Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 30922–30938.
DOI: 10.3934/math.20241492
Received: 07 August 2024
Revised: 27 September 2024
Accepted: 03 October 2024
Published: 30 October 2024

*Research article*

# A new double series space derived by factorable matrix and four-dimensional matrix transformations

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**Abstract:** In this study, we introduce a new double series space $\left|F_{a,b}^{u,\theta}\right|_k$ using the four dimensional factorable matrix $F$ and absolute summability method for $k \geq 1$. Also, examining some algebraic and topological properties of $\left|F_{a,b}^{u,\theta}\right|_k$, we show that it is norm isomorphic to the well-known double sequence space $\mathcal{L}_k$ for $1 \leq k < \infty$. Furthermore, we determine the $\alpha$-, $\beta(bp)$- and $\gamma$-duals of the spaces $\left|F_{a,b}^{u,\theta}\right|_k$ for $k \geq 1$. Additionally, we characterize some new four dimensional matrix transformation classes on double series space $\left|F_{a,b}^{u,\theta}\right|_k$. Hence, we extend some important results concerned on Riesz and Cesàro matrix methods to double sequences owing to four dimensional factorable matrix.

**Keywords:** 4D factorable matrix; dual spaces; 4D matrix transformations; double sequences; Pringsheim convergence
**Mathematics Subject Classification:** 46B45, 15A60, 40C05, 46A45, 40F05

## 1. Introduction

The investigation of the convergence of sequences and the generation of a new sequence space occupy a significant position within the fields of mathematical analysis, Fourier analysis, and approximation theory. A variety of novel single sequence spaces have been developed through the utilisation of intriguing matrix summation techniques, including Riesz [1, 2], Cesàro [3–5], Euler totient [6, 7], Nörlund [8], and factorable methods [9] in the literature. Nevertheless, research on the generation of novel double sequences or series spaces remains limited, despite the existence of significant studies on double sequences. Additionally, there has been a considerable amount of interest recently in the generalizations of single sequence spaces to double sequence spaces. The initial works on double sequences were done by Bromwich [10]. In her doctoral dissertation, Zeltser studied both the theory of topological double sequence spaces and the summability of double sequences [11]. The notion of convergence for double sequences has been the subject of work by Pringsheim in [12]. Hardy also introduced the notion of regular convergence for double sequences in the sense that a double sequence has a limit in Pringsheim's sense and has one-sided limits [13]. Later, the theory of double sequences was studied by Móricz [14], Başarır and Sonalcan [15], Demiriz and Duyar [16], Demiriz and Erdem [17,18] and many others. Moreover, the theory of double sequences has numerous applications in engineering and applied sciences. For example, Nayak and Baliarsingh [19] have defined the notion of difference double sequence spaces based on fractional order, which have been used to study the fractional derivatives of certain functions and their geometrical interpretations.

A double sequence $x = (x_{rs})$ is a double infinite array of elements $x_{rs}$ for all $r, s \in \mathbb{N}$. The set of all complex valued double sequences is denoted as

$$\Omega = \{x = (x_{mn}) : x_{mn} \in \mathbb{C}, \forall m, n \in \mathbb{N}\},$$

which is a vector space with coordinatewise addition and scalar multiplication of double sequences, where $\mathbb{C}$ is the complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. Any vector subspace of $\Omega$ is called *a double sequence space*. We denote the space of all bounded double sequences by $\mathcal{M}_u$, that is,

$$\mathcal{M}_u = \left\{x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty\right\},$$

which is a Banach space with the norm $\|.\|_\infty$. A double sequence $x = (x_{mn}) \in \Omega$ is called *convergent* to the limit point $L \in \mathbb{C}$ in *Pringsheim's sense* if for every given $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \epsilon$ for all $m, n > n_0$. Then, it is denoted by $p - \lim_{m,n\to\infty} x_{mn} = L$, where $L$ is called the Pringsheim limit of $x$. Further, $C_p$ shows the space of all convergent double sequences in Pringsheim's sense [12]. It is well known that every convergent single sequence is bounded, but this may not be the case for double sequences in general. To put it clearly, there are such double sequences that are convergent in Pringsheim's sense but not bounded. Namely, the set $C_p - \mathcal{M}_u$ is not empty. In fact, following Boos [20], consider the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} m; & n = 0, m \in \mathbb{N}, \\ n; & m = 0, n \in \mathbb{N}, \\ 0; & m, n \in \mathbb{N} \setminus \{0\}, \end{cases}$$

for all $m, n \in \mathbb{N}$. Then, it is clearly seen that $x \in C_p - \mathcal{M}_u$, since $p - \lim_{m,n\to\infty} x_{mn} = 0$ but $\|x\|_\infty = \infty$. Therefore, the set $C_{bp}$ of double sequences denotes both convergent in Pringsheim's sense and bounded, i.e., $C_{bp} = C_p \cap \mathcal{M}_u$. Hardy [13] showed that a sequence in the space $C_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index, and $C_r$ denotes the set of all such double sequences.

Here and afterwards, we assume that $\upsilon$ denotes any of the symbols $p$, $bp$, or $r$, and also $k'$ denotes the conjugate of $k$, that is, $\frac{1}{k} + \frac{1}{k'} = 1$ for $1 < k < \infty$, and $\frac{1}{k'} = 0$ for $k = 1$.

Let us consider a double sequence $x = (x_{mn})$ and define the sequence $s = (s_{mn})$ via $x$ by

$$s_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij}$$

for all $m, n \in \mathbb{N}$. Then, the pair of $(x, s)$ and the sequence $s = (s_{mn})$ are named as a double series and the sequence of partial sums of the double series, respectively. For the sake of brevity, here and in what follows we use the abbreviation $\sum_{i,j} x_{ij}$ in place of the summation $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij}$.

Let $\lambda$ be a space of the double sequences converging with respect to some linear convergence rule $\upsilon-\lim:\lambda\rightarrow\mathbb{C}$. If the double sequence $(s_{mn})$ is convergent in the $\upsilon$-sense with respect to this rule, then the double series $\sum_{i,j}x_{ij}$ is said to be convergent in the $\upsilon$-sense and it is denoted that $\upsilon-\sum_{i,j}x_{ij}=\upsilon-\lim_{m,n\rightarrow\infty}s_{mn}$.

Quite recently, Başar and Sever have introduced the Banach space $\mathcal{L}_k$ of double sequences as

$$\mathcal{L}_k=\left\{x=(x_{mn})\in\Omega:\sum_{m,n}|x_{mn}|^k<\infty\right\},$$

which corresponds to the well-known space $\ell_k$ of absolutely $k$-summable single sequences, and examined some properties of the space $\mathcal{L}_k$ in [21]. Also, for the special case $k=1$, the space $\mathcal{L}_k$ is reduced to the space $\mathcal{L}_u$, which was defined by Zeltser in [11].

The $\alpha$-*dual*, $\beta(\upsilon)$-*dual*, and $\gamma$-*dual* of the double sequence space $\lambda\subset\Omega$ are denoted by $\lambda^{\alpha}$, $\lambda^{\beta(\upsilon)}$, $\lambda^{\gamma}$ in regard to the $\upsilon$-convergence for $\upsilon\in\{p,bp,r\}$ and are defined respectively by

$$\lambda^{\alpha}:=\left\{\mu=(\mu_{kl})\in\Omega:\sum_{k,l}|\mu_{kl}x_{kl}|<\infty,\text{ for all }(x_{kl})\in\lambda\right\},$$

$$\lambda^{\beta(\upsilon)}:=\left\{\mu=(\mu_{kl})\in\Omega:\upsilon-\sum_{k,l}\mu_{kl}x_{kl}\text{ exists, for all }(x_{kl})\in\lambda\right\},$$

and

$$\lambda^{\gamma}:=\left\{\mu=(\mu_{kl})\in\Omega:\sup_{m,n\in\mathbb{N}}\left|\sum_{k,l=0}^{m,n}\mu_{kl}x_{kl}\right|<\infty,\text{ for all }(x_{kl})\in\lambda\right\}.$$

Now, we shall deal with the four-dimensional transformations. Let $X$ and $Y$ be two double sequence spaces that are converging with regard to the linear convergence rules $\upsilon_1-\lim$ and $\upsilon_2-\lim$, respectively, and $A=\left(a_{mnij}\right)$ be any four-dimensional complex infinite matrix. Then, $A$ defines a matrix transformation from $X$ into $Y$, if for every double sequence $x=\left(x_{ij}\right)\in X$, $Ax=\{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$, the $A$-transform of $x$, is in $Y$, where

$$(Ax)_{mn}=\upsilon-\sum_{i,j}a_{mnij}x_{ij}\tag{1.1}$$

provided that the double series exists for each $m,n\in\mathbb{N}$. By $(X,Y)$, we show the set of such all four-dimensional matrix transformations from the space $X$ into the space $Y$. Thus, $A=\left(a_{mnij}\right)\in(X,Y)$ if and only if the double series on the right side of (1.1) is convergent with regard to the linear convergence rule $\upsilon-\lim$ for each $m,n\in\mathbb{N}$ and $Ax\in Y$ for all $x\in X$.

The $\upsilon$-summability domain $\lambda_A^{(\upsilon)}$ of $A=\left(a_{mnij}\right)$ in a space $\lambda$ of double sequences is defined by

$$\lambda_A^{(\upsilon)}=\left\{x=\left(x_{ij}\right)\in\Omega:Ax=\left(\upsilon-\sum_{i,j}a_{mnij}x_{ij}\right)_{m,n\in\mathbb{N}}\text{ exists and is in }\lambda\right\}.\tag{1.2}$$

We write throughout for simplicity in notation for all $m,n,k,l\in\mathbb{N}$ that

$$\Delta_{10}x_{mn}\ =\ x_{mn}-x_{m+1,n},$$

$$\begin{aligned}\Delta_{01}x_{mn} &= x_{mn}-x_{m,n+1},\\ \Delta_{11}x_{mn} &= \Delta_{01}\left(\Delta_{10}x_{mn}\right)=\Delta_{10}\left(\Delta_{01}x_{mn}\right),\end{aligned}$$

and

$$\begin{aligned}\Delta_{10}^{kl}a_{mnkl} &= a_{mnkl}-a_{mn,k+1,l},\\ \Delta_{01}^{kl}a_{mnkl} &= a_{mnkl}-a_{mnk,l+1},\\ \Delta_{11}^{kl}a_{mnkl} &= \Delta_{01}^{kl}\left(\Delta_{10}^{kl}a_{mnkl}\right)=\Delta_{10}^{kl}\left(\Delta_{01}^{kl}a_{mnkl}\right).\end{aligned}$$

## 2. The double factorable series spaces

In this section, we introduce the new $\left|F_{a,b}^{u,\theta}\right|_k$ doubly summable sequence spaces using the four-dimensional factorable matrix $F$ and, absolute summability method for $k \geq 1$. Also, we investigate some algebraic and topological properties of $\left|F_{a,b}^{u,\theta}\right|_k$, and show that it is norm isomorphic to the well-known double sequence space $\mathcal{L}_k$ for $1 \leq k < \infty$. Furthermore, we determine the $\alpha$-, $\beta(bp)$-, and $\gamma$-duals of the spaces $\left|F_{a,b}^{u,\theta}\right|_k$ in regard to the $bp$-convergence for $k \geq 1$.

Prior to introducing the factorable matrix, it is necessary to provide definitions of the well-known four-dimensional matrices associated with the factorable matrix.

One of the fundamental four-dimensional matrices is the four-dimensional Cesàro matrix $C = \left(c_{mnij}\right)$ of order one, which is defined by

$$c_{mnij}=\begin{cases}\frac{1}{mn}, & 1\leq i\leq m,\ 1\leq j\leq n,\\ & 0,\ \text{otherwise},\end{cases}$$

for all $m,n,i,j\in\mathbb{N}$ [22]. In a recent study, some topological properties of double series space $\left|C_{1,1}\right|_k$ were investigated using this four dimensional matrix in [23].

Let $p = (p_k)$ and $q = (q_k)$ be two sequences of non-negative numbers that are not all zero, and $P_n = \sum_{k=0}^{n} p_k$, $p_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k$, $q_0 > 0$. The four-dimensional Riesz matrix $R^{pq} = \left(r_{mnij}^{pq}\right)$ is defined by

$$r_{mnij}^{pq}=\begin{cases}\dfrac{p_iq_j}{P_mQ_n}, & 0\leq i\leq m,\ 0\leq j\leq n,\\ & 0,\ \text{otherwise},\end{cases}$$

for all $m,n,i,j\in\mathbb{N}$ [24]. Note that in the case $p_k = q_k = 1$ for all $k \in \mathbb{N}$, the Riesz matrix $R^{pq}$ is reduced to the four-dimensional Cesàro matrix of order one.

Let $s = (s_{mn})$ be partial sums of the double series $\sum_{i,j} x_{ij}$. Then, the double series $\sum_{i,j} x_{ij}$ is called absolutely double weighted summable $\left|\bar{N}, p_n, q_n\right|_k$, $k \geq 1$ [25], if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left|\Delta_{11}R_{m-1,n-1}^{pq}(s)\right|^k<\infty,$$

where

$$R_{mn}^{pq}(s)=\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_js_{ij},\ \ (m,n\in\mathbb{N}),$$

and

$$\Delta_{11} R^{pq}_{-1,-1} = R^{pq}_{0,0},$$
$$\Delta_{11}\left(R^{pq}_{m-1,-1}\right) = R^{pq}_{m0} - R^{pq}_{m-1,0},\ (m \geq 1),$$
$$\Delta_{11}\left(R^{pq}_{-1,n-1}\right) = R^{pq}_{0n} - R^{pq}_{0,n-1},\ (n \geq 1),$$
$$\Delta_{11}\left(R^{pq}_{m-1,n-1}\right) = R^{pq}_{mn} - R^{pq}_{m-1,n} - R^{pq}_{m,n-1} + R^{pq}_{m-1,n-1},\ (m,n \geq 1).$$

Furthermore, some topological properties of double series space $\left|\bar{N}_{p,q}\right|_k$ have been investigated and also dual spaces of $\left|\bar{N}_{p,q}\right|_k$ are determined in [26].

This study aims to define a more general double series space $\left|F^{u,\theta}_{a,b}\right|_k$ with the help of the four-dimensional factorable matrix, which is more comprehensive and more widely used in mathematical analysis. To do this, we first state the factorable matrix or, by another name, the generalized weighted mean matrix $F\left(a,b,\hat{a},\hat{b}\right) = \left(f_{mnij}\left(a,b,\hat{a},\hat{b}\right)\right)$ which gives these two fundamental matrices in special cases and is used in many places in mathematical analysis and applied mathematics.

Let $C$ denote the set of all sequences $a = (a_i)$ such that $a_i \neq 0$ for all $i \in \mathbb{N}$, and $a = (a_i)$, $b = \left(b_j\right)$, $\hat{a} = (\hat{a}_m)$, $\hat{b} = \left(\hat{b}_n\right) \in C$. Then, the four-dimensional factorable matrix $F\left(a,b,\hat{a},\hat{b}\right) = \left(f_{mnij}\left(a,b,\hat{a},\hat{b}\right)\right)$ is defined by

$$f_{mnij}\left(a,b,\hat{a},\hat{b}\right) = \begin{cases} a_i b_j \hat{a}_m \hat{b}_n, & 0 \leq i \leq m \text{ and } 0 \leq j \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

for all $i, j, m, n \in \mathbb{N}$.

The four-dimensional factorable matrix $F\left(a,b,\hat{a},\hat{b}\right) = \left(f_{mnij}\left(a,b,\hat{a},\hat{b}\right)\right)$ is invertible, and its inverse $F^{-1}\left(a,b,\hat{a},\hat{b}\right) = \left(f^{-1}_{mnij}\left(a,b,\hat{a},\hat{b}\right)\right)$ is defined for all $i, j, m, n \in \mathbb{N}$ by

$$f^{-1}_{mnij}\left(a,b,\hat{a},\hat{b}\right) = \begin{cases} \dfrac{(-1)^{m+n-(i+j)}}{a_m b_n \hat{a}_i \hat{b}_j}, & m-1 \leq i \leq m \text{ and } n-1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we introduce a new doubly summable sequence space $\left|F^{u,\theta}_{a,b}\right|_k$ using the four-dimensional factorable matrix $F$ and, absolute summability method for $k \geq 1$. Hence, we extend double weighted series space $\left|\bar{N}_{p,q}\right|_k$ with the factorable matrix $F$ to double generalized weighted series space, or namely double factorable series space, $\left|F^{u,\theta}_{a,b}\right|_k$ as follows: Consider $\theta = (\theta_n)$, $u = (u_m)$ positive sequences. Then, we can define a new double factorable series space by

$$\left|F^{u,\theta}_{a,b}\right|_k = \left\{ x = (x_{mn}) \in \Omega : \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (u_m\theta_n)^{k-1} \left| \hat{a}_m \hat{b}_n \sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x_{ij} \right|^k \right\} < \infty.$$

If we define $\Psi(x) = (\Psi_{mn}(x))$ transformation of the sequence $x = (x_{mn})$ by

$$y_{mn} = \Psi_{mn}(x) = (u_m\theta_n)^{1/k'}\ \hat{a}_m \hat{b}_n \sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x_{ij};\ m,n \geq 0, \tag{2.1}$$

then $\left|F^{u,\theta}_{a,b}\right|_k$ double series space can be rewritten by

$$\left|F^{u,\theta}_{a,b}\right|_k = (\mathcal{L}_k)_\Psi$$

in view of relation (1.2). Throughout the paper, we will suppose that the terms of the double sequences $x = (x_{mn})$ and $y = (y_{mn})$ are connected with the relation (2.1).

Let us proceed with the following essential theorem, which gives us some algebraic and topological properties of $\left|F^{u,\theta}_{a,b}\right|_k$.

**Theorem 2.1.** The set $\left|F^{u,\theta}_{a,b}\right|_k$ becomes a linear space with the coordinatewise addition and scalar multiplication for double sequences, and $\left|F^{u,\theta}_{a,b}\right|_k$ is a Banach space with the norm

$$\|x\|_{\left|F^{u,\theta}_{a,b}\right|_k} = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|(u_m\theta_n)^{1/k'}\hat{a}_m\hat{b}_n\sum_{i=0}^{m}\sum_{j=0}^{n}a_ib_jx_{ij}\right|^k\right)^{1/k} < \infty, \tag{2.2}$$

and it is norm isomorphic to the well-known double sequence space $\mathcal{L}_k$ for $1 \leq k < \infty$.

*Proof.* As the initial assertion is routine verification, it can be omitted.

To confirm the fact that $\left|F^{u,\theta}_{a,b}\right|_k$ is norm isomorphic to the space $\mathcal{L}_k$, we need to show the existence of a linear and norm-preserving bijection transformation between the spaces $\left|F^{u,\theta}_{a,b}\right|_k$ and $\mathcal{L}_k$ for $1 \leq k < \infty$. In order to achieve this, into account the transformation $\Psi$ defined by

$$\Psi : \left|F^{u,\theta}_{a,b}\right|_k \to \mathcal{L}_k, \tag{2.3}$$

$$x \to y = \Psi(x),$$

where $\Psi(x) = (\Psi_{mn}(x)) = (y_{mn})$ is the same as in (2.1) for $m, n \geq 0$. The linearity of $\Psi$ is clear. Also, $x = \theta$ whenever $\Psi(x) = \theta$, where $\theta$ denotes the zero vector. This gives us that $\Psi$ is injective.

Let us consider $y = (y_{mn}) \in \mathcal{L}_k$ and define the sequence $x = (x_{mn})$ via $y$ by

$$x_{mn} = \frac{1}{a_mb_n}\Delta_{11}\left(\frac{y_{m-1,n-1}}{\hat{a}_{m-1}\hat{b}_{n-1}(u_{m-1}\theta_{n-1})^{1/k'}}\right), \tag{2.4}$$

$$x_{m0} = \frac{1}{a_mb_0}\bar{\Delta}_{10}\left(\frac{y_{m0}}{(u_m\theta_0)^{1/k'}\hat{a}_m\hat{b}_0}\right), \tag{2.5}$$

$$x_{0n} = \frac{1}{a_0b_n}\bar{\Delta}_{01}\left(\frac{y_{0n}}{(u_0\theta_n)^{1/k'}\hat{a}_0\hat{b}_n}\right) \tag{2.6}$$

for $m, n \geq 1$, and

$$x_{00} = \left(\frac{y_{00}}{(u_0\theta_0)^{1/k'}\hat{a}_0\hat{b}_0a_0b_0}\right), \tag{2.7}$$

where $\bar{\Delta}_{10}$ and $\bar{\Delta}_{01}$ refer to the back difference notations, that is, $\bar{\Delta}_{10}(x_{mn}) = x_{m,n} - x_{m-1,n}$, $\bar{\Delta}_{01}(x_{mn}) = x_{m,n} - x_{m,n-1}$ for all $m, n \in \mathbb{N}$. In that case, it is seen that

$$\|x\|_{\left|F^{u,\theta}_{a,b}\right|_k} = \|\Psi(x)\|_{\mathcal{L}_k} = \left(\sum_{m,n}|\Psi_{mn}(x)|^k\right)^{1/k} = \|y\|_{\mathcal{L}_k} < \infty,$$

where the double sequences $x = (x_{mn})$ and $y = (y_{mn})$ are connected with the relation (2.1) for $1 \leq k < \infty$. This implies that $\Psi$ is surjective and norm preserving. Consequently, $\Psi$ is a linear and norm-preserving bijection, which gives us that $\left|F^{u,\theta}_{a,b}\right|_k$ and $\mathcal{L}_k$ are norm-isomorphic for $1 \leq k < \infty$, as desired.

In the proof of the last part of the theorem, we show that $\left|F_{a,b}^{u,\theta}\right|_k$ is a Banach space with the norm defined by (2.2). To prove this, we can use the statement "Let $(X,\rho)$ and $(Y,\sigma)$ be semi-normed spaces and $\Phi : (X,\rho) \to (Y,\sigma)$ be an isometric isomorphism. Then, $(X,\rho)$ is complete if and only if $(Y,\sigma)$ is complete. In particular, $(X,\rho)$ is a Banach space if and only if $(Y,\sigma)$ is a Banach space", which can be found section (b) of Corollary 6.3.41 in [20]. Since the transformation $\Psi$ defined from $\left|F_{a,b}^{u,\theta}\right|_k$ into $\mathcal{L}_k$ by (2.3) is an isometric isomorphism and the double sequence space $\mathcal{L}_k$ is a Banach space from Theorem 2.1 in [21], we obtain that the space $\left|F_{a,b}^{u,\theta}\right|_k$ is a Banach space. This completes the proof.

Now we state the following significant lemmas giving some characterizations for any four-dimensional infinite matrices, which will be used in order to calculate the $\alpha$-, $\beta(bp)$-, and $\gamma$-duals of the spaces $\left|F_{a,b}^{u,\theta}\right|_k$ for $k \geq 1$.

**Lemma 2.2.** [27] Let $A = \left(a_{mnij}\right)$ be any four-dimensional infinite matrix. In that case, the following statements hold:

**(a)** Let $0 < k \leq 1$. Then, $A = \left(a_{mnij}\right) \in (\mathcal{L}_k, \mathcal{M}_u)$ iff

$$M_1 = \sup_{m,n,i,j\in\mathbb{N}} \left|a_{mnij}\right| < \infty. \tag{2.8}$$

**(b)** Let $1 < k < \infty$. Then, $A = \left(a_{mnij}\right) \in (\mathcal{L}_k, \mathcal{M}_u)$ iff

$$M_2 = \sup_{m,n\in\mathbb{N}} \sum_{i,j} \left|a_{mnij}\right|^{k'} < \infty. \tag{2.9}$$

**(c)** Let $0 < k \leq 1$ and $1 \leq k_1 < \infty$. Then, $A = \left(a_{mnij}\right) \in (\mathcal{L}_k, \mathcal{L}_{k_1})$ iff

$$\sup_{i,j\in\mathbb{N}} \sum_{m,n} \left|a_{mnij}\right|^{k_1} < \infty.$$

**(d)** Let $0 < k \leq 1$. Then, $A = \left(a_{mnij}\right) \in \left(\mathcal{L}_k, \mathcal{C}_{bp}\right)$ iff the condition (2.8) holds and there exists a $\left(\lambda_{ij}\right) \in \Omega$ such that

$$bp - \lim_{m,n\to\infty} a_{mnij} = \lambda_{ij}. \tag{2.10}$$

**(e)** Let $1 < k < \infty$. Then, $A = \left(a_{mnij}\right) \in \left(\mathcal{L}_k, \mathcal{C}_{bp}\right)$ iff (2.9) and (2.10) are satisfied.

**Lemma 2.3.** [28] Let $1 < k < \infty$ and $A = \left(a_{mnij}\right)$ be a four-dimensional infinite matrix of complex numbers. Define $W_k(A)$ and $w_k(A)$ by

$$W_k(A) = \sum_{r,s=0}^{\infty} \left( \sum_{m,n=0}^{\infty} |a_{mnrs}| \right)^k,$$

$$w_k(A) = \sup_{M\times N} \sum_{r,s=0}^{\infty} \left| \sum_{(m,n)\in M\times N} a_{mnrs} \right|^k,$$

where the supremum is taken through all finite subsets $M$ and $N$ of the natural numbers. Then, the following statements are equivalent:

$$i)\ W_{k'}(A) < \infty, \qquad ii)\ A \in (\mathcal{L}_k, \mathcal{L}_u),$$

$$iii)\ A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k'}) < \infty, \qquad iv)\ w_{k'}(A) < \infty.$$

Now, we prove the following theorems, which give the $\alpha$- and $\beta(bp)$-, and $\gamma$-duals of the spaces $\left|F_{a,b}^{u,\theta}\right|_k$ for $k \geq 1$. To shorten the theorems and their proofs, let us denote the sets $\Lambda_k$ with $k \in \{1,2,3,4,5\}$ as follows:

$$\Lambda_1 = \left\{\mu = (\mu_{mn}) \in \Omega : \sup_{i,j\in\mathbb{N}} \sum_{m,n} \left|g_{mnij}^{(1)}\right| < \infty\right\}, \tag{2.11}$$

$$\Lambda_2 = \left\{\mu = (\mu_{mn}) \in \Omega : \sum_{i,j} \left(\sum_{m,n} \left|g_{mnij}^{(k)}\right|\right)^{k'} < \infty\right\}, \tag{2.12}$$

$$\Lambda_3 = \left\{\mu = (\mu_{mn}) \in \Omega : bp - \lim_{m,n\to\infty} d_{mnij}^{(k)} \text{ exists for all } i,j \in \mathbb{N}\right\}, \tag{2.13}$$

$$\Lambda_4 = \left\{\mu = (\mu_{mn}) \in \Omega : \sup_{m,n,i,j\in\mathbb{N}} \left|d_{mnij}^{(1)}\right| < \infty\right\}, \tag{2.14}$$

$$\Lambda_5 = \left\{\mu = (\mu_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} \sum_{i,j} \left|d_{mnij}^{(k)}\right|^{k'} < \infty\right\}, \tag{2.15}$$

where the 4-dimensional matrices $D^{(k)} = \left(d_{mnij}^{(k)}\right)$ and $G^{(k)} = \left(g_{mnij}^{(k)}\right)$ are defined by

$$d_{mnij}^{(k)} = \begin{cases} \dfrac{1}{\hat{a}_0\hat{b}_0\left(u_0\theta_0\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{\mu_{ij}}{a_ib_j}\right), & m=n=i=j=0, \\ \dfrac{1}{a_m\hat{a}_m\hat{b}_0\left(u_m\theta_0\right)^{1/k'}}\Delta_{01}^{(ij)}\left(\dfrac{\mu_{mj}}{b_j}\right), & i=m,\ j=n=0, \\ \dfrac{1}{b_n\hat{a}_0\hat{b}_n\left(u_0\theta_n\right)^{1/k'}}\Delta_{10}^{(ij)}\left(\dfrac{\mu_{in}}{a_i}\right), & i=m=0,\ j=n, \\ \dfrac{1}{\hat{a}_i\hat{b}_0\left(u_i\theta_0\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{\mu_{ij}}{a_ib_j}\right), & 1\leq i\leq m-1,\ \ j=n=0, \\ \dfrac{1}{\hat{a}_0\hat{b}_j\left(u_0\theta_j\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{\mu_{ij}}{a_ib_j}\right), & i=m=0,\ 1\leq j\leq n-1, \\ \dfrac{1}{\hat{a}_i\hat{b}_j\left(u_i\theta_j\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{\mu_{ij}}{a_ib_j}\right), & 1\leq i\leq m-1,\ 1\leq j\leq n-1, \\ \dfrac{1}{b_n\hat{a}_i\hat{b}_n\left(u_i\theta_n\right)^{1/k'}}\Delta_{10}^{(ij)}\left(\dfrac{\mu_{ij}}{a_i}\right), & 1\leq i\leq m-1,\ j=n, \\ \dfrac{1}{a_m\hat{a}_m\hat{b}_j\left(u_m\theta_j\right)^{1/k'}}\Delta_{01}^{(ij)}\left(\dfrac{\mu_{ij}}{b_j}\right), & 1\leq j\leq n-1,\ i=m, \\ \dfrac{\mu_{mn}}{a_mb_n\hat{a}_m\hat{b}_n\left(u_m\theta_n\right)^{1/k'}}, & i=m,\ j=n, \end{cases} \tag{2.16}$$

and

$$g^{(k)}_{mnij}=\begin{cases}\dfrac{\mu_{00}}{(u_0\theta_0)^{1/k'}\,a_0b_0\hat{a}_0\hat{b}_0}, & m=n=0,\\ (-1)^{n-j}\,\dfrac{\mu_{0n}}{(u_0)^{1/k'}\,a_0b_n\hat{a}_0\left(\theta_j\right)^{1/k'}\,\hat{b}_j}, & m=0,\ n-1\le j\le n,\\ (-1)^{m-i}\,\dfrac{\mu_{m0}}{(\theta_0)^{1/k'}\,a_mb_0\hat{a}_i\,(u_i)^{1/k'}\,\hat{b}_0}, & n=0,\ m-1\le i\le m,\\ (-1)^{m+n-(i+j)}\,\dfrac{\mu_{mn}}{\left(u_i\theta_j\right)^{1/k'}\,a_mb_n\hat{a}_i\hat{b}_j}, & m-1\le i\le m,\ n-1\le j\le n,\end{cases} \tag{2.17}$$

respectively.

**Theorem 2.4.** Let the sets $\Lambda_1,\Lambda_2$ and the 4-dimensional matrix $G^{(k)}=\left(g^{(k)}_{mnij}\right)$ be defined as in (2.11), (2.12), and (2.17), respectively. Then, $\left(\left|F_{a,b}\right|_1\right)^{\alpha}=\Lambda_1$ and $\left(\left|F^{u,\theta}_{a,b}\right|_k\right)^{\alpha}=\Lambda_2$ for $1<k<\infty$.

*Proof.* Since the case of $1<k<\infty$ can be proved similarly using Lemma 2.3, we will prove the theorem for $k=1$. Let $x=(x_{mn})\in\left|F_{a,b}\right|_1$, $\mu=(\mu_{mn})\in\Omega$ . Taking account of the relations (2.4)–(2.7) for $m,n\geq 0$, we can compute the following equalities:

For $m=n=0$,

$$\mu_{00}x_{00}=\mu_{00}\frac{y_{00}}{(u_0\theta_0)^{1/k'}\,\hat{a}_0\hat{b}_0a_0b_0}=(Gy)_{00}\,,$$

for $m=0$, $n\geq 1$

$$\begin{aligned}\mu_{0n}x_{0n} &= \mu_{0n}\frac{1}{a_0b_n}\left(\frac{y_{0n}}{(u_0\theta_n)^{1/k'}\,\hat{a}_0\hat{b}_n}-\frac{y_{0,n-1}}{(u_0\theta_{n-1})^{1/k'}\,\hat{a}_0\hat{b}_{n-1}}\right)\\ &= \mu_{0n}\frac{1}{a_0b_n\hat{a}_0\,(u_0)^{1/k'}}\sum_{j=n-1}^{n}(-1)^{n-j}\,\frac{y_{0j}}{\left(\theta_j\right)^{1/k'}\,\hat{b}_j}=(Gy)_{0n}\,,\end{aligned}$$

for $n=0$, $m\geq 1$,

$$\begin{aligned}\mu_{m0}x_{m0} &= \mu_{m0}\frac{1}{a_mb_0}\left(\frac{y_{m0}}{(u_m\theta_0)^{1/k'}\,\hat{a}_m\hat{b}_0}-\frac{y_{m-1,0}}{(u_{m-1}\theta_0)^{1/k'}\,\hat{a}_{m-1}\hat{b}_0}\right)\\ &= \mu_{m0}\frac{1}{a_mb_0\hat{b}_0\,(\theta_0)^{1/k'}}\sum_{i=m-1}^{m}(-1)^{m-i}\,\frac{y_{i0}}{(u_i)^{1/k'}\,\hat{a}_i}=(Gy)_{m0}\,,\end{aligned}$$

and for $n,m\geq 1$,

$$\begin{aligned}\mu_{mn}x_{mn} &= \mu_{mn}\frac{1}{a_mb_n}\Big(\frac{y_{mn}}{(u_m\theta_n)^{1/k'}\,\hat{a}_m\hat{b}_n}-\frac{y_{m,n-1}}{(u_m\theta_{n-1})^{1/k'}\,\hat{a}_m\hat{b}_{n-1}}\\ &\quad -\frac{y_{m-1,n}}{(u_{m-1}\theta_n)^{1/k'}\,\hat{a}_{m-1}\hat{b}_n}+\frac{y_{m-1,n-1}}{(u_{m-1}\theta_{n-1})^{1/k'}\,\hat{a}_{m-1}\hat{b}_{n-1}}\Big)\\ &= \mu_{mn}\frac{1}{a_mb_n}\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}(-1)^{m+n-(i+j)}\,\frac{y_{ij}}{\left(u_i\theta_j\right)^{1/k'}\,\hat{a}_i\hat{b}_j}\\ &= (Gy)_{mn}\,,\end{aligned}$$

where the four-dimensional matrix $G^{(k)} = \left(g_{mnij}^{(k)}\right)$ is defined by (2.17). In this situation, we see that $\mu x = (\mu_{mn}x_{mn}) \in \mathcal{L}_u$ whenever $x \in \left|F_{a,b}\right|_1$ if and only if $G^{(k)}y \in \mathcal{L}_u$ whenever $y \in \mathcal{L}_u$. This gives us that $\mu = (\mu_{mn}) \in \left(\left|F_{a,b}\right|_1\right)^{\alpha}$ iff $G^{(k)} \in (\mathcal{L}_u, \mathcal{L}_u)$ . Thus, using (c) of Lemma 2.2 with $k_1 = k = 1$, we obtain

$$\sup_{i,j\in\mathbb{N}} \sum_{m,n} \left|g_{mnij}^{(1)}\right| < \infty.$$

Hence, we deduce that $\left(\left|F_{a,b}\right|_1\right)^{\alpha} = \Lambda_1$, as desired.

**Theorem 2.5.** Let the sets $\Lambda_3$, $\Lambda_4$, $\Lambda_5$ and the 4-dimensional matrix $D^{(k)} = \left(d_{mnij}^{(k)}\right)$ be given as in (2.13)–(2.16) , respectively. Then, we have $\left(\left|F_{a,b}\right|_1\right)^{\beta(bp)} = \Lambda_3 \cap \Lambda_4$ and $\left(\left|F_{a,b}^{u,\theta}\right|_k\right)^{\beta(bp)} = \Lambda_3 \cap \Lambda_5$ for $1 < k < \infty$.

*Proof.* To avoid the repetition of similar statements, we prove the second part of the theorem for $1 < k < \infty$. Let $\mu = (\mu_{mn}) \in \Omega$ and $x = (x_{mn}) \in \left|F_{a,b}^{u,\theta}\right|_k$ be given. Then, using Theorem 2.1, we can say that there exists a double sequence $y = \left(y_{ij}\right) \in \mathcal{L}_k$. Therefore, taking account of relations (2.4)–(2.7) , we can calculate that

$$\begin{aligned}
z_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \mu_{ij}x_{ij} \\
&= \left(\frac{\mu_{00}}{a_0b_0} - \frac{\mu_{10}}{a_1b_0} - \frac{\mu_{01}}{a_0b_1} + \frac{\mu_{11}}{a_1b_1}\right)\frac{y_{00}}{\hat{a}_0\hat{b}_0\left(u_0\theta_0\right)^{1/k'}} \\
&+ \left(\frac{\mu_{m0}}{a_mb_0} - \frac{\mu_{m1}}{a_mb_1}\right)\frac{y_{m0}}{\left(u_m\theta_0\right)^{1/k'}\hat{a}_m\hat{b}_0} + \left(\frac{\mu_{0n}}{a_0b_n} - \frac{\mu_{1n}}{a_1b_n}\right)\frac{y_{0n}}{\hat{a}_0\hat{b}_n\left(u_0\theta_n\right)^{1/k'}} \\
&+ \sum_{i=1}^{m-1}\left(\frac{\mu_{i0}}{a_ib_0} - \frac{\mu_{i+1,0}}{a_{i+1}b_0} - \frac{\mu_{i1}}{a_ib_1} + \frac{\mu_{i+1,1}}{a_{i+1}b_1}\right)\frac{y_{i0}}{\hat{a}_i\hat{b}_0\left(u_i\theta_0\right)^{1/k'}} \\
&+ \sum_{j=1}^{n-1}\left(\frac{\mu_{0j}}{a_0b_j} - \frac{\mu_{0,j+1}}{a_0b_{j+1}} - \frac{\mu_{1j}}{a_1b_j} + \frac{\mu_{1,j+1}}{a_1b_{j+1}}\right)\frac{y_{0j}}{\hat{a}_0\hat{b}_j\left(u_0\theta_j\right)^{1/k'}} \\
&+ \sum_{i=1}^{m-1}\Delta_{10}^{(in)}\left(\frac{\mu_{in}}{a_ib_n}\right)\frac{y_{in}}{\hat{a}_i\hat{b}_n\left(u_i\theta_n\right)^{1/k'}} + \sum_{j=1}^{n-1}\Delta_{01}^{(mj)}\left(\frac{\mu_{mj}}{a_mb_j}\right)\frac{y_{mj}}{\hat{a}_m\hat{b}_j\left(u_m\theta_j\right)^{1/k'}} \\
&+ \sum_{i,j=1}^{m-1,n-1}\Delta_{11}^{(ij)}\left(\frac{\mu_{ij}}{a_ib_j}\right)\frac{y_{ij}}{\hat{a}_i\hat{b}_j\left(u_i\theta_j\right)^{1/k'}} + \frac{\mu_{mn}y_{mn}}{a_mb_n\hat{a}_m\hat{b}_n\left(u_m\theta_n\right)^{1/k'}} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} d_{mnij}^{(k)}y_{ij} = \left(D^{(k)}(y)\right)_{mn}.
\end{aligned}$$

This implies that $\mu x = (\mu_{mn}x_{mn}) \in \mathcal{CS}_{bp}$ whenever $x = (x_{mn}) \in \left|F_{a,b}^{u,\theta}\right|_k$ if and only if $z = (z_{mn}) \in C_{bp}$ whenever $y = \left(y_{ij}\right) \in \mathcal{L}_k$. Thus, it is clear that $\mu = (\mu_{mn}) \in \left(\left|F_{a,b}^{u,\theta}\right|_k\right)^{\beta(bp)}$ if and only if $D^{(k)} \in \left(\mathcal{L}_k, C_{bp}\right)$, where the four-dimensional matrix $D^{(k)} = \left(d_{mnij}^{(k)}\right)$ is defined in (2.16) for every $m, n, i, j \in \mathbb{N}$. Hence, we obtain that $\left(\left|F_{a,b}\right|_1\right)^{\beta(bp)} = \Lambda_3 \cap \Lambda_4$ and $\left(\left|F_{a,b}^{u,\theta}\right|_k\right)^{\beta(bp)} = \Lambda_3 \cap \Lambda_5$ for $1 < k < \infty$ using parts (d) and (e) of Lemma 2.2, respectively.

**Theorem 2.6.** Let the sets $\Lambda_4$, $\Lambda_5$ and the 4-dimensional matrix $D^{(k)} = \left(d^{(k)}_{mnij}\right)$ be defined as in (2.14)–(2.16), respectively. Then, $\left(\left|F_{a,b}\right|_1\right)^{\gamma} = \Lambda_4$ and $\left(\left|F^{u,\theta}_{a,b}\right|_k\right)^{\gamma} = \Lambda_5$ for $1 < k < \infty$.
*Proof.* The proof of this theorem can be obtained similar to the proof of Theorem 2.5 using parts (a) and (b) of Lemma 2.2 in place of parts (d) and (e) of Lemma 2.2, respectively. To avoid the repetition of similar statements, we omit the details.

## 3. Characterizations of some classes of four-dimensional matrices

In the present section, we characterize some 4-dimensional matrix transformations from the double series spaces $\left|F_{a,b}\right|_1$ and $\left|F^{u,\theta}_{a,b}\right|_k$ to the double sequence spaces $\mathcal{M}_u$, $C_{bp}$, and $\mathcal{L}_k$ for $1 \le k < \infty$. Also, we characterize the 4-dimensional matrix transformations from classical double sequence spaces $\mathcal{L}_u$ and $\mathcal{L}_k$ to the double series spaces $\left|F^{u,\theta}_{a,b}\right|_k$ and $\left|F_{a,b}\right|_1$, respectively, for $k \ge 1$. Although we prove the theorem characterizing 4-dimensional matrix transformations from double series spaces $\left|F_{a,b}\right|_1$and $\left|F^{u,\theta}_{a,b}\right|_k$ to the double sequence space $\mathcal{M}_u$, we give theorems characterizing other 4-dimensional matrix transformations without proof since the proof techniques are similar.
**Theorem 3.1.** Assume that $A = \left(a_{mnij}\right)$ be an arbitrary 4-dimensional infinite matrix. In that case, the following statements hold:
**(a)** $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}\right|_1, \mathcal{M}_u\right)$ if and only if

$$A_{mn} \in \left(\left|F_{a,b}\right|_1\right)^{\beta(bp)} \tag{3.1}$$

and

$$\sup_{m,n,i,j\in\mathbb{N}} \left| \frac{1}{\hat{a}_i \hat{b}_j} \Delta^{(ij)}_{11} \left(\frac{a_{mnij}}{a_i b_j}\right)\right| < \infty. \tag{3.2}$$

**(b)** Let $1 < k < \infty$. Then, $A = \left(a_{mnij}\right) \in \left(\left|F^{u,\theta}_{a,b}\right|_k, \mathcal{M}_u\right)$ if and only if

$$A_{mn} \in \left(\left|F^{u,\theta}_{a,b}\right|_k\right)^{\beta(bp)} \tag{3.3}$$

and

$$\sup_{m,n\in\mathbb{N}} \sum_{i,j} \left| \frac{1}{\hat{a}_i \hat{b}_j \left(u_i \theta_j\right)^{1/k'}} \Delta^{(ij)}_{11} \left(\frac{a_{mnij}}{a_i b_j}\right)\right|^{k'} < \infty. \tag{3.4}$$

*Proof.* To avoid the repetition of similar statements, we give the proof only for $1 < k < \infty$. Let $x = \left(x_{ij}\right) \in \left|F^{u,\theta}_{a,b}\right|_k$. Then, there exists a double sequence $y = (y_{mn}) \in \mathcal{L}_k$. By using the equalities (2.4)–(2.7), for $(s,t)$th rectangular partial sum of the series $\sum_{i,j} a_{mnij} x_{ij}$, we have

$$\begin{aligned} (Ax)^{[s,t]}_{mn} &= \sum_{i,j}^{s,t} a_{mnij} x_{ij} \\ &= \left(\frac{a_{mn00}}{a_0 b_0} - \frac{a_{mn10}}{a_1 b_0} - \frac{a_{mn01}}{a_0 b_1} + \frac{a_{mn11}}{a_1 b_1}\right) \frac{y_{00}}{\hat{a}_0 \hat{b}_0 \left(u_0 \theta_0\right)^{1/k'}} \\ &+ \left(\frac{a_{mns0}}{a_s b_0} - \frac{a_{mns1}}{a_s b_1}\right) \frac{y_{s0}}{\left(u_s \theta_0\right)^{1/k'} \hat{a}_s \hat{b}_0} + \left(\frac{a_{mn0t}}{a_0 b_t} - \frac{a_{mn1t}}{a_1 b_t}\right) \frac{y_{0t}}{\hat{a}_0 \hat{b}_t \left(u_0 \theta_t\right)^{1/k'}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{s-1}\left(\frac{a_{mni0}}{a_ib_0}-\frac{a_{mn,i+1,0}}{a_{i+1}b_0}-\frac{a_{mni1}}{a_ib_1}+\frac{a_{mn,i+1,1}}{a_{i+1}b_1}\right)\frac{y_{i0}}{\hat{a}_i\hat{b}_0\left(u_i\theta_0\right)^{1/k'}}\\
&+\sum_{j=1}^{t-1}\left(\frac{a_{mn0j}}{a_0b_j}-\frac{a_{mn0,j+1}}{a_0b_{j+1}}-\frac{a_{mn1j}}{a_1b_j}+\frac{a_{mn1,j+1}}{a_1b_{j+1}}\right)\frac{y_{0j}}{\hat{a}_0\hat{b}_j\left(u_0\theta_j\right)^{1/k'}}\\
&+\sum_{i=1}^{s-1}\left(\frac{a_{mnit}}{a_ib_t}-\frac{a_{mn,i+1,t}}{a_{i+1}b_t}\right)\frac{y_{it}}{\hat{a}_i\hat{b}_t\left(u_i\theta_t\right)^{1/k'}}\\
&+\sum_{j=1}^{t-1}\left(\frac{a_{mnsj}}{a_sb_j}-\frac{a_{mns,j+1}}{a_sb_{j+1}}\right)\frac{y_{sj}}{\hat{a}_s\hat{b}_j\left(u_s\theta_j\right)^{1/k'}}\\
&+\sum_{i,j=1}^{s-1,t-1}\left(\frac{a_{mnij}}{a_ib_j}-\frac{a_{mni,j+1}}{a_ib_{j+1}}-\frac{a_{mn,i+1,j}}{a_{i+1}b_j}-\frac{a_{mn,i+1,j+1}}{a_{i+1}b_{j+1}}\right)\frac{y_{ij}}{\hat{a}_i\hat{b}_j\left(u_i\theta_j\right)^{1/k'}}\\
&+\frac{a_{mnst}y_{st}}{a_sb_t\hat{a}_s\hat{b}_t\left(u_s\theta_t\right)^{1/k'}}\\
&=\sum_{i,j}^{s,t}h_{stij}^{mn}y_{ij}=\left(H_{mn}^{(k)}y\right)_{[s,t]}
\end{aligned}
$$

for every $s, t, m, n \in \mathbb{N}$, where the 4−dimensional matrix $H_{mn}^{(k)} = \left(h_{stij}^{mn}\right)$ is defined by

$$
h_{stij}^{mn}=\begin{cases}
\dfrac{1}{\hat{a}_0\hat{b}_0\left(u_0\theta_0\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{a_{mnij}}{a_ib_j}\right), & i=j=s=t=0,\\
\dfrac{1}{a_s\hat{a}_s\hat{b}_0\left(u_s\theta_0\right)^{1/k'}}\Delta_{01}^{(ij)}\left(\dfrac{a_{mnij}}{b_j}\right), & i=s,\ j=t=0,\\
\dfrac{1}{b_t\hat{a}_0\hat{b}_t\left(u_0\theta_t\right)^{1/k'}}\Delta_{10}^{(ij)}\left(\dfrac{a_{mnij}}{a_i}\right), & j=t\ ,\ i=s=0,\\
\dfrac{1}{\hat{a}_i\hat{b}_0\left(u_i\theta_0\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{a_{mnij}}{a_ib_j}\right), & 1\le i\le s-1,\ j=t=0,\\
\dfrac{1}{\hat{a}_0\hat{b}_j\left(u_0\theta_j\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{a_{mnij}}{a_ib_j}\right), & 1\le j\le t-1,\ i=s=0,\\
\dfrac{1}{b_t\hat{a}_i\hat{b}_t\left(u_i\theta_t\right)^{1/k'}}\Delta_{10}^{(ij)}\left(\dfrac{a_{mnij}}{a_i}\right), & 1\le i\le s-1,\ j=t,\\
\dfrac{1}{a_s\hat{a}_s\hat{b}_j\left(u_s\theta_j\right)^{1/k'}}\Delta_{01}^{(ij)}\left(\dfrac{a_{mnij}}{b_j}\right), & 1\le j\le t-1\ ,\ i=s,\\
\dfrac{1}{\hat{a}_i\hat{b}_j\left(u_i\theta_j\right)^{1/k'}}\Delta_{11}^{(ij)}\left(\dfrac{a_{mnij}}{a_ib_j}\right), & 1\le i\le s-1,\ 1\le j\le t-1,\\
\dfrac{a_{mnst}}{a_sb_t\hat{a}_s\hat{b}_t\left(u_s\theta_t\right)^{1/k'}} & ,\ i=s \text{ and } j=t,
\end{cases}
$$

for every $s, t, i, j \in \mathbb{N}$. Then, we can write the equality as follows:

$$
(Ax)_{mn}^{[s,t]}=\left(H_{mn}^{(k)}y\right)_{[s,t]}. \tag{3.5}
$$

Thus, it follows from (3.5) that the $bp$-convergence of $(Ax)_{mn}^{[s,t]}$ and the statement $H_{mn}^{(k)} \in \left(\mathcal{L}_k, C_{bp}\right)$ are equivalent for all $x \in \left|F_{a,b}^{u,\theta}\right|_k$ and $m,n \in \mathbb{N}$. Therefore, the condition (3.3) is satisfied for each fixed $m,n \in \mathbb{N}$, that is, $A_{mn} \in \left(\left|F_{a,b}^{u,\theta}\right|_k\right)^{\beta(bp)}$ for each fixed $m,n \in \mathbb{N}$ and $1 < k < \infty$ .

If we take $bp$-limit in the terms of the matrix $H_{mn}^{(k)} = \left(h_{stij}^{mn}\right)$ while $s,t \to \infty$, we deduce that

$$bp - \lim_{s,t\to\infty} h_{stij}^{mn} = h_{mnij} = \frac{1}{\hat{a}_i \hat{b}_j \left(u_i \theta_j\right)^{1/k'}} \Delta_{11}^{(ij)} \left(\frac{a_{mnij}}{a_i b_j}\right). \tag{3.6}$$

Therefore, using the 4-dimensional matrix $H^{(k)} = \left(h_{mnij}^{(k)}\right)$, we obtain with the relations (3.5) and (3.6) that

$$bp - \lim_{s,t\to\infty} (Ax)_{mn}^{[s,t]} = bp - \lim \left(H^{(k)} y\right)_{mn}. \tag{3.7}$$

Thus, it can be written that $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}^{u,\theta}\right|_k, \mathcal{M}_u\right)$ if and only if $H^{(k)} \in (\mathcal{L}_k, \mathcal{M}_u)$, by having in mind of the relation (3.7).

Therefore, using Lemma 2.2 (b), we conclude that

$$\sup_{m,n\in\mathbb{N}} \sum_{i,j} \left| \frac{1}{\hat{a}_i \hat{b}_j \left(u_i \theta_j\right)^{1/k'}} \Delta_{11}^{(ij)} \left(\frac{a_{mnij}}{a_i b_j}\right) \right|^{k'} < \infty,$$

which satisfies the condition (3.4).

So, we obtain that $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}^{u,\theta}\right|_k, \mathcal{M}_u\right)$ if and only if the conditions (3.3) and (3.4) are satisfied.

Thus, the theorem is proved.

**Theorem 3.2.** Assume that $A = \left(a_{mnij}\right)$ be an arbitrary 4-dimensional infinite matrix. In that case, the following statements hold:

**(a)** $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}\right|_1, C_{bp}\right)$ if and only if (3.1) and (3.2) are satisfied, and there exists $\left(\alpha_{ij}^{(1)}\right) \in \Omega$ such that

$$bp - \lim_{m,n\to\infty} \Delta_{11}^{(ij)} \left(\frac{a_{mnij}}{a_i b_j}\right) \frac{1}{\hat{a}_i \hat{b}_j} = \alpha_{ij}^{(1)}.$$

**(b)** Let $1 < k < \infty$. Then, $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}^{u,\theta}\right|_k, C_{bp}\right)$ if and only if (3.3) and (3.4) are satisfied, and there exists $\left(\alpha_{ij}^{(k)}\right) \in \Omega$ such that

$$bp - \lim_{m,n\to\infty} \Delta_{11}^{(ij)} \left(\frac{a_{mnij}}{a_i b_j}\right) \frac{1}{\hat{a}_i \hat{b}_j \left(u_i \theta_j\right)^{1/k'}} = \alpha_{ij}^{(k)}.$$

*Proof.* This theorem can be proved by using Lemma 2.2 (d) and (e) in a similar way to that used in the proof of Theorem 3.1.

**Theorem 3.3.** Assume that $A = \left(a_{mnij}\right)$ be an arbitrary 4-dimensional infinite matrix. In that case, the following statements hold:

**(a)** Let $1 \leq k < \infty$. $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}\right|_1, \mathcal{L}_k\right)$ if and only if (3.1) and

$$\sup_{i,j\in\mathbb{N}} \sum_{m,n} \left| \frac{1}{\hat{a}_i \hat{b}_j} \Delta_{11}^{(ij)} \left( \frac{a_{mnij}}{a_i b_j} \right) \right|^k < \infty$$

hold.
**(b)** Let $1 < k < \infty$. $A = \left(a_{mnij}\right) \in \left(\left|F_{a,b}^{u,\theta}\right|_k, \mathcal{L}_u\right)$ if and only if (3.3) and

$$\sum_{i,j=0}^{\infty} \left( \sum_{m,n=0}^{\infty} \left| \frac{1}{\hat{a}_i \hat{b}_j \left(u_i \theta_j\right)^{1/k'}} \Delta_{11}^{(ij)} \left( \frac{a_{mnij}}{a_i b_j} \right) \right| \right)^{k'} < \infty$$

hold.
*Proof.* This theorem can be proved by using Lemma 2.2 (c) and Lemma 2.3 in a similar way to that used in the proof of Theorem 3.1.
**Lemma 3.4.** [27] Let $\lambda$ and $\mu$ be two double sequence spaces in $\Omega$, $A = \left(a_{mnij}\right)$ an arbitrary 4-dimensional infinite matrix and $B = \left(b_{mnij}\right)$ be a triangle 4-dimensional infinite matrix. Then, $A \in (\lambda, \mu_B)$ if and only if $BA \in (\lambda, \mu)$.

Now, we can give the final results of our work by considering the Lemmas 2.2, 2.3, and 3.4.
**Corollary 3.5.** Let $A = \left(a_{mnij}\right)$ and $T = \left(t_{mnij}\right)$ four-dimensional matrices be given by the relation

$$t_{mnij} = \sum_{u,v=1}^{m,n} \psi_{mnuv} a_{uvij},$$

where $\Psi = (\psi_{mnuv})$ is defined as

$$\psi_{mnuv} = \begin{cases} (u_m \theta_n)^{1/k'} \hat{a}_m \hat{b}_n a_u b_v, & 0 \leq u \leq m,\ 0 \leq v \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

by considering the relation (2.1). Then, the necessary and sufficient conditions for the classes $\left(\mathcal{L}_u, \left|F_{a,b}^{u,\theta}\right|_k\right)$ and $\left(\mathcal{L}_k, \left|F_{a,b}\right|_1\right)$ can be found for $1 \leq k < \infty$ as follows:
**(a)** $A = \left(a_{mnij}\right) \in \left(\mathcal{L}_u, \left|F_{a,b}^{u,\theta}\right|_k\right)$ if and only if

$$\sup_{i,j\in\mathbb{N}} \sum_{m,n} \left|t_{mnij}\right|^k < \infty$$

holds for $1 \leq k < \infty$.
**(b)** $A = \left(a_{mnij}\right) \in \left(\mathcal{L}_k, \left|F_{a,b}\right|_1\right)$ if and only if

$$\sum_{r,s=0}^{\infty} \left( \sum_{m,n=0}^{\infty} |t_{mnrs}| \right)^{k'} < \infty$$

holds for $1 < k < \infty$.

## 4. Conclusions

In this paper, a new double series space $\left|F_{a,b}^{u,\theta}\right|_k$ is defined by using the four-dimensional factorable matrix $F$ and the absolute summability method for $k \geq 1$. Also, some algebraic and topological properties of the space $\left|F_{a,b}^{u,\theta}\right|_k$ are given, and the $\alpha$-, $\beta\,(bp)$-, and $\gamma$-duals of this space are determined. Finally, the characterizations of some new four-dimensional matrix classes in the related spaces are presented and some important results concerned with Riesz and Cesàro matrix methods are extended to double sequences owing to the four-dimensional factorable matrix. By using the new series space defined the four-dimensional factorable matrix $F$, many impressive results can be obtained in the theory of series spaces and matrix transformations.

## Author contributions

Aslıhan ILIKKAN CEYLAN: Idea, conceptualization, methodology, investigation, solving methods, formal analysis, writing-original draft; Canan HAZAR GÜLEÇ: Idea, conceptualization, methodology, supervision, review editing. All authors have read and approved the final version of the manuscript for publication.

## Conflict of interest

The authors declare that there is no conflict of interest.

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