho_q > m_q > q$,

$$\varpi_\lambda^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2m_q+2}\right) < \frac{\varepsilon}{2\rho} \leq \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+1}\right),$$

that is, the expression (2.14) is proved. Therefore, from (2.1), we have

$$\urcorner\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{G}\lambda_{2\varrho_q}, \mathcal{R}\mathcal{G}\lambda_{2m_q}\right)\right) \leq \Gamma\begin{pmatrix} \chi\left(\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \mathcal{G}\lambda_{2m_q}\right)\right)\max\left\{\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \mathcal{G}\lambda_{2m_q}\right), \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \mathcal{G}\lambda_{2\varrho_q}\right),\right. \\ \left.\varpi_\lambda^{p_b}\left(\mathcal{G}\lambda_{2m_q}, \mathcal{R}\mathcal{G}\lambda_{2m_q}\right), \frac{\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_q}, \mathcal{R}\mathcal{G}\lambda_{2m_q}\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{G}\lambda_{2m_q}, \mathcal{G}\lambda_{2\varrho_q}\right)}{2\rho}\right\} \end{pmatrix}$$

$$= \Gamma\begin{pmatrix} \chi\left(\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+1}\right)\right)\max\left\{\varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+1}\right), \varpi_\lambda^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2\varrho_q+1}\right),\right. \\ \left.\varpi_\lambda^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2m_q+2}\right), \frac{\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_q}, \lambda_{2m_q+2}\right)+\varpi_{2\lambda}^{p_b}\left(\lambda_{2m_q+1}, \lambda_{2\varrho_q+1}\right)}{2\rho}\right\} \end{pmatrix}. \tag{2.15}$$

Thereupon, if we pass to the limit superior in (2.15), and using (2.11)–(2.13), we conclude that

$$\begin{aligned}
\urcorner\left(\tfrac{\rho\varepsilon}{2}\right) &\leq \limsup_{q\to\infty} \urcorner\left(\rho^3 \varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q+1}, \mathcal{A}_{2m_q+2}\right)\right) \\
&\leq \limsup_{q\to\infty} \Gamma\begin{pmatrix} \chi\left(\varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2m_q+1}\right)\right)\max\left\{\varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2m_q+1}\right), \varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2\varrho_q+1}\right),\right. \\ \left. \varpi_\lambda^{p_b}\left(\mathcal{A}_{2m_q+1}, \mathcal{A}_{2m_q+2}\right), \frac{\varpi_{2\lambda}^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2m_q+2}\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{A}_{2m_q+1}, \mathcal{A}_{2\varrho_q+1}\right)}{2\rho}\right\} \end{pmatrix} \\
&< \urcorner\left(\limsup_{q\to\infty}\begin{pmatrix} \chi\left(\varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2m_q+1}\right)\right)\max\left\{\varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2m_q+1}\right), \varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2\varrho_q+1}\right),\right. \\ \left. \varpi_\lambda^{p_b}\left(\mathcal{A}_{2m_q+1}, \mathcal{A}_{2m_q+2}\right), \frac{\varpi_{2\lambda}^{p_b}\left(\mathcal{A}_{2\varrho_q}, \mathcal{A}_{2m_q+2}\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{A}_{2m_q+1}, \mathcal{A}_{2\varrho_q+1}\right)}{2\rho}\right\} \end{pmatrix}\right) \\
&\leq \urcorner\left(\chi\left(\tfrac{\rho\varepsilon}{2}\right)\max\left\{\rho\varepsilon, 0, 0, \frac{\frac{\rho^2\varepsilon}{2}+\frac{\rho^2\varepsilon}{2}}{2\rho}\right\}\right) \\
&< \urcorner\left(\tfrac{\rho\varepsilon}{2}\right),
\end{aligned}$$

which results in a contradiction. Consequently, it yields that $\left\{\mathcal{A}_\varrho\right\}_{\varrho\in\mathbb{N}}$ is a $\omega-$Cauchy sequence in $\mathfrak{X}_\omega^*$. By Lemma 1.8 (*i*), $\left\{\mathcal{A}_\varrho\right\}_{\varrho\in\mathbb{N}}$ is also a $\varpi^{p_b}-$ Cauchy sequence in $\mathfrak{X}_{\varpi^{p_b}}^*$. Since $\mathfrak{X}_{\varpi^{p_b}}^*$ is a $\varpi^{p_b}-$complete $\mathcal{PM}_b\mathcal{MS}$, by Lemma 1.8 (*ii*), $\mathfrak{X}_\omega^*$ is also a $\omega-$complete modular $b-\mathcal{MS}$. Thus, there exists $\mathcal{A}^* \in \mathfrak{X}_\omega^*$ such that $\mathcal{A}_\varrho \to \mathcal{A}^*$, that is, $\lim_{\rho\to\infty} \omega_\lambda\left(\mathcal{A}_\varrho, \mathcal{A}^*\right) = 0$. By Lemma 1.8 (*iii*), we get

$$\lim_{\rho\to\infty} \varpi_\lambda^{p_b}\left(\mathcal{A}_\varrho, \mathcal{A}^*\right) = \varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{A}^*\right) = \lim_{\rho,m\to\infty} \varpi_\lambda^{p_b}\left(\mathcal{A}_\varrho, \mathcal{A}_m\right), \quad \forall\lambda > 0. \tag{2.16}$$

Because $\lim_{\rho,m\to\infty} \varpi_\lambda^{p_b}\left(\mathcal{A}_\varrho, \mathcal{A}_m\right) = 0$, we get $\varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{A}^*\right) = 0$. Thus, the sequence $\left\{\mathcal{A}_\varrho\right\}_{\varrho\in\mathbb{N}}$ converges to $\mathcal{A}^*$ in $\mathfrak{X}_{\varpi^{p_b}}^*$. If $\mathcal{G}$ is a continuous mapping, then we have

$$\varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right) = \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho}, \mathcal{G}\mathcal{A}_{2\varrho}\right) = 0 = \lim_{\varrho\to\infty} \varpi_\lambda^{p_b}\left(\mathcal{A}_{2\varrho}, \mathcal{A}_{2\varrho+1}\right),$$

which implies that $\mathcal{A}^*$ is the $\mathcal{FP}$ of $\mathcal{G}$. Assuming $\mathcal{A}^* \neq \mathcal{R}\mathcal{A}^*$, i.e., $\varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{R}\mathcal{A}^*\right) > 0$, we obtain, considering (2.1),

$$\frac{1}{2\rho}\min\left\{\varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right), \varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}^*, \mathcal{R}\mathcal{G}\mathcal{A}^*\right)\right\} \leq \varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right),$$

which yields that

$$\urcorner\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}^*, \mathcal{R}\mathcal{G}\mathcal{A}^*\right)\right) \leq \Gamma\left(\chi\left(\varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right)\right)\max\left\{\begin{array}{c} \varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right), \varpi_\lambda^{p_b}\left(\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right), \varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}^*, \mathcal{R}\mathcal{G}\mathcal{A}^*\right) \\ \frac{\varpi_{2\lambda}^{p_b}\left(\mathcal{A}^*, \mathcal{R}\mathcal{G}\mathcal{A}^*\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{G}\mathcal{A}^*, \mathcal{G}\mathcal{A}^*\right)}{2\rho} \end{array}\right\}\right).$$

Thus, the subsequent statement is derived. However, it represents a contradiction.

$$\begin{aligned}\daleth\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{R}\mathcal{A}^*\right)\right) &\leq \Gamma\left(\chi(0)\max\left\{0,0,\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{R}\mathcal{A}^*\right),\frac{\varpi_{2\lambda}^{p_b}(\mathcal{A}^*,\mathcal{R}\mathcal{A}^*)}{2\rho}\right\}\right)\\ &\leq \Gamma\left(\chi(0)\,\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{R}\mathcal{A}^*\right)\right)\\ &< \daleth\left(\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{R}\mathcal{A}^*\right)\right),\end{aligned}$$

that is, $\mathcal{A}^* = \mathcal{R}\mathcal{A}^*$. Hence, $\mathcal{A}^*$ is a common $\mathcal{FP}$ of the mappings $\mathcal{G}$ and $\mathcal{R}$ when the mapping $\mathcal{G}$ is continuous.

In conclusion, let us choose $\mathcal{A}^*$ and $\mathcal{A}_1^*$ to be two distinct common $\mathcal{FP}$s of $\mathcal{G}$ and $\mathcal{R}$. We conclude $\varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}^*,\mathcal{R}\mathcal{G}\mathcal{A}_1^*\right) = \varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right) > 0$ and, also,

$$0 = \frac{1}{2\rho}\min\{\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{G}\mathcal{A}^*\right),\varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}_1^*,\mathcal{R}\mathcal{G}\mathcal{A}_1^*\right)\} \leq \varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{G}\mathcal{A}_1^*\right) = \varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right).$$

Utilizing (2.1), we infer

$$\begin{aligned}\daleth\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}^*,\mathcal{R}\mathcal{G}\mathcal{A}_1^*\right)\right) \leq \Gamma\Big(\chi\left(\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{G}\mathcal{A}_1^*\right)\right)\max\Big\{&\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{G}\mathcal{A}_1^*\right),\\ &\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{G}\mathcal{A}^*\right),\varpi_\lambda^{p_b}\left(\mathcal{G}\mathcal{A}_1^*,\mathcal{R}\mathcal{G}\mathcal{A}_1^*\right),\frac{\varpi_{2\lambda}^{p_b}\left(\mathcal{A}^*,\mathcal{R}\mathcal{G}\mathcal{A}_1^*\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{G}\mathcal{A}_1^*,\mathcal{G}\mathcal{A}^*\right)}{2\rho}\Big\}\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\daleth\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right)\right) &\leq \Gamma\left(\chi\left(\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right)\right)\max\left\{\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right),0,0,\frac{\varpi_{2\lambda}^{p_b}(\mathcal{A}^*,\mathcal{A}_1^*)}{\rho}\right\}\right)\\ &\leq \Gamma\left(\chi\left(\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right)\right)\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right)\right)\\ &< \daleth\left(\varpi_\lambda^{p_b}\left(\mathcal{A}^*,\mathcal{A}_1^*\right)\right),\end{aligned}$$

which is a contradiction, so we have $\mathcal{A}^* = \mathcal{A}_1^*$. This authenticates the uniqueness of the common $\mathcal{FP}$ of $\mathcal{G}$ and $\mathcal{R}$. ◻

Theorem 2.2. *Presume that all the conditions of Theorem 2.1 are held without $\mathcal{G}$ being continuous. Then, the mappings $\mathcal{G}$ and $\mathcal{R}$ own a unique common $\mathcal{FP}$.*

Proof. As in the proof of Theorem 2.1, we say that the sequence $\left\{\mathcal{A}_\varrho\right\}_{\varrho\in\mathbb{N}}$ is a $\varpi^{p_b}-$ Cauchy sequence in $\mathfrak{X}^*_{\varpi^{p_b}}$ and there exists $\mathcal{A}^* \in \mathfrak{X}^*_{\varpi^{p_b}}$ such that $\mathcal{A}_\varrho \to \mathcal{A}^*$. If for infinite values $\varrho \in \mathbb{N}$, $\mathcal{G}\mathcal{A}_{2\varrho} = \mathcal{G}\mathcal{A}^*$, we arrive at

$$\mathcal{A}^* = \lim_{\varrho\to\infty}\mathcal{A}_{2\varrho+1} = \lim_{\varrho\to\infty}\mathcal{G}\mathcal{A}_{2\varrho} = \mathcal{G}\mathcal{A}^*,$$

thereby by proving $\mathcal{A}^*$ to be $\mathcal{FP}$ of $\mathcal{G}$. Since $\mathcal{G}\mathcal{A}_{2\varrho} = \mathcal{G}\mathcal{A}^* = \mathcal{A}^*$, we conclude that $\mathcal{R}\mathcal{G}\mathcal{A}_{2\varrho} = \mathcal{R}\mathcal{A}_{2\varrho+1} = \mathcal{R}\mathcal{A}^*$ and also get

$$\mathcal{A}^* = \lim_{\varrho\to\infty}\mathcal{A}_{2\varrho+2} = \lim_{\varrho\to\infty}\mathcal{R}\mathcal{A}_{2\varrho+1} = \mathcal{R}\mathcal{A}^*.$$

Thus, $\mathcal{R}$ admits an $\mathcal{FP}$ viz. λ^*.

Now, assume that $\lambda_{2\varrho+2} \neq \mathcal{G}\lambda^* \ \forall\ \varrho \in \mathbb{N}$. To prove $\lambda^* = \mathcal{G}\lambda^*$, let one of the subsequent inequalities hold:

$$\frac{1}{2\rho}\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+1}, \lambda_{2\varrho+2}\right) \leq \varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right), \tag{2.17}$$

or

$$\frac{1}{2\rho}\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+2}, \lambda_{2\varrho+3}\right) \leq \varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right). \tag{2.18}$$

Unlike if, for some $\varrho_0 \geq 0$, both of them are not provided, that is,

$$\frac{1}{2\rho}\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+1}, \lambda_{2\varrho+2}\right) \geq \frac{1}{2\rho}\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho+1}, \lambda_{2\varrho+2}\right) > \varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right),$$

or

$$\frac{1}{2\rho}\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+2}, \lambda_{2\varrho+3}\right) \geq \frac{1}{2\rho}\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho+2}, \lambda_{2\varrho+3}\right) > \varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right).$$

Hence, using (2.17) and (2.18), we conclude that

$$\begin{aligned}
\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_0+1}, \lambda_{2\varrho_0+2}\right) &\leq \rho\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho_0+1}, \lambda^*\right) + \rho\varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho_0+2}\right) - \varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda^*\right) \\
&< \tfrac{1}{2}\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_0+1}, \lambda_{2\varrho_0+2}\right) + \tfrac{1}{2}\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_0+2}, \lambda_{2\varrho_0+3}\right) \\
&< \tfrac{1}{2}\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_0+1}, \lambda_{2\varrho_0+2}\right) + \tfrac{1}{2}\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_0+1}, \lambda_{2\varrho_0+2}\right) = \varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho_0+1}, \lambda_{2\varrho_0+2}\right),
\end{aligned}$$

such that a contradictory situation arises, which causes our assertion to be true. Then, we refer to the following two cases.

Case (i): The inequality (2.17) satisfies for infinitely many $\varrho \geq 0$. In this case, for infinitely many $\varrho \geq 0$, we have

$$\begin{aligned}
\tfrac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}\left(\lambda^*, \mathcal{G}\lambda^*\right), \varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambda_{2\varrho}, \mathcal{R}\mathcal{G}\lambda_{2\varrho}\right)\right\} &= \tfrac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}\left(\lambda^*, \mathcal{G}\lambda^*\right), \varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+1}, \lambda_{2\varrho+2}\right)\right\} \\
&\leq \varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right).
\end{aligned}$$

Then, by (2.1), we get

$$\begin{aligned}
\daleth\left(\rho^3\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambda^*, \mathcal{R}\mathcal{G}\lambda_{2\varrho}\right)\right) \leq \Gamma\Big(\chi\left(\varpi_{\lambda}^{p_b}\left(\lambda^*, \mathcal{G}\lambda_{2\varrho}\right)\right)\max\Big\{&\varpi_{\lambda}^{p_b}\left(\lambda^*, \mathcal{G}\lambda_{2\varrho}\right), \\
&\varpi_{\lambda}^{p_b}\left(\lambda^*, \mathcal{G}\lambda^*\right), \varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambda_{2\varrho}, \mathcal{R}\mathcal{G}\lambda_{2\varrho}\right), \tfrac{\varpi_{2\lambda}^{p_b}\left(\lambda^*, \mathcal{R}\mathcal{G}\lambda_{2\varrho}\right) + \varpi_{2\lambda}^{p_b}\left(\mathcal{G}\lambda_{2\varrho}, \mathcal{G}\lambda^*\right)}{2\rho}\Big\}\Big)
\end{aligned}$$

and so, it implies that

$$\begin{aligned}
\daleth\left(\rho^3\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambda^*, \lambda_{2\varrho+2}\right)\right) \leq \Gamma\Big(\chi\left(\varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right)\right)\max\Big\{&\varpi_{\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+1}\right), \\
&\varpi_{\lambda}^{p_b}\left(\lambda^*, \mathcal{G}\lambda^*\right), \varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+1}, \lambda_{2\varrho+2}\right), \tfrac{\varpi_{2\lambda}^{p_b}\left(\lambda^*, \lambda_{2\varrho+2}\right) + \varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho+1}, \mathcal{G}\lambda^*\right)}{2\rho}\Big\}\Big).
\end{aligned} \tag{2.19}$$

Then, considering the upper semicontinuity of χ, we achieve

$$\lim_{\varrho\to\infty}\sup \chi\left(\varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \lambdabar_{2\varrho+1}\right)\right) \leq \chi(0).$$

Hence, taking the upper limit as $\varrho \to \infty$ in (2.19),

$$\begin{aligned}
\daleth\left(\rho^{3}\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambdabar^{*}, \lambdabar^{*}\right)\right) &\leq \Gamma\left(\limsup_{\varrho\to\infty}\left[\begin{array}{l} \chi\left(\varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \lambdabar_{2\varrho+1}\right)\right)\max\left\{\varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \lambdabar_{2\varrho+1}\right), \varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \mathcal{G}\lambdabar^{*}\right),\right. \\ \left.\varpi_{\lambda}^{p_b}\left(\lambdabar_{2\varrho+1}, \lambdabar_{2\varrho+2}\right), \frac{\varpi_{2\lambda}^{p_b}\left(\lambdabar^{*}, \lambdabar_{2\varrho+2}\right)+\varpi_{2\lambda}^{p_b}\left(\lambdabar_{2\varrho+1}, \mathcal{G}\lambdabar^{*}\right)}{2\rho}\right\}\end{array}\right]\right) \\
&\leq \Gamma\left(\chi(0)\,\varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \mathcal{G}\lambdabar^{*}\right)\right) \\
&< \daleth\left(\chi(0)\,\varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \mathcal{G}\lambdabar^{*}\right)\right),
\end{aligned}$$

is obtained. Since the mapping $\daleth$ is nondecreasing, we get

$$\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambdabar^{*}, \lambdabar^{*}\right) \leq \rho^{3}\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambdabar^{*}, \lambdabar^{*}\right) \leq \chi(0)\,\varpi_{\lambda}^{p_b}\left(\lambdabar^{*}, \mathcal{G}\lambdabar^{*}\right),$$

which yields $\lambdabar^{*} = \mathcal{G}\lambdabar^{*}$.

Similarly, taking $\lambdabar_{2\varrho+1} \neq \mathcal{R}\lambdabar^{*}$ $\forall\, \varrho \in \mathbb{N}$, we achieve $\mathcal{R}\lambdabar^{*} = \lambdabar^{*}$.

Case (ii): One can see that (2.17) merely holds for finite values $\varrho \geq 0$. Consequently, $\exists\, \varrho_0 \geq 0$ satisfies (2.18) for any $\varrho \geq \varrho_0$. As proved in Case (i), (2.18) also arrives at a contradiction unless $\lambdabar^{*}$ is a common $\mathcal{FP}$ of $\mathcal{G}$ and $\mathcal{R}$. Thus, $\lambdabar^{*}$ is the common $\mathcal{FP}$ of $\mathcal{G}$ and $\mathcal{R}$ in either of the cases. We can use the same approach as demonstrated in the preceding theorem to achieve uniqueness concisely. □

Now, we present an example illustrating the usability of the main theorem.

Example 2.3. Let $\mathfrak{X}^{*}_{\varpi^{p_b}} = [0, 1]$ and $\varpi^{p_b} : (0, \infty) \times \mathfrak{X} \times \mathfrak{X} \to [0, \infty]$ be defined by

$$\varpi_{\lambda}^{p_b}(\lambdabar, \hbar) = \begin{cases} \frac{[\max\{\lambdabar, \hbar\}]^2}{\lambda + [\max\{\lambdabar, \hbar\}]^2}, & \lambdabar \neq \hbar \\ 0, & \lambdabar = \hbar \end{cases},$$

for all $\lambdabar, \hbar \in \mathfrak{X}$. Then, we conclude that ϖ^{p_b} is a $\mathcal{PM}_b\mathcal{M}$ on $\mathfrak{X}$ with the coefficient $\rho = 2$. Consider the mappings $\mathcal{G}, \mathcal{R} : \mathfrak{X}^{*}_{\varpi^{p_b}} \to \mathfrak{X}^{*}_{\varpi^{p_b}}$ by $\mathcal{G}\lambdabar = \frac{\lambdabar}{8}$ and $\mathcal{R}\lambdabar = 2\lambdabar$ for all $\lambdabar \in \mathfrak{X}^{*}_{\varpi^{p_b}}$. Without loss of the generality, we assume that $\lambdabar > \hbar \geq 0$. Thereupon, it is clear that $\varpi_{\lambda}^{p_b}(\mathcal{G}\lambdabar, \mathcal{R}\mathcal{G}\hbar) = \varpi_{\lambda}^{p_b}\left(\frac{\lambdabar}{8}, \frac{\hbar}{4}\right) > 0$ such that

$$\begin{aligned}
\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}(\lambdabar, \mathcal{G}\lambdabar), \varpi_{\lambda}^{p_b}(\mathcal{G}\hbar, \mathcal{R}\mathcal{G}\hbar)\right\} &= \frac{1}{4}\min\left\{\varpi_{\lambda}^{p_b}\left(\lambdabar, \frac{\lambdabar}{8}\right), \varpi_{\lambda}^{p_b}\left(\frac{\hbar}{8}, \frac{\hbar}{4}\right)\right\} \\
&= \frac{1}{4}\min\left\{\frac{\lambdabar^2}{\lambda+\lambdabar^2}, \frac{\left(\frac{\hbar}{4}\right)^2}{\lambda+\left(\frac{\hbar}{4}\right)^2}\right\} \\
&= \frac{\hbar^2}{64\lambda+4\hbar^2} \leq \varpi_{\lambda}^{p_b}\left(\lambdabar, \frac{\hbar}{8}\right) = \frac{\lambdabar^2}{\lambda+\lambdabar^2}
\end{aligned}$$

implies

$$\daleth\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{G}\lambdabar,\mathcal{R}\mathcal{G}\hbar\right)\right)=\daleth\left(8\varpi_\lambda^{p_b}\left(\frac{\lambdabar}{8},\frac{\hbar}{4}\right)\right)=\daleth\left(8\frac{\left(\frac{\lambdabar}{8}\right)^2}{\lambda+\left(\frac{\lambdabar}{8}\right)^2}\right)=\daleth\left(\frac{8\lambdabar^2}{64\lambda+\lambdabar^2}\right)$$

$$\leq\Gamma\left(\chi\left(\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\hbar\right)\right)\max\left\{\begin{array}{l}\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\hbar\right),\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\lambdabar\right),\\ \varpi_\lambda^{p_b}\left(\mathcal{G}\hbar,\mathcal{R}\mathcal{G}\hbar\right),\\ \frac{\varpi_{2\lambda}^{p_b}\left(\lambdabar,\mathcal{R}\mathcal{G}\hbar\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{G}\hbar,\mathcal{G}\lambdabar\right)}{2\rho}\end{array}\right\}\right)$$

$$=\Gamma\left(\chi\left(\varpi_\lambda^{p_b}\left(\lambdabar,\frac{\hbar}{8}\right)\right)\max\left\{\begin{array}{l}\varpi_\lambda^{p_b}\left(\lambdabar,\frac{\hbar}{8}\right),\varpi_\lambda^{p_b}\left(\lambdabar,\frac{\lambdabar}{8}\right),\varpi_\lambda^{p_b}\left(\frac{\hbar}{8},\frac{\hbar}{4}\right)\\ \frac{\varpi_{2\lambda}^{p_b}\left(\lambdabar,\frac{\hbar}{4}\right)+\varpi_{2\lambda}^{p_b}\left(\frac{\hbar}{8},\frac{\lambdabar}{8}\right)}{4}\end{array}\right\}\right)$$

$$=\Gamma\left(\chi\left(\frac{\lambdabar^2}{\lambda+\lambdabar^2}\right)\max\left\{\begin{array}{l}\frac{\lambdabar^2}{\lambda+\lambdabar^2},\frac{\hbar^2}{16\lambda+\hbar^2}\\ \frac{1}{4}\left(\frac{\lambdabar^2}{2\lambda+\lambdabar^2}+\frac{\lambdabar^2}{128\lambda+\lambdabar^2}\right)\end{array}\right\}\right)$$

$$=\Gamma\left(\chi\left(\frac{\lambdabar^2}{\lambda+\lambdabar^2}\right)\frac{\lambdabar^2}{\lambda+\lambdabar^2}\right)$$

$$<\daleth\left(\chi\left(\frac{\lambdabar^2}{\lambda+\lambdabar^2}\right)\frac{\lambdabar^2}{\lambda+\lambdabar^2}\right).$$

Moreover, by the property (c_1) and considering the features of $\chi:\bar{P}\to\mathbb{R}^+$, we yield that the inequality

$$\frac{8\lambdabar^2}{64\lambda+\lambdabar^2}<\frac{\lambdabar^2}{\lambda+\lambdabar^2}$$

holds for all $\lambdabar\in(0,1]$. Also, even if $\hbar=0$, the result is still valid; that is, all of the conditions of Theorem 2.1 are satisfied.

We achieve the following consequence by taking $\mathcal{G}=\mathcal{R}$ in Theorem 2.1.

Corollary 2.4. *Let $\mathfrak{X}^*_{\varpi^{p_b}}$ be a ϖ^{p_b}–complete $\mathcal{PM}_b\mathcal{MS}$ with $\rho\geq 1$ and $\mathcal{G}:\mathfrak{X}^*_{\varpi^{p_b}}\to\mathfrak{X}^*_{\varpi^{p_b}}$ be a self-mapping. For all $\lambdabar,\hbar\in\mathfrak{X}^*_{\varpi^{p_b}}$ and all $\lambda>0$ with $\varpi_\lambda^{p_b}\left(\mathcal{G}\lambdabar,\mathcal{G}^2\hbar\right)>0$ such that*

$$\frac{1}{2\rho}\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\lambdabar\right)\leq\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\hbar\right)$$

implies

$$\daleth\left(\rho^3\varpi_\lambda^{p_b}\left(\mathcal{G}\lambdabar,\mathcal{G}^2\hbar\right)\right)\leq\Gamma\left(\chi\left(\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\hbar\right)\right)\max\left\{\begin{array}{l}\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\hbar\right),\varpi_\lambda^{p_b}\left(\lambdabar,\mathcal{G}\lambdabar\right),\varpi_\lambda^{p_b}\left(\mathcal{G}\hbar,\mathcal{G}^2\hbar\right),\\ \frac{\varpi_{2\lambda}^{p_b}\left(\lambdabar,\mathcal{G}^2\hbar\right)+\varpi_{2\lambda}^{p_b}\left(\mathcal{G}\hbar,\mathcal{G}\lambdabar\right)}{2\rho}\end{array}\right\}\right),$$

$\chi:\bar{P}\to\mathbb{R}^+$ as upper semicontinuous on $\bar{P}:=\left\{\varpi_\lambda^{p_b}\left(\lambdabar,\hbar\right):\lambdabar,\hbar\in\mathfrak{X}^*_{\varpi^{p_b}}\right\}$, and $\chi(t)<t$ for each $t\in\bar{P}$ and the functions $\daleth,\Gamma:(0,\infty)\to\mathbb{R}$, which hold the features of (c_1)–(c_3). If $\mathcal{G}$ is continuous (not necessary), then $\mathcal{G}$ admits a $\mathfrak{UFP}$ in $\mathfrak{X}^*_{\varpi^{p_b}}$, whenever the assumptions (Ξ_1) and (Ξ_2) are satisfied.

Theorem 2.5. *Consider $\mathfrak{X}^*_{\varpi^{p_b}}$ to be a ϖ^{p_b}–complete $\mathcal{PM}_b\mathcal{MS}$ with $1 \leq \rho$ and $\mathcal{G}, \mathcal{R} : \mathfrak{X}^*_{\varpi^{p_b}} \to \mathfrak{X}^*_{\varpi^{p_b}}$ be two self-maps. If (i)–(iii) are contended:*

(i) *there exist $\alpha \in (0,1)$ and the functions $\daleth, \Gamma : (0,\infty) \to \mathbb{R}$, which have the properties of (c_1)–(c_3) such that*

$$\frac{1}{2\rho}\min\left\{\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\lambda), \varpi_\lambda^{p_b}(\mathcal{G}\hbar, \mathcal{R}\mathcal{G}\hbar)\right\} \leq \varpi_\lambda^{p_b}(\lambda, \mathcal{G}\hbar)$$

implies

$$\daleth\left(\rho^3\varpi_\lambda^{p_b}(\mathcal{G}\lambda, \mathcal{R}\mathcal{G}\hbar)\right) \leq \Gamma\left(\alpha\max\left\{\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\hbar), \frac{\varpi_\lambda^{p_b}(\lambda,\mathcal{G}\lambda)+\varpi_\lambda^{p_b}(\mathcal{G}\hbar,\mathcal{R}\mathcal{G}\hbar)}{2}, \frac{\varpi_{2\lambda}^{p_b}(\lambda,\mathcal{R}\mathcal{G}\hbar)+\varpi_{2\lambda}^{p_b}(\mathcal{G}\hbar,\mathcal{G}\lambda)}{2\rho}\right\}\right)$$

for all $\lambda, \hbar \in \mathfrak{X}^*_{\varpi^{p_b}}$ and all $\lambda > 0$,

(ii) *the mapping $\mathcal{G}$ is continuous,*

(iii) *the conditions (Ξ_1) and (Ξ_2) hold.*

Then, $\mathcal{G}$ and $\mathcal{R}$ admit a unique common $\mathcal{FP}$ in $\mathfrak{X}^*_{\varpi^{p_b}}$.

Proof. The proof can be completed on similar lines as followed in Theorem 2.1. □

Remark 2. Note that we can acquire other consequences by taking $\mathcal{G} = \mathcal{R}$ in Theorem 2.5.

In what follows, we establish a new contraction mapping, which involves a quadratic term in the setting of $\mathcal{PM}_b\mathcal{MS}$.

Theorem 2.6. *Let $\mathfrak{X}^*_{\varpi^{p_b}}$ be a ϖ^{p_b}–complete $\mathcal{PM}_b\mathcal{MS}$ with $\rho \geq 1$ and $\mathcal{G}, \mathcal{R} : \mathfrak{X}^*_{\varpi^{p_b}} \to \mathfrak{X}^*_{\varpi^{p_b}}$ be two self-maps. If the underneath conditions are contented:*

(i) *there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < \frac{1}{\rho}$ such that*

$$\frac{1}{2\rho}\min\left\{\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\lambda), \varpi_\lambda^{p_b}(\hbar, \mathcal{R}\hbar)\right\} \leq \varpi_\lambda^{p_b}(\lambda, \hbar)$$

implies

$$\daleth\left(\rho^7\varpi_\lambda^2(\mathcal{G}\lambda, \mathcal{R}\hbar)\right) \leq \Gamma\left(\begin{array}{c}\alpha\left[\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\lambda)\varpi_\lambda^{p_b}(\hbar, \mathcal{R}\hbar) + \frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda, \mathcal{R}\hbar)\varpi_{2\lambda}^{p_b}(\hbar, \mathcal{G}\lambda)\right] \\ +\beta\left[\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\lambda)\varpi_{2\lambda}^{p_b}(\hbar, \mathcal{G}\lambda) + \frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda, \mathcal{R}\hbar)\varpi_\lambda^{p_b}(\hbar, \mathcal{R}\hbar)\right]\end{array}\right) \tag{2.20}$$

for all $\lambda, \hbar \in \mathfrak{X}^*_{\varpi^{p_b}}$ and all $\lambda > 0$ with $\varpi_\lambda^{p_b}(\mathcal{G}\lambda, \mathcal{R}\hbar) > 0$, where the functions $\daleth, \Gamma : (0,\infty) \to \mathbb{R}$ hold the features of (c_1)–(c_3),

(ii) *$\mathcal{G}$ is a mapping, which need not be continuous,*

(iii) *(Ξ_1) and (Ξ_2) are fulfilled.*

Then, $\mathcal{G}$ and $\mathcal{R}$ admit a unique-common $\mathcal{FP}$ in $\mathfrak{X}^*_{\varpi^{p_b}}$.

Proof. Let $\lambda_0 \in \mathfrak{X}^*_{\varpi^{p_b}}$ be arbitrary, and $\exists\ \lambda_1 \in \mathfrak{X}^*_{\varpi^{p_b}}$ with $\lambda_1 = \mathcal{G}\lambda_0$. Likewise, there exists $\lambda_2 \in \mathfrak{X}^*_{\varpi^{p_b}}$ such that $\lambda_2 = \mathcal{R}\lambda_1$. Continuing in the same manner, we can set up a sequence $\left\{\lambda_\varrho\right\}_{\varrho\in\mathbb{N}}$ in $\mathfrak{X}^*_{\varpi^{p_b}}$ such that

$$\lambda_{2\varrho+1} = \mathcal{G}\lambda_{2\varrho} \quad \text{and} \quad \lambda_{2\varrho+2} = \mathcal{R}\lambda_{2\varrho+1}.$$

Presume that $\varpi_{\lambda}^{p_b}\left(\lambda_{\varrho}, \lambda_{\varrho+1}\right) = 0, \forall\, \lambda > 0$. Now, taking $\varrho = 2i$ for some $i \in \mathbb{N}$ yields into $\varpi_{\lambda}^{p_b}(\lambda_{2i}, \lambda_{2i+1}) = 0$ for all $\lambda > 0$. So, we suppose $\varpi_{\lambda}^{p_b}(\lambda_{2i+1}, \lambda_{2i+2}) > 0$. Due to the fact that

$$\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}(\lambda_{2i}, \mathcal{G}\lambda_{2i}), \varpi_{\lambda}^{p_b}(\lambda_{2i+1}, \mathcal{R}\lambda_{2i+1})\right\} \leq \varpi_{\lambda}^{p_b}(\lambda_{2i}, \lambda_{2i+1}),$$

from (2.20), this implies that

$$\daleth\left(\rho^{7}\varpi_{\lambda}^{p_b}(\mathcal{G}\lambda_{2i}, \mathcal{R}\lambda_{2i+1})^{2}\right) \leq \Gamma\left(\begin{array}{c} \alpha\left[\begin{array}{c} \varpi_{\lambda}^{p_b}(\lambda_{2i}, \mathcal{G}\lambda_{2i})\,\varpi_{\lambda}^{p_b}(\lambda_{2i+1}, \mathcal{R}\lambda_{2i+1}) \\ +\frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda_{2i}, \mathcal{R}\lambda_{2i+1})\,\varpi_{2\lambda}^{p_b}(\lambda_{2i+1}, \mathcal{G}\lambda_{2i}) \end{array}\right] \\ +\beta\left[\begin{array}{c} \varpi_{\lambda}^{p_b}(\lambda_{2i}, \mathcal{G}\lambda_{2i})\,\varpi_{2\lambda}^{p_b}(\lambda_{2i+1}, \mathcal{G}\lambda_{2i}) \\ +\frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda_{2i}, \mathcal{R}\lambda_{2i+1})\,\varpi_{\lambda}^{p_b}(\lambda_{2i+1}, \mathcal{R}\lambda_{2i+1}) \end{array}\right] \end{array}\right).$$

Also, let $\eta_i = \varpi_{\lambda}^{p_b}(\lambda_i, \lambda_{i+1})$. Then, we get

$$\daleth\left(\rho^{7}{\eta_{2i+1}}^{2}\right) \leq \Gamma\left(\alpha\left[\eta_{2i}\eta_{2i+1}\right] + \beta\left[\frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda_{2i}, \lambda_{2i+2})\,\eta_{2i+1}\right]\right).$$

Note that $\varpi_{2\lambda}^{p_b}(\lambda_{2i}, \lambda_{2i+2}) \leq \rho\,(\eta_{2i} + \eta_{2i+1})$ and as $\eta_{2i} = \varpi_{\lambda}^{p_b}(\lambda_{2i}, \lambda_{2i+1}) = 0$, by (c_2), we obtain

$$\daleth\left(\rho^{7}{\eta_{2i+1}}^{2}\right) \leq \Gamma\left(\beta{\eta_{2i+1}}^{2}\right) < \daleth\left(\beta{\eta_{2i+1}}^{2}\right).$$

In view of the property (c_1), we determine $\rho^{7}{\eta_{2i+1}}^{2} < \beta{\eta_{2i+1}}^{2}$, a contradiction. Hence, $\lambda_{2i+1} = \lambda_{2i+2}$, we obtain $\lambda_{2i} = \mathcal{G}\lambda_{2i} = \mathcal{R}\lambda_{2i}$. This ensures λ_{2i} is a common $\mathcal{FP}$ of $\mathcal{G}$ and $\mathcal{R}$. In the rest of the analysis, we suppose that $\lambda_{\varrho} \neq \lambda_{\varrho+1}$. Utilizing (2.20), we derive

$$\daleth\left(\rho^{7}{\eta_{2\varrho+1}}^{2}\right) \leq \Gamma\left(\alpha\left[\eta_{2\varrho}\eta_{2\varrho+1}\right] + \beta\left[\left(\eta_{2\varrho} + \eta_{2\varrho+1}\right)\eta_{2\varrho+1}\right]\right)$$

$$= \Gamma\left(\left[\alpha + \beta\right]\eta_{2\varrho}\eta_{2\varrho+1} + \beta{\eta_{2\varrho+1}}^{2}\right).$$

By using the features of (c_1) and (c_2), we deduce that

$$\rho^{7}{\eta_{2\varrho+1}}^{2} < (\alpha + \beta)\,\eta_{2\varrho}\eta_{2\varrho+1} + \beta{\eta_{2\varrho+1}}^{2},$$

hence

$$\left(\rho^{7} - \beta\right)\eta_{2\varrho+1} < (\alpha + \beta)\,\eta_{2\varrho},$$

for all $\varrho \in \mathbb{N}$. Since $\alpha + \beta < \frac{1}{\rho}$, where $\rho \geq 1$, we obtain $\rho^{7} - \beta > 0$, and so

$$\eta_{2\varrho+1} < \left(\frac{\alpha + \beta}{\rho^{7} - \beta}\right)\eta_{2\varrho} < \eta_{2\varrho}.$$

Therefore, by following the same steps as in the proof of Theorem 2.1, the equality (2.4) is easily achieved.

Next, we will demonstrate that $\left\{\hbar_{\varrho}\right\}_{\varrho\in\mathbb{N}}$ is a $\varpi^{p_b}-$ Cauchy sequence in $\mathfrak{X}^{*}_{\varpi^{p_b}}$. Similarly, if we consider the same steps as in Theorem 2.1, then we obtain (2.11) and (2.13). On the other hand, it is clear that the inequality

$$\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\mathcal{G}\hbar_{2\varrho_q}\right),\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\mathcal{R}\hbar_{2m_q+1}\right)\right\}\leq\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2m_q+1}\right)$$

is fulfilled. Then, from (2.20), we have

$$\begin{aligned}\daleth\left(\rho^{7}\varpi_{\lambda}^{p_b}\left(\mathcal{G}\hbar_{2\varrho_q},\mathcal{R}\hbar_{2m_q+1}\right)^{2}\right)&\leq\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\mathcal{G}\hbar_{2\varrho_q}\right)\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\mathcal{R}\hbar_{2m_q+1}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\hbar_{2\varrho_q},\mathcal{R}\hbar_{2m_q+1}\right)\varpi_{2\lambda}^{p_b}\left(\hbar_{2m_q+1},\mathcal{G}\hbar_{2\varrho_q}\right)\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\mathcal{G}\hbar_{2\varrho_q}\right)\varpi_{2\lambda}^{p_b}\left(\hbar_{2m_q+1},\mathcal{G}\hbar_{2\varrho_q}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\hbar_{2\varrho_q},\mathcal{R}\hbar_{2m_q+1}\right)\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\mathcal{R}\hbar_{2m_q+1}\right)\end{array}\right]\end{array}\right)\\&=\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2\varrho_q+1}\right)\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2m_q+2}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2m_q+2}\right)\varpi_{2\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2\varrho_q+1}\right)\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2\varrho_q+1}\right)\varpi_{2\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2\varrho_q+1}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2m_q+2}\right)\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2m_q+2}\right)\end{array}\right]\end{array}\right).\end{aligned}$$

Hence, if we take the limit superior in the above inequality and consider the expressions (2.11) and (2.13), together with the property of (c_1), we gain

$$\begin{aligned}\daleth\left(\tfrac{\rho^{3}\varepsilon^{2}}{4}\right)=\daleth\left(\rho^{7}\left(\tfrac{\varepsilon}{2\rho^{2}}\right)^{2}\right)&\leq\limsup_{q\to\infty}\daleth\left(\rho^{7}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q+1},\hbar_{2m_q+2}\right)^{2}\right)\\&\leq\limsup_{q\to\infty}\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2\varrho_q+1}\right)\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2m_q+2}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2m_q+2}\right)\varpi_{2\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2\varrho_q+1}\right)\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2\varrho_q+1}\right)\varpi_{2\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2\varrho_q+1}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\hbar_{2\varrho_q},\hbar_{2m_q+2}\right)\varpi_{\lambda}^{p_b}\left(\hbar_{2m_q+1},\hbar_{2m_q+2}\right)\end{array}\right]\end{array}\right)\end{aligned}$$

$$< \daleth\left(\limsup_{q\to\infty}\left[\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{2\varrho_q},\text{\textit{A}}_{2\varrho_q+1}\right)\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{2m_q+1},\text{\textit{A}}_{2m_q+2}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}_{2\varrho_q},\text{\textit{A}}_{2m_q+2}\right)\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}_{2m_q+1},\text{\textit{A}}_{2\varrho_q+1}\right)\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{2\varrho_q},\text{\textit{A}}_{2\varrho_q+1}\right)\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}_{2m_q+1},\text{\textit{A}}_{2\varrho_q+1}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}_{2\varrho_q},\text{\textit{A}}_{2m_q+2}\right)\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{2m_q+1},\text{\textit{A}}_{2m_q+2}\right)\end{array}\right]\end{array}\right]\right)$$

$$\leq \daleth\left(\alpha\left[\frac{1}{\rho}\frac{\rho^2\varepsilon}{2}\frac{\rho^2\varepsilon}{2}\right]\right)=\daleth\left(\alpha\frac{\rho^3\varepsilon^2}{4}\right).$$

Owing to $\alpha+\beta<\frac{1}{\rho}$, the last inequality causes a contradiction, that is, we conclude that the sequence $\left\{\text{\textit{A}}_{\varrho}\right\}_{\varrho\in\mathbb{N}}$ is a $\varpi^{p_b}-$ Cauchy sequence in $\mathfrak{X}^*_{\varpi^{p_b}}$. Also, as in the proof of Theorem 2.1, considering the Lemma 1.8 (*ii-iii*), we acquire that

$$\lim_{\rho\to\infty}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{\varrho},\text{\textit{A}}^*\right)=\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\text{\textit{A}}^*\right)=\lim_{\rho,m\to\infty}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{\varrho},\text{\textit{A}}_{m}\right),\quad\forall\lambda>0$$

and $\left\{\text{\textit{A}}_{\varrho}\right\}_{\varrho\in\mathbb{N}}$ converges to $\text{\textit{A}}^*$ in $\varpi^{p_b}-$complete $\mathcal{PM}_b\mathcal{MS}$ $\mathfrak{X}^*_{\varpi^{p_b}}$.

Now, if $\mathcal{G}$ is continuous, then we have

$$\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right)=\lim_{\varrho\to\infty}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{2\varrho},\mathcal{G}\text{\textit{A}}_{2\varrho}\right)=\lim_{\varrho\to\infty}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_{2\varrho},\text{\textit{A}}_{2\varrho+1}\right)=0,$$

which implies that $\text{\textit{A}}^*$ is a $\mathcal{FP}$ of $\mathcal{G}$. Assume that $\text{\textit{A}}^*\neq\mathcal{R}\text{\textit{A}}^*$, that is, $\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)>0$. Then, because

$$\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right),\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\right\}\leq\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\text{\textit{A}}^*\right),$$

from (2.20), we get

$$\daleth\left(\rho^7\varpi_{\lambda}^{p_{b}}(\mathcal{G}\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*)^2\right)\leq\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right)\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right)\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right)\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\end{array}\right]\end{array}\right).$$

Note that $\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\leq\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)$, and by using (c_2), the above inequality turns into

$$\daleth\left(\rho^7\varpi_{\lambda}^{p_{b}}(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*)^2\right)\leq\Gamma\left(\beta\left[\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*\right)\right]\right)<\daleth\left(\beta\varpi_{\lambda}^{p_{b}}(\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}^*)^2\right),$$

such that this conclusion causes a contradiction due to $\alpha+\beta<\frac{1}{\rho}$, i.e., $\text{\textit{A}}^*=\mathcal{R}\text{\textit{A}}^*$. Finally, for the uniqueness, let $\text{\textit{A}}^*$ and $\text{\textit{A}}_1^*$ be two distinct common $\mathcal{FP}$s of $\mathcal{G}$ and $\mathcal{R}$. Hence, $\varpi_{\lambda}^{p_{b}}\left(\mathcal{G}\text{\textit{A}}^*,\mathcal{R}\text{\textit{A}}_1^*\right)=\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\text{\textit{A}}_1^*\right)>0$ and the expression

$$0=\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\mathcal{G}\text{\textit{A}}^*\right),\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}_1^*,\mathcal{R}\text{\textit{A}}_1^*\right)\right\}\leq\varpi_{\lambda}^{p_{b}}\left(\text{\textit{A}}^*,\text{\textit{A}}_1^*\right)$$

implies from the inequality (2.20):

$$\daleth\left(\rho^{7}\varpi_{\lambda}^{p_b}(\lambda^{*},\lambda_{1}^{*})^{2}\right)=\daleth\left(\rho^{7}\varpi_{\lambda}^{p_b}(\mathcal{G}\lambda^{*},\mathcal{R}\lambda_{1}^{*})^{2}\right)$$

$$\leq\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_b}(\lambda^{*},\mathcal{G}\lambda^{*})\,\varpi_{\lambda}^{p_b}(\lambda_{1}^{*},\mathcal{R}\lambda_{1}^{*})\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda^{*},\mathcal{R}\lambda_{1}^{*})\,\varpi_{2\lambda}^{p_b}(\lambda_{1}^{*},\mathcal{G}\lambda^{*})\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_b}(\lambda^{*},\mathcal{G}\lambda^{*})\,\varpi_{2\lambda}^{p_b}(\lambda_{1}^{*},\mathcal{G}\lambda^{*})\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda^{*},\mathcal{R}\lambda_{1}^{*})\,\varpi_{\lambda}^{p_b}(\lambda_{1}^{*},\mathcal{R}\lambda_{1}^{*})\end{array}\right]\end{array}\right)$$

$$=\Gamma\left(\alpha\left[\tfrac{1}{\rho}\varpi_{2\lambda}^{p_b}(\lambda^{*},\lambda_{1}^{*})^{2}\right]\right)<\daleth\left(\alpha\varpi_{\lambda}^{p_b}(\lambda^{*},\lambda_{1}^{*})^{2}\right).$$

This is a contradiction, that is, $\lambda^{*}=\lambda_{1}^{*}$. Consequently, it is asserted that the common fixed point of the mappings $\mathcal{G}$ and $\mathcal{R}$ possesses uniqueness, concluding the proof. □

Theorem 2.7. *In Theorem 2.6, if we ignore the continuity of $\mathcal{G}$, then, under the same conditions, we get a similar inference.*

Proof. As in the proof of Theorem 2.6, we say that $\left\{\lambda_{\varrho}\right\}_{\varrho\in\mathbb{N}}$ is a ϖ^{p_b}– Cauchy sequence in $\mathfrak{X}^{*}_{\varpi^{p_b}}$ and there exists $\lambda^{*}\in\mathfrak{X}^{*}_{\varpi^{p_b}}$ such that $\lambda_{\varrho}\to\lambda^{*}$. Thus, if $\mathcal{G}\lambda_{2\varrho}=\mathcal{G}\lambda^{*}$ for infinite values of $\varrho\in\mathbb{N}$, then we have

$$\lambda^{*}=\lim_{\varrho\to\infty}\lambda_{2\varrho+1}=\lim_{\varrho\to\infty}\mathcal{G}\lambda_{2\varrho}=\mathcal{G}\lambda^{*}.$$

This proves that λ^{*} is an $\mathcal{FP}$ of $\mathcal{G}$. Since $\mathcal{G}\lambda_{2\varrho}=\mathcal{G}\lambda^{*}=\lambda^{*}$, we conclude that $\mathcal{R}\mathcal{G}\lambda_{2\varrho}=\mathcal{R}\lambda_{2\varrho+1}=\mathcal{R}\lambda^{*}$. Then, we get

$$\lambda^{*}=\lim_{\varrho\to\infty}\lambda_{2\varrho+2}=\lim_{\varrho\to\infty}\mathcal{R}\lambda_{2\varrho+1}=\mathcal{R}\lambda^{*},$$

which means that λ^{*} is an $\mathcal{FP}$ of $\mathcal{R}$. We suppose that $\lambda_{2\varrho+2}\neq\mathcal{G}\lambda^{*}$ for all $\varrho\in\mathbb{N}$. Again, as in Theorem 2.1, we have

$$\frac{1}{2\rho}\min\left\{\varpi_{\lambda}^{p_b}(\lambda^{*},\mathcal{G}\lambda^{*}),\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambda_{2\varrho},\mathcal{R}\mathcal{G}\lambda_{2\varrho}\right)\right\}\leq\varpi_{\lambda}^{p_b}\left(\lambda^{*},\mathcal{G}\lambda_{2\varrho}\right).$$

Hence, by (2.20), we obtain

$$\daleth\left(\rho^{7}\varpi_{\lambda}^{p_b}\left(\mathcal{G}\lambda^{*},\mathcal{R}\lambda_{2\varrho+1}\right)^{2}\right)\leq\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_b}(\lambda^{*},\mathcal{G}\lambda^{*})\,\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+1},\mathcal{R}\lambda_{2\varrho+1}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\lambda^{*},\mathcal{R}\lambda_{2\varrho+1}\right)\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho+1},\mathcal{G}\lambda^{*}\right)\end{array}\right]\\+\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_b}(\lambda^{*},\mathcal{G}\lambda^{*})\,\varpi_{2\lambda}^{p_b}\left(\lambda_{2\varrho+1},\mathcal{G}\lambda^{*}\right)\\+\frac{1}{\rho}\varpi_{2\lambda}^{p_b}\left(\lambda^{*},\mathcal{R}\lambda_{2\varrho+1}\right)\varpi_{\lambda}^{p_b}\left(\lambda_{2\varrho+1},\mathcal{R}\lambda_{2\varrho+1}\right)\end{array}\right]\end{array}\right),$$

and so this implies that

$$\daleth\left(\rho^{7}\varpi_{\lambda}^{p_{b}}\left(\mathcal{G}\lambda^{*},\lambda_{2\varrho+2}\right)^{2}\right)\leq\Gamma\left(\begin{array}{c}\alpha\left[\begin{array}{c}\varpi_{\lambda}^{p_{b}}\left(\lambda^{*},\mathcal{G}\lambda^{*}\right)\varpi_{\lambda}^{p_{b}}\left(\lambda_{2\varrho+1},\lambda_{2\varrho+2}\right)\\ +\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\lambda^{*},\lambda_{2\varrho+2}\right)\varpi_{2\lambda}^{p_{b}}\left(\lambda_{2\varrho+1},\mathcal{G}\lambda^{*}\right)\end{array}\right]\\ +\beta\left[\begin{array}{c}\varpi_{\lambda}^{p_{b}}\left(\lambda^{*},\mathcal{G}\lambda^{*}\right)\varpi_{2\lambda}^{p_{b}}\left(\lambda_{2\varrho+1},\mathcal{G}\lambda^{*}\right)\\ +\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\lambda^{*},\lambda_{2\varrho+2}\right)\varpi_{\lambda}^{p_{b}}\left(\lambda_{2\varrho+1},\lambda_{2\varrho+2}\right)\end{array}\right]\end{array}\right). \tag{2.21}$$

Then, taking the limit as $\varrho\to\infty$ in (2.21) and using (c_2), the following expression is acquired;

$$\begin{aligned}\daleth\left(\rho^{7}\varpi_{\lambda}^{p_{b}}(\mathcal{G}\lambda^{*},\lambda^{*})^{2}\right)&\leq\lim_{\varrho\to\infty}\Gamma\left(\beta\left[\varpi_{\lambda}^{p_{b}}\left(\mathcal{G}\lambda^{*},\lambda^{*}\right)\left(\rho\varpi_{\lambda}^{p_{b}}\left(\lambda_{2\varrho+1},\lambda_{2\varrho+2}\right)+\rho\varpi_{\lambda}^{p_{b}}\left(\lambda_{2\varrho+2},\mathcal{G}\lambda^{*}\right)\right)\right]\right)\\ &<\daleth\left(\lim_{\varrho\to\infty}\left[\beta\left[\varpi_{\lambda}^{p_{b}}\left(\mathcal{G}\lambda^{*},\lambda^{*}\right)\left(\rho\varpi_{\lambda}^{p_{b}}\left(\lambda_{2\varrho+1},\lambda_{2\varrho+2}\right)+\rho\varpi_{\lambda}^{p_{b}}\left(\lambda_{2\varrho+2},\mathcal{G}\lambda^{*}\right)\right)\right]\right]\right)\\ &\leq\daleth\left(\beta\rho\varpi_{\lambda}^{p_{b}}(\mathcal{G}\lambda^{*},\lambda^{*})^{2}\right).\end{aligned}$$

This means that $\mathcal{G}\lambda^{*}=\lambda^{*}$. Similarly, taking $\lambda_{2\varrho+1}\neq\mathcal{R}\lambda^{*}$ for all $\varrho\in\mathbb{N}$, we also attain $\mathcal{R}\lambda^{*}=\lambda^{*}$.
Consequently, λ^{*} is a common $\mathcal{FP}$ of $\mathcal{G}$ and $\mathcal{R}$. □

The following result is procured in the case of $\mathcal{G}=\mathcal{R}$ in Theorem 2.6.

Corollary 2.8. *Let $\mathfrak{X}^{*}_{\varpi^{p_b}}$ be a $\varpi^{p_b}-$complete $\mathcal{PM}_b\mathcal{MS}$ with $\rho\geq1$ and $\mathcal{G}:\mathfrak{X}^{*}_{\varpi^{p_b}}\to\mathfrak{X}^{*}_{\varpi^{p_b}}$ be a self-mapping. All $\lambda,\hbar\in\mathfrak{X}^{*}_{\varpi^{p_b}}$ and all $\lambda>0$ with $\varpi_{\lambda}^{p_b}(\mathcal{G}\lambda,\mathcal{G}\hbar)>0$ such that*

$$\frac{1}{2\rho}\varpi_{\lambda}^{p_{b}}\left(\lambda,\mathcal{G}\lambda\right)\leq\varpi_{\lambda}^{p_{b}}\left(\lambda,\hbar\right) \tag{2.22}$$

implies

$$\daleth\left(\rho^{2}\varpi_{\lambda}^{2}\left(\mathcal{G}\lambda,\mathcal{G}\hbar\right)\right)\leq\Gamma\left(\begin{array}{c}\alpha\left[\varpi_{\lambda}^{p_{b}}\left(\lambda,\mathcal{G}\lambda\right)\varpi_{\lambda}^{p_{b}}\left(\hbar,\mathcal{G}\hbar\right)+\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\lambda,\mathcal{G}\hbar\right)\varpi_{2\lambda}^{p_{b}}\left(\hbar,\mathcal{G}\lambda\right)\right]\\ +\beta\left[\varpi_{\lambda}^{p_{b}}\left(\lambda,\mathcal{G}\lambda\right)\varpi_{2\lambda}^{p_{b}}\left(\hbar,\mathcal{G}\lambda\right)+\frac{1}{\rho}\varpi_{2\lambda}^{p_{b}}\left(\lambda,\mathcal{G}\hbar\right)\varpi_{\lambda}^{p_{b}}\left(\hbar,\mathcal{G}\hbar\right)\right]\end{array}\right), \tag{2.23}$$

where the functions $\daleth,\Gamma:(0,\infty)\to\mathbb{R}$ are held the features of (c_1)–(c_3). If $\mathcal{G}$ is continuous (not necessary), then under the conditions (Ξ_1) and (Ξ_2), $\mathcal{G}$ holds a $\mathcal{UFP}$ in $\mathfrak{X}^{*}_{\varpi^{p_b}}$.

In the ensuing discussion, we aim to present an illustrative example demonstrating that the prerequisites of Corollary 2.8 can be satisfied even in the absence of continuity in $\mathcal{G}$.

Example 2.9. Let $\mathfrak{X}=[0,1]$ and define the $\mathcal{PM}_b\mathcal{M}$ by $\varpi_{\lambda}^{p_b}=\frac{|\lambda-\hbar|^2}{\lambda}$. So, we clearly attain that $\mathfrak{X}^{*}_{\varpi^{p_b}}$ is a $\varpi^{p_b}-$complete $\mathcal{PM}_b\mathcal{MS}$ with $\rho=2$. Also, we introduced a self-mapping $\mathcal{G}:\mathfrak{X}^{*}_{\varpi^{p_b}}\to\mathfrak{X}^{*}_{\varpi^{p_b}}$ as indicated below:

$$\mathcal{G}\lambda=\begin{cases}0, & \text{if }\lambda=1\\ \frac{\lambda}{2}, & \text{otherwise}\end{cases}.$$

Upon careful analysis, it becomes apparent that the mapping denoted as $\mathcal{G}$ lacks continuity at the point $ƛ = 1$, given that $\mathcal{G}(1) = 0$. Conversely, within the interval $ƛ \in [0, 1)$, the mapping $\mathcal{G}$ exhibits continuous behavior, characterized by the relation $\mathcal{G}(ƛ) = \frac{ƛ}{2}$. Furthermore, it is pertinent to note that all prerequisites stipulated in Corollary 2.8 have been satisfactorily fulfilled. In our forthcoming analysis, we shall delve into two distinct scenarios. To facilitate a comprehensive discussion without compromising generality, it is posited under the assumption that $\hbar \geq 2ƛ$.

Case 1: For $ƛ \in \left[0, \frac{1}{2}\right)$ and $\hbar = 1$, the inequality (2.22) becomes

$$\frac{1}{4}\varpi_{\lambda}^{p_b}\left(ƛ, \frac{ƛ}{2}\right) = \frac{ƛ^2}{16\lambda} \leq \varpi_{\lambda}^{p_b}(ƛ, 1) = \frac{|ƛ - 1|^2}{\lambda},$$

which holds for all $ƛ \in \left[0, \frac{1}{2}\right)$. So, from the inequality (2.23), we get

$$\daleth\left(\frac{ƛ^4}{4\lambda^2}\right) = \daleth\left(4\varpi_{\lambda}^{p_b}\left(\frac{ƛ}{2}, 0\right)^2\right) \leq \Gamma\begin{pmatrix} \alpha\left[\varpi_{\lambda}^{p_b}\left(ƛ, \frac{ƛ}{2}\right)\varpi_{\lambda}^{p_b}(1, 0) + \frac{1}{2}\varpi_{2\lambda}^{p_b}(ƛ, 0)\varpi_{2\lambda}^{p_b}\left(1, \frac{ƛ}{2}\right)\right] \\ \beta\left[\varpi_{\lambda}^{p_b}\left(ƛ, \frac{ƛ}{2}\right)\varpi_{2\lambda}^{p_b}\left(1, \frac{ƛ}{2}\right) + \frac{1}{2}\varpi_{2\lambda}^{p_b}(ƛ, 0)\varpi_{\lambda}^{p_b}(1, 0)\right] \end{pmatrix}$$

$$\leq \Gamma\begin{pmatrix} \alpha\left[\frac{ƛ^2}{4\lambda^2} + \frac{ƛ^2(2-ƛ)^2}{8\lambda^2}\right] \\ \beta\left[\frac{ƛ^2(2-ƛ)^2}{16\lambda^2} + \frac{ƛ^2}{2\lambda^2}\right] \end{pmatrix}$$

$$< \daleth\left((\alpha + \beta)\left[\frac{ƛ^2}{2\lambda^2} + \frac{ƛ^2(2-ƛ)^2}{8\lambda^2}\right]\right),$$

which yields that $\frac{ƛ^4}{4\lambda^2} < (\alpha + \beta)\left[\frac{4ƛ^2 + ƛ^2(2-ƛ)^2}{8\lambda^2}\right]$. Thereby, considering the fact that $\alpha + \beta < \frac{1}{2}$, by simple calculations, it is obvious that the inequality (2.23) is fulfilled for all $ƛ \in \left[0, \frac{1}{2}\right)$ with a sufficiently large value of $\alpha + \beta$.

Case 2: Let $ƛ, \hbar \in [0, 1)$. Then, from (2.22), the inequality

$$\frac{1}{4}\varpi_{\lambda}^{p_b}\left(ƛ, \frac{ƛ}{2}\right) = \frac{ƛ^2}{16\lambda} \leq \frac{|ƛ - \hbar|^2}{\lambda} = \varpi_{\lambda}^{p_b}(ƛ, \hbar)$$

is fulfilled because of $\hbar \geq 2ƛ$. So, we have

$$\daleth\left(\frac{|ƛ-\hbar|^4}{4\lambda^2}\right) = \daleth\left(4\varpi_{\lambda}^{p_b}\left(\frac{ƛ}{2}, \frac{\hbar}{2}\right)^2\right) \leq \Gamma\begin{pmatrix} \alpha\left[\varpi_{\lambda}^{p_b}\left(ƛ, \frac{ƛ}{2}\right)\varpi_{\lambda}^{p_b}\left(\hbar, \frac{\hbar}{2}\right) + \frac{1}{2}\varpi_{2\lambda}^{p_b}\left(ƛ, \frac{\hbar}{2}\right)\varpi_{2\lambda}^{p_b}\left(\hbar, \frac{ƛ}{2}\right)\right] \\ \beta\left[\varpi_{\lambda}^{p_b}\left(ƛ, \frac{ƛ}{2}\right)\varpi_{2\lambda}^{p_b}\left(\hbar, \frac{ƛ}{2}\right) + \frac{1}{2}\varpi_{2\lambda}^{p_b}\left(ƛ, \frac{\hbar}{2}\right)\varpi_{\lambda}^{p_b}\left(\hbar, \frac{\hbar}{2}\right)\right] \end{pmatrix}$$

$$\leq \Gamma\begin{pmatrix} \alpha\left[\frac{ƛ^2\hbar^2}{16\lambda^2} + \frac{(2ƛ-\hbar)^2(2\hbar-ƛ)^2}{32\lambda^2}\right] \\ \beta\left[\frac{ƛ^2(2\hbar-ƛ)^2}{16\lambda^2} + \frac{(2ƛ-\hbar)^2\hbar^2}{32\lambda^2}\right] \end{pmatrix}$$

$$< \daleth\left((\alpha + \beta)\left[\frac{ƛ^2\hbar^2}{16\lambda^2} + \frac{ƛ^2(2\hbar-ƛ)^2}{16\lambda^2}\right]\right)$$

$$< \daleth\left((\alpha + \beta)\left[\frac{(2\hbar-ƛ)^4}{8\lambda^2}\right]\right).$$

Hence, considering the properties of $\daleth$, we conclude that the inequality $\frac{|\lambda-\hbar|^4}{4\lambda^2} < (\alpha+\beta)\left[\frac{(2\hbar-\lambda)^4}{8\lambda^2}\right]$ is satisfied for all $\lambda, \hbar \in [0,1)$ with $\hbar \geq 2\lambda$ and for the sufficiently closest value of $\alpha+\beta$ to $\frac{1}{2}$. Consequently, despite the discontinuous to $\lambda = 1$, the mapping $\mathcal{G}$ has a fixed point at $\lambda = 0$.

3. An application to a system of Fredholm integral equations

This section aims to show that our results can be applied to the existence of a common solution in the Fredholm integral equation system. Let us consider the following Fredholm integral equations:

$$\begin{cases} \lambda(t) = \varphi(t) + \int_{\hat{a}}^{\hat{b}} \mathcal{K}_1(t,s,\lambda(s))\,ds \\ \\ \hbar(t) = \varphi(t) + \int_{\hat{a}}^{\hat{b}} \mathcal{K}_2(t,s,\lambda(s))\,ds \end{cases}, \tag{3.1}$$

where $\hat{a}, \hat{b} \in \mathbb{R}$ with $\hat{a} < \hat{b}$, $\varphi : [\hat{a},\hat{b}] \to \mathbb{R}$, and $\lambda \in C([\hat{a},\hat{b}],\mathbb{R})$ and $\mathcal{K}_1, \mathcal{K}_2 : [\hat{a},\hat{b}] \times [\hat{a},\hat{b}] \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Also, let $\mathfrak{X}^*_{\varpi^{p_\mathfrak{b}}} = C([\hat{a},\hat{b}],\mathbb{R})$ and define $\varpi^{p_\mathfrak{b}} : (0,\infty) \times \mathfrak{X} \times \mathfrak{X} \to [0,\infty]$ by

$$\varpi^{p_\mathfrak{b}}_\lambda(\lambda,\hbar) = e^{-\lambda}|\lambda(t) - \hbar(t)|^2 + |\lambda(t)| + |\hbar(t)|,$$

for all $\lambda, \hbar \in \mathfrak{X}^*_{\varpi^{p_\mathfrak{b}}}$ and all $\lambda > 0$. Evidently, $\mathfrak{X}^*_{\varpi^{p_\mathfrak{b}}}$ is a $\varpi^{p_\mathfrak{b}}$–complete $\mathcal{PM}_\mathfrak{b}\mathcal{MS}$ with the constant $\rho = 2$. Furthermore, let $\mathcal{G}, \mathcal{G}^2 = \mathcal{G} \circ \mathcal{G} : \mathfrak{X}^*_{\varpi^{p_\mathfrak{b}}} \to \mathfrak{X}^*_{\varpi^{p_\mathfrak{b}}}$ be defined by

$$\mathcal{G}(\lambda(t)) = \int_{\hat{a}}^{\hat{b}} \mathcal{K}_1(t,s,\lambda(s))\,ds,$$

$$\mathcal{G}^2(\lambda(t)) = \int_{\hat{a}}^{\hat{b}} \mathcal{K}_2(t,s,\lambda(s))\,ds$$

for all $\lambda \in \mathfrak{X}^*_{\varpi^{p_\mathfrak{b}}}$ and $t \in [\hat{a},\hat{b}]$.

Theorem 3.1. *Consider the nonlinear integral equation (3.1). Presume that the following statements are satisfied:*

(*i*) $\mathcal{K}_1, \mathcal{K}_2 : [\hat{a},\hat{b}] \times [\hat{a},\hat{b}] \times \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing in the third order,*

(*ii*) *for each* $t, s \in [\hat{a},\hat{b}]$ *and* $\lambda, \hbar \in \mathfrak{X}^*_\varpi$ *with* $\lambda(r) \leq \hbar(r)$ *for all* $r \in [\hat{a},\hat{b}]$, *we have*

$$|\mathcal{K}_1(t,s,\lambda(s)) - \mathcal{K}_2(t,s,\hbar(s))| \leq \sigma(t,s)\begin{bmatrix} |\lambda(s) - \mathcal{G}\hbar(s)|^2 + e^{\lambda}(|\lambda(s)| + |\mathcal{G}\hbar(s)|) \\ -64e^{\lambda}\left(|\mathcal{G}\lambda(t)| + |\mathcal{G}^2\hbar(t)|\right) \end{bmatrix}^{\frac{1}{2}}, \tag{3.2}$$

where $\sigma : [\hat{a},\hat{b}] \times [\hat{a},\hat{b}] \to [0,\infty)$ *is a continuous function defined by*

$$\sup_{t\in[\hat{a},\hat{b}]}\left(\int_{\hat{a}}^{\hat{b}} \sigma(t,s)^2 ds\right) \leq \frac{1}{64}. \tag{3.3}$$

Then, the system of integral equations (3.1) *has a unique solution.*

Proof. From (3.2) and (3.3), for all $t \in \left[\hat{a}, \hat{b}\right]$, we have

$$\urcorner\left(\rho^3 \varpi_\lambda^{p_b}\left(\mathcal{G}\lambda, \mathcal{G}^2\hbar\right)\right) = 16\varpi_\lambda^{p_b}\left(\mathcal{G}\lambda, \mathcal{G}^2\hbar\right)$$

$$= 16 \sup_{t\in[\hat{a},\hat{b}]}\left[e^{-\lambda}\left|\mathcal{G}\lambda(t) - \mathcal{G}^2\hbar(t)\right|^2 + |\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right]$$

$$= 16 \sup_{t\in[\hat{a},\hat{b}]}\left[e^{-\lambda}\left|\int_{\hat{a}}^{\hat{b}} \mathcal{K}_1(t,s,\lambda(s))\,ds - \int_{\hat{a}}^{\hat{b}} \mathcal{K}_2(t,s,\hbar(s))\,ds\right|^2 + |\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right]$$

$$\leq 16 \sup_{t\in[\hat{a},\hat{b}]}\left[\left(e^{-\lambda}\int_{\hat{a}}^{\hat{b}} |\mathcal{K}_1(t,s,\lambda(s)) - \mathcal{K}_2(t,s,\hbar(s))|\,ds\right)^2 + |\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right]$$

$$\leq 16\left[\left(e^{-\lambda}\sup_{t\in[\hat{a},\hat{b}]}\int_a^b \sigma(t,s)\,ds\left[\begin{array}{c} |\lambda(s) - \mathcal{G}\hbar(s)|^2 + e^{\lambda}(|\lambda(s)| + |\mathcal{G}\hbar(s)|) \\ -64e^{\lambda}\left(|\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right)\end{array}\right]^{\frac{1}{2}}\right)^2 + |\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right]$$

$$\leq 16\left[e^{-\lambda}\sup_{t\in[\hat{a},\hat{b}]}\int_a^b \sigma(t,s)^2 ds\left[\begin{array}{c} |\lambda(s) - \mathcal{G}\hbar(s)|^2 + e^{\lambda}(|\lambda(s)| + |\mathcal{G}\hbar(s)|) \\ -64e^{\lambda}\left(|\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right)\end{array}\right] + |\mathcal{G}\lambda(t)| + \left|\mathcal{G}^2\hbar(t)\right|\right]$$

$$\leq \frac{1}{4}\left[e^{-\lambda}|\lambda(s) - \mathcal{G}\hbar(s)|^2 + |\lambda(s)| + |\mathcal{G}\hbar(s)|\right]$$

$$\leq \Gamma\left(\frac{1}{2}\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\hbar)\right)$$

$$\leq \Gamma\left(\chi\left(\varpi_\lambda^{p_b}(\lambda, \mathcal{G}\hbar)\right)\max\left\{\begin{array}{c} \varpi_\lambda^{p_b}(\lambda, \mathcal{G}\hbar), \varpi_\lambda^{p_b}(\lambda, \mathcal{G}\lambda), \varpi_\lambda^{p_b}\left(\mathcal{G}\hbar, \mathcal{G}^2\hbar\right) \\ \frac{\varpi_{2\lambda}^{p_b}\left(\lambda, \mathcal{G}^2\hbar\right) + \varpi_{2\lambda}^{p_b}(\mathcal{G}\hbar, \mathcal{G}\lambda)}{2\rho}\end{array}\right\}\right),$$

where $\chi : \bar{P} \to [0,1)$ and, also, $\urcorner(\iota) = \rho\iota$ and $\Gamma(\iota) = \frac{1}{\rho}\iota$ for all $\iota > 0$. Thereupon, we conclude that all the conditions of Corollary 2.4 are contended. Then, the system of nonlinear Fredholm integral equations (3.1) has a unique solution. □

4. Conclusions

This paper provides a method for solving a system of Fredholm integral equations that is based on Suzuki-type and Proinov-type contractions. To underscore the significance of the proposed methodology, an illustrative example is meticulously analyzed. Within the ambit of partial modular $b-$metric spaces, this study successfully derives new common fixed-point results through the application of Suzuki-type and Proinov-type contractions.

Furthermore, it presents an intriguing avenue for future research, suggesting the potential applicability of the findings to the domain of multivalued mappings. This prospect opens up fertile ground for exploration, possibly expanding the scope and utility of the current study's methodologies and outcomes.

Author contributions

A. Büyükkaya: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data Curation, Writing-Original Draft Preparation, Writing-Review and Editing, Visualization; M. Younis: Methodology, Formal analysis, Investigation, Writing-Review and Editing, Visualization, Supervision; D. Kesik: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources; M.Öztürk; Data Curation, Writing-Original Draft Preparation, Writing-Review and Editing, Visualization, Supervision, Project Administration, Funding Acquisition. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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