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Research article

Fractal barrier option pricing under sub-mixed fractional Brownian motion with jump processes

Chao Yue and Chuanhe Shen*

School of Economics, Shandong Women's University, Ji'nan 250300, Shandong, China

* **Correspondence:** Email: schuanhe@sina.com.

Abstract: In this work, we mainly focused on the pricing formula for fractal barrier options where the underlying asset followed the sub-mixed fractional Brownian motion with jump, including the down-and-out call option, the down-and-out put option, the down-and-in call option, the down-and-in put option, and so on. To start, the fractal Black-Scholes type partial differential equation was established by using the fractal Itô's formula and a self-financing strategy. Then, by transforming the partial differential equation to the Cauchy problem, we obtained the explicit pricing formulae for fractal barrier options. Finally, the effects of barrier price, fractal dimension, Hurst index, jump intensity, and volatility on the value of fractal barrier options were exhibited through numerical experiments.

Keywords: fractal barrier options; fractal dimension; sub-mixed fractional Brownian motion; jump diffusion
Mathematics Subject Classification: 91G20, 35R60, 35Q91

1. Introduction

Barrier option is a European option contract in which the value depends not only on the price of the underlying asset on the expiration date of the option, but also on whether the underlying asset price reaches a specified level (barrier) during the entire option validity period. It is cheaper than ordinary European options, and therefore attracts more attention from investors in the financial market. Barrier option also contributes to the research of many structured financial products, so barrier option pricing has always been a hot topic [1–4].

Merton [5] proposed a closed solution for European options, which was later extended by Reiner and Rubinstein [6] to pricing formulas for other European barrier options. However, these studies were carried out under the Black-Scholes (B-S) model [7] in which the underlying asset price assumed to obey the logarithmic normal distribution. However, later, a large number of subsequent financial empirical studies [8,9] revealed that financial assets have self-similarity and long-term dependence, which is inconsistent with the B-S model. To deal with this, subsequently following Kolmogorov's theory [10] that assets price is driven by fractional Brownian motion (fBm), many option pricing models with fBm have been extensively studied [11–14]. However, we can apply Wick-self-financing strategies to explore the fBm [15,16], but its application has tiny economic significance, which severely placed restrictions on its applicability in the financial market. As a result, alternative models have been suggested to account for the variation in financial assets, including the subfractional Brownian motion (sub-fBm) [17] and the sub-mixed fractional Brownian motion (sub-mixed fBm) [18].

The sub-fBm is similar to the fBm in most respects, but it differs in that it possesses a non-stationary second-order moment increment and converges more quickly [19]. Additionally, the sub-mixed fBm is a hybrid of the Brownian motion and the sub-fBm. The sub-mixed fBm transforms into a semi-martingale that is equivalent to the Brownian motion when the Hurst index $H \in [0.75, 1)$[20]. Meanwhile, enlightened by Merton [21] and some recent studies [22–25], this article considers jump diffusion processes to describe asset price jump points caused by some unsystematic risk factors, which are often overlooked in the pricing of barrier options.

Nowadays, the fractional calculus has extensive applications in mathematical finance [26,27] and other problems [28–32]. Considering the fractal structure of financial markets, [33] addressed a double-barrier-option pricing problem under the time-fractional B-S framework and presented a robust second-order numerical scheme to solve the discretely monitored double-barrier time-fractional B-S partial differential equation. However, the barrier options studied in this paper did not involve jump processes. The authors [34] investigated the methodology for hedging an up-out put lookback-barrier option with the floating strike price, taking into account the dynamics of the underlying asset as modeled within a framework based on mixed fBm. The conclusion section of this article mentioned that future work would focus on developing a jump-diffusion version of the mixed fBm model, which can accurately describe the leptokurtosis phenomenon and infinite small jump behaviors of asset return distribution. In view of this, we introduce fractal derivatives into barrier options to study its pricing in the sub-mixed fBm with jump environment.

The paper is organized as follows. In Section 2, we introduce the definitions, properties, and formulae of the sub-mixed fBm and fractal derivative. Section 3 presents the fractal Itô's formula of the asset price driven by the sub-mixed fBm with jump, as well as the explicit solution of underlying asset price. In Section 4, we obtain the fractal B-S PDE and the closed-form solutions of barrier options. Section 5 is devoted to discussing the influences of some parameters on barrier options. Section 6 concludes the paper.

2. Preparation knowledge

Definition 2.1. The sub-mixed fBm $\zeta_t^H = \{\zeta_t^H(a,b)\}_{t\geq 0}$ of parameters a, b and H, is a linear combination of the Brownian motion $\{B_t\}_{t\geq 0}$ and the sub-fBm $\{B_t^H\}_{t\geq 0}$, defined on the probability space $\{\Omega, F, P\}$ by

$$\zeta_t^H(a,b) = aB_t + bB_t^H, \forall t \geq 0,$$

where $\{B_t\}_{t\geq 0}$ and $\{B_t^H\}_{t\geq 0}$ are independent of each other.
Some properties of the sub-mixed fBm $\zeta_t^H = \{\zeta_t^H(a,b)\}_{t\geq 0}$ are presented as
(1) ζ_t^H is a central Gaussian process.
(2) $\zeta_0^H(a,b) = aB_0 + bB_0^H = 0, t = 0.$

(3) The covariance of $\zeta_t^H(a,b)$ and $\zeta_s^H(a,b)$ is

$$\mathrm{Cov}(\zeta_t^H(a,b), \zeta_s^H(a,b)) = a^2(t \wedge s) + \frac{b^2}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H}),$$

where $t \wedge s = \frac{1}{2}(t + s - \mid t - s \mid), \forall t, s \geq 0$.

(4) $E((\zeta_t^H(a,b))^2) = a^2 t + b^2(2 - 2^{2H-1})t^{2H}), \forall t \geq 0$.

Definition 2.2. The fractal derivative with respect to t is defined as [35–37]:

$$\frac{\partial u}{\partial t^\alpha}(t_0, x) = \Gamma(1+\alpha) \lim_{\substack{t - t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{u(t,x) - u(t_0,x)}{(t - t_0)^\alpha}, \tag{2.1}$$

where Δt is the smallest timescale, and α is the fractal dimension.

The following rules and formulae are very useful for practical applications:

(1) The chain rules:

$$\frac{\partial}{\partial t^\alpha}\Big(\frac{\partial u}{\partial t^\beta}\Big) = \frac{\partial}{\partial t^\beta}\Big(\frac{\partial u}{\partial t^\alpha}\Big), \tag{2.2}$$

$$\frac{\partial}{\partial t^\alpha}[\phi(u)] = \frac{\partial \phi}{\partial u}\Big(\frac{\partial u}{\partial t^\alpha}\Big). \tag{2.3}$$

(2) The differential and integration formulae:

$$\frac{\partial t^m}{\partial t^\alpha} = \frac{m}{\alpha} t^{m-\alpha}, \tag{2.4}$$

$$\int_{t_0^\alpha}^{t_1^\alpha} t^m dt^\alpha = \frac{\alpha}{m+\alpha}\Big[t_1^{\alpha(m+\alpha)} - t_0^{\alpha(m+\alpha)}\Big]. \tag{2.5}$$

3. Fractal asset pricing model

In this article, we combine classical financial stochastic analysis theory and fractal derivative knowledge to extend the B-S model. In addition, the following assumptions hold:

(1) There are two types of assets in the financial market: Risk-free assets (bonds) and risky assets (stocks).

(2) We suppose that the dynamics of stock price S_t is driven by the fractal sub-mixed fBm with jump:

$$\begin{aligned} d^\alpha S_t &= (\mu - q)S_t dt^\alpha + S_t d\zeta_t^H(\sigma_1, \sigma_2) + \sigma_3 S_t dJ_t \\ &= (\mu - q)S_t dt^\alpha + \sigma_1 S_t dB_t + \sigma_2 S_t dB_t^H + \sigma_3 S_t dJ_t, \end{aligned} \tag{3.1}$$

where μ represents the instantaneous expected return rate of the stock, q represents the stock dividend rate, $\sigma_i (i = 1, 2, 3)$ are the volatility of stock price, $\{J_t\}_{t \geq 0}$ is a compensated Poisson process with intensity λ, and $\{B_t\}_{t\geq 0}$, $\{B_t^H\}_{t \geq 0}$ and $\{J_t\}_{t\geq 0}$ are independent of each other.

(3) The return of risk-free assets in time period t are presented as follows:

$$d^\alpha M_t = r M_t dt^\alpha, \tag{3.2}$$

where constant r presents the risk-free interest rate.

(4) All assets can be freely and continuously traded without the need to pay transaction costs and taxes.

(5) There is no arbitrage opportunity in the financial market.
(6) Short selling is unrestricted.
(7) The option can only be exercised at maturity.

Theorem 3.1. *Suppose the initial value of $\xi_t = \zeta_t^H(\sigma_1, \sigma_2) + \sigma_3 J_t$ is zero, and $f(t, \xi_t)$ is second-order differentiable. Hence, the fractal Itô's formula of the sub-mixed fBm with jump can be given as:*

$$\begin{aligned}
f(t,\xi_t) =& f(0,0) + \int_0^{t^\alpha} \Big(\frac{\partial f}{\partial s} - \lambda\sigma_3 \frac{\partial f}{\partial \xi}\Big) ds^\alpha + \int_0^{t^\alpha} \Big[\frac{\sigma_1^2}{2} + \Big(2 - 2^{2H-1}\Big) H \sigma_2^2 s^{2H-1}\Big] \frac{\partial^2 f}{\partial \xi^2} ds^\alpha \\
&+ \sigma_1 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dB_s + \sigma_2 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dB_s^H + \sum_{s \le t} \Big[f(s, \xi_s) - f(s-, \xi_{s-})\Big] \\
=& f(0,0) + \int_0^{t^\alpha} \Big\{\frac{\partial f}{\partial s} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda \sigma_1^3}{2} + \Big(2 - 2^{2H-1}\Big) H \sigma_2^2 s^{2H-1}\Big] \frac{\partial^2 f}{\partial \xi^2} ds^\alpha \\
&+ \sigma_1 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dB_s + \sigma_2 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dB_s^H + \sigma_3 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dJ_s.
\end{aligned}$$

Proof. Based on the Itô's formula of the sub-mixed fBm [18], the jump process analysis method [38], and fractal derivative knowledge, we obtain

$$\begin{aligned}
f(t,\xi_t) =& f(0,0) + \int_0^{t^\alpha} \frac{\partial f}{\partial s} ds^\alpha + \int_0^{t^\alpha} \frac{\partial f}{\partial S} d\xi_s^c + \frac{1}{2} \int_0^{t^\alpha} \frac{\partial^2 f}{\partial S^2} d(\xi_s^c)^2 + \sum_{s \le t} \Big[f(s,\xi_s) - f(s-, \xi_{s-})\Big] \\
=& f(0,0) + \int_0^{t^\alpha} \Big(\frac{\partial f}{\partial s} - \lambda \sigma_3 \frac{\partial f}{\partial \xi}\Big) ds^\alpha + \int_0^{t^\alpha} \Big[\frac{\sigma_1^2}{2} + \Big(2 - 2^{2H-1}\Big) H \sigma_2^2 s^{2H-1}\Big] \frac{\partial^2 f}{\partial \xi^2} ds^\alpha \\
&+ \sigma_1 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dB_s + \sigma_2 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dB_s^H + \sum_{s \le t} \Big[f(s,\xi_s) - f(s-,\xi_{s-})\Big].
\end{aligned} \tag{3.3}$$

Take advantage of the identities:

$$d\xi_t^c = \sigma_1 dB_t + \sigma_2 dB_t^H - \lambda \sigma_3 dt^\alpha,$$

$$(d\xi_t^c)^2 = \Big[\sigma_1^2 + 2\Big(2 - 2^{2H-1}\Big) H \sigma_2^2 t^{(2H-1)\alpha}\Big] dt^\alpha,$$

where $\xi_t^c = \sigma_1 B_t + \sigma_2 B_t^H - \lambda \sigma_3 t^\alpha$ represents the continuous part of ξ_t.

Provided that $u(x)$ is second-order differentiable and the Poisson process $\{N_t\}_{t \ge 0}$ possesses second-order moment increment $< dN_t, dN_t >= \lambda dt^\alpha$, the generalized fractal Itô's formula gives

$$\sum_{s \le t} \Big[u(N_s) - u(u_{s-})\Big] = \int_0^{t^\alpha} \frac{\partial u}{\partial N} dN_s + \frac{\lambda}{2} \int_0^{t^\alpha} \frac{\partial^2 u}{\partial N^2} ds^\alpha.$$

Coupling $\xi_t = \zeta_t^H(\sigma_1, \sigma_2) + \sigma_3 J_t = \sigma_1 B_t + \sigma_2 B_t^H + \sigma_3 N_t - \lambda \sigma_3 t^\alpha$, we have

$$\sum_{s \le t} \Big[f(s,\xi_s) - f(s-,\xi_{s-})\Big] = \sigma_3 \int_0^{t^\alpha} \frac{\partial f}{\partial \xi} dN_s + \frac{\lambda \sigma_3^2}{2} \int_0^{t^\alpha} \frac{\partial^2 f}{\partial \xi^2} ds^\alpha. \tag{3.4}$$

Inserting (3.4) into (3.3), we arrive at

$$f(t,\xi_t) = f(0,0) + \int_0^{t^\alpha} \Big\{\frac{\partial f}{\partial s} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_1^3}{2} + \Big(2 - 2^{2H-1}\Big)H\sigma_2^2 s^{2H-1}\Big]\frac{\partial^2 f}{\partial \xi^2}\Big\}ds^\alpha + \sigma_1\int_0^{t^\alpha}\frac{\partial f}{\partial \xi}dB_s + \sigma_2\int_0^{t^\alpha}\frac{\partial f}{\partial \xi}dB_s^H + \sigma_3\int_0^{t^\alpha}\frac{\partial f}{\partial \xi}dJ_s.$$

Theorem 3.2. *The explicit solution of the stock price* (3.1) *is given by:*

$$S_t = S_0\texttt{exp}\Big[\Big(\mu - q - \frac{\sigma_1^2}{2} - \frac{\lambda\sigma_1^3}{2}\Big)t^\alpha - \Big(1 - 2^{2H-2}\Big)\sigma_2^2 t^{2H\alpha} + \sigma_1 B_t + \sigma_2 B_t^H + \sigma_3 J_t\Big].$$

Proof. Suppose $f(t,\xi_t) = S_0\texttt{exp}\Big[\Big(\mu - q - \frac{\sigma_1^2}{2} - \frac{\lambda\sigma_1^3}{2}\Big)t^\alpha - \Big(1 - 2^{2H-2}\Big)\sigma_2^2 t^{2H\alpha} + \xi_t\Big]$,
then by use of Theorem 3.1, we obtain

$$\begin{aligned} df(t,\xi_t) =& \Big\{\frac{\partial f}{\partial t^\alpha} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_1^3}{2} + \Big(2 - 2^{2H-1}\Big)H\sigma_2^2 t^{(2H-1)\alpha}\Big]\frac{\partial^2 f}{\partial \xi^2}\Big\}dt^\alpha + \frac{\partial f}{\partial \xi}d\xi_t \\ =& (\mu - q)f(t,\xi_t)dt^\alpha + f(t,\xi_t)d\xi_t \\ =& (\mu - q)f(t,\xi_t)dt^\alpha + f(t,\xi_t)d\zeta_t^H(\sigma_1,\sigma_2) + \sigma_3 f(t,\xi_t)dJ_t, \end{aligned} \tag{3.5}$$

where

$$\frac{\partial f}{\partial t^\alpha} = \Big[\mu - q - \frac{\sigma_1^2}{2} - \frac{\lambda\sigma_1^3}{2} - \Big(2 - 2^{2H-1}\Big)H\sigma_2^2 t^{(2H-1)\alpha}\Big]f(t,\xi_t),$$
$$\frac{\partial f}{\partial \xi} = f(t,\xi_t), \frac{\partial^2 f}{\partial \xi^2} = f(t,\xi_t).$$

Comparing (3.1) with (3.5), we have $dS_t = df(t,\xi_t)$, where $f(0,\xi_0) = S_0$. Thence,

$$S_t = S_0\texttt{exp}\Big[\Big(\mu - q - \frac{\sigma_1^2}{2} - \frac{\lambda\sigma_1^3}{2}\Big)t^\alpha - \Big(1 - 2^{2H-2}\Big)\sigma_2^2 t^{2H\alpha} + \sigma_1 B_t + \sigma_2 B_t^H + \sigma_3 J_t\Big].$$

4. Derivation of pricing formula for fractal barrier options

In this section, we will derive the pricing formula for fractal battier options with the help of the explicit solution of stock price S_t.

Theorem 4.1. *Suppose that the underlying asset price S_t complies with (3.1), then the value of contingent claims $W_t = W(t,S_t)$ is presented as:*

$$\frac{\partial W}{\partial t^\alpha} - (r-q)S_t\frac{\partial W}{\partial S} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2 - 2^{2H-1})H\sigma_2^2 t^{(2H-1)\alpha}\Big]S_t^2\frac{\partial^2 W}{\partial S^2} - rW_t = 0.$$

Proof. Applying self-financing strategy $v_t = (v_t^1, v_t^2)$, we hold many v_t^1 bonds and v_t^2 stocks to construct the wealth process, and its value at time t is given as

$$W_t = v_t^1 M_t + v_t^2 S_t. \tag{4.1}$$

Using (3.1) and (3.2), we have

$$\begin{aligned} dW_t &= v_t^1 dM_t + v_t^2 dS_t + v_t^2 q S_t dt^\alpha \\ &= (rv_t^1 M_t + \mu v_t^2 S_t)dt^\alpha + v_t^2 S_t(\sigma_1 dB_t + \sigma_2 dB_t^H + \sigma_3 dJ_t). \end{aligned} \tag{4.2}$$

Meanwhile, combining Theorems 3.1 and 3.2, we obtain

$$
\begin{aligned}
dW_t &= \frac{\partial W}{\partial t^{\alpha}}dt^{\alpha} + \frac{\partial W}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}(dS_t)^2 \\
&= \Big\{\frac{\partial W}{\partial t^{\alpha}} + (\mu - q)S_t\frac{\partial W}{\partial S} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2 - 2^{2H-1})H\sigma_2^2 t^{(2H-1)\alpha}\Big]S_t^2\frac{\partial^2 W}{\partial S^2}\Big\}dt^{\alpha} \\
&+ S_t\frac{\partial W}{\partial S}(\sigma_1 dB_t + \sigma_2 dB_t^H + \sigma_3 dJ_t),
\end{aligned}
\tag{4.3}
$$

where $(dS_t)^2 = S_t^2[\sigma_1^2 + \lambda\sigma_3^2 + (4 - 4^H)H\sigma_2^2 t^{(2H-1)\alpha}]dt^{\alpha}$.

By using (4.2) and (4.3), v_t^1 and v_t^2 are presented as

$$
\begin{cases}
v_t^1 = (rM_t)^{-1}\Big\{\frac{\partial W}{\partial t^{\alpha}} - qS_t\frac{\partial W}{\partial S} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2 - 2^{2H-1})H\sigma^2 t^{(2H-1)\alpha}\Big]S_t^2\frac{\partial^2 W}{\partial S^2}\Big\}, \\
v_t^2 = \frac{\partial W}{\partial S}.
\end{cases}
\tag{4.4}
$$

In addition, according to formula (4.1), we have

$$
v_t^1 = \frac{W_t - v_t^2 S_t}{M_t},
\tag{4.5}
$$

then combining (4.4) and (4.5) yields the result.

Theorem 4.2. *Consider that the underlying asset price S_t complies with (3.1), then the value of the down-and-out call option $V_{do}(t, S_t)$ at time t, with the fixed strike price K, the fixed barrier R, and the maturity time T, is expressed as follows:*

$$
\begin{aligned}
V_{do}(t, S_t) &= S_t e^{-q(T^{\alpha} - t^{\alpha^2})}N(l_1) - Ke^{-r(T^{\alpha} - t^{\alpha^2})}N(l_2) \\
&- \Big(\frac{S_t}{R}\Big)^{h(t)}\Big[\frac{R^2}{S_t}e^{-q(T^{\alpha} - t^{\alpha^2})}N(l_3) - Ke^{-r(T^{\alpha} - t^{\alpha^2})}N(l_4)\Big],
\end{aligned}
$$

where $N(\cdot)$ stands for the cumulative probability of standard normal distribution, and

$$
l_1 = \frac{ln\frac{S_t}{K} + \Big(r - q + \frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2}\Big)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(1 - 2^{2H-2})(T^{2H\alpha} - t^{2H\alpha^2})}{\sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})}},
$$

$$
l_2 = l_1 - \sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})},
$$

$$
l_3 = \frac{ln\frac{R^2}{KS_t} + \Big(r - q + \frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2}\Big)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(1 - 2^{2H-2})(T^{2H\alpha} - t^{2H\alpha^2})}{\sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})}},
$$

$$
l_4 = l_3 - \sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})},
$$

$$
h(t) = 1 - \frac{2(r - q)(T^{\alpha} - t^{\alpha^2})}{(\sigma_1^2 + \lambda\sigma_3^2)(T^{\alpha} - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})}.
$$

Proof. For convenience, let $W_t(t, S_t) = V_{do}(t, S_t) = V_{do}$, then in terms of Theorem 4.1, the value of the down-and-out call option $V_{do}(t, S_t)$ is expressed as follows

$$\frac{\partial V_{do}}{\partial t^\alpha} + (r-q)S_t\frac{\partial V_{do}}{\partial S} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2-2^{2H-1})H\sigma_2^2 t^{(2H-1)\alpha}\Big]S_t^2\frac{\partial^2 V_{do}}{\partial S^2} - rV_{do} = 0,$$

along with the initial condition $V_{do}(T, S_T) = (S_T - K)^+, \quad R < S_t < +\infty$, as well as the boundary condition $V_{do}(t, R) = 0, \quad 0 \le t \le T$.

Suppose

$$x = ln\frac{S_t}{R}, \quad V_{do}(t, S_t) = R\hat{V}(t, x). \tag{4.6}$$

We have

$$\frac{\partial V_{do}}{\partial t^\alpha} = R\frac{\partial \hat{V}}{\partial t^\alpha}, \quad \frac{\partial V_{do}}{\partial S} = R\frac{\partial \hat{V}}{\partial x}\frac{\partial x}{\partial S} = \frac{R}{S_t}\frac{\partial \hat{V}}{\partial x}, \quad \frac{\partial^2 V_{do}}{\partial S^2} = \frac{R}{S_t^2}\Big(\frac{\partial^2 \hat{V}}{\partial x^2} - \frac{\partial \hat{V}}{\partial x}\Big).$$

Then, we obtain

$$\frac{\partial \hat{V}}{\partial t^\alpha} + (r-q)\frac{\partial \hat{V}}{\partial x} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2-2^{2H-1})H\sigma_2^2 t^{(2H-1)\alpha}\Big]\Big(\frac{\partial^2 \hat{V}}{\partial x^2} - \frac{\partial \hat{V}}{\partial x}\Big) - r\hat{V} = 0,$$

along with the initial condition $\hat{V}\Big(T, ln\frac{S_T}{R}\Big) = \Big(e^x - \frac{K}{R}\Big)^+, \quad 0 < x < +\infty$, as well as the boundary condition $\hat{V}(0, t) = 0, \quad 0 \le t \le T$.

Furthermore, we let

$$\delta(\rho, \iota) = \hat{V}(x, t)e^{k_2(t)}, \rho = k_3(t), \iota = x + k_1(t), \tag{4.7}$$

where $k_i(t)(i = 1, 2, 3)$ are functions to be determined about t. Then we have

$$\frac{\partial \hat{V}}{\partial t^\alpha} = e^{-k_2(t)}\Big[\frac{dk_1(t)}{dt^\alpha}\frac{\partial \delta}{\partial \iota} + \frac{dk_3(t)}{dt^\alpha}\frac{\partial \delta}{\partial \rho} - \frac{dk_2(t)}{dt^\alpha}\delta\Big],$$

$$\frac{\partial \hat{V}}{\partial x} = e^{-k_2(t)}\frac{\partial \delta}{\partial \iota}, \frac{\partial^2 \hat{V}}{\partial x^2} = e^{-k_2(t)}\frac{\partial^2 \delta}{\partial \iota^2}$$

and

$$\frac{dk_3(t)}{dt^\alpha}\frac{\partial \delta}{\partial \rho} + \kappa(t)\frac{\partial^2 \delta}{\partial \iota^2} + [r - q + \frac{dk_1(t)}{dt^\alpha} - \kappa(t)]\frac{\partial \hat{V}}{\partial x} - [r + \frac{dk_1(t)}{dt^\alpha}]\delta = 0, \tag{4.8}$$

where $\kappa(t) = \frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2-2^{2H-1})H\sigma_2^2 t^{(2H-1)\alpha}$.

In order to find the solution, let

$$\begin{cases} \frac{dk_3(t)}{dt^\alpha} + \kappa(t) = 0, \\ r - q + \frac{dk_1(t)}{dt^\alpha} - \kappa(t) = 0, \\ r + \frac{dk_2(t)}{dt^\alpha} = 0, \\ k_1(T) = k_2(T) = k_3(T) = 0, \end{cases} \tag{4.9}$$

to transform (4.8) into the heat equation. According to (4.9), $k_i(t)(i=1,2,3)$ are presented as

$$\begin{cases} k_1(t) = \int_{t^\alpha}^{T} r - q - \kappa(s) ds^\alpha = \Big(r - q - \frac{\sigma_1^2}{2} - \frac{\lambda\sigma_3^2}{2}\Big)(T^\alpha - t^{\alpha^2}) - \sigma_2^2\Big(1 - 2^{2H-2}\Big)\Big(T^{2H\alpha} - t^{2H\alpha^2}\Big), \\ k_2(t) = \int_{t^\alpha}^{T} r ds^\alpha = r(T^\alpha - t^{\alpha^2}), \\ k_3(t) = \int_{t^\alpha}^{T} \kappa(s) ds^\alpha = \Big(\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2}\Big)(T^\alpha - t^{\alpha^2}) + \sigma_2^2\Big(1 - 2^{2H-2}\Big)\Big(T^{2H\alpha} - t^{2H\alpha^2}\Big). \end{cases} \tag{4.10}$$

Inserting (4.10) into (4.8), we obtain the value of the down-and-out call option $V_{do}(t, S_t)$ presented by

$$\frac{\partial \delta}{\partial \rho} = \frac{\partial^2 \delta}{\partial \iota^2}, \tag{4.11}$$

along with the initial condition $\delta(0,\iota) = (e^\iota - K)^+, \quad 0 < \iota < +\infty$, and the boundary condition $\delta(\rho, k_1(t)) = 0, \quad 0 \le t \le T$.

To begin, considering the above equation with initial condition, we obtain the following solution through Poisson formula

$$\delta(\rho,\iota) = \frac{1}{2\sqrt{\pi\rho}} \int_{-\infty}^{+\infty} \varphi(y) e^{-\frac{(\iota-y)^2}{4\rho}} dy. \tag{4.12}$$

Next, we handle the boundary conditions and let $\Phi(y) = \varphi(y)e^{-\frac{[k_1(t)-y]^2}{4\iota}} (y > 0)$, Then $\Phi(y)$ is extended to an odd function in the entire real field

$$\Phi(y) = \begin{cases} \varphi(y) e^{-\frac{[k_1(t)-y]^2}{4\iota}}, & y > 0, \\ -\varphi(-y) e^{\frac{[k_1(t)+y]^2}{4\iota}}, & y \le 0. \end{cases}$$

Consider the above equation and the original initial condition in (4.11), then the extended initial condition, including the boundary condition, can be presented as follows:

$$\varphi(y) = \begin{cases} \Big(e^y - \frac{K}{R}\Big)^+, & y > 0, \\ -\Big(e^{-y} - \frac{K}{R}\Big)^+ e^{-\frac{k_1(t)y}{\iota}}, & y \le 0. \end{cases}$$

Then, (4.6) becomes a Cauchy problem

$$\frac{\partial \delta}{\partial \rho} = \frac{\partial^2 \delta}{\partial \iota^2}, \tag{4.13}$$

along with the initial condition $\delta(0,\iota)=\varphi(\iota), \quad 0<\iota<+\infty$.

In terms of (4.12), we have

$$
\begin{aligned}
\delta(\rho,\iota) =& \frac{1}{2\sqrt{\pi\rho}}\int_{-\infty}^{+\infty}\varphi(y)e^{-\frac{(\iota-y)^2}{4\rho}}dy. \\
=& \frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}\left(e^{y}-\frac{K}{R}\right)e^{-\frac{(\iota-y)^2}{4\rho}}dy - \frac{1}{2\sqrt{\pi\rho}}\int_{-\infty}^{-ln\frac{K}{R}}\left(e^{-y}-\frac{K}{R}\right)e^{-\frac{(\iota-y)^2+4k_1(t)y}{4\rho}}dy \\
=& \frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}e^{y-\frac{(\iota-y)^2}{4\rho}}dy - \frac{1}{2\sqrt{\pi\rho}}\frac{K}{R}\int_{ln\frac{K}{L}}^{+\infty}e^{-\frac{(\iota-y)^2}{4\rho}}dy - \frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}e^{y-\frac{(\iota+y)^2-4k_1(t)y}{4\rho}}dy \\
&+ \frac{1}{2\sqrt{\pi\rho}}\frac{K}{R}\int_{ln\frac{K}{R}}^{+\infty}e^{\frac{(\iota+y)^2-4k_1(t)y}{4\rho}}dy \\
=& A_1+A_2+A_3+A_4.
\end{aligned}
$$

Consider A_1,

$$
A_1=\frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}e^{y-\frac{(\iota-y)^2}{4\rho}}dy = e^{\rho+\iota}\frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}e^{-\frac{(y-\iota-2\rho)^2}{4\rho}}dy.
$$

Now, let $t=\frac{y-\iota-2\rho}{\sqrt{2\rho}}$, then we obtain

$$
A_1=e^{\rho+\iota}\frac{1}{\sqrt{2\pi}}\int_{\frac{ln\frac{K}{R}-\iota-2\rho}{\sqrt{2\rho}}}^{+\infty}e^{-\frac{t^2}{2}}dt = e^{\rho+\iota}N(l_1),
$$

where $N(\cdot)$ stands for the cumulative probability of standard normal distribution, and $l_1=\frac{\iota+2\rho-ln\frac{K}{R}}{\sqrt{2\rho}}$.

Then, in the similar way, denote $t=\frac{y-\iota}{\sqrt{2\rho}}$, and we have

$$
A_2=-\frac{1}{2\sqrt{\pi\rho}}\frac{K}{R}\int_{ln\frac{K}{R}}^{+\infty}e^{-\frac{(\iota-y)^2}{4\rho}}dy = -\frac{K}{R}\frac{1}{\sqrt{2\pi}}\int_{\frac{ln\frac{K}{R}-\iota}{\sqrt{2\rho}}}^{+\infty}e^{-\frac{t^2}{2}}dt = -\frac{K}{R}N(l_2),
$$

where $l_2=\frac{\iota-ln\frac{K}{R}}{\sqrt{2\rho}}=l_1-\sqrt{2\rho}$.

For A_3,

$$
A_3=-\frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}e^{y-\frac{(\iota+y)^2-4k_1(t)y}{4\rho}}dy = -e^{\frac{[\rho+k_1(t)][\rho+k_1(t)-\iota]}{\rho}}\frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty}e^{-\frac{[y+\iota-2k_1(t)-2\rho]^2}{4\rho}}dy.
$$

Making the change of variable $t=\frac{y+\iota-2k_1(t)-2\rho}{\sqrt{2\rho}}$,

$$
A_3=-e^{\frac{[\rho+k_1(t)][\rho+k_1(t)-\iota]}{\rho}}\frac{1}{\sqrt{2\pi}}\int_{\frac{ln\frac{K}{R}+\iota-2k_1(t)-2\rho}{\sqrt{2\rho}}}^{+\infty}e^{-\frac{t^2}{2}}dt = -e^{\frac{[\rho+k_1(t)][\rho+k_1(t)-\iota]}{\rho}}N(l_3),
$$

with $l_3 = \frac{2k_1(t)+2\rho-\iota-ln\frac{K}{R}}{\sqrt{2\rho}}$.

We put $t = \frac{y+\iota-2k_1(t)}{\sqrt{2\rho}}$, then

$$\begin{aligned}A_4 &= \frac{1}{2\sqrt{\pi\rho}}\frac{K}{R}\int_{ln\frac{K}{R}}^{+\infty} e^{-\frac{(\iota+y)^2-4k_1(t)y}{4\rho}}dy = \frac{K}{R}e^{\frac{k_1(t)[k_1(t)-\iota]}{\rho}}\frac{1}{2\sqrt{\pi\rho}}\int_{ln\frac{K}{R}}^{+\infty} e^{\frac{[y+\iota-2k_1(t)]^2}{4\rho}}dy \\ &= e^{\frac{k_1(t)[k_1(t)-\iota]}{\rho}}\frac{1}{\sqrt{2\pi}}\int_{\frac{ln\frac{K}{R}+\iota-2k_1(t)}{\sqrt{2\rho}}}^{+\infty}\frac{K}{R}e^{-\frac{t^2}{2}}dt = \frac{K}{R}e^{\frac{k_1(t)[k_1(t)-\iota]}{\rho}}N(l_4),\end{aligned}$$

where $l_4 = \frac{2k_1(t)-\iota-ln\frac{K}{R}}{\sqrt{2\rho}} = l_3 - \sqrt{2\rho}$.

Insert (4.6) and (4.7) into them, we have

$$\begin{aligned}A_1 &= \frac{S_t}{R}e^{(r-q)(T^\alpha-t^{\alpha^2})N(l_1)},\\ A_2 &= -\frac{K}{R}N(l_2),\\ A_3 &= -e^{\frac{(r-q)(T^\alpha-t^{\alpha^2})\left(\rho-ln\frac{S_t}{R}\right)}{\rho}}N(l_3) = -e^{(r-q)(T^\alpha-t^{\alpha^2})+\left[1-\frac{(r-q)(T^\alpha-t^{\alpha^2})}{\rho}\right]ln\frac{S_t}{R}-ln\frac{S_t}{R}}N(l_3)\\ &= -e^{(r-q)(T^\alpha-t^{\alpha^2})}\Big(\frac{S_t}{R}\Big)^{1-\frac{(r-q)(T^\alpha-t^{\alpha^2})}{\rho}}\frac{R}{S_t}N(l_3),\end{aligned}$$

$$A_4 = \frac{K}{R}e^{\frac{[\rho-(r-q)(T^\alpha-t^{\alpha^2})]ln\frac{S_t}{R}}{\rho}}N(l_4) = \frac{K}{R}\Big(\frac{S_t}{R}\Big)^{1-\frac{(r-q)(T^\alpha-t^{\alpha^2})}{\rho}}N(l_4).$$

By using $A_i (i = 1, 2, 3, 4)$, one has

$$\begin{aligned}V_{do}(t, S_t) &= R\hat{V}(t, x) = Re^{-r(T^\alpha-t^{\alpha^2})}\delta(\rho, \iota) = Re^{-r(T^\alpha-t^{\alpha^2})}(A_1 + A_2 + A_3 + A_4)\\ &= S_te^{-q(T^\alpha-t^{\alpha^2})}N(l_1) - Ke^{-r(T^\alpha-t^{\alpha^2})}N(l_2)\\ &-\Big(\frac{S_t}{R}\Big)^{h(t)}\Big[\frac{R^2}{S_t}e^{-q(T^\alpha-t^{\alpha^2})}N(l_3) - Ke^{-r(T^\alpha-t^{\alpha^2})}N(l_4)\Big],\end{aligned}$$

where $h(t) = 1 - \frac{2(r-q)(T^\alpha-t^{\alpha^2})}{(\sigma_1^2+\lambda\sigma_3^2)(T^\alpha-t^{\alpha^2})+\sigma_2^2(2-2^{2H-1})(T^{2H\alpha}-t^{2H\alpha^2})}$.

Corollary 4.1. *Assuming that the underlying asset price S_t meets (3.1), we have the value of the vanilla call option $V_{vanilla}(t, S_t)$ at time t, along with a fixed strike price K and the maturity time T as follows:*

$$V_{vanilla}(t, S_t) = S_te^{-q(T^\alpha-t^{\alpha^2})}N(l_1) - Ke^{-r(T^\alpha-t^{\alpha^2})}N(l_2),$$

among them, $N(\cdot), l_1$, and l_2 are the same as Theorem 4.2.

Proof. We can prove it using a process similar to Theorem 4.2. Let

$$\bar{x} = ln\frac{S_t}{R}, \quad V_{vanilla}(t, S_t) = R\bar{V}(t, \bar{x}).$$

$$\bar{\delta}(\bar{\rho},\bar{\iota}) = \bar{V}(t,\bar{x})e^{k_2(t)}, \bar{\rho} = k_3(t), \bar{\iota} = \bar{x} + k_1(t),$$

where $k_i(t)(i = 1, 2, 3)$ are shown in (4.10).

Then, we can obtain the value of vanilla call option $V_{vanilla}(t, S_t)$ by analyzing the Cauchy problem below

$$\frac{\partial \bar{\delta}}{\partial \bar{\rho}} = \frac{\partial^2 \bar{\delta}}{\partial \bar{\iota}^2},$$

along with the initial condition $\bar{\delta}(0,\bar{\iota}) = (e^{\bar{\iota}} - K)^+, \quad 0 < \bar{\iota} < +\infty$. Then, we can use a process similar to (4.13) to prove the subsequent parts of this corollary.

Corollary 4.2. *Assuming that the underlying asset price S_t meets (3.1), we obtain the value of the vanilla put option $G_{vanilla}(t, S_t)$ at time t, along with a fixed strike price K and the maturity time T*

$$G_{vanilla}(t, S_t) = Ke^{-r(T^\alpha - t^{\alpha^2})}N(-l_2) - S_t e^{-q(T^\alpha - t^{\alpha^2})}N(-l_1),$$

among them, $N(\cdot), l_1$, and l_2 are presented in Theorem 4.2.

Proof. The remaining proof process is similar to Corollary 4.1 after changing the condition to $(K-S_T)^+$.

Theorem 4.3. *Consider that the underlying asset price S_t complies with (3.1). If the options possess the same fixed strike price K, fixed barrier R, and maturity time T, then at time t, there exists the parity formula between the value of the down-and-out call option $V_{do}(t, S_t)$ and the value of the down-and-out put option $G_{do}(t, S_t)$ as follows:*

$$\begin{aligned} &V_{do}(t, S_t) + Ke^{-r(T^\alpha - t^{\alpha^2})}\Big[N(l_6) - \Big(\frac{S_t}{R}\Big)^{h(t)}N(l_8)\Big] \\ &= G_{do}(t, S_t) + S_t e^{-q(T^\alpha - t^{\alpha^2})}\Big[N(l_5) - \Big(\frac{S_t}{R}\Big)^{h(t)-2}N(l_7)\Big], \end{aligned}$$

where $N(\cdot)$ stands for the cumulative probability of standard normal distribution, and

$$l_5 = \frac{ln\frac{S_t}{R} + \Big(r - q + \frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2}\Big)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(1 - 2^{2H-2})(T^{2H\alpha} - t^{2H\alpha^2})}{\sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})}},$$

$$l_6 = l_5 - \sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})\sigma_2^2(T^{2H\alpha} - t^{2H\alpha^2})},$$

$$l_7 = \frac{ln\frac{R}{S_t} + \Big(r - q + \frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2}\Big)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(1 - 2^{2H-2})\sigma_2^2(T^{2H\alpha} - t^{2H\alpha^2})}{\sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})}},$$

$$l_8 = l_7 - \sqrt{(\sigma_1^2 + \lambda\sigma_3^2)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})\sigma_2^2(T^{2H\alpha} - t^{2H\alpha^2})},$$

$$h(t) = 1 - \frac{2(r - q)(T^\alpha - t^{\alpha^2})}{(\sigma_1^2 + \lambda\sigma_3^2)(T^\alpha - t^{\alpha^2}) + \sigma_2^2(2 - 2^{2H-1})(T^{2H\alpha} - t^{2H\alpha^2})}.$$

Proof. To start, let

$$C_{do}(t,S_t) = V_{do}(t,S_t) - G_{do}(t,S_t), \tag{4.14}$$

which denotes the difference between the value of $V_{do}(t,S_t)$ and $G_{do}(t,S_t)$ at time t and meets

$$\frac{\partial C_{do}}{\partial t^{\alpha}} + (r-q)S_t\frac{\partial C_{do}}{\partial S} + \Big[\frac{\sigma_1^2}{2} + \frac{\lambda\sigma_3^2}{2} + (2-2^{2H-1})H\sigma_2^2 t^{(2H-1)\alpha}\Big]S_t^2\frac{\partial^2 C_{do}}{\partial S^2} - rC_{do} = 0,$$

along with the initial condition $C_{do}(T,S_T) = (S_T - K), \quad R < S_t < +\infty$, as well as the boundary condition $C_{do}(t,R) = 0, \quad 0 \le t \le T$.

Similar to the solving process of (4.11), we can obtain

$$\begin{aligned} C_{do}(t,S_t) =& S_t e^{-q(T^{\alpha}-t^{\alpha^2})}N(l_5) - Ke^{-r(T^{\alpha}-t^{\alpha^2})}N(l_6) \\ &- \Big(\frac{S_t}{R}\Big)^{h(t)}\Big[\frac{R^2}{S_t}e^{-q(T^{\alpha}-t^{\alpha^2})}N(l_7) - Ke^{-r(T^{\alpha}-t^{\alpha^2})}N(l_8)\Big], \end{aligned}$$

then combining the above result and (4.14) yields the Theorem 4.3.

Theorem 4.4. *Consider that the underlying asset price S_t complies with (3.1), then the value of the down-and-out put option $G_{do}(t,S_t)$ at time t, with the fixed strike price K, the fixed barrier R, and the maturity time T, is expressed as follows:*

$$\begin{aligned} G_{do}(t,S_t) =& S_t e^{-q(T^{\alpha}-t^{\alpha^2})}[N(l_1) - N(l_5)] - Ke^{-r(T^{\alpha}-t^{\alpha^2})}[N(l_2) - N(l_6)] \\ &- \Big(\frac{S_t}{R}\Big)^{h(t)}\Big\{\frac{R^2}{S_t}e^{-q(T^{\alpha}-t^{\alpha^2})}[N(l_3) - N(l_7)] - Ke^{-r(T^{\alpha}-t^{\alpha^2})}[N(l_4) - N(l_8)]\Big\}, \end{aligned}$$

where $N(\cdot), l_i(i = 1,2,\ldots,8)$, and $h(t)$ are presented in Theorems 4.2 and 4.3.

Proof. Theorem 4.4 can be easily proved by using Theorems 4.2 and 4.3.

Theorem 4.5. *Consider that the underlying asset price S_t complies with (3.1), then the value of the down-and-in call option $V_{di}(t,S_t)$ and the value of the down-and-in put option $G_{di}(t,S_t)$ at time t, with the fixed strike price K, the fixed barrier R, and the maturity time T, is*

$$V_{di}(t,S_t) = \Big(\frac{S_t}{R}\Big)^{h(t)}\Big[\frac{R^2}{S_t}e^{-q(T^{\alpha}-t^{\alpha^2})}N(l_3) - Ke^{-r(T^{\alpha}-t^{\alpha^2})}N(l_4)\Big],$$

$$\begin{aligned} G_{di}(t,S_t) =& Ke^{-r(T^{\alpha}-t^{\alpha^2})}N(-l_6) - S_t e^{-q(T^{\alpha}-t^{\alpha^2})}N(-l_5)] \\ &+ \Big(\frac{S_t}{R}\Big)^{h(t)}\Big\{\frac{R^2}{S_t}e^{-q(T^{\alpha}-t^{\alpha^2})}[N(l_3) - N(l_7)] - Ke^{-r(T^{\alpha}-t^{\alpha^2})}[N(l_4) - N(l_8)]\Big\}, \end{aligned}$$

where $N(\cdot), l_i(i = 3,4,\ldots,8)$ and $h(t)$ are shown in Theorems 4.2 and 4.3.

Proof. Investment portfolio with both out option and corresponding in option tend to always perform one of their option rights when other conditions are the same. In this case, it is equivalent to a vanilla option

$$W_{vanilla}(t,S_t) = W_{do}(t,S_t) + W_{di}(t,S_t) = W_{uo}(t,S_t) + W_{ui}(t,S_t),$$

where $W_{vanilla}(t, S_t)$ means the European option, and $W_{do}(t, S_t), W_{di}(t, S_t), W_{uo}(t, S_t)$, and $W_{ui}(t, S_t)$ stand for the value of the down-and-out option, the down-and-in option, the up-and-out option, and the up-and-in option. Then, we have

$$V_{di}(t, S_t) = V_{vanilla}(t, S_t) - V_{do}(t, S_t), \quad G_{di}(t, S_t) = G_{vanilla}(t, S_t) - G_{do}(t, S_t).$$

Combining Corollaries 4.1 and 4.2 and Theorems 4.2 and 4.4, Theorem 4.5 is proved.

So far, we have obtained the pricing formulas for all four fractal downward barrier options. Of course, using a similar process, pricing formulas corresponding to the four fractal upward barrier options can also be derived. Obviously, the aforementioned are closed-form solutions of barrier options. Due to the difficulty in obtaining general analytical expressions for barrier options under the jump-diffusion framework, a significant amount of work has focused on numerical or the Monte Carlo simulation algorithm. For example, S. A. Metwally and A. F. Atiya [39] put forward a fast and unbiased Monte Carlo approach for pricing barrier options when the underlying security adheres to a simple jump-diffusion process with constant parameters and a continuously monitored barrier. Two algorithms were founded on the Brownian bridge concept. Both methods remarkably reduced bias and accelerated convergence compared to the standard Monte Carlo simulation approach. Based on this comparative analysis approach, we will discuss the impact of different parameter values on barrier options under sub-mixed fBm in three different cases in the next section.

5. Numerical experiment

In this section, we take the down-and-out call option as an example to discuss the impacts of the fractal dimension α, the barrier price R, the Hurst index H, the jump intensity λ, and volatility $\sigma_1, \sigma_2, \sigma_3$ on barrier options.

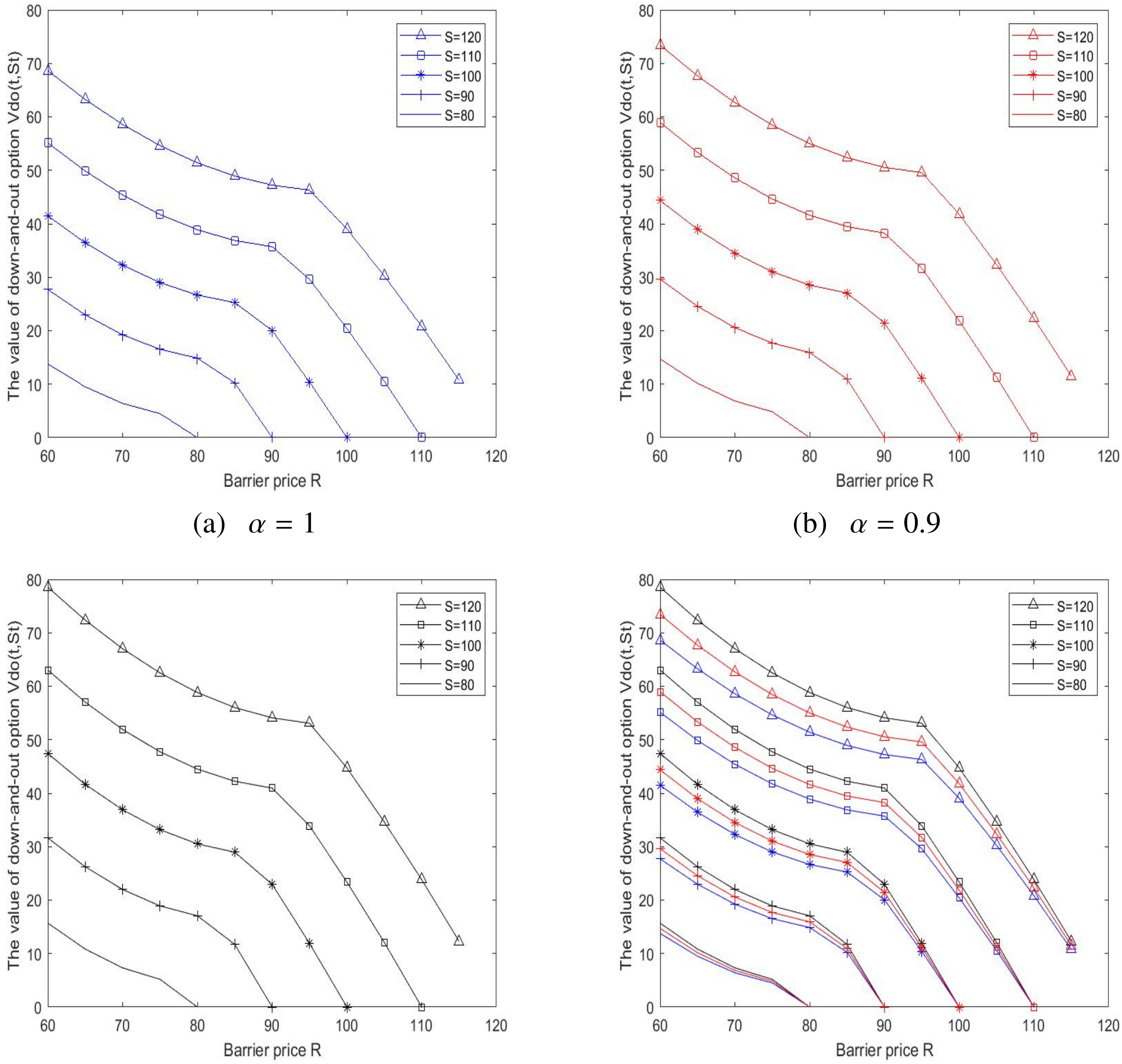
Case 1. Numerical analysis of barrier prices under different fractal dimensions

According to Theorem 4.2, assume that the parameter selection is as follows:

$$t = 0, T = 0.5, K = 100, H = 0.95, \sigma_1 = \sigma_2 = \sigma_3 = 0.4, \lambda = 1.$$

Then the trend of option value $V_{do}(t, S_t)$ affected by different barrier prices $R = 60, 65, \ldots, 115$, and different stock prices $S = 80, 90, \ldots, 120$, with different fractal dimensions $\alpha = 1, 0.9, 0.8$ is given in Figure 1(a)–(d), respectively.

From Figure 1, it can be seen that when the stock price is fixed, the relationships between the value of down-and-out call option and the barrier price is always negative as fractal dimension α changes. Under other unchanged conditions, as the barrier price rises, the possibility of down-and-out call option termination due to the option hitting the barrier price during its validity period will increase, and therefore the value of the option will continue to decline. Especially when the barrier price rises to the initial stock price, the option will be knocked out immediately, which means it no longer has value. On the other hand, for each fixed stock price and barrier price, the value of down-and-out call option increases with the decrease of fractal dimension, and the larger the stock price, the greater the difference in option value corresponding to the same barrier price.



(a) $\alpha = 1$ (b) $\alpha = 0.9$

(c) $\alpha = 0.8$ (d) $\alpha = 1, 0.9, 0.8$

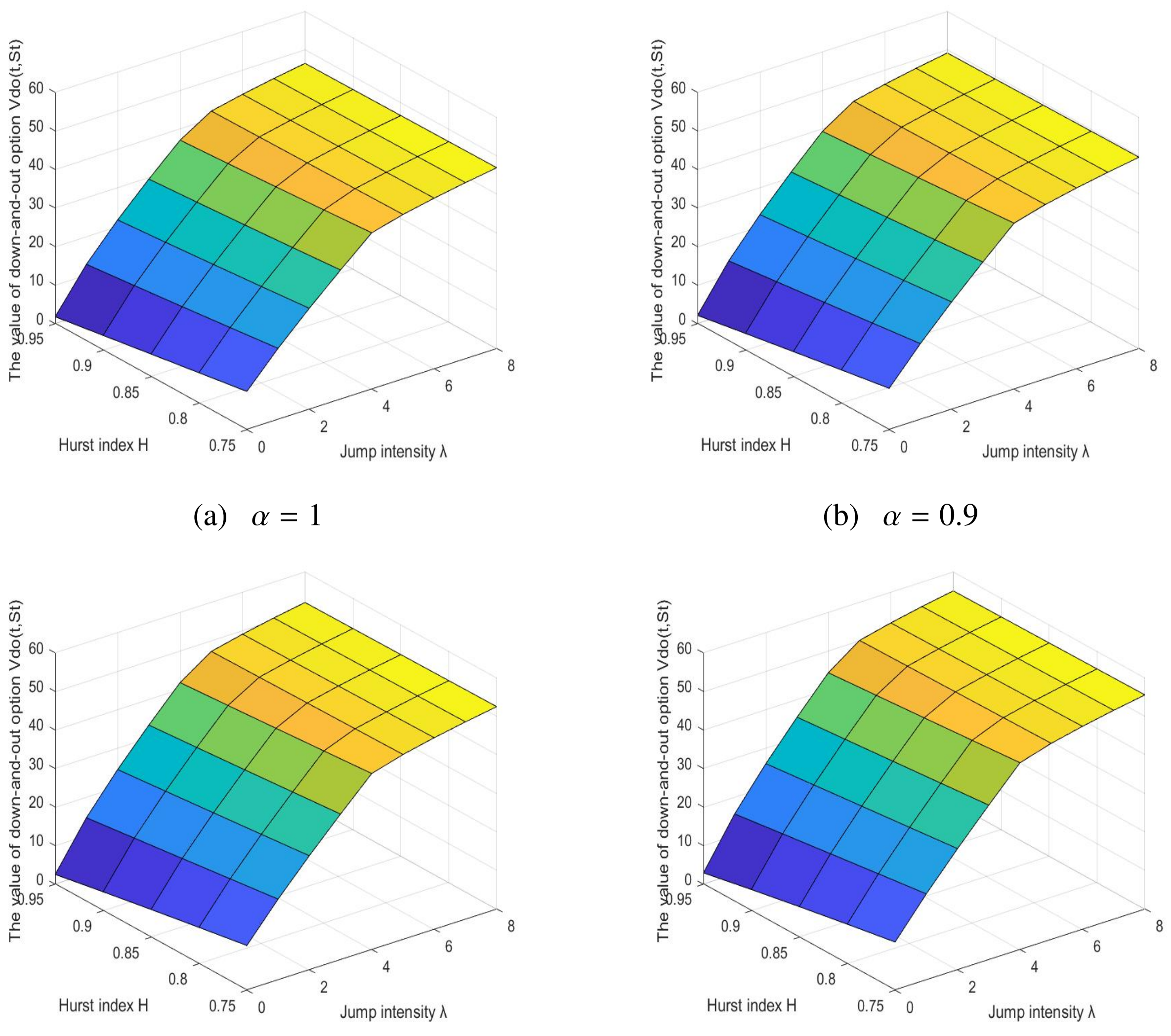
Figure 1. The value of down-and-out options for different barrier prices, stock prices, and fractal dimensions.

Case 2. Numerical comparisons for different Hurst index and jump intensity values under different fractal dimensions.

In order to analyze the impact of the fractal dimension α, the Hurst index H, and the jump intensity λ on the option price, some parameters are chosen as follows:

$$t = 0, T = 0.5, S_0 = 85, K = 100, R = 70, \sigma_1 = \sigma_2 = \sigma_3 = 0.4.$$

Figure 2(a) shows the variation of the value of down-and-out call option with the different Hurst index and jump intensity when $\alpha = 1$. As the Hurst index rises, the value of the down-and-out call option continues to decline. This change is mainly due to the fact that a larger Hurst index represents a smoother and more stable price of the underlying asset, which means that its price fluctuation will also be smaller, ultimately resulting in a smaller corresponding option value. In addition, it can be seen that the value of options and the jumps intensity vary in the same direction. The jump intensity reflects the unsystematic risk. As it increases, the underlying asset will experience more drastic fluctuations, which implies higher upper limit and a constant lower limit. Hence, the value of options will increase. Figure 2(b)–(d) depicts the trend of the value of down-and-out call option affected by different fractal dimensions $\alpha = 0.9, 0.8$, and 0.7, respectively. From Figure 2(a)–(d), it can be seen that under the same other conditions, the value of down-and-out call option gradually increases as the fractal dimension α decreases, which indicates a negative correlation between them.



(a) $\alpha = 1$ (b) $\alpha = 0.9$

(c) $\alpha = 0.8$ (d) $\alpha = 0.7$

Figure 2. The value of down-and-out options for different Hurst index, jump intensity values, and fractal dimensions.

Case 3. Numerical results of different volatilities and fractal dimensions.

The parameter values are given as

$$t = 0, T = 0.5, R = 70, K = 100, H = 0.95, \lambda = 2, \alpha = 1.$$

Set $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, let $\bar{\sigma}_1 = (0.1, 0.15, 0.2), \bar{\sigma}_2 = (0.2, 0.25, 0.3), \bar{\sigma}_3 = (0.3, 0.35, 0.4), \bar{\sigma}_4 = (0.4, 0.45, 0.5)$. In terms of Theorem 4.2, we present the results for a comparison of the value of down-and-out call option under different volatility across different S_0 between 75 and 120 in Table 1. It can be clearly seen that the value of the down-and-out call option increases with the growth of the volatility, which is consistent with the fact.

Table 1. The value of down-and-out option against the volatility of the underlying asset.

S	$\bar{\sigma}_1 = (0.1, 0.15, 0.2)$	$\bar{\sigma}_2 = (0.2, 0.25, 0.3)$	$\bar{\sigma}_3 = (0.3, 0.35, 0.4)$	$\bar{\sigma}_4 = (0.4, 0.45, 0.5)$
75	0.9617	2.8239	3.7386	11.8228
80	1.6937	5.4347	10.5512	21.4360
85	2.1655	8.1616	19.1924	31.2478
90	2.3682	15.3213	27.9598	41.2370
95	6.6820	22.5143	36.8354	51.3861
100	11.5698	29.7287	45.8048	61.6805
105	16.4553	36.9562	54.8566	72.1077
110	21.3405	44.1912	63.9814	82.6571
115	26.2281	51.4299	73.1715	93.3198
120	31.1214	58.6699	82.4206	104.0877

In addition, let

$$t = 0, T = 0.5, R = 70, K = 100, H = 0.95, \lambda = 2, \bar{\sigma}_1 = (0.1, 0.15, 0.2).$$

Table 2 shows that the value of the down-and-out call option is decreasing as the fractal dimension increasing with other parameters remains unchanged, which means a negative relationship between them.

Table 2. The value of down-and-out option against different fractal dimension α.

S	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$	$\alpha = 0.6$
75	1.0735	1.1924	1.3187	1.4523
80	1.8954	2.1100	2.3378	2.5789
85	2.4341	2.7199	3.0232	3.3443
90	2.6809	3.0137	3.3668	3.7405
95	7.3313	8.0241	8.7620	9.5467
100	12.5828	13.6637	14.8154	16.0409
105	17.8296	19.2957	20.8582	22.5214
110	23.0736	24.9227	26.8935	28.9916
115	28.3184	30.5486	32.9256	35.4567
120	33.5677	36.1775	38.9595	41.9220

6. Conclusions

Considering that the price change of the underlying is regarded as a fractal transmission system, the fractal derivative is introduced into the barrier option under sub-mixed fBm with jump to try to achieve the ideal expectation of market justice. This paper mainly investigates the pricing formula for fractal barrier options under sub-mixed fBm with jump, including the down-and-out call option, the down-and-out put option, the down-and-in call option, the down-and-in put option, and so on. To start, the B-S type PDE is established by using the fractal Itô's formula and a self-financing strategy. Then, by transforming the PDE to the Cauchy problem, we obtain the explicit pricing formulae for fractal barrier options. Besides, the value of the fractal vanilla call option, the value of the fractal vanilla put option, and the parity formula between fractal barrier call option and fractal barrier put option are obtained by a similar method. Finally, taking the down-and-out call option as an example, numerical experiments show that barrier price, fractal dimension, and Hurst index are negatively correlated with the value of down-and-out call option, while jump intensity and volatility are positively correlated with it. In numerical experiments, using real data and achieving the calibration of the model to real-time market data will be an important topic of future research. This is beneficial in enhancing the degree of fit between the model and the actual market and providing directions for model improvement, so as to help investors analyze and control the risks associated with barrier options more intuitively and effectively and enrich the research content of barrier options.

Author contributions

Chao Yue: Conceptualization, formal analysis, methodology, software, resources, writing-original draft; Chuanhe Shen: Funding acquisition, investigation, supervision, validation, visualization, editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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