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Application of fixed point theory to synaptic delay differential equations in neural networks

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Abstract: The objective of this research is to propose a new concept known as rational ($\alpha\eta$-ψ)-contractions in the framework of $\mathcal{F}$-metric spaces and to establish several fixed point theorems. These theorems help to generalize and unify various established fixed point results from the existing literature. To demonstrate the practical effectiveness of our approach, we provide a significant example that confirms our findings. In addition, we introduce a generalized multivalued (α-ψ)-contraction concept in $\mathcal{F}$-metric spaces and use it to prove fixed point theorems specifically designed for multivalued mappings. To demonstrate the practical utility of our findings, we apply our main results to the solution of synaptic delay differential equations in neural networks.

Keywords: fixed point; rational ($\alpha\eta$-ψ)-contractions; $\mathcal{F}$-metric spaces; multivalued mappings; synaptic delay differential equations
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1. Introduction

Fixed point (FP) theory, a key area in both topology and analysis, is a powerful tool used in many fields of science. Establishing essential concepts and schemes, it acts as a driving force for continuous exploration and progress. The foundation of this field rests on the analysis of metric spaces (MSs), where distances within sets are precisely defined. This fundamental notion finds myriad elegant and impactful applications across diverse scientific domains [1, 2]. To expand its utility, mathematicians have extended the scope of MSs, giving rise to fruitful advancements in this theory. Branciari [3] proposed a novel concept: the generalized metric space. This framework departs from the traditional triangle inequality, adopting a more adaptable four-point "rectangular inequality". Bakhtin [4] pioneered the notion of b-metric space (b-MS), that Czerwik [5] further elaborated on in 1993. A key distinction between b-MSs and classical metrics is the absence of a continuity requirement within their defined topology. The most recent development came in 2018 with Jleli et al.'s introduction of the $\mathcal{F}$-metric space ($\mathcal{F}$-MS) [6], encompassing these prior advancements. By generalizing the concept of metrics, these novel approaches provide a more adaptable framework for modeling real-world phenomena, tackling issues beyond the reach of traditional metric structures.

Stefan Banach [7] was the pioneer researcher in FP theory, having given the conception of contraction and proved a result known as Banach contraction principle (BCP). A lot of research work in this field has been given to the improvement and generalization of BCP in different ways. Samet et al. [8] introduced the idea of α-admissibility which is a crucial concept in FP theory. This advancement incorporates a "weighting" function, $\alpha(\mathfrak{e}, \mathfrak{c})$, which assigns flexible significance to the contractive nature between various point pairings. These innovative functions enhance adaptability and allow for FP analysis within a wider range of mathematical structures. In this direction, they gave the notion of (α-ψ)-contraction and proved some results to generalize certain FP theorems. Later on, Salimi et al. [9] gave the notion of α-admissible mapping with respect to η and modified the concept of α-admissibility in the context of MS. Building on the concept of (α-ψ)-contractions, Hussain et al. [10] introduced the generalized idea of Ćirić type (α-ψ)-contractions within the surroundings of $\mathcal{F}$-MS. Subsequently, Al-Mezel et al. [11] proved FP results for generalized (α-ψ)-contraction and discussed the solution of the differential equation by applying their leading theorem. Faraji et al. [12] presented some emerging theorems for self mappings satisfying (α,β)-admissibility in an $\mathcal{F}$-MS and they applied their results to solve integral equations.

Nadler [13] proved an FP theorem for multivalued mapping and extended the BCP. Latif et al. [14] presented FP theorems for multivalued mappings satisfying generalized contraction conditions in the context of metric type spaces equipped with a generalized distance. Asl et al. [15] generalized Nadler's FP theorem by introducing (α-ψ)-contractive multifunctions in the MSs. Recently, Isık et al. [16] extended the above concept to the setting of $\mathcal{F}$-MS and obtained end point theorems for weakly contractive mappings. Mlaiki et al. [17] established some end point results for generalized contractions in the context of $\mathcal{F}$-MS and discussed the solution of the integral equation as application. Mudhesh et al. [18] defined the notion of α_*-ψ-Λ-contraction multivalued mappings in an $\mathcal{F}$-MS and proved FP results along with an application of existence and uniqueness of a solution of a functional equation. Zhao [19] investigated the existence of solutions and generalized Ulam-Hyers stability for a nonlinear fractional coupled system involving the Atangana-Baleanu-Caputo (ABC) fractional derivative and a Laplacian operator, utilizing F-contractive mapping techniques. Zhao et al. [20] presented a unified framework for investigating the solvability and stability of multipoint boundary value problems (BVPs) for Langevin and Sturm-Liouville equations involving Caputo-Hadamard (CH) fractional derivatives and impulses, utilizing coincidence theory. For a thorough investigation of these developments and their consequences, a review of the cited references [21–26] is recommended.

On the other hand, FP theory has emerged as a powerful tool for analyzing and solving synaptic delay differential equations. Synaptic delay differential equations are employed to model the behavior of neurons, where synaptic delays play a pivotal role. These delays, arising from the finite transmission time of neurotransmitters across the synaptic gap, significantly impact the dynamics of neural networks. Understanding synaptic delays is crucial for comprehending the stability, oscillatory behavior, and synchronization of neuron firing patterns. Bocharov et al. [27] investigated the numerical modeling in biosciences using delay differential equations involving simulating biological processes that depend on past states, such as population dynamics or neural activity with synaptic delays, to better understand complex temporal interactions. Spek et al. [28] conducted a comprehensive investigation into neural field models that incorporate both transmission delays and diffusion, exploring how the interplay between delayed signal propagation and spatial spread of neural activity affects patterns of brain dynamics, such as wave propagation, stability, and synchronization in neural networks. Rehan et al. [29] presented a detailed analysis of the applications of delay differential equations in biological systems, highlighting their significance in understanding complex biological phenomena.

Motivated by the main results of Faraji et al. [12] and Mudhesh et al. [18], this paper presents two new concepts: rational ($\alpha\eta$-ψ)-contractions and generalized multivalued (α, ψ)-contractions. We then utilize these concepts within the framework of $\mathcal{F}$-MS to establish FP theorems. As consequences of leading results, we deduce some well-known FP results of the literature. We also supply a meaningful example to validate our established results.

2. Preliminaries

This section presents the necessary preliminary definitions, theorems, and lemmas that will be used in the paper. Throughout this article, we will use the following notation: $\mathbb{N}$ for the set of natural numbers, $\mathbb{R}^+$ for the set of positive real numbers, and $\mathbb{R}$ for the set of real numbers.

Bakhtin [4] defined the concept of b-metric space in this way.

Definition 1. *([4]) Let $\mathfrak{P}$ be a nonempty set and $s \geq 1$ be a constant. A function $d : \mathfrak{P} \times \mathfrak{P} \to [0, \infty)$ is called a b-metric if the following assertions hold:*

$(b1)$ $d(\mathfrak{e}, \mathfrak{c}) = 0 \Leftrightarrow \mathfrak{e} = \mathfrak{c}$;
$(b2)$ $d(\mathfrak{e}, \mathfrak{c}) = d(\mathfrak{c}, \mathfrak{e})$;
$(b3)$ $d(\mathfrak{e}, \nu) \leq s[d(\mathfrak{e}, \mathfrak{c}) + d(\mathfrak{c}, \nu)]$,

for all $\mathfrak{e}, \mathfrak{c}, \nu \in \mathfrak{P}$.

The pair $(\mathfrak{P}, d)$ is then said to be a b-metric space.

Very recently, Jleli and Samet [6] introduced an interesting generalization of metric space and b-metric space in the following way.

Let $\mathcal{F}$ be the set of continuous functions $\curlywedge : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

$(\mathcal{F}_1)$ $\curlywedge$ is nondecreasing, i.e., $0 < \mathfrak{e} < \mathfrak{c}$ implies $\curlywedge(\mathfrak{e}) \leq \curlywedge(\mathfrak{c})$,
$(\mathcal{F}_2)$ for every sequence $\{\mathfrak{e}_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \mathfrak{e}_n = 0 \Longleftrightarrow \lim_{n\to\infty} \curlywedge(\mathfrak{e}_n) = -\infty$.

Definition 2. *([6]) Let $\mathfrak{P} \neq \emptyset$ and let $d_{\mathcal{F}} : \mathfrak{P} \times \mathfrak{P} \to [0, +\infty)$. There exists $(\curlywedge, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ such that*

(D$_1$) $(\mathfrak{e}, \mathfrak{c}) \in \mathfrak{P} \times \mathfrak{P}$, $d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}) = 0 \Leftrightarrow \mathfrak{e} = \mathfrak{c}$,
(D$_2$) $d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}) = d_{\mathcal{F}}(\mathfrak{c}, \mathfrak{e})$, for all $(\mathfrak{e}, \mathfrak{c}) \in \mathfrak{P} \times \mathfrak{P}$,
(D$_3$) for each $(\mathfrak{e}, \mathfrak{c}) \in \mathfrak{P} \times \mathfrak{P}$ and $(\mathfrak{e}_1, \mathfrak{e}_2, ..., \mathfrak{e}_n)$ in $\mathfrak{P}$ such that $\mathfrak{e}_1 = \mathfrak{e}$ and $\mathfrak{e}_n = \mathfrak{c}$, we have

$$d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}) > 0 \Rightarrow \curlywedge(d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})) \leq \curlywedge(\sum_{i=1}^{n-1} d_{\mathcal{F}}(\mathfrak{e}_i, \mathfrak{e}_{i+1})) + \mathfrak{h},$$

where $n \geq 2$ is natural number. Then, $d_{\mathcal{F}}$ is said to be an $\mathcal{F}$-metric on $\mathfrak{P}$, and the pair $(\mathfrak{P}, d_{\mathcal{F}})$ is said to be an $\mathcal{F}$-MS.

Remark 1. *They showed that any metric space is an $\mathcal{F}$-MS but the converse is not true in general, which confirms that this concept is more general than the standard metric concept.*

Example 1. *([6]) Let* $\mathfrak{P}=\mathbb{N}$. *Define* $d_{\mathcal{F}}:\mathbb{N}\times\mathbb{N}\to[0,+\infty)$ *by*

$$d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})=\begin{cases}(\mathfrak{e}-\mathfrak{c})^2 & \text{if } (\mathfrak{e},\mathfrak{c})\in[0,3]\times[0,3]\\ |\mathfrak{e}-\mathfrak{c}| & \text{if } (\mathfrak{e},\mathfrak{c})\notin[0,3]\times[0,3]\end{cases}$$

with $\curlywedge(t)=\ln(t)$ and $\mathfrak{h}=\ln(3)$, then $(\mathfrak{P},d_{\mathcal{F}})$ is an $\mathcal{F}$-MS.

Definition 3. *([6]) Consider* $(\mathfrak{P},d_{\mathcal{F}})$ *as an* $\mathcal{F}$*-MS and* $\{\mathfrak{e}_n\}$ *is a sequence in* $\mathfrak{P}$.

(i) $\{\mathfrak{e}_n\}$ exhibits $\mathcal{F}$-convergence to an element $\mathfrak{e}$ in $\mathfrak{P}$ if it converges to $\mathfrak{e}$ regarding $d_{\mathcal{F}}$.

(ii) $\{\mathfrak{e}_n\}$ is considered an $\mathcal{F}$-Cauchy if the limit of $d_{\mathcal{F}}(\mathfrak{e}_n,\mathfrak{e}_m)$ approaches zero as n and m tend to infinity.

(iii) The $\mathcal{F}$-completeness of $(\mathfrak{P},d_{\mathcal{F}})$ implies that any $\mathcal{F}$-Cauchy sequence within $\mathfrak{P}$ converges to a specific element within $\mathfrak{P}$ using the $\mathcal{F}$-metric $d_{\mathcal{F}}$.

Theorem 1. *([6]) Let* O *be self-mapping on* $\mathcal{F}$*-complete* $\mathcal{F}$*-MS* $(\mathfrak{P},d_{\mathcal{F}})$. *Suppose that there exists* $k\in(0,1)$ *such that*

$$d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c})\leq kd_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}).$$

Then, O has a unique fixed point $\mathfrak{e}^*\in\mathfrak{P}$.

Samet et al. [8] proposed the concepts of α-ψ-contractive and α-admissible mappings, utilizing them to demonstrate fixed point theorems within complete metric spaces.

Let Ψ represent the collection of nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$, for all $t>0$.

To establish our main result, we will leverage the following well-established lemma:

Lemma 1. *([8]) If* ψ *belongs to* Ψ*, then the following conditions are satisfied:*

(i) The sequence $(\psi^n(t))_{n\in\mathbb{N}}$ approaches 0 as $n\to\infty$,

(ii) $\psi(t)<t$, for all $t>0$,

(iii) $\psi(t)=0$, if and only if, $t=0$.

Samet et al. [8] introduced the concept of an α-admissibility of the self mapping in such fashion:

Definition 4. *([8]) Let* $\alpha:\mathfrak{P}\times\mathfrak{P}\to[0,+\infty)$ *be a function. A function* $O:\mathfrak{P}\to\mathfrak{P}$ *is considered as an* α*-admissible if it satisfies the following condition:*

$$\alpha(\mathfrak{e},\mathfrak{c})\geq 1\Longrightarrow\alpha(O\mathfrak{e},O\mathfrak{c})\geq 1$$

for all $\mathfrak{e},\mathfrak{c}\in\mathfrak{P}$.

Salimi et al. [9] generalized the aforementioned idea of an α-admissible mapping as follows.

Definition 5. *([9]) Let* $\alpha,\eta:\mathfrak{P}\times\mathfrak{P}\to[0,+\infty)$. *A function* $O:\mathfrak{P}\to\mathfrak{P}$ *is termed as an* α*-admissible with respect to* η *if*

$$\alpha(\mathfrak{e},\mathfrak{c}) \geq \eta(\mathfrak{e},\mathfrak{c}) \Longrightarrow \alpha(O\mathfrak{e},O\mathfrak{c}) \geq \eta(O\mathfrak{e},O\mathfrak{c})$$

for all $\mathfrak{e},\mathfrak{c} \in \mathfrak{P}$.

If $\eta(\mathfrak{e},\mathfrak{c}) = 1$, then Definition 5 reduces to Definition 4.

Recently, the concept of a Ćirić type (α-ψ)-contraction in the background of $\mathcal{F}$-MS was presented by Hussain et al. [10] and established the corresponding fixed point result.

Theorem 2. *([10]) Let O be self-mapping on $\mathcal{F}$-complete $\mathcal{F}$-MS $(\mathfrak{P}, d_{\mathcal{F}})$ provided that the following conditions have been met:*

(i) $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible mapping,

(ii) there exist two functions $\alpha : \mathfrak{P} \times \mathfrak{P} \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(\mathfrak{e},\mathfrak{c}) d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) \leq \psi(M(\mathfrak{e},\mathfrak{c})),$$

where

$$M(\mathfrak{e},\mathfrak{c}) = \max\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}), d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e}), d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})\}$$

for $\mathfrak{e},\mathfrak{c} \in \mathfrak{P}$,

(iii) there exists $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, O\mathfrak{e}_0) \geq 1$.

Then, O has a fixed point $\mathfrak{e}^* \in \mathfrak{P}$.

3. Results and discussions

This section is devoted to presenting some new fixed point results for self-mappings satisfying generalized contractions within the framework of $\mathcal{F}$-complete $\mathcal{F}$-MS.

Definition 6. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS and $O : \mathfrak{P} \to \mathfrak{P}$. Then, O is defined as a rational ($\alpha\eta$-ψ)-contraction if there exist $\alpha, \eta : \mathfrak{P} \times \mathfrak{P} \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(\mathfrak{e}, O\mathfrak{e})\alpha(\mathfrak{c}, O\mathfrak{c}) \geq \eta(\mathfrak{e}, O\mathfrak{e})\eta(\mathfrak{c}, O\mathfrak{c})$ implies*

$$d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}), \min\left\{\frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}, \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{e})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}\right\}\right\}\right) \tag{3.1}$$

for all $\mathfrak{e},\mathfrak{c} \in \mathfrak{P}$.

Theorem 3. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS. Suppose that $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible mapping with respect to η and rational ($\alpha\eta$-ψ)-contraction such that the following assertions hold:*

(i) there exists $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, O\mathfrak{e}_0) \geq \eta(\mathfrak{e}_0, O\mathfrak{e}_0)$,

(ii) if either O is continuous or $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq \eta(\mathfrak{e}_n, \mathfrak{e}_{n+1})$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\alpha(\mathfrak{e}, O\mathfrak{e}) \geq \eta(\mathfrak{e}, O\mathfrak{e})$.

Then, there exists $\mathfrak{e}^* \in \mathfrak{P}$ such that $\mathfrak{e}^* = O\mathfrak{e}^*$.

Proof. Let $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, \mathfrak{e}_1) = \alpha(\mathfrak{e}_0, O\mathfrak{e}_0) \geq \eta(\mathfrak{e}_0, O\mathfrak{e}_0) = \eta(\mathfrak{e}_0, \mathfrak{e}_1)$. Define a sequence $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ by $\mathfrak{e}_{n+1} = O^n\mathfrak{e}_0 = O\mathfrak{e}_n$, for all $n \in \mathbb{N}$. Since the mapping $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible mapping with respect to η, we have

$$\alpha(\mathfrak{e}_1, \mathfrak{e}_2) = \alpha(O\mathfrak{e}_0, O\mathfrak{e}_1) \geq \eta(O\mathfrak{e}_0, O\mathfrak{e}_1) = \eta(\mathfrak{e}_1, \mathfrak{e}_2).$$

Continuing in this way, we get

$$\alpha(\mathfrak{e}_{n-1},\mathfrak{e}_n)=\alpha(O\mathfrak{e}_{n-2},O\mathfrak{e}_{n-1})\geq\eta(O\mathfrak{e}_{n-2},O\mathfrak{e}_{n-1})=\eta(\mathfrak{e}_{n-1},\mathfrak{e}_n),$$

and

$$\alpha(\mathfrak{e}_n,\mathfrak{e}_{n+1})=\alpha(O\mathfrak{e}_{n-1},O\mathfrak{e}_n)\geq\eta(O\mathfrak{e}_{n-1},O\mathfrak{e}_n)=\eta(\mathfrak{e}_n,\mathfrak{e}_{n+1}),$$

for all $n\in\mathbb{N}$. We can express these inequalities in this way

$$\alpha(\mathfrak{e}_{n-1},\mathfrak{e}_n)=\alpha(\mathfrak{e}_{n-1},O\mathfrak{e}_{n-1})\geq\eta(\mathfrak{e}_{n-1},O\mathfrak{e}_{n-1})=\eta(\mathfrak{e}_{n-1},\mathfrak{e}_n),\tag{3.2}$$

and

$$\alpha(\mathfrak{e}_n,\mathfrak{e}_{n+1})=\alpha(\mathfrak{e}_n,O\mathfrak{e}_n)\geq\eta(\mathfrak{e}_n,O\mathfrak{e}_n)=\eta(\mathfrak{e}_n,\mathfrak{e}_{n+1}),\tag{3.3}$$

for all $n\in\mathbb{N}$. By (3.2) and (3.3), we have

$$\alpha(\mathfrak{e}_{n-1},O\mathfrak{e}_{n-1})\alpha(\mathfrak{e}_n,O\mathfrak{e}_n)\geq\eta(\mathfrak{e}_{n-1},O\mathfrak{e}_{n-1})\eta(\mathfrak{e}_n,O\mathfrak{e}_n),\tag{3.4}$$

for all $n\in\mathbb{N}$. Clearly, if there exists $n_0\in\mathbb{N}$ for which $\mathfrak{e}_{n_0+1}=\mathfrak{e}_{n_0}$, then $O\mathfrak{e}_{n_0}=\mathfrak{e}_{n_0}$, and we have proven the claim. Consequently, we assume that $\mathfrak{e}_{n+1}\neq\mathfrak{e}_n$ or $d_{\mathcal{F}}(O\mathfrak{e}_{n-1},O\mathfrak{e}_n)>0$, for all $n\in\mathbb{N}$. Since O satisfies the properties of a rational ($\alpha\eta$-ψ)-contraction, we obtain the following inequality:

$$\begin{aligned}d_{\mathcal{F}}(\mathfrak{e}_n,\mathfrak{e}_{n+1})&=d_{\mathcal{F}}(O\mathfrak{e}_{n-1},O\mathfrak{e}_n)\\&\leq\psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e}_{n-1},\mathfrak{e}_n),\min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e}_{n-1},O\mathfrak{e}_{n-1})d_{\mathcal{F}}(\mathfrak{e}_n,O\mathfrak{e}_n)}{1+d_{\mathcal{F}}(\mathfrak{e}_{n-1},\mathfrak{e}_n)},\\\frac{d_{\mathcal{F}}(\mathfrak{e}_{n-1},O\mathfrak{e}_n)d_{\mathcal{F}}(\mathfrak{e}_n,O\mathfrak{e}_{n-1})}{1+d_{\mathcal{F}}(\mathfrak{e}_{n-1},\mathfrak{e}_n)}\end{array}\right\}\right\}\right)\\&=\psi\left(d_{\mathcal{F}}(\mathfrak{e}_{n-1},\mathfrak{e}_n)\right),\end{aligned}$$

for all $n\in\mathbb{N}$, that is,

$$d_{\mathcal{F}}(\mathfrak{e}_n,\mathfrak{e}_{n+1})\leq\psi\left(d_{\mathcal{F}}(\mathfrak{e}_{n-1},\mathfrak{e}_n)\right),\tag{3.5}$$

for all $n\in\mathbb{N}$. Proceeding in this manner, we find

$$d_{\mathcal{F}}(\mathfrak{e}_n,\mathfrak{e}_{n+1})\leq\psi^n\left(d_{\mathcal{F}}(\mathfrak{e}_0,\mathfrak{e}_1)\right),\tag{3.6}$$

for all $n\in\mathbb{N}$. Given a pair $(\curlywedge,\mathfrak{h})\in\mathcal{F}\times[0,+\infty)$ satisfying condition (D_3), let ϵ be a positive constant. By virtue of property ($\mathcal{F}_2$), we can identify a positive constant δ such that

$$0<t<\delta\Longrightarrow\curlywedge(t)<\curlywedge(\delta)-\mathfrak{h}.\tag{3.7}$$

Let $n(\epsilon)$ be a natural number such that $0<\sum_{n\geq n(\epsilon)}\psi^n(d_{\mathcal{F}}(\mathfrak{e}_0,\mathfrak{e}_1))<\delta$. Therefore, by (3.7) and conditions ($\mathcal{F}_1$), we obtain

$$\curlywedge(\sum_{i=n}^{m-1}\psi^i(d_{\mathcal{F}}(\mathfrak{e}_0,\mathfrak{e}_1)))\leq\curlywedge(\sum_{n\geq n(\epsilon)}\psi^n(d_{\mathcal{F}}(\mathfrak{e}_0,\mathfrak{e}_1)))<\curlywedge(\epsilon)-\mathfrak{h},\tag{3.8}$$

for $m>n\geq n(\epsilon)$. Leveraging the triangle inequality (D_3) and the inequality (3.8), we can deduce $d_{\mathcal{F}}(\mathfrak{e}_n,\mathfrak{e}_m)>0$, for $m>n\geq n(\epsilon)$. It yields

$$\curlywedge(d_{\mathcal{F}}(\mathfrak{e}_n,\mathfrak{e}_m))\leq\curlywedge(\sum_{i=n}^{m-1}d_{\mathcal{F}}(\mathfrak{e}_i,\mathfrak{e}_{i+1}))+\mathfrak{h}$$

$$
\begin{aligned}
&\leq \curlywedge(\sum_{i=n}^{m-1} \psi^i(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1))) + \mathfrak{h} \\
&\leq \curlywedge(\sum_{n \geq n(\epsilon)} \psi^n(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1))) + \mathfrak{h} \\
&< \curlywedge(\epsilon).
\end{aligned}
$$

By $(\mathcal{F}_1)$, we have $d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}_m) < \epsilon$, for $m > n \geq n(\epsilon)$. This establishes that the sequence $\{\mathfrak{e}_n\}$ is $\mathcal{F}$-Cauchy. The $\mathcal{F}$-completeness of $(\mathfrak{P}, d_{\mathcal{F}})$ guarantees the existence of an element $\mathfrak{e}^* \in \mathfrak{P}$ such that $\{\mathfrak{e}_n\}$ converges to $\mathfrak{e}^*$ under the $\mathcal{F}$-metric, which can be written as

$$
\lim_{n\to\infty} d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}^*) = 0. \tag{3.9}
$$

Next, as $\mathfrak{e}_n \to \mathfrak{e}^*$ and $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq \eta(\mathfrak{e}_n, \mathfrak{e}_{n+1})$, then $\alpha(\mathfrak{e}^*, O\mathfrak{e}^*) \geq \eta(\mathfrak{e}^*, O\mathfrak{e}^*)$. Thus,

$$
\alpha(\mathfrak{e}^*, O\mathfrak{e}^*)\alpha(\mathfrak{e}_n, O\mathfrak{e}_n) \geq \eta(\mathfrak{e}^*, O\mathfrak{e}^*)\eta(\mathfrak{e}_n, O\mathfrak{e}_n).
$$

To establish a contradiction, let's assume that $d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*) > 0$. Based on properties $(\mathcal{F}_1)$ and (D_3), we obtain

$$
\begin{aligned}
\curlywedge(d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*)) &\leq \curlywedge(d_{\mathcal{F}}(O\mathfrak{e}^*, O\mathfrak{e}_n) + d_{\mathcal{F}}(O\mathfrak{e}_n, \mathfrak{e}^*)) + \mathfrak{h} \\
&= \curlywedge(d_{\mathcal{F}}(O\mathfrak{e}^*, O\mathfrak{e}_n) + d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}^*)) + \mathfrak{h}.
\end{aligned}
$$

By the inequality (3.1), we have

$$
\begin{aligned}
\curlywedge(d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*)) &\leq \curlywedge(d_{\mathcal{F}}(O\mathfrak{e}^*, O\mathfrak{e}_n) + d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}^*)) + \mathfrak{h} \\
&\leq \curlywedge\left[\begin{array}{c} \psi\left[\max\left\{d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n), \min\left(\begin{array}{c} \frac{d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}^*)d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}_n)}{1+d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n)}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}_n)d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}^*)}{1+d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n)} \end{array}\right)\right\}\right] \\ +d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}*) \end{array}\right] + \mathfrak{h}, \\
&= \curlywedge\left[\begin{array}{c} \psi\left[\max\left\{d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n), \min\left(\begin{array}{c} \frac{d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}^*)d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}_{n+1})}{1+d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n)}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_{n+1})d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}^*)}{1+d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n)} \end{array}\right)\right\}\right] \\ +d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}*) \end{array}\right] + \mathfrak{h}, \\
&= \curlywedge(\psi\,(d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n)) + d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}*)) + \mathfrak{h} \\
&< \curlywedge(d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n) + d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}*)) + \mathfrak{h},
\end{aligned}
$$

for $n \in \mathbb{N}$. Upon taking the limit as n approaches infinity and using the Eq (3.9), we derive

$$
\lim_{n\to\infty} \curlywedge(d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*)) \leq \lim_{n\to\infty} \curlywedge(d_{\mathcal{F}}(\mathfrak{e}^*, \mathfrak{e}_n) + d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}*)) + \mathfrak{h} = -\infty.
$$

Then, by $(\mathcal{F}_2)$, we get that $d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*) = 0$, which is a contradiction to the assumption that $d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*) > 0$. Consequently, $d_{\mathcal{F}}(O\mathfrak{e}^*, \mathfrak{e}^*) = 0$, implying $O\mathfrak{e}^* = \mathfrak{e}^*$. As a result, $\mathfrak{e}^* \in \mathfrak{P}$ serves as the fixed point of O. □

Example 2. *Considering set of natural numbers* ($\mathbb{N}$)*, we equip it with an $\mathcal{F}$-metric denoted by $d_{\mathcal{F}}$, defined as follows:*

$$d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}) = \begin{cases} (\mathfrak{e}-\mathfrak{c})^2, & (\mathfrak{e},\mathfrak{c}) \in [0,3]\times[0,3], \\ |\mathfrak{e}-\mathfrak{c}|, & (\mathfrak{e},\mathfrak{c}) \notin [0,3]\times[0,3] \end{cases}$$

with $\curlywedge(t) = \ln(t)$ and $\mathfrak{h} = \ln(3)$. Then, $(\mathfrak{P}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete $\mathcal{F}$-MS. Define the continuous mapping $O : \mathfrak{P} \to \mathfrak{P}$ by

$$O\mathfrak{e} = \begin{cases} \frac{\mathfrak{e}}{2} + 1, & \mathfrak{e} \geq 0, \\ 0, & \mathfrak{e} < 0, \end{cases}$$

and $\alpha, \eta : \mathfrak{P} \times \mathfrak{P} \to [0,\infty)$ by

$$\alpha(\mathfrak{e},\mathfrak{c}) = \eta(\mathfrak{e},\mathfrak{c}) = \begin{cases} 1, & \mathfrak{e},\mathfrak{c} \in [0,\infty), \\ 0, & \text{otherwise}. \end{cases}$$

Define $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = \frac{1}{2}t$. Now, we need to show that O is an α-admissible mapping with respect to η.

Now, we have to discuss the following two cases.

Case 1. If $\mathfrak{e},\mathfrak{c} \in [0,3]$, then $\alpha(\mathfrak{e},\mathfrak{c}) = \eta(\mathfrak{e},\mathfrak{c}) = 1$, then $O\mathfrak{e}, O\mathfrak{c} \in [0,\infty)$. Hence, it is obvious to prove that the mapping O is an α-admissible mapping with respect to η and $\alpha(\mathfrak{e},O\mathfrak{e})\alpha(\mathfrak{c},O\mathfrak{c}) \geq \eta(\mathfrak{e},O\mathfrak{e})\eta(\mathfrak{c},O\mathfrak{c})$. Now, we prove that the mapping $O : \mathfrak{P} \to \mathfrak{P}$ is a rational $(\alpha\eta\text{-}\psi)$-contraction.

$$\begin{aligned} d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) &= \left(\frac{\mathfrak{e}}{2}+1-\frac{\mathfrak{c}}{2}-1\right)^2 = \left(\frac{\mathfrak{e}}{2}-\frac{\mathfrak{c}}{2}\right)^2 \\ &= \left(\frac{1}{2}\right)^2(\mathfrak{e}-\mathfrak{c})^2 = \frac{1}{4}(\mathfrak{e}-\mathfrak{c})^2 \\ &\leq \frac{1}{2}\max\left\{(\mathfrak{e}-\mathfrak{c})^2, \min\left\{\begin{array}{c} \frac{(\mathfrak{e}-\frac{\mathfrak{e}}{2}-1)^2(\mathfrak{c}-\frac{\mathfrak{c}}{2}-1)^2}{1+(\mathfrak{e}-\mathfrak{c})^2}, \\ \frac{(\mathfrak{e}-\frac{\mathfrak{c}}{2}-1)^2(\mathfrak{c}-\frac{\mathfrak{e}}{2}-1)^2}{1+(\mathfrak{e}-\mathfrak{c})^2} \end{array}\right\}\right\}. \end{aligned}$$

Hence,

$$d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}), \min\left\{\begin{array}{c} \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{e})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})} \end{array}\right\}\right\}\right).$$

Case 2. If $\mathfrak{e},\mathfrak{c} \notin [0,3]$, then $\alpha(\mathfrak{e},\mathfrak{c}) = \eta(\mathfrak{e},\mathfrak{c}) = 1$ implies that

$$\begin{aligned} d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) &= \left|\frac{\mathfrak{e}}{2}+1-\frac{\mathfrak{c}}{2}-1\right| = \left|\frac{\mathfrak{e}}{2}-\frac{\mathfrak{c}}{2}\right| = \frac{1}{2}|\mathfrak{e}-\mathfrak{c}| \\ &\leq \frac{1}{2}\max\left\{|\mathfrak{e}-\mathfrak{c}|, \min\left\{\begin{array}{c} \frac{\left|\mathfrak{e}-\frac{\mathfrak{e}}{2}-1\right|\left|\mathfrak{c}-\frac{\mathfrak{c}}{2}-1\right|}{1+|\mathfrak{e}-\mathfrak{c}|}, \\ \frac{\left|\mathfrak{e}-\frac{\mathfrak{c}}{2}-1\right|\left|\mathfrak{c}-\frac{\mathfrak{e}}{2}-1\right|}{1+|\mathfrak{e}-\mathfrak{c}|} \end{array}\right\}\right\}. \end{aligned}$$

Hence,

$$d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}), \min\left\{\begin{array}{c} \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{e})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})} \end{array}\right\}\right\}\right).$$

Moreover, there exists $\mathfrak{e}_0 = 0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, O\mathfrak{e}_0) = 1 = \eta(\mathfrak{e}_0, O\mathfrak{e}_0)$. Thus, all the conditions of Theorem 3 are satisfied and the mapping O has a fixed point, which is 2.

Corollary 1. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS. Suppose that $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible such that $\alpha(\mathfrak{e}, O\mathfrak{e})\alpha(\mathfrak{c}, O\mathfrak{c}) \geq 1$ implies*

$$d_{\mathcal{F}}(O\mathfrak{e}, O\mathfrak{c}) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}), \min\left\{\frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{1 + d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})}, \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{1 + d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})}\right\}\right\}\right)$$

for $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$ such that for all $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$. Let's proceed under the following assumptions:

(i) There exists $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, O\mathfrak{e}_0) \geq 1$.

(ii) If either O is continuous or $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\alpha(\mathfrak{e}, O\mathfrak{e}) \geq 1$.

Then, there exists $\mathfrak{e}^* \in \mathfrak{P}$ such that $\mathfrak{e}^* = O\mathfrak{e}^*$.

Proof. Within Theorem 3, we introduce a function $\eta : \mathfrak{P} \times \mathfrak{P} \to [0, +\infty)$ such that $\eta(\mathfrak{e}, \mathfrak{c}) = 1$, for all $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$. □

Building upon Theorem 3, the following corollaries can be directly inferred.

Corollary 2. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS. Suppose that $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible such that*

$$(d_{\mathcal{F}}(O\mathfrak{e}, O\mathfrak{c}) + l)^{\alpha(\mathfrak{e}, O\mathfrak{e})\alpha(\mathfrak{c}, O\mathfrak{c})} \leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}), \min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})}\end{array}\right\}\right\}\right) + l$$

for all $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$, where $l > 0$. Let's proceed under the following hypotheses

(i) There exists $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, O\mathfrak{e}_0) \geq 1$.

(ii) If either O is continuous or $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq 1$ for all sequences $\mathfrak{e}_n$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\alpha(\mathfrak{e}, O\mathfrak{e}) \geq 1$.

Then, there exists $\mathfrak{e}^* \in \mathfrak{P}$ such that $\mathfrak{e}^* = O\mathfrak{e}^*$.

Corollary 3. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS. Suppose that the mapping $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible such that*

$$(\alpha(\mathfrak{e}, O\mathfrak{e})\alpha(\mathfrak{c}, O\mathfrak{c}) + 1)^{d_{\mathcal{F}}(O\mathfrak{e}, O\mathfrak{c})} \leq 2^{\psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}), \min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})}\end{array}\right\}\right\}\right)}$$

for all $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$. Suppose that these conditions are met:

(i) There exists $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\alpha(\mathfrak{e}_0, O\mathfrak{e}_0) \geq 1$.

(ii) If either O is continuous or $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\alpha(\mathfrak{e}, O\mathfrak{e}) \geq 1$.

Then, there exists $\mathfrak{e}^* \in \mathfrak{P}$ such that $\mathfrak{e}^* = O\mathfrak{e}^*$.

Corollary 4. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS. Suppose that the mapping $O : \mathfrak{P} \to \mathfrak{P}$ is an α-admissible such that*

$$\alpha(\mathfrak{e},O\mathfrak{e})\alpha(\mathfrak{c},O\mathfrak{c})d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) \leq k\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}),\min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})},\\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}\end{array}\right\}\right\}$$

for all $\mathfrak{e},\mathfrak{c}\in\mathfrak{P}$. Suppose that these conditions are met:

(i) There exists $\mathfrak{e}_0\in\mathfrak{P}$ such that $\alpha(\mathfrak{e}_0,O\mathfrak{e}_0)\geq 1$.

(ii) If either O is continuous or $\alpha(\mathfrak{e}_n,\mathfrak{e}_{n+1})\geq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\alpha(\mathfrak{e},O\mathfrak{e})\geq 1$.

Then, there exists $\mathfrak{e}^*\in\mathfrak{P}$ such that $\mathfrak{e}^*=O\mathfrak{e}^*$.

In the special case where $\alpha(\mathfrak{e},\mathfrak{c})=1$, the following corollaries hold.

Corollary 5. *Let* $(\mathfrak{P},d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-MS. Suppose that the mapping* $O:\mathfrak{P}\to\mathfrak{P}$ *is* η*-sub-admissible such that* $\eta(\mathfrak{e},O\mathfrak{e})\eta(\mathfrak{c},O\mathfrak{c})\leq 1$ *implies*

$$d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c}) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}),\min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})},\\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}\end{array}\right\}\right\}\right)$$

for all $\mathfrak{e},\mathfrak{c}\in\mathfrak{P}$. Assume that the following assertions hold:

(i) There exists $\mathfrak{e}_0\in\mathfrak{P}$ such that $\eta(\mathfrak{e}_0,O\mathfrak{e}_0)\leq 1$.

(ii) If either O is continuous or $\eta(\mathfrak{e}_n,\mathfrak{e}_{n+1})\leq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\eta(\mathfrak{e},O\mathfrak{e})\leq 1$.

Then, there exists $\mathfrak{e}^*\in\mathfrak{P}$ such that $\mathfrak{e}^*=O\mathfrak{e}^*$.

Corollary 6. *Let* $(\mathfrak{P},d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-MS. Suppose that the mapping* $O:\mathfrak{P}\to\mathfrak{P}$ *is* η*-sub-admissible such that*

$$d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c})+l \leq \left[\psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}),\min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})},\\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}\end{array}\right\}\right\}\right)+l\right]^{\eta(\mathfrak{e},O\mathfrak{e})\eta(\mathfrak{c},O\mathfrak{c})}$$

for all $\mathfrak{e},\mathfrak{c}\in\mathfrak{P}$, where $l>0$. Suppose that these hypotheses are met:

(i) There exists $\mathfrak{e}_0\in\mathfrak{P}$ such that $\eta(\mathfrak{e}_0,O\mathfrak{e}_0)\leq 1$.

(ii) If either O is continuous or $\eta(\mathfrak{e}_n,\mathfrak{e}_{n+1})\leq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\eta(\mathfrak{e},O\mathfrak{e})\leq 1$.

Then, there exists $\mathfrak{e}^*\in\mathfrak{P}$ such that $\mathfrak{e}^*=O\mathfrak{e}^*$.

Corollary 7. *Let* $(\mathfrak{P},d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-MS. Suppose that the mapping* $O:\mathfrak{P}\to\mathfrak{P}$ *is* η*-sub-admissible such that*

$$2^{d_{\mathcal{F}}(O\mathfrak{e},O\mathfrak{c})} \leq \left(\eta(\mathfrak{e},O\mathfrak{e})\eta(\mathfrak{c},O\mathfrak{c})+1\right)^{\psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c}),\min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})},\\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}\end{array}\right\}\right\}\right)}$$

for all $\mathfrak{e},\mathfrak{c}\in\mathfrak{P}$. Suppose that these hypotheses are met:

(i) There exists $\mathfrak{e}_0\in\mathfrak{P}$ such that $\eta(\mathfrak{e}_0,O\mathfrak{e}_0)\leq 1$.

(ii) If either O is continuous or $\eta(\mathfrak{e}_n,\mathfrak{e}_{n+1})\leq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\eta(\mathfrak{e},O\mathfrak{e})\leq 1$.

Then, there exists $\mathfrak{e}^*\in\mathfrak{P}$ such that $\mathfrak{e}^*=O\mathfrak{e}^*$.

Corollary 8. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS. Suppose that the mapping $O : \mathfrak{P} \to \mathfrak{P}$ is η-sub-admissible such that*

$$d_{\mathcal{F}}(O\mathfrak{e}, O\mathfrak{c}) \leq \psi\left(\eta(\mathfrak{e}, O\mathfrak{e})\eta(\mathfrak{c}, O\mathfrak{c}) \max\left\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}), \min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{e})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}, \\ \frac{d_{\mathcal{F}}(\mathfrak{e},O\mathfrak{c})d_{\mathcal{F}}(\mathfrak{c},O\mathfrak{c})}{1+d_{\mathcal{F}}(\mathfrak{e},\mathfrak{c})}\end{array}\right\}\right\}\right)$$

for all $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$. Suppose that these hypotheses are met:

(i) There exists $\mathfrak{e}_0 \in \mathfrak{P}$ such that $\eta(\mathfrak{e}_0, O\mathfrak{e}_0) \leq 1$.

(ii) If either O is continuous or $\eta(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \leq 1$ for all sequences $\{\mathfrak{e}_n\}$ in $\mathfrak{P}$ converging to $\mathfrak{e}$, then $\eta(\mathfrak{e}, O\mathfrak{e}) \leq 1$.

Then, there exists $\mathfrak{e}^* \in \mathfrak{P}$ such that $\mathfrak{e}^* = O\mathfrak{e}^*$.

4. Fixed point results for multivalued mappings

In FP theory, we know that the famous theorem, which is the BCP, has aroused the interest of significant mathematical scientists because of its importance and impressive achievements. It has been expanded and extended to a larger scope, called multivalued mapping, by Nadler [13]. However, significant authors have established different theorems to extend, unify, and generalize multivalued mapping under different contractive conditions.

Definition 7. *([16]) Let $\mathfrak{P}$ and Y be two nonempty sets. A mapping O is said to be multivalued mapping from $\mathfrak{P}$ to Y if O is a function for $\mathfrak{P}$ to the power set of Y. We denote a multivalued mapping by $O : \mathfrak{P} \to 2^Y$. A point $\mathfrak{e} \in \mathfrak{P}$ is said to be a fixed point of the multivalued mapping $O : \mathfrak{P} \to 2^{\mathfrak{P}}$ if $\mathfrak{e} \in O\mathfrak{e}$.*

Definition 8. *([16]) Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS. A Hausdorff metric $H_{\mathcal{F}}$ on $CB(\mathfrak{P})$ induced by the $\mathcal{F}$-metric $d_{\mathcal{F}}$ is defined by*

$$H_{\mathcal{F}}(A, B) = \max\left\{\sup_{\mathfrak{e}\in A} d_{\mathcal{F}}(\mathfrak{e}, B), \sup_{\mathfrak{c}\in A} d_{\mathcal{F}}(\mathfrak{c}, A)\right\}$$

for all $A, B \in CB(\mathfrak{P})$, where $CB(\mathfrak{P})$ denotes the family of all nonempty closed and bounded subsets of $\mathfrak{P}$ and $d_{\mathcal{F}}(\mathfrak{e}, B) = \inf\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}) : \mathfrak{c} \in B\}$, for all $\mathfrak{e} \in \mathfrak{P}$.

Theorem 4. *([16]) Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS and let $O : \mathfrak{P} \to CB(\mathfrak{P})$ be a multivalued mapping. If there exists a constant $k \in (0, 1)$ such that*

$$H_{\mathcal{F}}(O\mathfrak{e}, O\mathfrak{c}) \leq k d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})$$

for all $\mathfrak{e}, \mathfrak{c} \in \mathfrak{P}$, then O has a FP.

Definition 9. *([16]) A multivalued mapping $O : \mathfrak{P} \to 2^{\mathfrak{P}}$ is said to be multivalued α-admissible if there exists a function $\alpha : \mathfrak{P} \times \mathfrak{P} \to [0, \infty)$ such that*

$$\alpha(\mathfrak{e}, \mathfrak{c}) \geq 1 \text{ implies } \alpha(u, v) \geq 1,$$

for $u \in O\mathfrak{e}$ and $v \in O\mathfrak{c}$.

Lemma 2. *([16]) Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS and $B \in CL(\mathfrak{P})$. Then, for each $\mathfrak{e} \in \mathfrak{P}$ with $d_{\mathcal{F}}(\mathfrak{e}, B) > 0$ and $r > 1$, there exists an element $\mathfrak{c} \in B$ such that*

$$d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}) \leq r d_{\mathcal{F}}(\mathfrak{e}, B).$$

In this way, we define the notion of generalized multivalued (α, ψ)-contraction in the framework of $\mathcal{F}$-MS.

Definition 10. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS and let Y be a nonempty set of $\mathfrak{P}$. A mapping $O : \mathfrak{P} \to CB(\mathfrak{P})$ is said to be a generalized multivalued (α, ψ)-contraction on the set Y if there exists $\alpha : Y \times Y \to [0, \infty)$ and ψ is a strictly increasing function satisfying the following conditions:*

(i) $O\mathfrak{e} \cap Y \neq \theta$,for all $\mathfrak{e} \in Y$.
(ii) For each $\mathfrak{e}, \mathfrak{c} \in Y$, we have

$$\alpha(\mathfrak{e}, \mathfrak{c}) H_{\mathcal{F}}(O\mathfrak{e} \cap Y, O\mathfrak{c} \cap Y) \leq \psi(R(\mathfrak{e}, \mathfrak{c})), \tag{4.1}$$

where

$$R(\mathfrak{e}, \mathfrak{c}) = \max\left\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}), \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{e}) + d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{c}) + d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{e})}{2}\right\}. \tag{4.2}$$

Theorem 5. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS, $Y \subseteq \mathfrak{P}$ is $\mathcal{F}$-complete in accordance with the metric derived from the $\mathcal{F}$-metric, denoted as $d_{\mathcal{F}}$, and let $O : \mathfrak{P} \to CB(\mathfrak{P})$ be a generalized multivalued (α, ψ)-contraction on Y. Let us consider the fulfillment of the following axioms:*

(i) O is a multivalued α-admissible mapping;
(ii) There exist $\mathfrak{e}_0 \in Y$ and $\mathfrak{e}_1 \in O\mathfrak{e}_0 \cap Y$ such that $\alpha(\mathfrak{e}_0, \mathfrak{e}_1) \geq 1$;
(iii) O is continuous.
Then, O has a FP.

Proof. By the supposition (ii), there exist $\mathfrak{e}_0 \in Y$ and $\mathfrak{e}_1 \in O\mathfrak{e}_0 \cap Y$ such that $\alpha(\mathfrak{e}_0, \mathfrak{e}_1) \geq 1$. If $\mathfrak{e}_0 = \mathfrak{e}_1$, then $\mathfrak{e}_0$ is the required fixed point, and we have nothing to prove. So, we suppose that $\mathfrak{e}_0 \neq \mathfrak{e}_1$. If $\mathfrak{e}_1 \in O\mathfrak{e}_1 \cap Y$, then $\mathfrak{e}_1$ is a fixed point. Let $\mathfrak{e}_1 \notin O\mathfrak{e}_1 \cap Y$. Now, by the inequality (4.1), we have

$$\begin{aligned} 0 &< \alpha(\mathfrak{e}_0, \mathfrak{e}_1) H_{\mathcal{F}}(O\mathfrak{e}_0 \cap Y, O\mathfrak{e}_1 \cap Y) \\ &\leq \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1), \\ \frac{d_{\mathcal{F}}(\mathfrak{e}_0, O\mathfrak{e}_0) + d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}_0, O\mathfrak{e}_1) + d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_0)}{2} \end{array}\right\}\right) \\ &\leq \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1), \\ \frac{d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1) + d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}_0, O\mathfrak{e}_1)}{2} \end{array}\right\}\right) \\ &\leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1), d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)\right\}\right), \end{aligned} \tag{4.3}$$

because

$$\frac{d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1) + d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)}{2} \leq \max\left\{d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1), d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)\right\},$$

and

$$\frac{d_{\mathcal{F}}(\mathfrak{e}_0, O\mathfrak{e}_1)}{2} \leq \max\left\{d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1), d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)\right\}.$$

Suppose that $\max\left\{d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1), d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)\right\} = d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1)$. Then, by the inequality (4.3), we have

$$0 < d_{\mathcal{F}}(\mathfrak{e}_1, O\mathfrak{e}_1 \cap Y) \leq \alpha(\mathfrak{e}_0, \mathfrak{e}_1) H_{\mathcal{F}}(O\mathfrak{e}_0 \cap Y, O\mathfrak{e}_1 \cap Y)$$

$$
\begin{aligned}
&\le \psi\left(d_{\mathcal{F}}(\mathrm{e}_1, O\mathrm{e}_1)\right) \\
&< d_{\mathcal{F}}(\mathrm{e}_1, O\mathrm{e}_1), \qquad (4.4)
\end{aligned}
$$

a contradiction to the supposition. Thus, $\max\{d_{\mathcal{F}}(\mathrm{e}_0, \mathrm{e}_1), d_{\mathcal{F}}(\mathrm{e}_1, O\mathrm{e}_1)\} = d_{\mathcal{F}}(\mathrm{e}_0, \mathrm{e}_1)$. Hence, by (4.3), we have

$$0 < d_{\mathcal{F}}(\mathrm{e}_1, O\mathrm{e}_1 \cap Y) \le \psi\left(d_{\mathcal{F}}(\mathrm{e}_0, \mathrm{e}_1)\right).$$

Now, for $r > 1$ by Lemma (2), there exists $\mathrm{e}_2 \in O\mathrm{e}_1 \cap Y$ such that

$$0 < d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2) < rd_{\mathcal{F}}(\mathrm{e}_1, O\mathrm{e}_1 \cap Y) \le r\psi\left(d_{\mathcal{F}}(\mathrm{e}_0, \mathrm{e}_1)\right). \qquad (4.5)$$

Applying ψ in the inequality (4.5), we have

$$0 < \psi\left(d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2)\right) < \psi\left(r\psi\left(d_{\mathcal{F}}(\mathrm{e}_0, \mathrm{e}_1)\right)\right).$$

Put $r_1 = \frac{\psi\left(r\psi\left(d_{\mathcal{F}}(\mathrm{e}_0,\mathrm{e}_1)\right)\right)}{\psi\left(d_{\mathcal{F}}(\mathrm{e}_1,\mathrm{e}_2)\right)}$, then $r_1 > 1$. Now, since $\alpha(\mathrm{e}_0, \mathrm{e}_1) \ge 1$ and the mapping O is an α-admissible mapping, we have $\alpha(\mathrm{e}_1, \mathrm{e}_2) \ge 1$. If $\mathrm{e}_2 \in O\mathrm{e}_2 \cap Y$, then e_2 is an FP of mapping O. Let $\mathrm{e}_2 \notin O\mathrm{e}_2 \cap Y$. Now, from (4.1), we have

$$
\begin{aligned}
0 &< \alpha(\mathrm{e}_1, \mathrm{e}_2) H_{\mathcal{F}}(O\mathrm{e}_1 \cap Y, O\mathrm{e}_2 \cap Y) \\
&\le \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), \\ \frac{d_{\mathcal{F}}(\mathrm{e}_1,O\mathrm{e}_1)+d_{\mathcal{F}}(\mathrm{e}_2,O\mathrm{e}_2)}{2}, \frac{d_{\mathcal{F}}(\mathrm{e}_1,O\mathrm{e}_2)+d_{\mathcal{F}}(\mathrm{e}_2,O\mathrm{e}_1)}{2} \end{array}\right\}\right) \\
&\le \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), \\ \frac{d_{\mathcal{F}}(\mathrm{e}_1,\mathrm{e}_2)+d_{\mathcal{F}}(\mathrm{e}_2,O\mathrm{e}_2)}{2}, \frac{d_{\mathcal{F}}(\mathrm{e}_1,O\mathrm{e}_2)}{2} \end{array}\right\}\right) \\
&\le \psi\left(\max\left\{d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)\right\}\right), \qquad (4.6)
\end{aligned}
$$

because

$$\frac{d_{\mathcal{F}}(\mathrm{e}_1,\mathrm{e}_2)+d_{\mathcal{F}}(\mathrm{e}_2,O\mathrm{e}_2)}{2} \le \max\left\{d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)\right\},$$

and

$$\frac{d_{\mathcal{F}}(\mathrm{e}_1,O\mathrm{e}_2)}{2} \le \max\left\{d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)\right\}.$$

Suppose that $\max\{d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)\} = d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)$. Then, by the inequality (4.6), we have

$$
\begin{aligned}
0 &< d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2 \cap Y) \le \alpha(\mathrm{e}_1, \mathrm{e}_2) H_{\mathcal{F}}(O\mathrm{e}_1 \cap Y, O\mathrm{e}_2 \cap Y) \\
&\le \psi\left(d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)\right) \\
&< d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2), \qquad (4.7)
\end{aligned}
$$

a contradiction to the supposition. Thus, $\max\{d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2), d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2)\} = d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2)$. Hence, by (4.6), we have

$$0 < d_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2 \cap Y) \le \psi\left(d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2)\right).$$

Now, for $r_1 > 1$ by Lemma (2), there exists $\mathrm{e}_3 \in O\mathrm{e}_2 \cap Y$ such that

$$0 < d_{\mathcal{F}}(\mathrm{e}_2, \mathrm{e}_3) < rd_{\mathcal{F}}(\mathrm{e}_2, O\mathrm{e}_2 \cap Y) \le r_1\psi\left(d_{\mathcal{F}}(\mathrm{e}_1, \mathrm{e}_2)\right) = \psi\left(r\psi\left(d_{\mathcal{F}}(\mathrm{e}_0, \mathrm{e}_1)\right)\right). \qquad (4.8)$$

Applying ψ in the inequality (4.8), we have

$$0 < \psi\left(d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3)\right) < \psi^2\left(r\psi\left(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)\right)\right).$$

Put $r_2 = \frac{\psi^2(r\psi(d_{\mathcal{F}}(\mathfrak{e}_0,\mathfrak{e}_1)))}{\psi(d_{\mathcal{F}}(\mathfrak{e}_1,\mathfrak{e}_2))}$, then $r_2 > 1$. Now, since $\alpha(\mathfrak{e}_0, \mathfrak{e}_1) \geq 1$ and the mapping $\mathcal{O}$ is an α-admissible mapping, we have $\alpha(\mathfrak{e}_2, \mathfrak{e}_3) \geq 1$. If $\mathfrak{e}_3 \in \mathcal{O}\mathfrak{e}_3 \cap Y$, then $\mathfrak{e}_3$ is an FP of mapping $\mathcal{O}$. Let $\mathfrak{e}_3 \notin \mathcal{O}\mathfrak{e}_3 \cap Y$. Now, from (4.1), we have

$$\begin{aligned}
0 &< \alpha(\mathfrak{e}_2, \mathfrak{e}_3) H_{\mathcal{F}}(\mathcal{O}\mathfrak{e}_2 \cap Y, \mathcal{O}\mathfrak{e}_3 \cap Y) \\
&\leq \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), \\ \frac{d_{\mathcal{F}}(\mathfrak{e}_2,\mathcal{O}\mathfrak{e}_2)+d_{\mathcal{F}}(\mathfrak{e}_3,\mathcal{O}\mathfrak{e}_3)}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}_2,\mathcal{O}\mathfrak{e}_3)+d_{\mathcal{F}}(\mathfrak{e}_3,\mathcal{O}\mathfrak{e}_2)}{2} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), \\ \frac{d_{\mathcal{F}}(\mathfrak{e}_2,\mathfrak{e}_3)+d_{\mathcal{F}}(\mathfrak{e}_3,\mathcal{O}\mathfrak{e}_2)}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}_2,\mathcal{O}\mathfrak{e}_3)}{2} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)\right\}\right),
\end{aligned} \tag{4.9}$$

because

$$\frac{d_{\mathcal{F}}(\mathfrak{e}_2,\mathfrak{e}_3)+d_{\mathcal{F}}(\mathfrak{e}_3,\mathcal{O}\mathfrak{e}_2)}{2} \leq \max\left\{d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)\right\},$$

and

$$\frac{d_{\mathcal{F}}(\mathfrak{e}_2,\mathcal{O}\mathfrak{e}_3)}{2} \leq \max\left\{d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)\right\}.$$

Suppose that $\max\left\{d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)\right\} = d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)$. Then, by the inequality (4.9), we have

$$\begin{aligned}
0 &< d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3 \cap Y) \leq \alpha(\mathfrak{e}_2, \mathfrak{e}_3) H_{\mathcal{F}}(\mathcal{O}\mathfrak{e}_2 \cap Y, \mathcal{O}\mathfrak{e}_3 \cap Y) \\
&\leq \psi\left(d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)\right) \\
&< d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3),
\end{aligned} \tag{4.10}$$

a contradiction to the supposition. Thus, $\max\left\{d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3), d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3)\right\} = d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3)$. Hence, by (4.6), we have

$$0 < d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3 \cap Y) \leq \psi\left(d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3)\right).$$

Now, for $r_2 > 1$, by Lemma (2), there exists $\mathfrak{e}_4 \in \mathcal{O}\mathfrak{e}_3 \cap Y$ such that

$$0 < d_{\mathcal{F}}(\mathfrak{e}_3, \mathfrak{e}_4) < r_2 d_{\mathcal{F}}(\mathfrak{e}_3, \mathcal{O}\mathfrak{e}_3 \cap Y) \leq r_2\psi\left(d_{\mathcal{F}}(\mathfrak{e}_2, \mathfrak{e}_3)\right) = \psi^2\left(r\psi\left(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)\right)\right). \tag{4.11}$$

Applying ψ in the inequality (4.8), we have

$$0 < \psi\left(d_{\mathcal{F}}(\mathfrak{e}_3, \mathfrak{e}_4)\right) < \psi^3\left(r\psi\left(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)\right)\right).$$

Continuing in this way, we generate a sequence $\{\mathfrak{e}_n\}$ in Y such that $\mathfrak{e}_{n+1} \in \mathcal{O}\mathfrak{e}_n \cap Y$ with $\mathfrak{e}_n \neq \mathfrak{e}_{n+1}$, $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \geq 1$, and

$$d_{\mathcal{F}}(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) < \psi^n\left(r\psi\left(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)\right)\right) \tag{4.12}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, suppose that $(\curlywedge, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow \curlywedge(t) < \curlywedge(\delta) - \mathfrak{h}. \tag{4.13}$$

Let $n(\epsilon)$ be some natural number such that $0 < \sum_{n\geq n(\epsilon)} \psi^n(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)) < \delta$. Therefore, by the inequalities (4.12) and (4.13), and the conditions $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, we have

$$\begin{aligned}\curlywedge\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\mathfrak{e}_i, \mathfrak{e}_{i+1})\right) &\leq \curlywedge\left(\sum_{i=n}^{m-1} \psi^i\left(r\psi\left(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)\right)\right)\right) \\ &\leq \curlywedge\left(\sum_{n\geq n(\epsilon)} \psi^n\left(r\psi\left(d_{\mathcal{F}}(\mathfrak{e}_0, \mathfrak{e}_1)\right)\right)\right) < \curlywedge(\epsilon) - \mathfrak{h} \qquad (4.14)\end{aligned}$$

for $m > n \geq n(\epsilon)$. Utilizing the condition (D_3) and the inequality (4.14), we get $d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}_m) > 0$, $m > n \geq n(\epsilon)$ implying

$$\curlywedge(d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}_m)) \leq \curlywedge\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\mathfrak{e}_i, \mathfrak{e}_{i+1})\right) + \mathfrak{h} < \curlywedge(\epsilon),$$

which yields by $(\mathcal{F}_1)$ that $d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}_m) < \epsilon$, $m > n \geq n(\epsilon)$. This confirms the sequence $\{\mathfrak{e}_n\}$ is $\mathcal{F}$-Cauchy. Given that $(\mathfrak{P}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, there is an element denoted by $\mathfrak{e}^* \in \mathfrak{P}$ such that $\{\mathfrak{e}_n\}$ is $\mathcal{F}$-convergent to $\mathfrak{e}^*$. Now, by $(\mathcal{F}_1)$, the continuity of the function $d_{\mathcal{F}}$ and the mapping O, we have

$$\curlywedge\left(d_{\mathcal{F}}\left(\mathfrak{e}^*, O\mathfrak{e}^*\right)\right) = \lim_{n\to\infty} \curlywedge\left(d_{\mathcal{F}}\left(\mathfrak{e}_{n+1}, O\mathfrak{e}^*\right)\right) \leq \lim_{n\to\infty} \curlywedge\left(H\left(O\mathfrak{e}_n, O\mathfrak{e}^*\right)\right) = -\infty,$$

which implies by $(\mathcal{F}_2)$ that we have $d_{\mathcal{F}}\left(\mathfrak{e}^*, O\mathfrak{e}^*\right) = 0$. Now, since O is closed, $\mathfrak{e}^* \in O\mathfrak{e}^*$. □

Theorem 6. *Let $(\mathfrak{P}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS, $Y \subseteq \mathfrak{P}$ is $\mathcal{F}$-complete in accordance with the metric derived from the $\mathcal{F}$-metric $d_{\mathcal{F}}$, and let $O : \mathfrak{P} \to CB(\mathfrak{P})$ be a generalized (α, ψ)-contraction on Y. Let us consider the fulfillment of the following axioms:*

(i) O is a multivalued α-admissible mapping;
(ii) There exist $\mathfrak{e}_0 \in Y$ and $\mathfrak{e}_1 \in O\mathfrak{e}_0 \cap Y$ such that $\alpha(\mathfrak{e}_0, \mathfrak{e}_1) \geq 1$;
(iii) $\alpha\left(\mathfrak{e}_n, \mathfrak{e}_{n+1}\right) \geq 1$ for any sequence $\{\mathfrak{e}_n\}$ in Y converging to $\mathfrak{e}$, then $\alpha\left(\mathfrak{e}_n, \mathfrak{e}\right) \geq 1$, for each $n \in \mathbb{N}\cup\{0\}$.
Then, O has a fixed point.

Proof. Following the proof of Theorem 5, there exist a Cauchy sequence $\{\mathfrak{e}_n\}$ in Y with $\mathfrak{e}_n \to \mathfrak{e}^*$ as $n \to \infty$, and $\alpha\left(\mathfrak{e}_n, \mathfrak{e}_{n+1}\right) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Then, by the assumption (iii), we have $\alpha\left(\mathfrak{e}_n, \mathfrak{e}^*\right) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Assume that $d_{\mathcal{F}}\left(\mathfrak{e}^*, O\mathfrak{e}^*\right) \neq 0$. Now, by (4.1), we have

$$\begin{aligned}0 &< d_{\mathcal{F}}(\mathfrak{e}_{n+1}, O\mathfrak{e}^* \cap Y) \leq \alpha(\mathfrak{e}_n, \mathfrak{e}^*) H_{\mathcal{F}}(O\mathfrak{e}_n \cap Y, O\mathfrak{e}^* \cap Y) \\ &\leq \psi\left(\max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}^*), \\ \frac{d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}_n) + d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}^*)}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}^*) + d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}_n)}{2} \end{array}\right\}\right) \\ &< \max\left\{\begin{array}{c} d_{\mathcal{F}}(\mathfrak{e}_n, \mathfrak{e}^*), \\ \frac{d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}_n) + d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}^*)}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}_n, O\mathfrak{e}^*) + d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}_n)}{2} \end{array}\right\}. \qquad (4.15)\end{aligned}$$

Taking the limit as $n \to \infty$ in (4.15), we have

$$d_{\mathcal{F}}\left(\mathfrak{e}^*, O\mathfrak{e}^*\right) \leq d_{\mathcal{F}}(\mathfrak{e}^*, O\mathfrak{e}^* \cap Y) \leq \frac{d_{\mathcal{F}}\left(\mathfrak{e}^*, O\mathfrak{e}^*\right)}{2},$$

which is impossible. Thus, $d_{\mathcal{F}}\left(\mathfrak{e}^*, O\mathfrak{e}^*\right) = 0$. □

Example 3. *Let $\mathfrak{P} = (-\infty, -6) \cup [0, +\infty)$ be endowed with the $\mathcal{F}$-metric defined by*

$$d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}) = \begin{cases} e^{(|\mathfrak{e}-\mathfrak{c}|)}, & \text{if } \mathfrak{e} \neq \mathfrak{c}, \\ 0, & \text{otherwise,} \end{cases}$$

with $\curlywedge(t) = -\frac{1}{t}$ and $\mathfrak{h}$ =1. Then, ($\mathfrak{P}$,$d_{\mathcal{F}}$) is $\mathcal{F}$-complete $\mathcal{F}$-MS. Define $O : \mathfrak{P} \to CL(\mathfrak{P})$ by

$$O\mathfrak{e} = \begin{cases} [0, \frac{\mathfrak{e}}{3}], & \text{if } 0 \leq \mathfrak{e} < 3, \\ \{0\}, & \text{if } \mathfrak{e} = 3, \\ (-\infty, -3] \cup [x, x^2], & \text{if } \mathfrak{e} > 3, \end{cases}$$

and $\alpha : \mathfrak{P} \times \mathfrak{P} \to [0, +\infty)$ by

$$\alpha(\mathfrak{e}, \mathfrak{c}) = \begin{cases} 1, & \text{if } \mathfrak{e}, \mathfrak{c} \in [0, 3], \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = [0, +\infty)$. Then evidently, $O\mathfrak{e} \cap Y \neq \emptyset$ for all $\mathfrak{e} \in \mathfrak{P}$. Let $\psi(t) = \frac{t}{2}$ for each $t \geq 0$. To show that O is a generalized multivalued (α, ψ)-contraction on Y, we consider these cases.
Case 1. If $\mathfrak{e}, \mathfrak{c} \in [0, 3)$, we have

$$\alpha(\mathfrak{e}, \mathfrak{c}) H_{\mathcal{F}}(O\mathfrak{e} \cap Y, O\mathfrak{c} \cap Y) = e^{(|\frac{\mathfrak{e}}{3} - \frac{\mathfrak{c}}{3}|)} \leq e^{(\frac{|\mathfrak{e}-\mathfrak{c}|}{2})} = \psi(d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c})) \leq \psi(R(\mathfrak{e}, \mathfrak{c})).$$

Case 2. If $\mathfrak{e} \in [0, 3)$ and $\mathfrak{c} = v$, we have

$$\alpha(\mathfrak{e}, \mathfrak{c}) H_{\mathcal{F}}(O\mathfrak{e} \cap Y, O\mathfrak{c} \cap Y) = e^{(|\frac{\mathfrak{e}}{3}|)} \leq \psi(\frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{e}) + d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{2}) \leq \psi(R(\mathfrak{e}, \mathfrak{c})).$$

Case 3. Otherwise, we have

$$\alpha(\mathfrak{e}, \mathfrak{c}) H_{\mathcal{F}}(O\mathfrak{e} \cap Y, O\mathfrak{c} \cap Y) = 0 \leq \psi(R(\mathfrak{e}, \mathfrak{c})).$$

For $\alpha(\mathfrak{e}, \mathfrak{c}) \geq 1$, we have $\mathfrak{e}, \mathfrak{c} \in [0, 3]$. Then, $O\mathfrak{e} \cap Y = \left[0, \frac{\mathfrak{e}}{3}\right] \subseteq [0, 1]$ and $O\mathfrak{c} \cap Y = \left[0, \frac{\mathfrak{c}}{3}\right] \subseteq [0, 1]$. Thus, $\alpha(\mathfrak{u}, \mathfrak{v}) \geq 1$, for every $\mathfrak{u} \in O\mathfrak{e} \cap Y$ and $\mathfrak{v} \in O\mathfrak{c} \cap Y$. Moreover, for any sequence $\{\mathfrak{e}_n\}$ in Y such that $\mathfrak{e}_n \to \mathfrak{e}$ as $n \to \infty$ and $\alpha(\mathfrak{e}_n, \mathfrak{e}_{n+1}) = 1$ for every $n \in \mathbb{N} \cup \{0\}$, $\lim \alpha(\mathfrak{e}_n, \mathfrak{e}) = 1$. Consequently, all the hypotheses of Theorem 6 are satisfied, ensuring that O possesses a fixed point.

Corollary 9. *Let $(\mathfrak{P}, \preceq, d_{\mathcal{F}})$ be an ordered $\mathcal{F}$-MS, and let $(Y, \preceq) \subseteq \mathfrak{P}$ be an $\mathcal{F}$-complete in accordance with the metric derived from the $\mathcal{F}$-metric $d_{\mathcal{F}}$. Let $O : \mathfrak{P} \to CB(\mathfrak{P})$ be a multivalued mapping such that $O\mathfrak{e} \cap Y \neq \emptyset$ for every $\mathfrak{e} \in Y$, and*

$$H_{\mathcal{F}}(O\mathfrak{e} \cap Y, O\mathfrak{c} \cap Y) \leq \psi(R(\mathfrak{e}, \mathfrak{c})),$$

where

$$R(\mathfrak{e}, \mathfrak{c}) = \max\left\{d_{\mathcal{F}}(\mathfrak{e}, \mathfrak{c}), \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{e}) + d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{c})}{2}, \frac{d_{\mathcal{F}}(\mathfrak{e}, O\mathfrak{c}) + d_{\mathcal{F}}(\mathfrak{c}, O\mathfrak{e})}{2}\right\}$$

for every $\mathfrak{e}, \mathfrak{c} \in Y$ with $\mathfrak{e} \preceq \mathfrak{c}$. Furthermore, let it be assumed that the subsequent conditions are met:

(i) There exist $\mathfrak{e}_0 \in Y$ and $\mathfrak{e}_1 \in O\mathfrak{e}_0 \cap Y$ such that $\mathfrak{e}_0 \preceq \mathfrak{e}_1$;

(ii) If $\mathfrak{e} \preceq \mathfrak{c}$, then $O\mathfrak{e} \cap Y \prec O\mathfrak{c} \cap Y$;

(iii) Either O is continuous or for $\mathfrak{e}_n \preceq \mathfrak{e}_{n+1}$ for any $\{\mathfrak{e}_n\} \subseteq Y$ converging to $\mathfrak{e}$, implying $\mathfrak{e}_n \preceq \mathfrak{e}$ for each $n \in \mathbb{N} \cup \{0\}$.

Then, O has a FP.

Proof. Define $\alpha : Y \times Y \to [0, +\infty)$ by

$$\alpha(\mathfrak{e}, \mathfrak{c}) = \begin{cases} 1, & \text{if } \mathfrak{e} \preceq \mathfrak{c}, \\ 0, & \text{otherwise.} \end{cases}$$

By using the assumption (i) and the definition of the function $\alpha : Y \times Y \to [0, +\infty)$, we have $\alpha(\mathfrak{e}_0, \mathfrak{e}_1) = 1$. Also, from the assumption (ii), we have that $\mathfrak{e} \preceq \mathfrak{c}$ implies $\mathcal{O}\mathfrak{e} \cap Y \prec \mathcal{O}\mathfrak{c} \cap Y$; by using the definition of the function $\alpha : Y \times Y \to [0, +\infty)$ and the partial order $\prec$, we have that $\alpha(\mathfrak{e}, \mathfrak{c}) = 1$ implies $\alpha(u, v) = 1$ for each $u \in \mathcal{O}\mathfrak{e} \cap Y$ and $v \in \mathcal{O}\mathfrak{c} \cap Y$. Moreover, it is simple to check that the mapping $\mathcal{O}$ is a generalized (α, ψ)-contraction on Y. Therefore, all the assertions of Theorems 5 and 6 are satisfied and, hence, the mapping $\mathcal{O}$ has a fixed point. □

5. Applications

Delay differential equations (DDEs), also known as differential equations with retarded argument, are a specific type of functional differential equation (FDE) where the current rate of change depends on the system's state at earlier times. Neutral DDEs serve as a common tool for characterizing dynamic systems reliant on both current and past states. Instances of such systems span various domains, encompassing biological models depicting the growth of single species [27], applications in thermal processes like steam or water pipes, heat exchanges [28], ecological models in population dynamics [29], and various engineering systems [28]. To explore neutral stochastic delay differential systems, the reader is directed to references [16–20].

This section focuses on finding the solution to the following differential equation:

$$\mathfrak{e}^{/}(t) = -a(t)\mathfrak{e}(t) + b(t)g(\mathfrak{e}(t - j(t))) + c(t)\mathfrak{e}^{/}(t - j(t)). \tag{5.1}$$

The lemma proposed by Djoudi et al. [26] proves to be highly useful in establishing the validity of our theorem.

Lemma 3. *([26]) Suppose that $j^{/}(t) \neq 1$, for all $t \in \mathbb{R}$. Then, $\mathfrak{e}(t)$ is a solution of (5.1) if*

$$\begin{aligned} \mathfrak{e}(t) = {} & \left(\mathfrak{e}(0) - \frac{c(0)}{1 - j^{/}(0)}\mathfrak{e}(-j(0))\right)\rho^{-\int_0^t \alpha(j)dj} + \frac{c(t)}{1 - j^{/}(t)}\mathfrak{e}(t - j(t)) \\ & - \int_0^t (h(u))\mathfrak{e}(u - j(u))) - b(u)g\left(\mathfrak{e}(u - j(u))\right)\rho^{-\int_u^t \alpha(j)dj} du, \end{aligned} \tag{5.2}$$

where

$$h(u) = \frac{j^{//}(u)c(u) + \left(c^{/}(u) + c(u)a(u)\right)(1 - j^{/}(u))}{(1 - j^{/}(u))^2}. \tag{5.3}$$

Now, let's consider $\varrho : (-\infty, 0] \to \mathbb{R}$ as a continuous bounded initial function. Then, if $\mathfrak{e}(t) = \mathfrak{e}(t, 0, \varrho)$, it represents a solution of (5.1) when $\mathfrak{e}(t) = \varrho(t)$ for $t \leq 0$, and satisfies (5.1) for $t \geq 0$. Let $\mathfrak{C}$ denote the set of $\varpi : (-\infty, +\infty) \to (-\infty, +\infty)$ comprising continuous functions. Define $\mathcal{B}_\varrho$ by

$$\mathcal{B}_\varrho = \{\varpi \in \mathfrak{C} \text{ such that } \varrho(t) = \varpi(t) \text{ if } t \leq 0, \varpi(t) \to 0 \text{ as } t \to \infty\}.$$

Thus, $\mathcal{B}_\varrho$ forms a Banach space under the supremum norm $\|\cdot\|$.

Lemma 4. *([10]) The space $(\mathcal{B}_{\varrho}, \|\cdot\|)$ equipped with the distance function $d_{\mathcal{F}}$, defined by*

$$d_{\mathcal{F}}(t, t^*) = \|t - t^*\| = \sup_{\mathfrak{e}\in I} |t(\mathfrak{e}) - t^*(\mathfrak{e})|$$

for $t, t^* \in \mathcal{B}_{\varrho}$, is an $\mathcal{F}$-MS.

Theorem 7. *Let $O : \mathcal{B}_{\varrho} \to \mathcal{B}_{\varrho}$ be a mapping defined by*

$$\begin{aligned}(O\varpi)(t) = &\left(\varpi(0) - \frac{c(0)}{1 - j^{/}(0)}\varpi(-j(0))\right)\rho^{-\int_0^t \alpha(j)dj} + \frac{c(t)}{1 - j^{/}(t)}\varpi(t - j(t)) \\ &- \int_0^t (h(u)\varpi(u - j(u)) - b(u)g\left(\varpi(u - j(u))\right))\rho^{-\int_u^t \alpha(j)dj}du, t \geq 0 \end{aligned} \tag{5.4}$$

for all $\varpi \in \mathcal{B}_{\varrho}$. Let's consider that these statements hold true:
(i) There exist $\mu \geq 0$ and $\varrho \in \Psi$ so that

$$\int_0^t |h(u)(\varpi(u - j(u))) - \omega(u - j(u))|\rho^{-\int_u^t \alpha(j)dj}du$$

$$\leq \frac{\mu}{2}\varrho\left(\max\left\{\|\varpi - \omega\|, \min\left\{\frac{\|\varpi - O\varpi\|\,\|\omega - O\omega\|}{1 + \|\varpi - \omega\|}, \frac{\|\varpi - O\omega\|\,\|\omega - O\varpi\|}{1 + \|\varpi - \omega\|}\right\}\right\}\right), \tag{5.5}$$

and

$$\int_0^t |(b(u))g(\varpi(u - j(u))) - g(\omega(u - j(u)))|\rho^{-\int_u^t \alpha(j)dj}du$$

$$\leq \frac{\mu}{2}\varrho\left(\max\left\{\|\varpi - \omega\|, \min\left\{\frac{\|\varpi - O\varpi\|\,\|\omega - O\omega\|}{1 + \|\varpi - \omega\|}, \frac{\|\varpi - O\omega\|\,\|\omega - O\varpi\|}{1 + \|\varpi - \omega\|}\right\}\right\}\right) \tag{5.6}$$

for all $\varpi, \omega \in \mathcal{B}_{\varrho}$.
Then, O has a fixed point.

Proof. Define $\alpha, \eta : \mathfrak{C} \times \mathfrak{C} \to \mathbb{R}$ by

$$\alpha(\varpi, \omega) = \eta(\varpi, \omega) = \begin{cases} 1, & \text{if } \varpi, \omega \in \mathcal{B}_{\varrho}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for $\varpi, \omega \in \mathcal{B}_{\varrho}$ such that $\alpha(\varpi, \omega) = \eta(\varpi, \omega) \geq 1$. It follows from (5.3) that $O(\varpi), O(\omega) \in \mathcal{B}_{\varrho}$. Therefore, $\alpha(O(\varpi), O(\omega)) = \eta(O(\varpi), O(\omega)) \geq 1$. Since, (5.4)–(5.6) hold, then for $\varpi, \omega \in \mathcal{B}_{\varrho}$, we have

$$\begin{aligned}|(O\varpi)(t) - (O\omega)(t)| \leq &\left|\frac{c(t)}{1 - j^{/}(t)}\right| \|\varpi - \omega\| \\ &+ \int_0^t |h(u)(\varpi(u - j(u))) - \omega(u - j(u))|\rho^{-\int_u^t \alpha(s)ds}du \\ &+ \int_0^t |(b(u))g(\varpi(u - j(u))) - g(\omega(u - j(u)))|\rho^{-\int_u^t \alpha(s)ds}du\end{aligned}$$

$$\leq \left|\frac{c(t)}{1-j'(t)}\right| \|\varpi-\omega\| + \mu\varrho\left(\max\left\{\begin{array}{c}\|\varpi-\omega\|, \\ \min\left\{\begin{array}{c}\frac{\|\varpi-O\varpi\|\|\omega-O\omega\|}{1+\|\varpi-\omega\|}, \\ \frac{\|\varpi-O\omega\|\|\omega-O\varpi\|}{1+\|\varpi-\omega\|}\end{array}\right\}\end{array}\right\}\right)$$

$$\leq \left\{\left|\frac{c(t)}{1-j'(t)}\right| + \mu\right\}\varrho\left(\max\left\{\begin{array}{c}\|\varpi-\omega\|, \\ \min\left\{\begin{array}{c}\frac{\|\varpi-O\varpi\|\|\omega-O\omega\|}{1+\|\varpi-\omega\|}, \\ \frac{\|\varpi-O\omega\|\|\omega-O\varpi\|}{1+\|\varpi-\omega\|}\end{array}\right\}\end{array}\right\}\right)$$

$$\leq \varrho\left(\max\left\{\begin{array}{c}\|\varpi-\omega\|, \\ \min\left\{\begin{array}{c}\frac{\|\varpi-O\varpi\|\|\omega-O\omega\|}{1+\|\varpi-\omega\|}, \\ \frac{\|\varpi-O\omega\|\|\omega-O\varpi\|}{1+\|\varpi-\omega\|}\end{array}\right\}\end{array}\right\}\right).$$

Hence,

$$d_{\mathcal{F}}(O\varpi, O\omega) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\varpi,\omega), \min\left\{\begin{array}{c}\frac{d_{\mathcal{F}}(\varpi,O\varpi)d_{\mathcal{F}}(\omega,O\omega)}{1+d_{\mathcal{F}}(\varpi,\omega)}, \\ \frac{d_{\mathcal{F}}(\varpi,O\omega)d_{\mathcal{F}}(\omega,O\varpi)}{1+d_{\mathcal{F}}(\varpi,\omega)}\end{array}\right\}\right\}\right)$$

implies that O is a rational ($\alpha\eta$-ψ)-contraction. Building upon Theorem 3, we can conclude that the mapping O possesses a unique fixed point within the set $\mathcal{B}_{\varrho}$. This fixed point satisfies the differential equation (5.1). □

6. Conclusions

In this research article, we have made significant contributions to the field of FP theory in $\mathcal{F}$-MS. By introducing innovative contraction conditions, we have established several new FP theorems. Our principal result provides a unified framework for various existing theorems. Additionally, we have presented a concrete example to illustrate the applicability of our findings. To demonstrate the practical utility of our findings, we have applied our main results to the solution of synaptic DDEs in neural networks. Future research will focus on extending these results to the domain of generalized contractive mappings, with a particular emphasis on solving fractional differential inclusion problems.

Author contributions

All authors contributed equally in this work. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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