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Research article

Novel wave solutions for the sixth-order Boussinesq equation arising in nonlinear lattice dynamics

Ali Althobaiti*

Department of Mathematics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: aa.althobaiti@tu.edu.sa.

Abstract: This study examines a class of Boussinesq equations with sixth-order using two promising analytical methods. The equation in question is among the frontier evolution equations with significant relevance in nonlinear lattice dynamics. To study this model, the Kudryashov method and the modified auxiliary equation method are employed due to their analytical precision in constructing several exact wave solutions for the model under examination. As expected, the methods yield many valid solution sets that satisfy all the underlying assumptions of the model. Finally, some of the obtained wave solutions are graphically illustrated, taking into account the parameter values of the model.

Keywords: higher-order evolution equations; Boussinesq equation; exact wave solutions; Kudryashov method; modified auxiliary equation method
Mathematics Subject Classification: 35A09, 35A24, 35C07, 35C08

1. Introduction

Due to recent technological advancements, nonlinear evolution equations have become crucial in modeling various physical processes across different fields, including fluid dynamics, pulse propagation, optical media, and the telecommunication sector, to mention a few [1, 2]. One notable equation on which this study focuses on is the classical Boussinesq equation, introduced by Joseph Boussinesq in 1871 [3]. This equation is renowned for representing the propagation of surface waves in water under conditions of long wavelength and small amplitude. Additionally, it has been widely utilized in literature to model concepts related to water, including coastal and harbor engineering, tide and tsunami simulations, and others [4]. Moreover, several variants of the Boussinesq equation exist, including the fourth-order Boussinesq equation [4], the sixth-order Boussinesq equation [4, 5], the coupled variant of Boussinesq equations [6], the $(2 + 1)$-dimensional Boussinesq equation [7], and the class of higher-order Boussinesq Burgers equations [8, 9], among others. In this context, it is worth noting that the Boussinesq equation is the first nonlinear evolution equation to mathematically explain the concept of solitons, or solitary waves, which were first described in the 1830s by Scottish naval architect and civil engineer John Scott Russell [10].

The exploration of nonlinear evolution equations cannot be completed without considering the mathematical methods proposed to solve the governing equations. This requires a thorough investigation of the inherent physical and significant theoretical features underlying these equations. In light of this, many scientists have proposed different methods, including analytical, semi-analytical, and numerical techniques, to solve various equations. The Kudryashov method [11], the modified auxiliary equation method [12, 13], the modified direct algebraic method [14], the bilinear transformation method [15, 16], the exponential ansatz method [17], and the modified tan method [18] are examples of analytical methods. Additionally, some well-known semi-analytical methods include the Adomian decomposition methods [19, 20], the modified decomposition approach [21], the semi-inverse variational principle [22], the collective variable method [23], and the variational iteration approach [24]. Relevant numerical approaches for evolution equations can be found in [25–27].

The importance of the Boussinesq equations, specifically the sixth-order Boussinesq equation [4, 5], has been discussed by several authors; see the work by Christov et al. [28], which presents insightful research on the sixth-order Boussinesq equation, demonstrating its stability and accuracy in modeling water wave propagation and nonlinear elastic crystal media evolution. This study utilizes two promising analytical techniques, the Kudryashov method [11] and the modified auxiliary equation method [12], to seek various exact wave solutions of the governing sixth-order Boussinesq equation. The present study aims to expand the existing literature by providing additional diverse exact solutions for computational validation, addressing a current gap in the research. Additionally, references [29, 30] and the studies cited within them provide various investigations related to the sixth-order Boussinesq equation, covering topics such as well-posedness, blow-up phenomena, global roughness, and inversion dynamics, which do not concentrate on a range of exact solutions. Additionally, complete solution sets for each approach will be determined for the model, and several obtained wave solutions will be graphically illustrated, taking into account the fixed parameter values of the model. Furthermore, the selection of these two methods is linked to their reliability and effectiveness in revealing various wave solutions for a wide range of both real and complex valued evolution equations; see [31–34]. The current paper is organized as follows: Section 2 presents the governing model, while Sections 3 and 4 outline the proposed methods. Section 5 is devoted for the application of the adopted methods. Section 6 provides the graphical depictions and discussion, and concluding remarks are given in Section 7.

2. Governing nonlinear equation

The current study aims to extensively examine the sixth-order Boussinesq equation. This equation is free from ill-posedness and effectively models the propagation of water waves and the evolution of lattice nonlinear elastic crystal media, to name a few. The explicit expression for the equation is given as follows [4, 5]:

$$u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} + (u^2)_{xx} = 0, \qquad t > 0, \tag{2.1}$$

where β is a real constant. Certainly, this equation is among the famous evolution equations used in modelling nonlinear lattice dynamics and the movement of water waves. The equation originates from the classical Boussinesq equation [6], which is given by

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = 0, \tag{2.2}$$

as introduced in 1871 by Joseph Boussinesq. It models the propagation of surface waves in water under conditions of long wavelengths and small amplitude.

This equation has been highly examined by various scientists and subsequently modified to yield several interesting models, such as the "good" fourth-order Boussinesq equation, proposed by Li et al. [4], which is expressed as

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, \tag{2.3}$$

and the coupled Boussinesq-Burgers' equations which are expressed as follows [8, 9]

$$\begin{cases} u_t - \frac{1}{2}v_x + 2uu_x = 0, \\ v_t - \frac{1}{2}u_{xxx} + 2(vu)_x = 0. \end{cases} \tag{2.4}$$

In this regard, Eq (2.1) is a higher-order equation, where higher-order equations are generally characterized by several underlying physical assumptions. Therefore, the current study examines the sixth-order Boussinesq equation expressed in (2.1) using two analytical techniques: the Kudryashov method [11] and the modified auxiliary equation method [12]. Furthermore, this study aims to contribute to the limited literature on diverse exact solutions for computational validation. Studies in [28, 29] regarding the relevance of the sixth-order Boussinesq equation, including aspects such as well-posedness, blow-up, global roughness, and inversion dynamics, focus on the model's analysis. Additionally, other researchers have examined several Boussinesq-like equations from different perspectives [35, 36] Complete solution sets for each of the employed approaches will be determined and further graphically examined for some fixed parameters of interest, providing insight into the effects of the involved parameters.

3. Kudryashov method (KM)

The steps of the KM are summarized here. Consider a nonlinear partial differential equation (NPDE) in the form

$$\Omega(v, v_x, v_t, v_{xt}, v_{xx}, ...) = 0, \tag{3.1}$$

such that Ω is a polynomial in v and its partial derivatives.

Step 1. We start by applying the transformation

$$v(x,t) = U(\xi), \qquad \xi = kx - ct, \tag{3.2}$$

where ξ is a new variable and k, c are constants. The transformation (3.2) converts the NPDE (3.1) to the nonlinear ordinary differential equation (ODE) as follows

$$\Phi(U, U'', ...) = 0, \tag{3.3}$$

and Φ is a polynomial in U and its derivatives.

Step 2. The solution of Eq (3.3) is assumed to be in the form

$$U(\xi) = \sum_{i=0}^{n} \rho_i \, \phi^i(\xi), \tag{3.4}$$

where n is a positive integer, and ρ_i are arbitrary constants (not all equal to zero) to be determined. Moreover, $\phi(\xi)$ satisfies the following ODE

$$\phi'(\xi) = \phi^2(\xi) - \phi(\xi). \tag{3.5}$$

Further, Eq (3.5) has the following solution

$$\phi(\xi) = \frac{1}{1 + \mu e^{\xi}}, \tag{3.6}$$

where μ is a non-zero arbitrary constant known as the Kudryashov-index. In addition, when $\mu > 0$, stable solutions are obtained; while for $\mu < 0$, one gets singular unstable solitons.

Step 3. Next, the value of n, which appears in the summation of (3.4), can be obtained using the balancing principle.

Step 4. By substituting (3.4) together with (3.5) into (3.3) and putting all terms with the same power of $\phi(\xi)$ to zero, this yields a set of over-determined systems of algebraic equations for ρ_i.

Step 5. Consequently, one then solves the obtained set of over-determined system of algebraic equations for ρ_0, ρ_j, for $j = 1, 2, ..., n$, to get hold of the possible solution sets for ρ_j, for $j = 0, 1, 2, ..., n$.

Step 6. Lastly, the wave transformation in (3.2) is reversed, along with the application of Eq (3.1), to derive the exact solutions for the governing nonlinear partial differential equations. In fact, such procedures are implemented using Mathematica software in the present study.

4. Modified auxiliary equation method (MAEM)

We start by considering the nonlinear PDE expressed in (3.1) and proceed to apply the **Step 1.** in the previous section.

Step 2. The solution of Eq (3.3) is assumed to be in the form

$$U(\xi) = \sum_{i=0}^{n} \rho_i \, \phi^i(\xi), \tag{4.1}$$

where ρ_i are unknown constants (not all equal to zero, to be determined), and $\phi(\xi)$ satisfies the following differential equation

$$\phi'^2(\xi) = \mu_2 \phi^2(\xi) + \mu_4 \phi^4(\xi) + \mu_6 \phi^6(\xi), \tag{4.2}$$

where μ_2, μ_4, and μ_6 are constants. The solution of Eq (4.2) takes the following forms:

Case 1. If $\mu_2 > 0$, then Eq (4.2) has solutions in the forms

$$\phi_1(\xi) = \sqrt{\frac{-\mu_2 \, \mu_4 \, \mathrm{sech}^2(\sqrt{\mu_2}\xi)}{\mu_4^2 - \mu_2 \, \mu_6 (1 + \epsilon \tanh(\sqrt{\mu_2}\xi)^2}}, \tag{4.3}$$

$$\phi_2(\xi) = \sqrt{\frac{-\mu_2\,\mu_4\,\mathrm{csch}^2(\sqrt{\mu_2}\xi)}{\mu_4^2 - \mu_2\,\mu_6(1+\epsilon\,\coth(\sqrt{\mu_2}\xi)^2}}, \tag{4.4}$$

$$\phi_3(\xi) = 4\sqrt{\frac{\mu_2 e^{2\epsilon\sqrt{\mu_2}\xi}}{(e^{2\epsilon\sqrt{\mu_2}\xi} - 4\mu_4)^2 - 64\mu_2\mu_6}}, \tag{4.5}$$

where $\epsilon = \pm 1$.

Case 2. If $\mu_2 > 0, \Delta > 0$, then (4.2) has a solution in the form

$$\phi_4(\xi) = \sqrt{\frac{2\mu_2}{\epsilon\,\sqrt{\Delta}\,\cosh(2\sqrt{\mu_2}\,\xi) - \mu_4}}, \tag{4.6}$$

where $\epsilon = \pm 1$ and $\Delta = \mu_4^2 - 4\mu_2\mu_6$.

Case 3. If $\mu_2 < 0, \Delta > 0$, then (4.2) has solutions in the forms

$$\phi_5(\xi) = \sqrt{\frac{2\mu_2}{\epsilon\,\sqrt{\Delta}\,\cos(2\sqrt{-\mu_2}\,\xi) - \mu_4}}, \tag{4.7}$$

$$\phi_6(\xi) = \sqrt{\frac{2\mu_2}{\epsilon\,\sqrt{\Delta}\,\sin(2\sqrt{-\mu_2}\,\xi) - \mu_4}}, \tag{4.8}$$

where $\epsilon = \pm 1$ and $\Delta = \mu_4^2 - 4\mu_2\mu_6$.

Step 3. Next, the value of n can be obtained using the balancing principle.

Step 4. By substituting (4.1) together with (4.2) into (3.3) and putting all terms with the same power of $\phi(\xi)$ to zero, yields a set of over-determined systems of algebraic equations for ρ_i.

Step 5. Consequently, one then solves the obtained set of over-determined systems of algebraic equations for ρ_j, where $j = 0, 1, 2, ..., n$.

Step 6. Finally, one can reverse the wave transformation used in Eq (3.2) and apply Eq (4.1) along with the solutions from Cases 1–3 to derive the exact solutions for the governing nonlinear partial differential equations.

5. Applications

This section shows the application of the Kudryashov and modified auxiliary methods for constructing distinct types of solutions for the sixth-order Boussinesq equation. Thus, to begin with, the wave transformation given in **Step 1.** is used on the governing sixth-order Boussinesq equation in (2.1) to obtain the corresponding nonlinear ODE as follows

$$\left(c^2 - k^2\right)U'' + \beta k^4 U^{(iv)} + k^6 U^{(vi)} + k^2(U^2)'' = 0. \tag{5.1}$$

Based on the balance principle, the value of n is given by

$$n + 6 = 2n + 2, \implies n = 4. \tag{5.2}$$

Therefore, with the determination of $n = 4$ above, the assumed solution from (3.4) and (4.1) concerning the application of both the Kudryashov and modified auxiliary methods for the governing Boussinesq equation takes the following solution form

$$U(\xi) = \rho_0 + \rho_1\phi(\xi) + \rho_2\phi^2(\xi) + \rho_3\phi^3(\xi) + \rho_4\phi^4(\xi), \tag{5.3}$$

where $\rho_0, \rho_1, \rho_2, \rho_3$ and ρ_4 are constants to be determined later.

5.1. Application of Kudryashov method

Now, with the use of the present Kudryashov method, Eq (5.3) is substituted into (5.1) to obtain the following over-determine system of algebraic equations

$$\begin{aligned}
&c^2\rho_1 - \rho_1k^6 + \beta\rho_1k^4 + 2\rho_0\rho_1k^2 - \rho_1k^2 = 0,\\
&-3c^2\rho_1 + 4c^2\rho_2 + 63\rho_1k^6 - 64\rho_2k^6 - 15\beta\rho_1k^4 + 16\beta\rho_2k^4 +\\
&4\rho_1^2k^2 - 6\rho_0\rho_1k^2 + 3\rho_1k^2 + 8\rho_0\rho_2k^2 - 4\rho_2k^2 = 0,\\
&2c^2\rho_1 - 10c^2\rho_2 + 9c^2\rho_3 - 602\rho_1k^6 + 1330\rho_2k^6 - 729\rho_3k^6 + 50\beta\rho_1k^4\\
&-130\beta\rho_2k^4 + 81\beta\rho_3k^4 - 10\rho_1^2k^2 + 4\rho_0\rho_1k^2 - 2\rho_1k^2 - 20\rho_0\rho_2k^2\\
&+18\rho_1\rho_2k^2 + 10\rho_2k^2 + 18\rho_0\rho_3k^2 - 9\rho_3k^2 = 0,\\
&6c^2\rho_2 - 21c^2\rho_3 + 16c^2\rho_4 + 2100\rho_1k^6 - 8106\rho_2k^6 + 10101\rho_3k^6 - 4096\rho_4k^6 - 60\beta\rho_1k^4 +\\
&330\beta\rho_2k^4 - 525\beta\rho_3k^4 + 256\beta\rho_4k^4 + 6\rho_1^2k^2 + 16\rho_2^2k^2 + 12\rho_0\rho_2k^2 -\\
&42\rho_1\rho_2k^2 - 6\rho_2k^2 - 42\rho_0\rho_3k^2 + 32\rho_1\rho_3k^2 + 21\rho_3k^2 + 32\rho_0\rho_4k^2 - 16\rho_4k^2 = 0,\\
&12c^2\rho_3 - 36c^2\rho_4 - 3360\rho_1k^6 + 21840\rho_2k^6 - 48972\rho_3k^6 + 46116\rho_4k^6 + 24\beta\rho_1k^4\\
&-336\beta\rho_2k^4 + 1164\beta\rho_3k^4 - 1476\beta\rho_4k^4 - 36\rho_2^2k^2 + 24\rho_1\rho_2k^2 + 24\rho_0\rho_3k^2\\
&-72\rho_1\rho_3k^2 + 50\rho_2\rho_3k^2 - 12\rho_3k^2 - 72\rho_0\rho_4k^2 + 50\rho_1\rho_4k^2 + 36\rho_4k^2 = 0,\\
&20c^2\rho_4 + 2520\rho_1k^6 - 29400\rho_2k^6 + 113400\rho_3k^6 - 195020\rho_4k^6 + 120\beta\rho_2k^4 - 1080\beta\rho_3k^4 +\\
&3020\beta\rho_4k^4 + 20\rho_2^2k^2 + 36\rho_3^2k^2 + 40\rho_1\rho_3k^2 - 110\rho_2\rho_3k^2 + 40\rho_0\rho_4k^2 - 110\rho_1\rho_4k^2\\
&+72\rho_2\rho_4k^2 - 20\rho_4k^2 = 0,\\
&-720\rho_1k^6 + 19440\rho_2k^6 - 136800\rho_3k^6 + 409200\rho_4k^6 + 360\beta\rho_3k^4 - 2640\beta\rho_4k^4\\
&-78\rho_3^2k^2 + 60\rho_2\rho_3k^2 + 60\rho_1\rho_4k^2 - 156\rho_2\rho_4k^2 + 98\rho_3\rho_4k^2 = 0,\\
&-5040\rho_2k^6 + 83160\rho_3k^6 - 457800\rho_4k^6 + 840\beta\rho_4k^4 + 42\rho_3^2k^2 + 64\rho_4^2k^2 +\\
&84\rho_2\rho_4k^2 - 210\rho_3\rho_4k^2 = 0, \quad 72k^2\rho_4^2 - 60480k^6\rho_4 = 0\\
&-20160\rho_3k^6 + 262080\rho_4k^6 - 136\rho_4^2k^2 + 112\rho_3\rho_4k^2 = 0.
\end{aligned} \tag{5.4}$$

Hence, on solving the resulting algebraic system expressed in (5.4), one gets the following solution set:

Set 1.

$$\begin{aligned}
&\rho_0 = \frac{-36\beta^3 + 169\beta - 2197c^2}{338\beta}, \quad \rho_1 = 0, \quad \rho_2 = \frac{840\beta^2}{169},\\
&\rho_3 = -\frac{1}{169}\left(1680\beta^2\right), \quad \rho_4 = \frac{840\beta^2}{169}, \quad k = \pm\frac{\sqrt{\beta}}{\sqrt{13}}, \quad c = c.
\end{aligned} \tag{5.5}$$

Therefore, with the above solution set, the governing one-dimensional sixth-order Boussinesq equation admits the following solution

$$u_{1_\pm}(x,t) = \frac{-36\beta^3 + 169\beta - 2197c^2}{338\beta} + \frac{840\beta^2}{169(1+\mu e^{\xi})^2} - \frac{1680\beta^2}{169(1+\mu e^{\xi})^3} + \frac{840\beta^2}{169(1+\mu e^{\xi})^4}, \tag{5.6}$$

where

$$\xi = \pm\frac{\sqrt{\beta}}{\sqrt{13}}x - ct. \tag{5.7}$$

Remarkably, the solution set described above is not the only one uncovered by the adopted Kudryashov method. However, it is considered the most physically relevant because it satisfies all the underlying assumptions of the model. In fact, several complex-valued solutions were also revealed by the method, but these complex-valued solutions violate the physical assumptions of the model. Additionally, various solitonic solutions can be constructed using modified methods of the standard Kudryashov method; see [31, 32]. In this regard, the nonlinear ODE in (3.5) can be modified to either [31].

$$\phi'(\xi) = [\phi^2(\xi) - \phi(\xi)]\ln(\eta), \qquad \eta \neq 1, \tag{5.8}$$

or [32]

$$\phi'^2(\xi) = \phi^2(\xi) - \zeta\phi^4(\xi). \tag{5.9}$$

The latter ODEs are admitted the following exact exponential solutions, respectively,

$$\phi(\xi) = \frac{1}{1+\mu\eta^{\xi}}, \tag{5.10}$$

or

$$\phi(\xi) = \frac{4\mu}{4\mu^2 e^{\xi} + \zeta e^{-\xi}}, \tag{5.11}$$

where ζ is a non-zero arbitrary constant, while μ is the Kudryashov-index. Furthermore, based on the modifications made to the Kudryashov method, the following supplementary solution sets are obtained.

Set 2.

$$\begin{aligned} \rho_0 &= \frac{-36\beta^3 + 169\beta - 2197c^2\ln^2(\eta)}{338\beta}, \quad \rho_1 = 0, \quad \rho_2 = \frac{840\beta^2}{169}, \\ \rho_3 &= -\frac{1}{169}\left(1680\beta^2\right), \quad \rho_4 = \frac{840\beta^2}{169}, \quad k = \pm\frac{\sqrt{\beta}}{\sqrt{13}\ln(\eta)}. \end{aligned} \tag{5.12}$$

Thus, the solution for the sixth-order Boussinesq equation is given by

$$u_{2_\pm}(x,t) = \frac{-36\beta^3 + 169\beta - 2197c^2\ln^2(\eta)}{338\beta} + \frac{840\beta^2}{169(1+\mu\eta^{\xi})^2} - \frac{1680\beta^2}{169(1+\mu\eta^{\xi})^3} + \frac{840\beta^2}{169(1+\mu\eta^{\xi})^4}, \tag{5.13}$$

where

$$\xi = \pm\frac{\sqrt{\beta}}{\sqrt{13}\ln(\eta)}x - ct, \qquad \text{and} \qquad \eta \neq 1. \tag{5.14}$$

Set 3.

$$\rho_0 = \frac{-36\beta^3 + 169\beta - 8788c^2}{338\beta}, \quad \rho_1 = 0, \quad \rho_2 = 0, \quad \rho_3 = 0, \quad \rho_4 = \frac{105\beta^2\zeta^2}{338}, \quad k = \pm\frac{\sqrt{\beta}}{2\sqrt{13}}. \quad (5.15)$$

This gives the following solution

$$u_{3_\pm}(x,t) = \frac{-36\beta^3 + 169\beta - 8788c^2}{338\beta} + \frac{13440\beta^2\zeta^2\mu^4}{169(4\mu^2 e^{\xi} + \zeta e^{-\xi})^4}, \quad (5.16)$$

where

$$\xi = \pm\frac{\sqrt{\beta}}{2\sqrt{13}}x - ct. \quad (5.17)$$

Moreover, the solutions in (5.16) can also be seen as bright solitonic solution (see [32]), which are significant in optical media.

Similarly, it is worth to mention the strong connection - or rather similarity- between the employed Kudryashov method used here and the tanh-coth method [37–39], which is an important analytical method primarily used to construct various periodic and dark solitonic solutions. In fact, both methods complement each other when manipulating the constant coefficients $\{\zeta_1, \zeta_2, \zeta_3\}$ of the associated Riccati equation, expressed as $\phi'(\xi) = \zeta_1 + \zeta_2\phi(\xi) + \zeta_3\phi^2(\xi)$. For further details, we refer interested reader(s) to the work of Kudryashov and Shilnikov [40], which deeply analyzed all the possible of the involving Riccati equation to reveal various exact analytical solutions.

5.2. *Application of modified auxiliary equation method*

Accordingly, substitution of (5.3) together with (4.2) into (5.1) and putting the coefficients of $\phi(\xi)$ to zero gives the following system of algebraic equations

$$\begin{aligned}
&\beta\rho_1\mu_2^2 + c^2\rho_1\mu_2 - \rho_1\mu_2^3 + 2\rho_0\rho_1\mu_2 - \rho_1\mu_2 = 0,\\
&16\beta\rho_2\mu_2^2 + 4c^2\rho_2\mu_2 - 64\rho_2\mu_2^3 + 4\rho_1^2\mu_2 + 8\rho_0\rho_2\mu_2 - 4\rho_2\mu_2 = 0,\\
&81\beta\rho_3\mu_2^2 + 20\beta\rho_1\mu_4\mu_2 + 9c^2\rho_3\mu_2 + 2c^2\rho_1\mu_4 - 729\rho_3\mu_2^3 - 182\rho_1\mu_4\mu_2^2\\
&\ + 18\rho_1\rho_2\mu_2 + 18\rho_0\rho_3\mu_2 - 9\rho_3\mu_2 + 4\rho_0\rho_1\mu_4 - 2\rho_1\mu_4 = 0,\\
&256\beta\rho_4\mu_2^2 + 120\beta\rho_2\mu_4\mu_2 + 16c^2\rho_4\mu_2 + 6c^2\rho_2\mu_4 - 4096\rho_4\mu_2^3 - 2016\rho_2\mu_4\mu_2^2\\
&\ + 16\rho_2^2\mu_2 + 32\rho_1\rho_3\mu_2 + 32\rho_0\rho_4\mu_2 - 16\rho_4\mu_2 + 6\rho_1^2\mu_4 + 12\rho_0\rho_2\mu_4 - 6\rho_2\mu_4 = 0,\\
&24\beta\rho_1\mu_4^2 + 408\beta\rho_3\mu_2\mu_4 + 12c^2\rho_3\mu_4 - 840\rho_1\mu_2\mu_4^2\\
&\ + 50\rho_2\rho_3\mu_2 + 50\rho_1\rho_4\mu_2 - 11172\rho_3\mu_2^2\mu_4 + 24\rho_1\rho_2\mu_4 + 24\rho_0\rho_3\mu_4 - 12\rho_3\mu_4 = 0,\\
&120\beta\rho_2\mu_4^2 + 1040\beta\rho_4\mu_2\mu_4 + 20c^2\rho_4\mu_4 - 6720\rho_2\mu_2\mu_4^2 + 36\rho_3^2\mu_2 + 72\rho_2\rho_4\mu_2\\
&\ + 20\rho_2^2\mu_4 - 42560\rho_4\mu_2^2\mu_4 + 40\rho_1\rho_3\mu_4 + 40\rho_0\rho_4\mu_4 - 20\rho_4\mu_4 = 0,\\
&360\beta\rho_3\mu_4^2 - 720\rho_1\mu_4^3 - 29880\rho_3\mu_2\mu_4^2 + 60\rho_2\rho_3\mu_4 + 60\rho_1\rho_4\mu_4 + 98\rho_3\rho_4\mu_2 = 0,\\
&840\beta\rho_4\mu_4^2 - 5040\rho_2\mu_4^3 - 97440\rho_4\mu_2\mu_4^2 + 42\rho_3^2\mu_4 + 84\rho_2\rho_4\mu_4 + 64\rho_4^2\mu_2 = 0,\\
&112\rho_3\rho_4\mu_4 - 20160\rho_3\mu_4^3 = 0,\\
&72\rho_4^2\mu_4 - 60480\rho_4\mu_4^3 = 0.
\end{aligned} \quad (5.18)$$

Consequently, solving the above system gives the following set:

Set 1.

$$\rho_0 = \frac{1}{2}\left(-c^2 - 576\mu_2^2 + 1\right), \rho_1 = \rho_2 = \rho_3 = 0, \rho_4 = 840\mu_4^2, \beta = 52\mu_2, \mu_6 = 0, k = 1. \tag{5.19}$$

Hence, on substituting these values into Eq (5.3) through (4.3)–(4.5) as enshrined in the modified auxiliary method procedure, various exact solutions can be contracted as in the following cases:

Case 1. If $\mu_2 > 0$, then Eq. (2.1) has solutions in the form

$$u_1(x,t) = \frac{1}{2}\left(-c^2 - 576\,\mu_2^2 + 1\right) + 840\mu_2^2\,\text{sech}^4\left(\sqrt{\mu_2}(x-ct)\right), \tag{5.20}$$

$$u_2(x,t) = \frac{1}{2}\left(-c^2 - 576\,\mu_2^2 + 1\right) + 840\mu_2^2\,\text{csch}^4\left(\sqrt{\mu_2}(x-ct)\right), \tag{5.21}$$

$$u_3(x,t) = \frac{1}{2}\left(-c^2 - 576\mu_2^2 + 1\right) + \frac{215040\,\mu_2^2\,\mu_4^2\,e^{4\sqrt{\mu_2}(x-ct)}}{\left(e^{2\sqrt{\mu_2}(x-ct)} - 4\mu_4\right)^4}. \tag{5.22}$$

Case 2. If $\mu_2 > 0, \Delta > 0$, then Eq. (2.1) has a solution in the form

$$u_4(x,t) = \frac{1}{2}\left(-c^2 - 576\,\mu_2^2 + 1\right) + \frac{3360\,\mu_2^2\,\mu_4^2}{\left(\mu_4\,\cosh\left(2\sqrt{\mu_2}(x-ct)\right) - \mu_4\right)^2}. \tag{5.23}$$

Case 3. If $\mu_2 < 0, \Delta > 0$, then Eq. (2.1) has solutions in the form

$$u_5(x,t) = \frac{1}{2}\left(-c^2 - 576\mu_2^2 + 1\right) + \frac{3360\mu_2^2\mu_4^2}{\left(\mu_4\cos\left(2\sqrt{-\mu_2}(x-ct)\right) - \mu_4\right)^2}, \tag{5.24}$$

$$u_6(x,t) = \frac{1}{2}\left(-c^2 - 576\mu_2^2 + 1\right) + \frac{3360\mu_2^2\mu_4^2}{\left(\mu_4\sin\left(2\sqrt{-\mu_2}(x-ct)\right) - \mu_4\right)^2}. \tag{5.25}$$

Notably, the application of the modified auxiliary method revealed some interesting solutions, including bright solitonic solutions, exponential function solutions, and periodic function solutions. Indeed, by fully implementing the method, one can derive a wealth of exact solutions; see [12] for more details on the diverse solutions provided by the method. Additionally, the implementation of the classical auxiliary method, which reveals fewer exact solutions, can be enhanced by considering the right-hand side of (4.2) as a polynomial of order 2 while maintaining the same procedure.

6. Graphical depictions and discussion

This section provides the graphical representation and discussion of the exact analytical solutions obtained for the governing sixth-order Boussinesq equation. The Kudryashov and modified auxiliary methods have yielded several exponential, periodic, and solitonic solutions, which are reported and examined in this section.

6.1. *Depictions and discussion of Kudryashov method's solutions*

Three sets of solutions are obtained, in which both the modified and enhanced Kudryashov methods were deployed to construct more exact solutions for the governing sixth-order Boussinesq equation. In this regard, Figures 1-7 depict various graphical illustrations, consisting mainly of the two-dimensional (2D) and three-dimensional (3D) depictions. To begin with, Figure 1 gives the 3D illustration for the exponential solution (u_{1_+}) obtained in (5.6), which is a kink shape. In addition, Figures 2 and 3 show the corresponding 2D plots for solution (u_{1_+}) in 5.6 with variation in the temporal variable and the Kudryashov-index parameter, respectively. Indeed, Figure 2 examines the evolution of the solution (u_{1_+}) in (5.6) with time variation, where it is noted that an increase in time accelerates the movement of the wave in the governing medium. Moreover, Figure 3 analyses the impact of the Kudryashov-index μ on the solution (5.6), where it is observed that an increase in the index decelerates the propagation of waves. Additionally, when $\beta = -1$, one obtains solution (u_{1_-}) from (5.6), which is a complex-valued solution; see Figure 4 (a), (b) for the graphical depiction of both the real and imaginary components of the solution.

Concurrently, Figure 5 analyses the variation of η in the u_{2_+} solution in (5.13), where it is noted from the figure that η decreases the wave profile. Moreover, when $\eta = e$, where e is the Euler's constant, the modified Kudryashov method [31] reduces to the classical Kudryashov method. Furthermore, Figures 6 and 7 examine the 3D and 2D plots, respectively, for the obtained solution u_{3_+} in (5.16), which further differs from the classical Kudryashov method's solution due to the parameters η and ζ. This solution is specially posed by an enhanced version of the Kudryashov method that works with two parameters η and ζ [32]. Thus, in light of this, Figure 6 shows a kink-type profile while Figures 7 (a),(b) examine the effects of these added parameters. Certainly, one notes from Figure 7 (a) that an increase in μ, the Kudryashov-index opposes the wave's movement, while an increase in ζ positively alters the propagation of the wave.

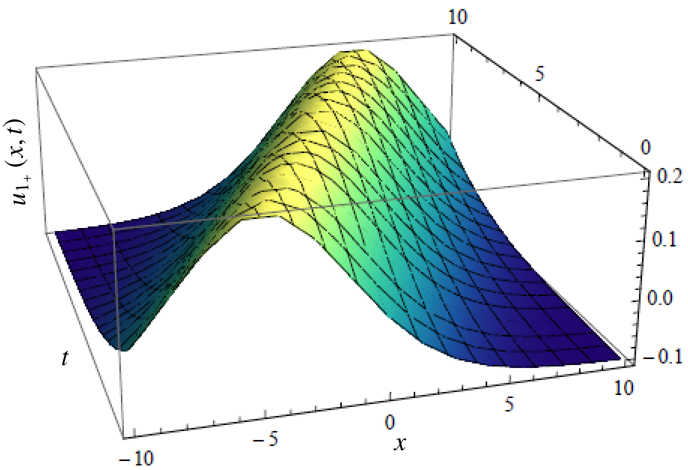


Figure 1. 3D plot for $u_{1_+}(x,t)$ determined in (5.6) when $\beta = 1$, $c = k$ and $\mu = 4$.

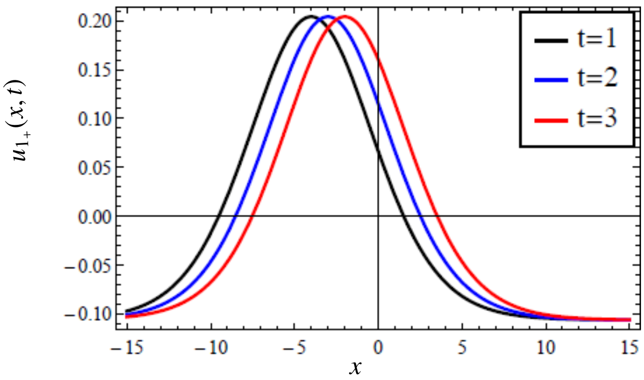


Figure 2. 2D plot for the $u_{1_+}(x,t)$ determined in (5.6) with variation in x when $\beta = 1$, $c = k$ and $\mu = 4$.

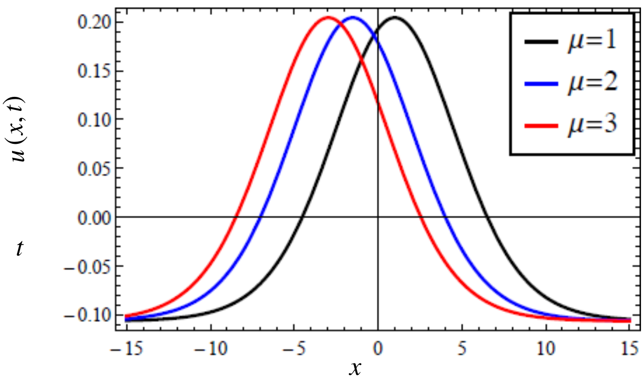
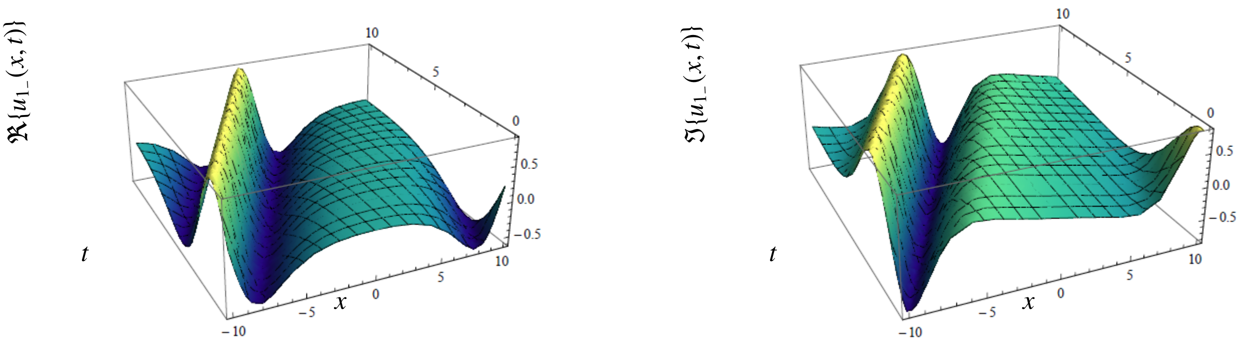


Figure 3. 2D plot for $u_{1_+}(x,t)$ determined in (5.6) with variation in x for different values of Kudryashov-index μ when $\beta = 1$, $c = k$, $t = 1$ and $\mu = 4$.



(a) 3D plot for $\Re\{u_{1\_}(x,t)\}$ in (5.6).

(b) 3D plot for $\Im\{u_{1\_}(x,t)\}$ in (5.6).

Figure 4. 3D plots for the real and imaginary parts of $u_{1\_}(x,t)$ determined in (5.6) when $\beta = -1$, $c = k$ and $\mu = 4$.

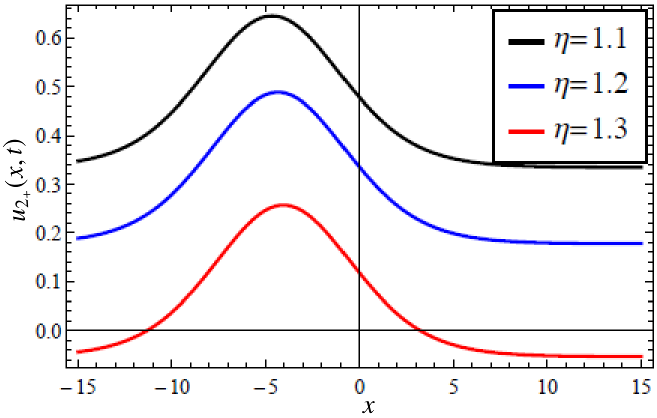


Figure 5. 2D plots for the $u_{2_+}(x,t)$ determined in (5.13) for different values of η when $t = 1, \beta = 1$, $c = k$ and $\mu = 4$.

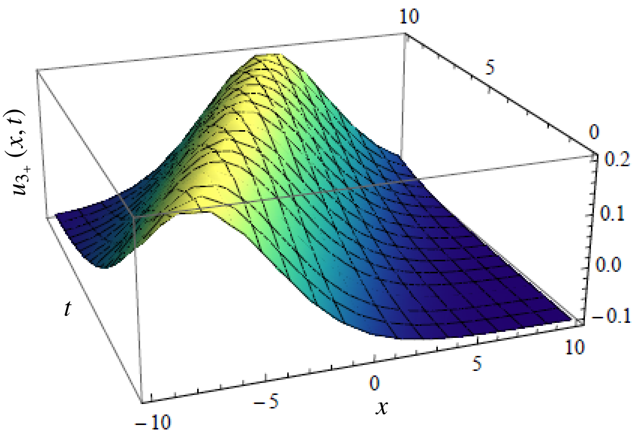
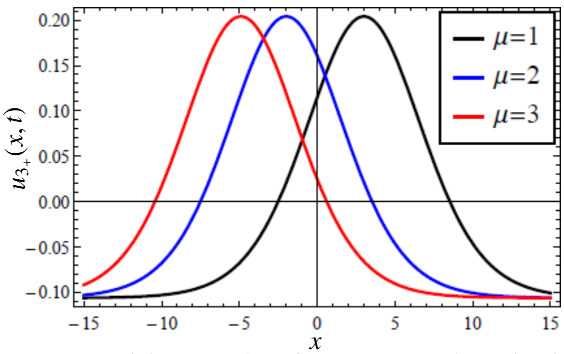
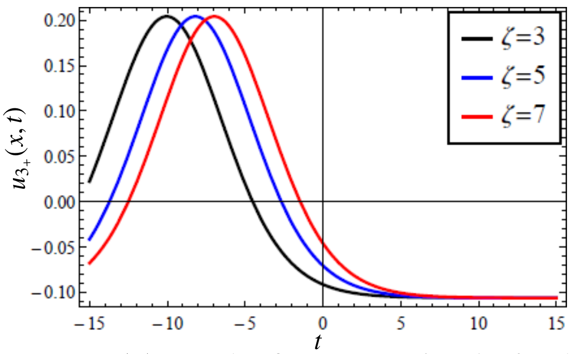


Figure 6. 3D plot for $u_{3_+}(x,t)$ determined in (5.16) when $\beta = 1$, $\zeta = 7$, $c = k$ and $\mu = 4$.



(a) 2D plot for $u_{3_+}(x,t)$ in (5.16) when $\zeta = 1$.



(b) 2D plot for $u_{3_+}(x,t)$ in (5.16) when $\mu = 4$.

Figure 7. 2D plots for the $u_{3_+}(x,t)$ determined in (5.16) for different values of Kudryashov-index μ in (a) and different values of ζ in (b) when $\beta = 1$, $t = 1$ and $c = k$.

6.2. Depictions and discussion of modified auxiliary method's solutions

This subsection graphically examines some of the obtained solutions for the model with the help of the modified auxiliary method [12]. This method revealed several exponential, periodic, and hyperbolic (solitonic) function solutions; see (5.20)–(5.25). Thus, Figures 8–10 show 3D plots for the selected solutions in (5.20)–(5.25), and their 2D plots can be obtained as in the case of the Kudryashov method. Therefore, without much delay, the bright solitonic solution $u_1(x,t)$ determined in (5.20) is plotted in Figure 8, which happens to be a kink-type shape. Further, Figures 9 and 10 show the obtained periodic solitonic solutions, earlier determined in (5.24), and (5.25), respectively, for $u_5(x,t)$ and $u_6(x,t)$. In the same way, one may equally plot the remaining solutions put forward by the modified auxiliary method. Moreover, upon implementing the full generalized auxiliary method [12, 13], several other exact solutions can be determined which shed more light on unearthing the governing model's dynamics; besides, these exact solutions can be used to ascertain both the experimental and numerical results.

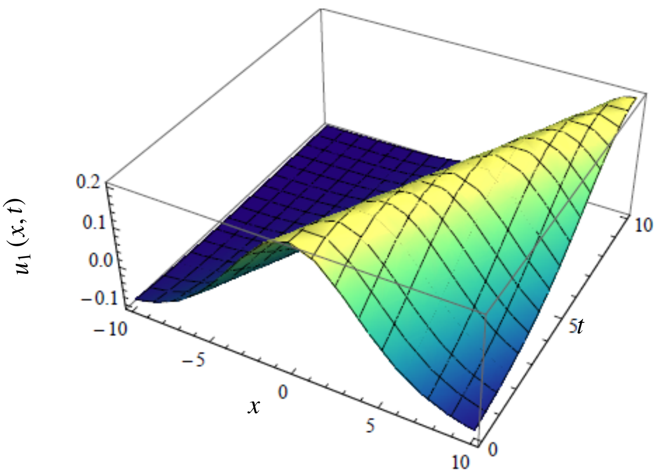


Figure 8. 3D plot for $u_1(x,t)$ determined in (5.20) when $c = 1$ and $\mu_2 = 1/52$.

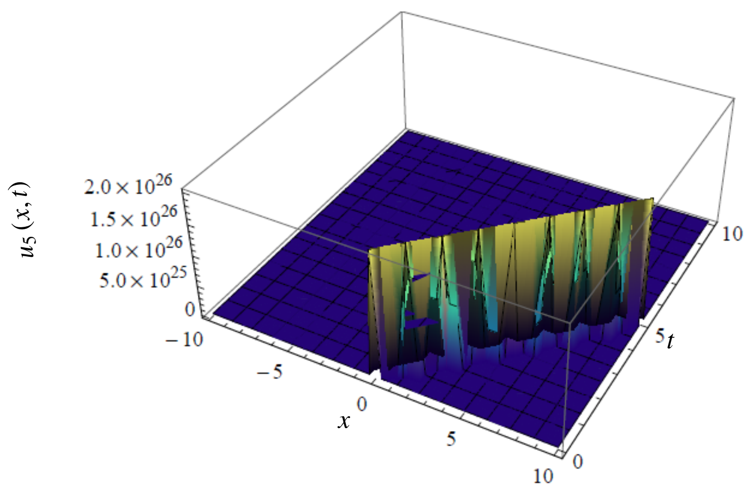


Figure 9. 3D plot for $u_5(x,t)$ determined in (5.24) when $c = 2$, $\mu_2 = 1/52$ and $\mu_4 = 1$.

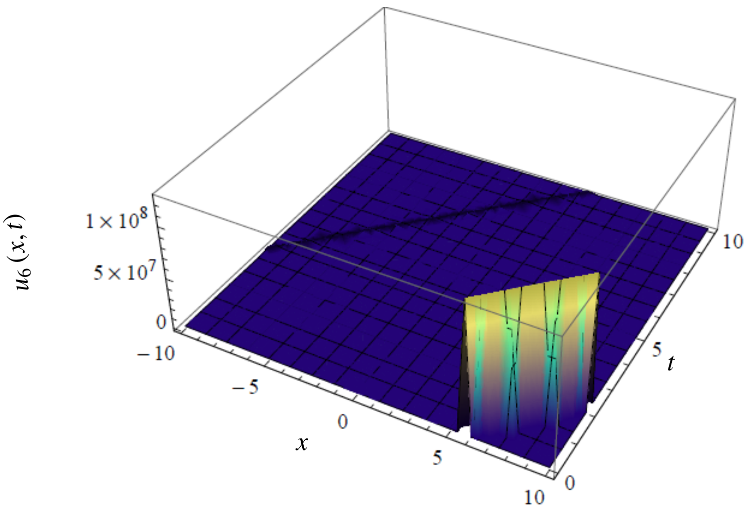


Figure 10. 3D plot for $u_6(x,t)$ determined in (5.25) when $c = 2, k = 1$, $\mu_2 = -1/52$ and $\mu_4 = 1$.

7. Conclusions

In conclusion, the Kudryashov method and the modified auxiliary equation method have been used due to their analytical precision in deriving several exact wave solutions for the sixth-order Boussinesq equation. These approaches have produced a wealth of valid solution sets, resulting in a diverse of exact wave solutions. Additionally, the study provides graphical illustrations based on specific fixed parameter values. Given the limited literature on the governing model, this research significantly expands the existing knowledge. It also offers insights into the global and local roughness, blow-up, and well-posedness of the model's valid solutions when appropriate initial and boundary conditions are applied. The obtained solutions could serve as benchmark solutions for numerical examinations of the model. Finally, due to the high precision of the employed analytical methods, this study recommends their use for higher-order evolution equations when seeking efficiency, reliability, and robustness.

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Conflict of interest

The author declares that there is no conflicts of interest.

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