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Research article

Accelerating the convergence of a two-dimensional periodic nonuniform sampling series through the incorporation of a bivariate Gaussian multiplier

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Abstract: Recently, in the field of periodic nonuniform sampling, researchers (Wang et al., 2019; Asharabi, 2023) have investigated the incorporation of a Gaussian multiplier in the one-dimensional series to improve its convergence rate. Building on these developments, this paper aimed to accelerate the convergence of the two-dimensional periodic nonuniform sampling series by incorporating a bivariate Gaussian multiplier. This approach utilized a complex-analytic technique and is applicable to a wide range of functions. Specifically, it applies to the class of bivariate entire functions of exponential type that satisfy a decay condition, as well as to the class of bivariate analytic functions defined on a bivariate horizontal strip. The original convergence rate of the two-dimensional periodic nonuniform sampling is given by $O(N^{-p})$, where $p \geq 1$. However, through the implementation of this acceleration technique, the convergence rate improved drastically and followed an exponential order, specifically $e^{-\alpha N}$, where $\alpha > 0$. To validate the theoretical analysis presented, the paper conducted rigorous numerical experiments.

Keywords: two-dimensional periodic nonuniform sampling; sinc approximation; Gaussian regularization; entire functions of exponential type; error bounds
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1. Introduction

The Bernstein space $B^p_\Omega(\mathbb{R})$ is a set of functions in $L^p(\mathbb{R})$ that can be extended to entire functions of exponential type Ω. According to Schwartz's theorem [1,2], $B^p_\Omega(\mathbb{R})$ can be defined as the collection of functions f in $L^p(\mathbb{R})$ such that the Fourier transform $\widehat{f}$ is supported in the interval $[-\Omega, \Omega]$. In simpler terms, we can describe it as follows:

$$B^p_\Omega(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}) : \quad \operatorname{supp} \widehat{f} \subset [-\Omega, \Omega] \right\}, \tag{1.1}$$

where $\widehat{f}$ represents the Fourier transform of f in the sense of generalized functions. Consider a set of arbitrary points $0 \leq x_1 < x_2 < \ldots < x_J < \kappa J$ in $\mathbb{R}$, which are not necessarily equidistant. We define the sampling points as follows:

$$\tau_{j,n,\kappa} := x_j + n\kappa J, \quad n \in \mathbb{Z}, ; j = 1, \ldots, J, \tag{1.2}$$

where $\kappa \in (0, \pi/\Omega]$, Ω is a positive number, and J is a positive integer. In this sampling scheme, the points are grouped into sets of J points. The one-dimensional periodic nonuniform sampling theorem states that if f belongs to the space $B^p_\Omega(\mathbb{R})$, where $1 \leq p < \infty$, then f can be expressed in the following form, as proved in [3],

$$f(z) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{J} f(\tau_{j,n,\frac{\pi}{\Omega}})\, \psi_{j,n,\frac{\pi}{\Omega}}(z), \qquad z \in \mathbb{C}, \tag{1.3}$$

where

$$\psi_{j,n,\kappa}(z) := \operatorname{sinc}\left(\frac{\pi}{J\kappa}(z - \tau_{j,n,\kappa})\right) \prod_{k=1,k\neq j}^{J} \frac{\sin\left(\frac{\pi}{J\kappa}(z - \tau_{k,n,\kappa})\right)}{\sin\left(\frac{\pi}{J\kappa}(x_j - x_k)\right)}. \tag{1.4}$$

The sinc function is defined by

$$\operatorname{sinc}(x) := \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

The series on the right-hand side of (1.3) converges uniformly and absolutely over $\mathbb{R}$ as well as on any compact subset of $\mathbb{C}$, as stated in [4]. The series (1.3) has garnered significant interest in both the fields of mathematics and engineering. It has been the subject of extensive attention, as evidenced by works such as [4–8] and other related references. The authors in [9] introduced a periodic nonuniform sampling approach that involves derivatives. In a different context, the authors of [6] extended the expansion given in Eq (1.3) to bandlimited functions in the fractional Fourier transform domains. In a related work, [10] modified the series presented in Eq (1.3) by incorporating a Gaussian function using a Fourier-analytic method. Additionally, in [11], Asharabi accelerated the series in Eq (1.3) by incorporating a Gaussian function based on a complex-analytic approach, specifically for two classes of analytic functions. The work in [11] is generalized in [12] for periodic nonuniform sampling involving higher-order derivatives.

Consider the class $E_\Omega(\varphi)$, where $\Omega > 0$, defined as follows:

$$E_\Omega(\varphi) := \left\{ f : \mathbb{C} \to \mathbb{C} \mid \text{ is entire and } |f(z)| \leq \phi\left(|\Re z|\right) \exp\left(\Omega |\Im z|\right),\ z \in \mathbb{C} \right\}, \tag{1.5}$$

where ϕ is a continuous, non-decreasing, and non-negative function defined on $\mathbb{R}^+$. It is worth noting that the space $E_\Omega(\phi)$, introduced in [13], is larger than the Bernstein space $B^p_\Omega(\mathbb{R})$. Consider the class $\mathcal{E}_{L^p(\mathbb{R})}$, which consists of entire functions that belong to $L^p(\mathbb{R})$ when restricted to the real line. In [11], the first author introduced the nonuniform sinc-Gauss localization operator $\mathcal{G}_{h,J,N} : E_\Omega(\phi) \to \mathcal{E}_{L^p(\mathbb{R})}$ for every $1 \leq p \leq \infty$ as follows:

$$\mathcal{G}_{h,J,N}[f](z) = \sum_{n\in\mathbb{Z}_N(z)} \sum_{j=1}^{J} f(\tau_{j,n,h})\, \psi_{j,n,h}(z) \exp\left(-\frac{\alpha}{NJh^2}\left(z - \tau_{j,n,h}\right)^2\right), \tag{1.6}$$

where the function $\psi_{j,n,h}$ is defined in (1.4), $\alpha := (\pi - h\Omega)/2$, $h \in (0, \pi/\Omega]$, and

$$\mathbb{Z}_N(z) := \left\{ n \in \mathbb{Z} : |\lfloor J^{-1}h^{-1}\Re z + 1/2 \rfloor - n| \le N \right\}.$$

Using the complex analytic technique presented in [13], the first author established in [11] that if $f \in E_\Omega(\phi$, then the following estimate is valid:

$$\left| f(z) - \mathcal{G}_{h,J,N}[f](z) \right| \le 2^{J-1} \left| \prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - x_k) \right) \right| \phi\left(|\Re z| + Jh(N+2) \right) \chi_{N,J}\left(\Im z \right) \frac{\mathrm{e}^{-\alpha JN}}{\sqrt{\pi\alpha JN}}, \tag{1.7}$$

where $|\Im z| < JhN$, J is a positive integer, and $h \in (x_J/J, \pi/\Omega]$. The function $\chi_{J,N}$ is given by

$$\begin{aligned} \chi_{N,J}(t) &:= \frac{2\mathrm{e}^{\alpha t^2/N}}{h\sqrt{\pi\alpha JN}\left(1 - (t/JhN)^2\right)} + \frac{\mathrm{e}^{-2\alpha t}}{(1 - \mathrm{e}^{-\frac{2\pi}{Jh}(JhN+t)})^J} + \frac{\mathrm{e}^{2\alpha t}}{(1 - \mathrm{e}^{-\frac{2\pi}{Jh}(JhN-t)})^J} \\ &= 2\cosh(2\alpha t) + O(N^{-1/2}), \quad \text{as} \quad N \to \infty. \end{aligned} \tag{1.8}$$

The author in [11] relaxed the condition of $f \in E_\Omega(\phi)$ in the previous result and considered f to belong to a class of analytic functions in an infinite horizontal strip $\mathcal{D}_d := \left\{ z \in \mathbb{C} : \left| \Im z \right| < d \right\}$. This class is denoted as $A_d(\phi)$ and is defined as follows:

$$A_d(\phi) := \left\{ f : \mathcal{D}_d \to \mathbb{C} \mid \text{ is analytic in } \mathcal{D}_d \text{ and } |f(z)| \le \phi\left(\left| \Re z \right| \right), \quad z \in \mathcal{D}_d \right\}, \tag{1.9}$$

where ϕ is a continuous, non-decreasing, and non-negative function on $\mathbb{R}^+$. This class was initially introduced in [13] and has been utilized in various studies, such as [14, 15]. In [11], an estimation for the error $\left| f(z) - \mathcal{G}_{h,J,N} f \right|$ was derived when f belongs to the class $A_d(\phi)$. If $f \in A_d(\phi)$, the following estimate is valid:

$$\left| f(z) - \mathcal{G}_{\frac{d}{N},J,N}[f](z) \right| \le 2^{J+1/2} \left| \prod_{k=1}^{J} \sin\left(\frac{\pi N}{Jd}(z - x_k) \right) \right| \phi\left(|\Re z| + \rho_N \right) \gamma_{N,J}\left(\Im z/d \right) \frac{e^{-\frac{\pi}{2}\left(JN - \frac{2|\Im z|}{d} \right)}}{\pi\sqrt{N}}, \tag{1.10}$$

where $z \in \mathcal{D}_{d/4}$ and $\rho_{N,J} := Jd(1 + \frac{2}{N})$. The function $\gamma_{N,J}$ is given by

$$\gamma_{N,J}(t) \;\; := \;\; \frac{1}{1-t}\left[\frac{1}{(1 - e^{-2\pi N})^J} + \frac{2\sqrt{2}}{\pi\sqrt{JN}(1+t)} \right] = \frac{1}{1-t}\left[1 + O\left(N^{-1/2} \right) \right], \quad \text{as} \quad N \to \infty.$$

The two-dimensional nonuniform periodic sampling series goes back to Butzer and Hinsen, as mentioned in [16, 17]. First, we introduce the definition of the Bernstein space for functions of two variables, denoted as $B^p_\Omega(\mathbb{R}^2)$, where $1 \le p < \infty$. This space encompasses all entire functions of two variables that exhibit exponential type Ω and belong to $L^p(\mathbb{R}^2)$ when their domain is restricted to $\mathbb{R}^2$. Schwartz's theorem, as presented in references such as [1, 2], provides further insights into this space.

$$B^p_\Omega(\mathbb{R}^2) = \left\{ f \in L^p(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset [-\Omega, \Omega]^2 \right\}, \tag{1.11}$$

where $\hat{f}$ is the Fourier transform of f in the sense of generalized functions. Let $0 \le x_{11} < x_{12} < \ldots < x_{1J_1} < J_1 h$ and $0 \le x_{21} < x_{22} < \ldots < x_{2J_2} < J_2 h$ be arbitrary points that are not necessarily equidistant in $\mathbb{R}$. We define the sampling points $(\tau_{j,n,h}, \nu_{k,m,h})$ in $\mathbb{R}^2$ as follows:

$$\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h} \right) := \left(x_{1j_1} + n_1 J_1 h, x_{2j_2} + n_2 J_2 h \right) \quad j_1 = 1, \ldots, J_1, \quad j_2 = 1, \ldots, J_2, \tag{1.12}$$

where $(n_1, n_2) \in \mathbb{Z}^2$, $h \in (0, \pi/\Omega]$, Ω is a positive number, and J_l, $l = 1, 2$, are positive integers. Butzer and Hinsen established the two-dimensional nonuniform sampling iterated expansion in [16, 17], and the two-dimensional nonuniform periodic sampling series is a special case of this expansion. We can express this special case in a new form as follows, as mentioned in [17, p. 78]. If $f \in B^p_\Omega(\mathbb{R}^2)$ with $1 \leq p < \infty$, then f can be represented as follows:

$$f(z) = \sum_{n\in\mathbb{Z}^2} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(z_l), \qquad z = (z_1, z_2) \in \mathbb{C}^2, \tag{1.13}$$

where $n = (n_1, n_2)$ and the function $\psi_{j_l,n_l,h}$ is given in (1.4). The series on the right-hand side of (1.13) converges uniformly and absolutely over $\mathbb{R}^2$ as well as on any compact subset of $\mathbb{C}^2$, as stated in [17].

The convergence rate of the expansion in (1.13) is relatively slow, on the order of $O(N^{-p})$ with $p \geq 1$ (see Section 2 below). As far as we know, no previous research has focused on accelerating the convergence of this expansion by incorporating a bivariate Gaussian kernel. By applying this acceleration technique, the convergence rate significantly improves to an exponential order, specifically $e^{-\alpha N}$, where $\alpha > 0$. In this study, we build upon the technique proposed in [11] to accelerate the convergence of the two-dimensional periodic nonuniform sampling series (1.13) by incorporating a bivariate Gaussian multiplier. The approach employed in this paper utilizes complex-analytic techniques and is applicable to a broad range of functions. Specifically, it applies to two classes of functions. The first class includes bivariate entire functions of exponential type that satisfy a decay condition. The second class comprises bivariate analytic functions defined on a bivariate horizontal strip.

The remaining sections of this paper are structured as follows. In Section 2, we establish the convergence rate of the two-dimensional periodic nonuniform sampling series (1.13). Section 3 is dedicated to accelerating the convergence of the series (1.13) by incorporating a bivariate Gaussian multiplier for a broader range of bivariate entire functions of exponential type that satisfy a decay condition. We relax the condition imposed on f in the previous section and consider it to belong to a class of bivariate analytic functions in a two-dimensional horizontal strip. In Section 5, we present numerical examples to illustrate the applicability and effectiveness of the proposed approach. Finally, Section 6 provides a summary and conclusion of the paper.

2. Convergence rate

This section is dedicated to examining the rate at which the sampling series (1.13) converges. We demonstrate that the series in (1.13) has a slow convergence rate, which is of order $O(N^{-1/p})$ where $P > 1$. To illustrate our approach, we examine the truncation error of the series in (1.13) based on localized sampling without a decay assumption. To achieve this, we truncate the series in (1.13) as follows:

$$T_{J_1,J_2,N}[f](x) := \sum_{n\in\mathbf{Z}^2_N(x)} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(z_l), \quad x := (x_1, x_2) \in \mathbb{R}^2, \tag{2.1}$$

where

$$\mathbf{Z}^2_N(x) := \Big\{(n_1, n_2) \in \mathbb{Z}^2 : -N < \frac{\Omega\, x_l}{\pi} - n_l \leq N, \quad l = 1, 2\Big\}. \tag{2.2}$$

That is, if we want to estimate f, we only sum over values of f on a part of $(\pi/\Omega)\mathbb{Z}^2$ near x. In the following two lemmas, we introduce auxiliary results that will be utilized to estimate the upper bound of $\left|f(x) - T_{J_1,J_2,N}[f](x)\right|$ for $(x, y) \in \mathbb{R}^2$.

Lemma 2.1. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\Omega > 0$. Then, we have*

$$\left(\sum_{n\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)} \left| \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l) \right|^q \right)^{1/q} \leq C_{p,\Omega} \prod_{l=1}^{2} \delta_{J_l} N^{-1/p}, \tag{2.3}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Here $C_{p,\Omega}$ is a positive constant dependent only on p, Ω and the constant δ_{J_l}, $l = 1, 2$, is defined as

$$\delta_{J_l} := \prod_{k=1,k\neq j}^{J_l} \frac{1}{\left|\sin\left(\frac{\Omega}{J}(x_{lj} - x_{lk})\right)\right|}. \tag{2.4}$$

Proof. It is evident from definition (2.2) of $\mathbf{Z}_N^2(x)$ that

$$\begin{aligned} \sum_{n\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)} \left| \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l) \right|^q \leq & \sum_{n_1=-\infty}^{\infty} \left|\psi_{j_1,n_1,\frac{\pi}{\Omega}}(x_1)\right|^q \sum_{\left|\frac{\Omega x_2}{\pi} - n_2\right| > N} \left|\psi_{j_2,n_2,\frac{\pi}{\Omega}}(x_2)\right|^q \\ & + \sum_{\left|\frac{\Omega x_1}{\pi} - n_1\right| > N} \left|\psi_{j_1,n_1,\frac{\pi}{\Omega}}(x_1)\right|^q \sum_{n_2=-\infty}^{\infty} \left|\psi_{j_2,n_2,\frac{\pi}{\Omega}}(x_2)\right|^q . \end{aligned} \tag{2.5}$$

The following inequality was derived by the first author in [11, Eq (16)]:

$$\left(\sum_{\left|\frac{\Omega x_l}{\pi} - n_l\right| > N} \left|\psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l)\right|^q \right)^{1/q} \leq c_{p,\Omega}\, \delta_{J_l}\, N^{-1/p}, \quad x_l \in \mathbb{R}, \tag{2.6}$$

where $c_{p,\Omega}$ is a positive constant dependent only on p and Ω, and the constant δ_{J_l}, is defined in (2.4). In [4, Lemma 2.2], the authors derived the subsequent inequality:

$$\left(\sum_{n_l=-\infty}^{\infty} \left|\psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l)\right|^q \right)^{1/q} \leq p\, \delta_{J_l}, \quad x_l \in \mathbb{R}. \tag{2.7}$$

Combining (2.7), (2.6), and (2.5), we get (2.3) and the proof is complete. □

Lemma 2.2. *For $f \in B_\Omega^p(\mathbb{R}^2)$ with $1 < p < \infty$, we have the following inequality:*

$$\sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \left(\sum_{n\in\mathbb{Z}^2} \left| f(\tau_{j_1,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \right|^p \right)^{1/p} \leq A_{p,\Omega,J_1,J_2} \|f\|_p, \tag{2.8}$$

where A_{p,Ω,J_1,J_2} is a constant that depends only on p, Ω, J_1, and J_2.

Proof. For $g \in B^p_\Omega(\mathbb{R})$, defined as (1.1), and for any increasing sequence λ_n, with the condition $\lambda_n - \lambda_n \geq \delta > 0$, we have, as demonstrated in [18, Theorem 6.7.15], the following inequality:

$$\sum_{n=-\infty}^{\infty} |g(\lambda_n|^p \leq a^p_{p,\Omega,\delta} \|g\|^p_p, \tag{2.9}$$

where $a_{p,\Omega,\delta}$ is a constant that depends solely on p, Ω, and δ. As $f \in B^p_\Omega(\mathbb{R}^2)$, the function $g(y) := f(\tau_{j_1,n_1,\frac{\pi}{\Omega}}, y)$ also belongs to the space $B^p_\Omega(\mathbb{R})$. Consequently, we can utilize (2.9) to obtain the following result:

$$\sum_{n_2=-\infty}^{\infty} \left| f(\tau_{j_1,n_1,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \right|^p \leq a^p_{p,\Omega,J_2} \|f(\tau_{j_1,n_1,\frac{\pi}{\Omega}}, \cdot)\|^p_{L^p(\mathbb{R})}, \tag{2.10}$$

where we have employed $\lambda_n := \tau_{j_2,n_2,\frac{\pi}{\Omega}}$. Given that $f \in B^p_\Omega(\mathbb{R}^2)$, the function $\left(\int_{-\infty}^{\infty} |f(x,y)|^p dy\right)^{1/p}$ also belongs to the space $B^p_\Omega(\mathbb{R})$. By applying (2.9) once again to the function $\left(\int_{-\infty}^{\infty} |f(x,y)|^p dy\right)^{1/p}$, we obtain the following inequality:

$$\sum_{n_1=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f(\tau_{j_1,n_1,\frac{\pi}{\Omega}}, y) \right|^p dy \leq a^p_{p,\Omega,J_1} \|f\|^p_p, \tag{2.11}$$

where a_{p,Ω,J_1} is a constant that depends solely on p, Ω, and J_1. By combining (2.11) and (2.9), we arrive at the following inequality:

$$\left(\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left| f(\tau_{j_1,n_1,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \right|^p \right)^{1/p} \leq \prod_{l=1}^{2} a_{p,\Omega,J_l} \|f\|_p. \tag{2.12}$$

Finally, by summing over j_1 and j_2, we deduce (2.8). □

In the following theorem, we demonstrate that the convergence rate of the sampling series (1.13) cannot be faster than $O(1/N)$.

Theorem 2.3. *Let f be a function in the Bernstein space $B^p_\Omega(\mathbb{R}^2)$ with $1 < p < \infty$. Then, the following inequality holds:*

$$\left| f(x) - T_{J_1,J_2,N}[f](x) \right| \leq \mathcal{D}_{p,\Omega,J_1,J_2} \prod_{l=1}^{2} \delta_{J_l} \|f\|_p \, N^{-1/p}, \tag{2.13}$$

for all $x \in \mathbb{R}^2$, and $\mathcal{D}_{p,\Omega,J_1,J_2}$ is a constant that depends only on p, Ω, J_1, and J_2.

Proof. Given that f belongs to the Bernstein space $B^p_\Omega(\mathbb{R}^2)$, we can utilize the expansion (1.13). By combining it with (2.1) and applying the triangle inequality, we derive the following expression:

$$\left| f(x) - T_{J_1,J_2,N}[f](x) \right| \leq \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \sum_{n \in \mathbb{Z}^2 \setminus \mathbf{Z}^2_N(x)} \left| f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l) \right|. \tag{2.14}$$

The reason for being able to interchange the sums in the last step is the absolute convergence of the series in (1.13). By applying Hölder's inequality, we acquire the following result:

$$\sum_{n \in \mathbb{Z}^2 \setminus \mathbf{Z}^2_N(x)} \left| f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l) \right| \leq \left(\sum_{n \in \mathbb{Z}^2 \setminus \mathbf{Z}^2_N(x)} \left| f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \right|^p \right)^{1/p}$$

$$\times \left(\sum_{n\in\mathbb{Z}^2\backslash\mathbf{Z}_N^2(x)} \left| \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l) \right|^q \right)^{1/q}, \tag{2.15}$$

with $p, q > 1$ and $1/p + 1/q = 1$. Substituting from (2.3) and (2.8) into (2.15), we obtain the following:

$$\sum_{n\in\mathbb{Z}^2\backslash\mathbf{Z}_N^2(x)} \left| f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(x_l) \right| \le C_{p,\Omega} \prod_{l=1}^{2} \delta_{J_l} a_{p,\Omega,J_l} \|f\|_p N^{-1/p}. \tag{2.16}$$

By combining (2.16) with (2.12), we obtain (2.13), and thus the proof is concluded. □

3. Bivariate nonuniform sinc-Gauss formula

In this section, we modify the two-dimensional periodic nonuniform sampling series (1.13) by incorporating a bivariate Gaussian multiplier using the complex-analytic approach. We consider the class $E^2_\Omega(\varphi)$ defined as follows:

$$E^2_\Omega(\varphi) := \left\{ f : \mathbb{C}^2 \to \mathbb{C} \mid \text{ is entire and } |f(z)| \le \varphi\left(\left|\Re z_1\right|, \left|\Re z_2\right| \right) \exp\left(\sigma \sum_{l=1}^{2} |\Im z_l| \right) \right\}, \tag{3.1}$$

where $z := (z_1, z_2) \in \mathbb{C}^2$. The function φ is continuous, non-negative, and non-decreasing in both variables $\left|\Re z_j\right|$, $j = 1, 2$. This class was first introduced in [15] and used in some studies, cf. e.g., [19]. It is important to note that the space $E^2_\Omega(\varphi)$, introduced in [15], is larger than the Bernstein space $B^p_\Omega(\mathbb{R}^2)$. The class $E^2_\Omega(C)$, with C being a constant, encompasses entire functions of exponential type Ω that may not necessarily belong to $L^p(\mathbb{R}^2)$ when restricted to $\mathbb{R}^2$. Additionally, we consider the class $\mathcal{E}_{L^p(\mathbb{R}^2)}$, which consists of entire functions of two variables that belong to $L^p(\mathbb{R}^2)$ when their real domain is considered. Consider the bivariate localization sampling operator $\mathcal{G}_{h,J_1,J_2,N} : E^2_\Omega(\varphi) \to \mathcal{E}_{L^p(\mathbb{R}^2)}$ defined as follows:

$$\mathcal{G}_{h,J_1,J_2,N}[f](z) = \sum_{n\in\mathbb{Z}_N^2(z)} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} f(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}) \prod_{l=1}^{2} \psi_{j_l,n_l,h}(z_l)\, \mathrm{e}^{-\frac{\alpha\left(z_l-\tau_{j_l,n_l,h}\right)^2}{NJ_l h^2}}, \tag{3.2}$$

where $n := (n_1, n_2)$ and $z := (z_1, z_2) \in \mathbb{C}^2$. The function $\psi_{j_l,n_l,h}$ is defined in (1.4), $\alpha := (\pi - h\Omega)/2$, $h \in (0, \pi/\Omega]$, and

$$\mathbb{Z}_N^2(z) := \left\{ n \in \mathbb{Z}^2 : |\lfloor J_l^{-1} h^{-1} \Re z_l + 1/2 \rfloor - n_l| \le N, \quad l = 1, 2 \right\}.$$

On the class $E^2_\Omega(\varphi)$, the first author and Prestin introduced the following two-dimensional uniform sampling operator, cf. [15]:

$$\mathcal{G}_{h,N}[f](z) := \sum_{n\in\mathbb{Z}_N^2(z)} f(n_1 h, n_n h) \prod_{j=1}^{2} \operatorname{sinc}\left(\pi h^{-1} z_j - n_j \pi\right) \exp\left(-\frac{\alpha\left(z_j - n_j h\right)^2}{Nh^2} \right). \tag{3.3}$$

The operator (3.3) can be considered as a special case of the operator in (3.2) when $J_1 = J_2 = 1$ and $x_{11} = x_{21} = 0$. Now, let us denote the periodic nonuniform sampling expansion (1.3) as $\mathcal{L}_{\Omega,J_1,J_2} f$, where

$\mathcal{L}_{\Omega,J_1,J_2} : B^p_\Omega\left(\mathbb{R}^2\right) \to B^p_\Omega\left(\mathbb{R}^2\right)$. The key question here is: What is the relationship between the operators $\mathcal{L}_{\Omega,J_1,J_2}$ and $\mathcal{G}_{h,J_1,J_2,N}$? The following lemma addresses this question and is specifically applicable to the Bernstein space $B^p_\Omega\left(\mathbb{R}^2\right)$.

Lemma 3.1. *For any $f \in B^p_\Omega\left(\mathbb{R}^2\right)$, we have*

$$\lim_{N\to\infty} \mathcal{G}_{h,J_1,J_2,N} f = \mathcal{L}_{\Omega,J_1,J_2} f = f.$$

Proof. By setting $h = \pi/\Omega$ in the operator (3.2) and taking the limit as $N \to \infty$, we obtain the right-hand side of the expansion (1.13) because $\alpha = 0$ and $\lim_{N\to\infty} \mathbb{Z}^2_N(z) = \mathbb{Z}^2$. Since $f \in B^p_\Omega\left(\mathbb{R}^2\right)$, this expansion converges uniformly on any compact subset of $\mathbb{C}^2$, and we have $\mathcal{L}_{\Omega,J_1,J_2} f = f$. □

Consider the kernel function

$$\mathcal{K}_z(\zeta) := \frac{\mathcal{S}_z(\zeta)\ \prod_{l=1}^{2} \mathrm{e}^{-\frac{\alpha(z_l-\zeta_l)^2}{NJ_lh^2}}}{\prod_{l=1}^{2}(\zeta_l - z_l)\prod_{j_l=1}^{J_l}\sin\left(\frac{\pi}{J_lh}(\zeta_l - \tau_{j_l,n_l,h})\right)} \tag{3.4}$$

where $\zeta := (\zeta_1, \zeta_2)$, $z := (z_1, z_2)$, $z \in \mathbb{C}^2\backslash\{(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h})\}$, $j_l = 1,\ldots,J_l$, $l = 1, 1$. The points $\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$ are given in (1.12) and the function $\mathcal{S}_z$ is defined as

$$\begin{aligned}\mathcal{S}_z(\zeta) \ := \ & \prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(z_1 - \tau_{j_1,n_1,h})\right)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(\zeta_2 - \tau_{j_2,n_2,h})\right)\\ + & \prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(\zeta_1 - \tau_{j_1,n_1,h})\right)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(z_2 - \tau_{j_2,n_2,h})\right)\\ - & \prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(z_1 - \tau_{j_1,n_1,h})\right)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(z_2 - \tau_{j_2,n_2,h})\right).\end{aligned}$$

The kernel $\mathcal{K}_z(\zeta)$, as a function of variables ζ_1 and ζ_2, has a singularity of order one at all points belonging to the sets $\{(z_1,\mathbb{C}),(\mathbb{C},z_2)\ : z_1, z_2 \in \mathbb{C}\}$ and $\left\{\left(\tau_{j_1,n_1,h},\mathbb{C}\right),\left(\mathbb{C},\tau_{j_2,n_2,h}\right) : (n_1,n_2) \in \mathbb{Z}^2\right\}$ where $j_l = 1,\ldots,J_l$, $l = 1, 2$. These sets are subsets of $\mathbb{C}^2$ and can be interpreted as the Cartesian product of the ζ_l-planes for $l = 1, 2$.

In the following result, we demonstrate that the difference between a function $f \in E^2_\Omega(\varphi)$ and the operator $\mathcal{G}_{h,J_1,J_2,N}[f]$ can be expressed as the integration of $\mathcal{K}_z f$ over a hyperrectangle $\prod_{l=1}^{2} R_{z_l}$. Here, R_{z_l} is a rectangle in the ζ_l-plane, oriented positively and defined by its vertices at $\pm J_lh(N + 3/2) + J_lhN_{z_l/Jh} + \mathrm{i}(\Im z_l \pm J_lhN)$, where $N_{z_l} := \lfloor \Re z_l + 1/2\rfloor$ and $l = 1, 2$. The hyperrectangle $\prod_{l=1}^{2} R_{z_l}$ depends on the point $z = (z_1, z_2)$.

Lemma 3.2. *For all $z \in \mathbb{C}^2$ and $f \in E^2_\Omega(\varphi)$, we have*

$$f(z) - \mathcal{G}_{h,J_1,J_2,N}[f](z) = \begin{cases} \dfrac{1}{(2\pi i)^2}\displaystyle\oint_{R_{z_2}}\oint_{R_{z_1}} \mathcal{K}_z(\zeta) f(\zeta)\, d\zeta, & z \neq \left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right),\\ 0, & z = \left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right),\end{cases} \tag{3.5}$$

where $\zeta = (\zeta_1, \zeta_2)$, and $\prod_{l=1}^{2} R_{z_l}$ represents the hyperrectangle described earlier.

Proof. By replacing the values of $\tau_{j_l,n_l,h}$, as defined in Eq (1.12), in the function $\psi_{j_l,n_l,h}$, we obtain

$$\psi_{j_l,n_l,h}(\tau_{j_k,n_k,h}) := \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \tag{3.6}$$

Substituting $z = \left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$ into (3.2) and using (3.6), we obtain $\mathcal{G}_{h,J_1,J_2,N}[f]\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right) = f\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$ for all $(n_1, n_2) \in \mathbb{Z}_N^2(z)$, $j_l = 1, \ldots, J_l$, $j = 1, 2$. Hence, the second part of the equality in (3.5) holds. To establish the first part of the equality in (3.5), we will apply the residue theorem. For convenience, let us define $F(\zeta) := \mathcal{K}z(\zeta)f(\zeta)$. We can denote the residue of the function $\zeta_1 \mapsto F(\zeta_1, \zeta_2)$ at $\zeta_1 = \lambda_1 \in \mathbb{C}$, where ζ_2 is a complex parameter, as $\operatorname{Res}_1 F(\lambda_1, \zeta_2)$. Assuming that $\operatorname{Res}_1 F(\lambda_1, \zeta_2)$ has already been defined, we can define the residue of the function $\zeta_2 \mapsto \operatorname{Res}_1 F(\lambda_1, \zeta_2)$ at $\zeta_2 = \lambda_2 \in \mathbb{C}$ by $\operatorname{Res}_2 F(\lambda_1, \lambda_2)$. Calculating the residue $\operatorname{Res}_2 F$, we obtain, for all $z := (z_1, z_2) \in \mathbb{C}^2$,

$$\operatorname{Res}_2 F(z) = f(z), \tag{3.7}$$

and for all $(n_1, n_2) \in \mathbb{Z}_N^2(z)$,

$$\operatorname{Res}_2 F\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right) = -f(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}) \prod_{l=1}^{2} \psi_{j_l,n_l,h}(z_l)\, \mathrm{e}^{-\frac{\alpha\left(z_l - \tau_{j_l,n_l,h}\right)^2}{N J_l h^2}}. \tag{3.8}$$

Therefore, we have

$$\frac{1}{(2\pi\mathrm{i})^2} \oint_{R_{z_2}} \oint_{R_{z_1}} \mathcal{K}_z(\zeta) f(\zeta)\, d\zeta = \operatorname{Res}_2 F(z) + \sum_{n \in \mathbb{Z}_N^2(z)} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \operatorname{Res}_2 F\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right), \tag{3.9}$$

where $z \in \mathbb{C}^2 \backslash \{(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h})\}$, and $j_l = 1, \ldots, J_l$, $l = 1, 2$. By combining (3.7), (3.8), and (3.9), we obtain the first part of the equality in (3.5). □

We can extend the condition of Lemma 3.2 by relaxing the requirement of $f \in E_\Omega^2(\varphi)$ to a broader class of functions. Instead, we consider functions that belong to a class of analytic functions defined in an infinite bivariate horizontal strip given by

$$\mathcal{S}_d^2 := \left\{ z \in \mathbb{C}^2 : \left|\Im z_l\right| < d, \quad l = 1, 2 \right\}. \tag{3.10}$$

In particular, let $A_d(\varphi)$ be the class defined as

$$A_d^2(\varphi) := \left\{ f : \mathcal{S}_d^2 \to \mathbb{C} \mid \text{ is analytic in } \mathcal{S}_d^2 \text{ and } |f(z)| \leq \varphi\left(\left|\Re z_1\right|, \left|\Re z_2\right|\right), \quad z \in \mathcal{S}_d^2 \right\}, \tag{3.11}$$

where φ is a continuous, non-negative, and non-decreasing function in both variables $\left|\Re z_l\right|$, $l = 1, 2$. The class $A_d^2(\varphi)$ was initially introduced in [15] and has been utilized in various studies, cf. e.g., [19]. Within the class $A_d^2(\varphi)$, we define the special case of the operator $\mathcal{G}_{h,J_1,J_2,N}$ with specific parameters, where we set $h := h_l = d/J_l N$ and $\alpha := \pi/2$. In this case, the operator (3.2) takes the form:

$$\mathcal{G}_{\frac{d}{N},J_1,J_2,N}[f](z) = \sum_{n \in \mathbb{Z}_N^2(z)} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} f(\tau_{j_1,n_1,h_1}, \tau_{j_2,n_2,h_2}) \prod_{l=1}^{2} \psi_{j_l,n_l,h_l}(z_l)\, \mathrm{e}^{-\pi N \frac{\left(z_l - \tau_{j_l,n_l,h_l}\right)^2}{2 J_l d^2}}. \tag{3.12}$$

The general operator $\mathcal{G}_{h,J_1,J_2,N}$ allows for independent selection of parameters N and h. However, in the specific case $\mathcal{G}_{\frac{d}{N},J_1,J_2,N}$, these parameters become correlated, implying that their values are interdependent.

In the following result, we demonstrate that the difference between a function $f \in A^2_\Omega(\varphi)$ and the operator $\mathcal{G}_{\frac{d}{N},J_1,J_2,N}$ can be represented as the integration of $\mathcal{K}zf$ over a hyperrectangle denoted as $\prod_{l=1}^2 \mathcal{R}_{z_l}$. Each $\mathcal{R}_{z_l}$ corresponds to a rectangle in the ζ_l-plane and is positively oriented. The vertices of $\mathcal{R}_{z_l}$ are determined by the expressions $\pm J_l h_l(N+3/2)+J_l h_l N z_{l/J_l h_l}+\mathrm{i}J_l d$ and $\pm J_l h_l(N+3/2)+J_l h N_{z_l/J_l h_l}+\mathrm{i}(dJ_l-\Im z_l)$, where $N_{z_l} := \lfloor \Re z_l + 1/2 \rfloor$ with $l=1,2$. The proof will not be presented because it is similar to the proof of Lemma 3.2.

Lemma 3.3. *For all $z \in \mathcal{S}^2_d$ and $f \in A^2_d(\varphi)$, we have*

$$f(z) - \mathcal{G}_{\frac{d}{N},J_1,J_2,N}[f](z) = \begin{cases} \dfrac{1}{(2\pi i)^2} \displaystyle\oint_{\mathcal{R}_{z_2}} \oint_{\mathcal{R}_{z_1}} \mathcal{K}_z(\zeta) f(\zeta)\, d\zeta, & z \neq \left(\tau_{j_1,n_1,h_1}, \nu_{j_2,n_2,h_2}\right), \\ 0, & z = \left(\tau_{j_1,n_1,h_1}, \nu_{j_2,n_2,h_2}\right), \end{cases} \tag{3.13}$$

where $\zeta = (\zeta_1, \zeta_2)$, and $\prod_{l=1}^2 \mathcal{R}_{z_l}$ represents the hyperrectangle described earlier.

The operator $\mathcal{G}_{h,J_1,J_2,N}$ provides a piecewise analytic approximation for functions from the class $E^2_\Omega(\varphi)$ or the class $A^2_d(\varphi)$ on each of the bivariate strips defined as follows:

$$\left\{ z \in \mathbb{C}^2 : \left(j_l - \frac{1}{2}\right) h_l J_l \leq \Re z_l \leq \left(j_l + \frac{1}{2}\right) h_l J_l, \quad l = 1,2 \right\}. \tag{3.14}$$

4. Error bound for $E^2_\Omega(\varphi)$-functions

In this section, we will derive bounds for the error $\left| f(z) - \mathcal{G}_{h,J_1,J_2,N} f \right|$ when f belongs to the class $E^2_\Omega(\varphi)$. We will consider special cases based on the characteristics of φ, which are commonly encountered in practical situations. These cases correspond to three familiar growth patterns of φ: Constant, polynomial, and exponential. The main result of this section is presented in the following theorem.

Theorem 4.1. *For $f \in E^2_\Omega(\varphi)$ with $\Omega > 0$, and $|\Im z_l| < J_l h N$ for $l = 1,2$, the error between f and $\mathcal{G}_{h,J_1,J_2,N}[f]$ can be bounded as follows:*

$$\begin{aligned} \left| f(z) - \mathcal{G}_{h,J_1,J_2,N}[f](z) \right| &\leq \varphi(\eta(z)) \sum_{l=1}^{2} 2^{J_l - 1} \mathrm{e}^{\Omega |\Im z_{3-l}|} \omega_{h,J_l}(z_l) \chi_{N,J_l}(\Im z_l) \frac{e^{-\alpha J_l N}}{\sqrt{\pi \alpha J_l N}} \\ &+ \varphi(\eta(z)) \prod_{k=1}^{2} 2^{J_k - 1} \omega_{h,J_k}(z_k) \chi_{N,J_k}(\Im z_k) \frac{e^{-\alpha J_k N}}{\sqrt{\pi \alpha J_k N}}, \end{aligned} \tag{4.1}$$

where $\eta(z) := (\beta_{J_1}(z_1), \beta_{J_2}(z_2))$ and $\beta_{J_l}(z_l) = |\Re z_l| + hJ_l(N+2)$, $l = 1,2$. Here, φ is the previously defined function, χ_{N,J_l} is given in (1.8), and $\omega_{h,J_l}(z_l)$ is defined as

$$\omega_{h,J_l}(z_l) := \prod_{j_l=1}^{J_l} \sin\left(\frac{\pi}{J_l h} (z_l - x_{l j_l}) \right), \quad l = 1,2. \tag{4.2}$$

Proof. Expanding the integral in (3.5) using the definition of $\mathcal{K}_z$ in (3.4), we have the following expression:

$$\oint_{R_{z_2}}\oint_{R_{z_1}} \mathcal{K}_z(\zeta) f(\zeta)\, d\zeta = \omega_{h,J_1}(z_1) \oint_{R_{z_2}}\oint_{R_{z_1}} \frac{f(\zeta_1,\zeta_2)\prod_{l=1}^{2} \mathrm{e}^{-\frac{\alpha(z_l-\zeta_l)^2}{NJ_lh^2}}\, d\zeta}{\prod_{l=1}^{2}(\zeta_l - z_l)\prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(\zeta_1 - x_{1j_1})\right)}$$
$$+\ \omega_{h,J_2}(z_2) \oint_{R_{z_2}}\oint_{R_{z_1}} \frac{f(\zeta_1,\zeta_2)\ \prod_{l=1}^{2} \mathrm{e}^{-\frac{\alpha(z_l-\zeta_l)^2}{NJ_lh^2}}\, d\zeta}{\prod_{l=1}^{2}(\zeta_l - z_l)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(\zeta_2 - x_{2j_2})\right)}$$
$$-\ \prod_{k=1}^{2}\omega_{h,J_k}(z_k) \oint_{R_{z_2}}\oint_{R_{z_1}} \frac{f(\zeta_1,\zeta_2)\ \prod_{l=1}^{2} \mathrm{e}^{-\frac{\alpha(z_l-\zeta_l)^2}{NJ_lh^2}}\, d\zeta}{\prod_{l=1}^{2}(\zeta_l - z_l)\prod_{j_l=1}^{J_l}\sin\left(\frac{\pi}{J_lh}(\zeta_l - x_{lj_l})\right)}, \tag{4.3}$$

where ω_{h,J_l} is defined in (4.2), R_{z_l} for $l = 1, 2$ denotes the rectangles that were described previously, and $d\zeta := d\zeta_1 d\zeta_2$. Applying the Cauchy integral formula in one dimension, we can express the integral in (4.3) as follows:

$$\oint_{R_{z_2}}\oint_{R_{z_1}} \mathcal{K}_z(\zeta) f(\zeta)\, d\zeta = \omega_{h,J_1}(z_1) \oint_{R_{z_1}} \frac{f(\zeta_1,z_2)\ \mathrm{e}^{-\frac{\alpha(z_1-\zeta_1)^2}{NJ_1h^2}}\, d\zeta_1}{(\zeta_1 - z_1)\prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(\zeta_1 - x_{1j_1})\right)}$$
$$+\ \omega_{h,J_2}(z_2) \oint_{R_{z_2}} \frac{f(z_1,\zeta_2)\ \mathrm{e}^{-\frac{\alpha(z_2-\zeta_2)^2}{NJ_2h^2}}\, d\zeta_2}{(\zeta_2 - z_2)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(\zeta_2 - x_{2j_2})\right)}$$
$$-\ \prod_{k=1}^{2}\omega_{h,J_k}(z_k) \oint_{R_{z_2}}\oint_{R_{z_1}} \frac{f(\zeta_1,\zeta_2)\ \prod_{l=1}^{2} \mathrm{e}^{-\frac{\alpha(z_l-\zeta_l)^2}{NJ_lh^2}}\, d\zeta}{\prod_{l=1}^{2}(\zeta_l - z_l)\prod_{j_l=1}^{J_l}\sin\left(\frac{\pi}{J_lh}(\zeta_l - x_{lj_l})\right)}. \tag{4.4}$$

Given that f belongs to the class $E^2_\Omega(\varphi)$, according to (3.1), we can conclude that for any point (ζ_1, ζ_2) within the hyperrectangle $\prod_{l=1}^{2} R_{z_l}$, the following holds:

$$|f(\zeta_1,\zeta_2)| \le \varphi\left(\eta(z)\right) \prod_{l=1}^{2} \mathrm{e}^{\Omega|\Im\zeta_l|}, \tag{4.5}$$

where $\eta(z)$ was defined earlier. Additionally, when either z_1 or z_2 is a fixed point, the following results hold:

$$|f(z_1,\zeta_2)| \le \varphi\left(\eta(z)\right) \mathrm{e}^{\Omega|\Im z_1|}\ \mathrm{e}^{\Omega|\Im\zeta_2|}, \quad z_1 \in R_{z_1}, \tag{4.6}$$

$$|f(\zeta_1,z_2)| \le \varphi\left(\eta(z)\right) \mathrm{e}^{\Omega|\Im\zeta_1|}\ \mathrm{e}^{\Omega|\Im z_2|}, \quad z_2 \in R_{z_2}. \tag{4.7}$$

These results are derived from the assumption that the function φ is non-decreasing with respect to all variables ζ_l, $l = 1, 2$. Substituting (4.5)–(4.7) into (4.4), we can deduce that:

$$\oint_{R_{z_2}}\oint_{R_{z_1}} |\mathcal{K}_z(\zeta) f(\zeta)|\,|d\zeta| \le \varphi\left(\eta(z)\right) \mathrm{e}^{\Omega|\Im z_2|}\omega_{h,J_1}(z_1) \oint_{R_{z_1}} \left|\frac{\mathrm{e}^{\Omega|\Im\zeta_1|}\ \mathrm{e}^{-\frac{\alpha(z_1-\zeta_1)^2}{NJ_lh^2}}}{(\zeta_1 - z_1)\prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(\zeta_1 - x_{1j_1})\right)}\right| |d\zeta_1|$$

$$
\begin{aligned}
&+\ \varphi(\eta(z))\, \mathrm{e}^{\Omega|\Im z_1|}\omega_{h,J_2}(z_2) \oint_{R_{z_2}} \left| \frac{\mathrm{e}^{\Omega|\Im\zeta_2|}\ \mathrm{e}^{-\frac{\alpha(z_2-\zeta_2)^2}{NJ_2h^2}}}{(\zeta_2-z_2)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(\zeta_2-x_{2j_2})\right)} \right| |d\zeta_2| \\
&+\ \varphi(\eta(z))\, \omega_{h,J_1}(z_1) \oint_{R_{z_1}} \left| \frac{\mathrm{e}^{\Omega|\Im\zeta_1|}\ \mathrm{e}^{-\frac{\alpha(z_1-\zeta_1)^2}{NJ_lh^2}}}{(\zeta_1-z_1)\prod_{j_1=1}^{J_1}\sin\left(\frac{\pi}{J_1h}(\zeta_1-x_{1j_1})\right)} \right| |d\zeta_1| \\
&\times\ \omega_{h,J_2}(z_2) \oint_{R_{z_2}} \left| \frac{\mathrm{e}^{\Omega|\Im\zeta_2|}\ \mathrm{e}^{-\frac{\alpha(z_2-\zeta_2)^2}{NJ_2h^2}}}{(\zeta_2-z_2)\prod_{j_2=1}^{J_2}\sin\left(\frac{\pi}{J_2h}(\zeta_2-x_{2j_2})\right)} \right| |d\zeta_2|. \qquad (4.8)
\end{aligned}
$$

The integrals in (4.8) can be approximated by dividing the contour integral over R_{z_l} into four individual integrals along line segments and converting them into ordinary integrals, using a similar approach as demonstrated in [11, Eq (32)].

$$
\oint_{R_{z_l}} \left| \frac{\mathrm{e}^{\Omega|\Im\zeta_l|}\ \mathrm{e}^{-\frac{\alpha(z_l-\zeta_1)^2}{NJ_lh^2}}}{(\zeta_l-z_l)\prod_{j_l=1}^{J_l}\sin\left(\frac{\pi}{J_lh}(\zeta_l-x_{lj_l})\right)} \right| |d\zeta_l| \le 2^{J_l}\pi\chi_{N,J_l}(\Im z_l)\, \frac{\mathrm{e}^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}}, \qquad (4.9)
$$

where the function χ_{N,J_l} is defined in (1.8) and $l = 1,2$. Substituting from (4.9) into (4.8) and using Lemma 3.2, we finally get (4.1). □

The bound presented in inequality (4.1) exhibits an exponential order and is directly affected by the characteristics of the functions χ_{N,J_l}, ω_{h,J_l}, $\mathrm{e}^{\Omega|\Im z_l|}$, and φ. To illustrate how different characteristics of φ can impact this bound, we present three specific cases based on its growth patterns: constant, polynomial, and exponential. These cases are useful in practical applications and provide insights into the behavior of the bound.

Case I. This corresponds to the case where the function φ exhibits constant growth, meaning $\varphi\left(\left|\Re z_1\right|, \left|\Re z_2\right|\right) := C$ for all $z \in \mathbb{C}^2$. In this case, the growth condition in (3.1) becomes:

$$
|f(z)| \le C \prod_{j=1}^{2} e^{\Omega|\Im z_j|}, \quad z \in \mathbb{C}^2, C > 0, \qquad (4.10)
$$

which implies that f is an entire function of exponential type Ω. The space $E^2_\Omega(C)$ is more inclusive than the Bernstein space $B^p_\Omega(\mathbb{R}^2)$ because the functions defined in (4.10) are not necessarily required to belong to $L^p(\mathbb{R}^2)$ when restricted to $\mathbb{R}^2$. The following corollary illustrates this particular case.

Corollary 4.2. *If f belongs to the space $E^2_\Omega(C)$ with C as a positive constant, then for all $z \in \mathbb{C}^2$ and $|\Im z_l| < N$ for $l = 1,2$, we have the following bound for the error:*

$$
\begin{aligned}
|f(z) - \mathcal{G}_{h,J_1,J_2,N}[f](z)| \ \le\ & C \sum_{l=1}^{2} 2^{J_l-1}\mathrm{e}^{\Omega|\Im z_{3-l}|}\omega_{h,J_l}(z_l)\chi_{N,J_l}(\Im z_l)\frac{e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}} \\
&+ \prod_{k=1}^{2} 2^{J_k-1}\omega_{h,J_k}(z_k)\chi_{N,J_k}(\Im z_k)\frac{e^{-\alpha J_k N}}{\sqrt{\pi\alpha J_k N}}, \qquad (4.11)
\end{aligned}
$$

where the functions χ_{N,J_l} and $\omega_{h,J_l}(z_l)$ are given in (4.2) *and* (1.8)*, respectively.*

Proof. This result can be immediately deduced from Theorem 4.1. □

In the real domain, the bound in inequality (4.11) will be uniform. This is because χ_{N,J_l} and ω_{h,J_l} are bounded functions on the real domain. The following corollary provides a uniform bound for the error $\left|f(x) - \mathcal{G}_{h,J_1,J_2,N}[f](x)\right|$ for all $x \in \mathbb{R}^2$.

Corollary 4.3. *For all $x \in \mathbb{R}^2$ and $f \in E^2_\Omega(C)$ with $C > 0$, the following uniform bound holds:*

$$\left|f(x) - \mathcal{G}_{h,J_1,J_2,N}[f](x)\right| \leq C \sum_{l=1}^{2} 2^{J_l-1} \chi_{N,J_l}(0) \frac{e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}} + \prod_{k=1}^{2} 2^{J_k-1} \chi_{N,J_k}(0) \frac{e^{-\alpha J_k N}}{\sqrt{\pi\alpha J_k N}}, \tag{4.12}$$

where the function χ_{N,J_l} is defined as previously mentioned.

Case II. This deals with the case where the function φ exhibits polynomial growth, which means that $\varphi\left(\left|\Re z_1\right|, \left|\Re z_2\right|\right) = C \prod_{l=1}^{2} (1 + |\Re z_l|)^{\nu_l}$ for all $z \in \mathbb{C}^2$. In this case, the growth condition specified in (3.1) can be reformulated as follows:

$$|f(z)| \leq C \prod_{l=1}^{2} (1 + |\Re z_l|)^{\nu_l} \, \mathrm{e}^{\Omega|\Im z_l|}, \tag{4.13}$$

where $z := (z_1, z_2) \in \mathbb{C}^2$, $C > 0$, and $\nu_l \in \mathbb{N}_\circ$. The subsequent corollary demonstrates this specific case.

Corollary 4.4. *Let g belong to the space $B^\infty_{\Omega'}(\mathbb{R}^2)$, and define $f(z) := \prod_{l=1}^{2}(1 + z_l)^{\nu_l} g(z)$, where $z \in \mathbb{C}^2$ and ν_l is a non-negative integer. Then $f \in E^2_\Omega(\varphi)(\mathbb{R}^2)$ for all $\Omega > \Omega'$ and the following estimate holds:*

$$\begin{aligned} |f(z) - \mathcal{G}_{h,J_1,J_2,N}[f](z)| \quad &\leq \quad C \sum_{l=1}^{2} 2^{J_l-1} \mathrm{e}^{\Omega|\Im z_{3-l}|} \omega_{h,J_l}(z_l) \mathcal{A}_{N,J_l}(\Im z_l) \frac{e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}} \\ &+ \quad \prod_{k=1}^{2} 2^{J_k-1} \omega_{h,J_k}(z_k) \mathcal{A}_{N,J_k}(\Im z_k) \frac{e^{-\alpha J_k N}}{\sqrt{\pi\alpha J_k N}}, \end{aligned} \tag{4.14}$$

where the function $\omega_{h,J_l}(z_l)$ is given in (4.2), and the function $\mathcal{A}_{N,J_l}$ is defined as

$$\mathcal{A}_{N,J_l}(z) := \left(1 + |\Re z_l| + J_l h(N + 2)\right)^{\nu_l}. \tag{4.15}$$

Proof. Considering $f(z) = \prod_{l=1}^{2}(1 + z_l)^{\nu_l} g(z)$ and $g \in B^\infty_{\Omega'}(\mathbb{R}^2)$, it is straightforward to find a positive constant C such that:

$$\begin{aligned} |f(z)| \quad &\leq \quad \|g\|_\infty \prod_{l=1}^{2}(1 + |z_l|)^{\nu_l} \mathrm{e}^{\Omega'|\Im z_l|} \leq \|g\|_\infty \prod_{l=1}^{2}(1 + |\Re z_l|)^{\nu_l}(1 + |\Im z_l|)^{\nu_l} \mathrm{e}^{\Omega'|\Im z_l|} \\ &\leq \quad C \prod_{l=1}^{2}(1 + |\Re z_l|)^{\nu_l} \mathrm{e}^{\Omega|\Im z_l|}, \end{aligned}$$

for all $\Omega > \Omega'$. Hence, f is an entire function of polynomial growth on the real domain, and $f \in E^2_\Omega(\varphi)(\mathbb{R}^2)$ with $\varphi(x) = \prod_{l=1}^{2}(1 + x_l^2)^{\nu_l}$, $x := (x_1, x_2) \in \mathbb{R}^2_+$. By substituting $\varphi(x) = \prod_{l=1}^{2}(1 + x_l^2)^{\nu_l}$ into (4.1), we obtain (4.14). □

Case III. This pertains to the situation where the function φ exhibits exponential growth on the real domain, which means $\varphi(|\Re z_1|, |\Re z_2|) = C\prod_{j=1}^{2} e^{\kappa|\Re z_l|}$, $\kappa > 0$, $C > 0$, for all $z \in \mathbb{C}^2$. In this situation, the growth condition specified in (3.1) can be expressed as follows:

$$|f(z)| \leq C\prod_{l=1}^{2} e^{\kappa|\Re z_l| + \Omega|\Im z_l|}, \quad z \in \mathbb{C}^2, C > 0, \tag{4.16}$$

where $\kappa > 0$ and $\Omega \geq 0$. The subsequent corollary addresses this particular case.

Corollary 4.5. *Suppose f is an entire function that satisfies the exponential growth condition (4.16). For $h \in (0, \pi/(\Omega + 2\kappa))$ and $|\Im z_l| < J_l hN$ with $l = 1, 2$, the following estimate holds:*

$$\begin{aligned} |f(z) - \mathcal{G}_{h,J_1,J_2,N}[f](z)| &\leq C\left(\prod_{j=1}^{2} \mathrm{e}^{\kappa|\Re z_j|}\right)\sum_{l=1}^{2} 2^{J_l-1}\omega_{h,J_l}(z_l)\chi_{N,J_l}(\Im z_l)\frac{e^{-(\alpha-\kappa h)J_l N}}{\sqrt{\pi\alpha J_l N}} \\ &+ C\prod_{k=1}^{2} 2^{J_k-1}\mathrm{e}^{\kappa|\Re z_k|}\omega_{h,J_k}(z_k)\chi_{N,J_k}(\Im z_k)\frac{e^{-(\alpha-\kappa h)J_k N}}{\sqrt{\pi\alpha J_k N}}, \end{aligned} \tag{4.17}$$

where the functions $\chi_{J_l,N}$ and ω_{h,J_l} are defined in (1.8) and (4.2), respectively.

Proof. By considering the function $\varphi(x) = C\prod_{l=1}^{2} \mathrm{e}^{\kappa x_l}$ with $x := (x_1, x_2) \in \mathbb{R}_+^2$ in Theorem 4.1, we can readily deduce (4.17) by restricting h to the interval $(0, \pi/(\Omega + 2\kappa))$. □

5. Error bound for $A_d^2(\varphi)$-functions

In this section, our main objective is to estimate the error $\left|f(z) - \mathcal{G}_{\frac{d}{N},J1,J2,N}[f]\right|$ for functions f belonging to the class $A_d^2(\varphi)$, as defined in (3.11). The operator $\mathcal{G}_{\frac{d}{N},J_1,J_2,N}$ is precisely defined in (3.12). In this section, we will represent the function ω_{h,J_l} given in Eq (4.2) by the notation $\omega_{\frac{d}{N},J_l}$ when h is equal to $d/J_l N$.

Theorem 5.1. *For $f \in A_d^2(\varphi)$, the following inequality holds:*

$$\begin{aligned} \left|f(z) - \mathcal{G}_{\frac{d}{N},J_1,J_2,N}[f](z)\right| &\leq \varphi(\rho(z))\sum_{l=1}^{2} 2^{J_l+1/2}\omega_{\frac{d}{N},J_l}(z_l)\gamma_{N,l}(\Im z_l/d)\,\frac{e^{-\frac{\pi}{2}\left(J_l N - \frac{2|\Im z_l|}{d}\right)}}{\pi\sqrt{N}} \\ &+ \varphi(\rho(z))\prod_{l=1}^{2} 2^{J_l+1/2}\omega_{\frac{d}{N},J_l}(z_l)\gamma_{N,l}(\Im z_l/d)\,\frac{e^{-\frac{\pi}{2}\left(J_l N - \frac{2|\Im z_l|}{d}\right)}}{\pi\sqrt{N}}, \end{aligned} \tag{5.1}$$

where $z \in \mathcal{S}_{d/4}^2$, and $\rho(z) := \left(|\Re z_1| + J_1 d(1 + \frac{2}{N}), |\Re z_2| + J_2 d(1 + \frac{2}{N})\right)$. The function γ_{N,J_l} is defined as (1.11).

Proof. By expanding the integral in (3.13) and utilizing the definition of $\mathcal{K}_z$ in (3.4) with $\alpha = \pi/2$ and $h := h_l = d/J_l N$, we can apply the Cauchy integral formula in one dimension, resulting in the following expression:

$$\oint_{\mathcal{R}_{z_2}}\oint_{\mathcal{R}_{z_1}} |\mathcal{K}_z(\zeta)f(\zeta)|\,|d\zeta| \leq \omega_{\frac{d}{N},J_1}(z_1)\oint_{R_{z_1}}\left|\frac{f(\zeta_1, z_2)\,\mathrm{e}^{-\frac{\pi N(z_l-\zeta_1)^2}{2J_l d^2}}\,d\zeta_1}{(\zeta_1 - z_1)\prod_{j_1=1}^{J_1}\sin\left(\frac{\pi N}{J_1 d}(\zeta_1 - x_{1j_1})\right)}\right|$$

$$+ \quad \omega_{\frac{d}{N},J_1}(z_2) \oint_{R_{z_2}} \left| \frac{f(z_1,\zeta_2)\, \mathrm{e}^{-\frac{\pi N (z_2-\zeta_2)^2}{2J_l d^2}} d\zeta_2}{(\zeta_2 - z_2)\prod_{j_2=1}^{J_2} \sin\left(\frac{\pi N}{J_2 d}(\zeta_2 - x_{2j_2})\right)} \right|$$

$$+ \quad \prod_{k=1}^{2} \omega_{\frac{d}{N},J_k}(z_k) \oint_{R_{z_2}} \oint_{R_{z_1}} \left| \frac{f(\zeta_1,\zeta_2)\ \prod_{l=1}^{2} \mathrm{e}^{-\frac{\pi N (z_l-\zeta_l)^2}{2J_l d^2}} d\zeta}{\prod_{l=1}^{2}(\zeta_l - z_l)\prod_{j_l=1}^{J_l} \sin\left(\frac{\pi N}{J_l d}(\zeta_l - x_{lj_l})\right)} \right|. \tag{5.2}$$

Since f belongs to the space $A_d^2(\varphi)$, then f satisfies the growth condition in (3.11). Therefore, we have

$$|f(\zeta)| \le \varphi(\rho(z)), \quad \zeta \in \prod_{l=1}^{2} \mathcal{R}_{z_l}. \tag{5.3}$$

By combining (5.2) and (5.3), we get the following result:

$$\begin{aligned}
\oint_{\mathcal{R}_{z_2}} \oint_{\mathcal{R}_{z_1}} |\mathcal{K}_z(\zeta) f(\zeta)|\ |d\zeta| \ &\le\ \varphi(\rho(z))\, \omega_{\frac{d}{N},J_1}(\ddagger_1) \oint_{\mathcal{R}_{z_1}} \left| \frac{\mathrm{e}^{-\frac{\pi N}{2J_1 d^2}(z_1-\zeta_1)^2} d\zeta_1}{(\zeta_1 - z_1)\prod_{j_1=1}^{J_1} \sin\left(\frac{\pi N}{J_1 d}(\zeta_1 - x_{1j_1})\right)} \right| \\
&+\ \varphi(\rho(z))\, \omega_{\frac{d}{N},J_1}(z_2) \oint_{\mathcal{R}_{z_2}} \left| \frac{\mathrm{e}^{-\frac{\pi N}{2J_2 d^2}(z_2-\zeta_2)^2} d\zeta_2}{(\zeta_2 - z_2)\prod_{j_2=1}^{J_2} \sin\left(\frac{\pi N}{J_2 d}(\zeta_2 - x_{2j_2})\right)} \right| \\
&+\ \varphi(\rho(z))\, \omega_{\frac{d}{N},J_1}(z_1) \oint_{\mathcal{R}_{z_1}} \left| \frac{\mathrm{e}^{-\frac{\pi N}{2J_1 d^2}(z_1-\zeta_1)^2} d\zeta_1}{(\zeta_1 - z_1)\prod_{j_1=1}^{J_1} \sin\left(\frac{\pi N}{J_1 d}(\zeta_1 - x_{1j_1})\right)} \right| \\
&\times\ \omega_{\frac{d}{N},J_1}(z_2) \oint_{\mathcal{R}_{z_2}} \left| \frac{\mathrm{e}^{-\frac{\pi N}{2J_2 d^2}(z_2-\zeta_2)^2} d\zeta_2}{(\zeta_2 - z_2)\prod_{j_2=1}^{J_2} \sin\left(\frac{\pi N}{J_2 d}(\zeta_2 - x_{2j_2})\right)} \right|.
\end{aligned} \tag{5.4}$$

The integrals in Eq (5.4) can be estimated by dividing the contour integral over $\mathcal{R}_{zl}$ into four separate integrals along line segments and converting them into ordinary integrals, following a similar approach as shown in [11, Eq (44)],

$$\oint_{\mathcal{R}_{z_l}} \left| \frac{\mathrm{e}^{-\frac{\pi N}{2J_l d^2}(z_l-\zeta_l)^2}}{(\zeta_l - z_l)\prod_{j_l=1}^{J_l} \sin\left(\frac{\pi N}{J_l d}(\zeta_l - x_{lj_l})\right)} \right| |d\zeta_l| \le 2^{J_l+3/2} \gamma_{N,l}(\Im z_l/d)\, \frac{e^{-\frac{\pi}{2}\left(J_l N - \frac{2|\Im z_l|}{d}\right)}}{\sqrt{N}}, \tag{5.5}$$

where the function γ_{N,J_l} is defined in (1.11) and $l = 1,2$. By substituting the expression from (5.5) into (5.4) and utilizing Lemma 3.3, we eventually arrive at Eq (5.1). □

The bound presented in inequality (5.1) in the complex domain is influenced by the behavior of the functions $w_{d,J_l}(z_l)$, $\gamma_{N,l}(\Im z_l/d)$, and $e^{\pi|\Im z_l|/d}$, as well as the growth of the function φ within the domain $S_{d/4}^2$. However, in the real domain and φ is a constant function, the bound in inequality (5.1) will be uniform. This is because $w_{d,J_l}(z_l)$ and $\gamma_{N,l}(\Im z_l/d)$ are bounded functions on the real domain. It is clear that the bound in (5.1) will be of an exponential order within the real domain and will only depend on the growth of the function φ. The following corollary provides a uniform bound for the error $\left|f(x) - \mathcal{G}_{\frac{d}{N},J_1,J_2,N}[f](x)\right|$ for all $x \in \mathbb{R}^2$ and $f \in A_d^2(\varphi)$ where φ is a constant function.

Corollary 5.2. *For all $x \in \mathbb{R}^2$ and $f \in A_d^2(C)$ with $C > 0$, the following uniform bound holds:*

$$\left| f(x) - \mathcal{G}_{\frac{d}{N},J_1,J_2,N}[f](x) \right| \le C \sum_{l=1}^{2} 2^{J_l+1/2}\gamma_{N,l}(0)\,\frac{e^{-\pi J_l N/2}}{\pi\sqrt{N}} + C \prod_{l=1}^{2} 2^{J_l+1/2}\gamma_{N,l}(0)\,\frac{e^{-\pi J_l N/2}}{\pi\sqrt{N}}, \tag{5.6}$$

where the function $\gamma_{N,l}$ is defined as previously mentioned.

6. Numerical experiments

In this section, we utilize the bivariate nonuniform sinc-Gauss sampling operator $\mathcal{G}_{h,J_1,J_2,N}[f]$ to approximate five different functions from diverse classes. In the first example, we compare the approximations of a function belonging to the Bernstein space $B_\Omega^p(\mathbb{R}^2)$. This comparison is made using both the two-dimensional periodic nonuniform sampling series (1.13) and its modification in (3.2), the sampling operator $\mathcal{G}_{h,J_1,J_2,N}$. As expected from theoretical analysis, the sampling operator $\mathcal{G}_{h,J_1,J_2,N}$ offers a substantial improvement over the original series (1.13). Achieving this improvement is one of the main goals of this study. Each of the last four examples corresponds to a specific case that was presented in Sections 4 and 5. The second example deals with Case I, where f is an entire function of exponential type Ω and is bounded on the real domain $\mathbb{R}^2$. In the third example, we consider Case II, where f is an entire function of exponential type with polynomial growth along both axes in $\mathbb{R}^2$. The fourth example focuses on Case III, where f is an entire function satisfying a specific condition with exponential growth along the axes of $\mathbb{R}^2$. Lastly, we approximate an analytic function $f \in A_d^2(\varphi)$ in the fourth example. The numerical results are summarized in tables and illustrated using figures. It is worth noting that the accuracy of our formula $\mathcal{G}_{h,J_1,J_2,N}[f]$ improves as we fix N and decrease h, without incurring any additional cost, except for the fact that the step size hJ becomes smaller. All computations were performed using Mathematica 13 on a personal computer. In the examples, we denote the bound in Eq (5.1) by $B_{h,J_1,J_2,N}$ and use the notation $\mathcal{R}_{h,J_1,J_2,N}$ to represent the relative bound associated with the bound $\mathcal{B}_{h,J_1,J_2,N}$, i.e.,

$$\mathcal{R}_{h,J_1,J_2,N}[f](x) := \mathcal{B}_{h,J_1,J_2,N}[f](x)/f(x), \quad x \in \mathbb{R}^2.$$

During this section, we let $x_{[J_l]} := (x_{l1}, x_{l2}, \ldots, x_{lJ_l})$ where x_{lj_l} is defined as (1.12) and $(\zeta_k, \zeta_j) := \left(\left(k - \frac{1}{2}\right) hJ_1, \left(j - \frac{1}{2}\right) hJ_2\right)$ where $(k, j) \in \mathbb{N}^2$. Let $\mathcal{T}_{J_1,J_2,N}[f]$ denote the truncated version of the classical expansion in (1.13), defined as

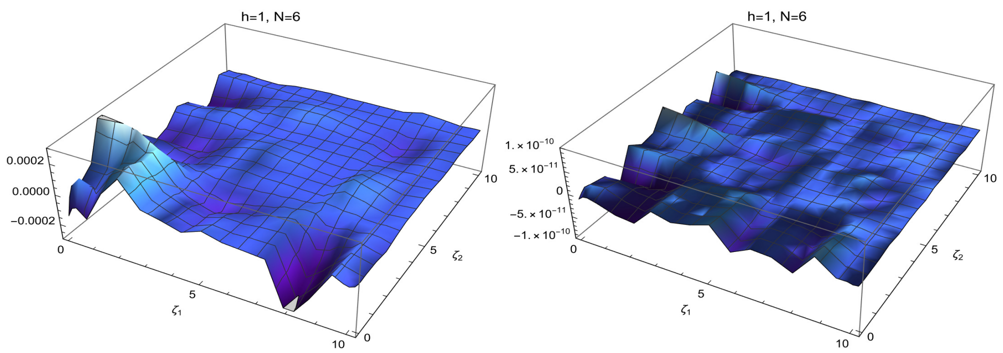
$$\mathcal{T}_{J_1,J_2,N}[f](z) = \sum_{n_2=-N}^{N} \sum_{n_1=-N}^{N} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} f(\tau_{j_l,n_l,\frac{\pi}{\Omega}}, \tau_{j_2,n_2,\frac{\pi}{\Omega}}) \prod_{l=1}^{2} \psi_{j_l,n_l,\frac{\pi}{\Omega}}(z_l), \qquad z = (z_1, z_2) \in \mathbb{C}^2. \tag{6.1}$$

This truncated series will be applied in Example 6.1.

Example 6.1. *Consider the function $f(z) = \prod_{j=1}^{2} \operatorname{sinc}\left(\sqrt{1 + z_j^2}\right)$, where $z = (z_1, z_2) \in \mathbb{C}^2$. This function belongs to the Bernstein space $B_1^2(\mathbb{R}^2)$, which allows us to approximate f using both the two-dimensional nonuniform periodic sampling series* (1.13) *and its modified version in (3.2), the sampling operator $\mathcal{G}_{h,J_1,J_2,N}$. Table 2 provides a comparison of the approximations of f at points $\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$, where $x_{[J_1]} = (0.1, 0.6, 1.2)$ and $x_{[J_2]} = (0.2, 0.8, 1.3)$ with $h = 1$ and $N = 6$, using both the truncated* *original sampling series* (6.1) *and its modification in (3.2). Furthermore, Figures 1(a) and 1(b) as well as Table* 1 *visually demonstrate the comparison of the approximations. The numerical results confirm the substantial improvement achieved by the sampling operator* $\mathcal{G}_{h,J_1,J_2,N}$*, aligning with our theoretical predictions.*

Table 1. The absolute errors associated with approximating f in Example 6.1 at the points (ζ_j, ζ_k) with parameters $j, k = 1, 5, 9$, $h = 1$, $J_1 = J_2 = 3$, and $N = 6$, using the truncated series from (6.1) and its modification in (3.2).

(ζ_k, ζ_l)	$\lvert f(\zeta) - \mathcal{T}_{3,3,6}[f](\zeta)\rvert$	$\lvert f(\zeta) - \mathcal{G}_{1,3,3,6}[f](\zeta)\rvert$
(ζ_1, ζ_1)	9.30909×10^{-5}	1.19794×10^{-10}
(ζ_1, ζ_5)	5.25264×10^{-4}	3.88436×10^{-10}
(ζ_1, ζ_9)	3.31706×10^{-4}	2.29679×10^{-10}
(ζ_5, ζ_1)	2.62136×10^{-4}	2.70596×10^{-10}
(ζ_5, ζ_5)	9.06053×10^{-5}	7.28933×10^{-11}
(ζ_5, ζ_9)	3.02992×10^{-5}	1.88964×10^{-11}
(ζ_9, ζ_1)	1.74133×10^{-4}	1.55210×10^{-10}
(ζ_9, ζ_5)	5.30675×10^{-6}	7.25985×10^{-12}
(ζ_9, ζ_9)	1.35891×10^{-5}	1.06303×10^{-11}



(a) **(b)**

Figure 1. Illustrations related to Example 6.1. The figure in (a) illustrates the error $f(\zeta) - \mathcal{T}_{3,3,6}[f](\zeta)$ with $\zeta \in [0, 10]^2$. The figure in (b) illustrates the error $f(\zeta) - \mathcal{G}_{1,3,3,6}[f](\zeta)$ with $\zeta \in [0, 10]^2$. We used the same parameters $h = 1$, $J_1 = J_2 = 3$, and $N = 6$, to generate the figures in (a) and (b).

Example 6.2. *Consider the function* $f(z_1, z_2) = \cos(z_1 + z_2)$*, where* $(z_1, z_2) \in \mathbb{C}^2$*. Obviously* $|\cos(z)| \le e^{|\Im z_1| + |\Im z_2|}$ *for every* $z = (z_1, z_2) \in \mathbb{C}^2$*. Thus the function belongs to the space* $E_1^2(\varphi)$ *and has a growth constant with* $\varphi = 1$*. Therefore, we will apply Corollary* 4.2 *by employing the sampling points* $\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$*, where* $x_{[J_1]} = (0.1, 0.6, 1.2, 1.9)$ *and* $x_{[J_2]} = (0.2, 0.8, 1.3, 1.8)$ *with* $h = 3/2, 1$ *and* $N = 6$*. Table* 2 *presents a comparison of the approximations of the function* f *at points* (ζ_j, ζ_k) *using the periodic sinc-Gauss sampling formula* $\mathcal{G}_{h,J_1,J_2,N}$ *with* $h = 3/2, 1$*,* $J_1 = J_2 = 4$*, and* $N = 6$*. Additionally, Figures 2(a) and 2(b) illustrate the comparison of the approximations of the function* f *using* $\mathcal{G}_{h,J_1,J_2,N}$

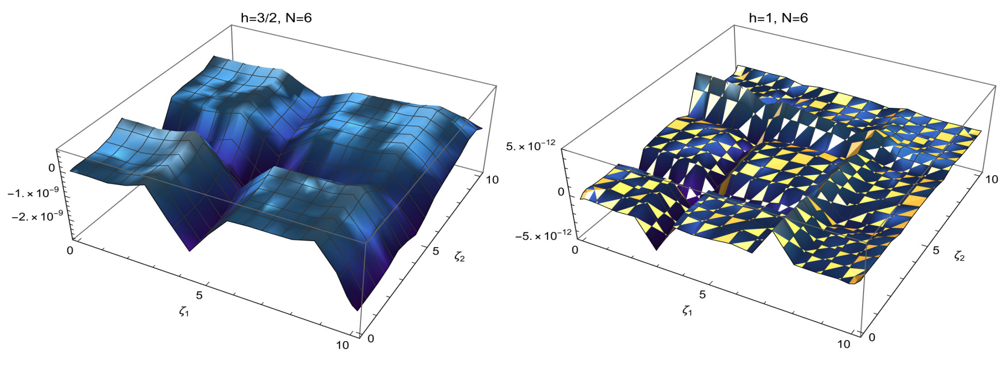
with $h = 3/2$ and $h = 1$, respectively, on the interval $[0, 10]^2$. By chance, the values of the bound $\mathcal{B}_{h,J1,J_2,N}$ happen to be the same at the points (ζ_k, ζ_j) for all $k, j = 1, 5, 9$ when h is held constant, as shown in Table 2. This coincidence may lead to a misleading perception of the bound. Consequently, Figures 3(a) and 3(b) demonstrate the accurate behavior of the bound to avoid any misconceptions. In this example, we will denote the real-valued bound of Case I, expressed in Eq (4.11), as:

$$\mathcal{B}_{h,J_1,J_2,N}[f](x) = \sum_{l=1}^{2} 2^{J_l-1}\omega_{h,J_l}(x_l)\chi_{N,J_l}(0)\frac{e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}} + \prod_{l=1}^{2} 2^{J_l-1}\omega_{h,J_l}(x_l)\chi_{N,J_l}(0)\frac{e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}},$$

where $x \in \mathbb{R}^2$ and the functions χ_{N,J_l}, and $\omega_{h,J_l}(z_l)$ are provided in (1.8) and (4.2), respectively.

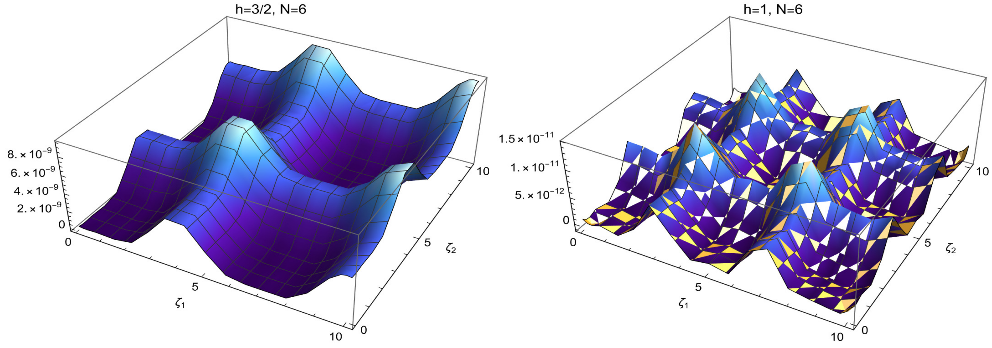
Table 2. Absolute errors associated with approximating f in Example 6.2 and their bounds at the points (ζ_j, ζ_k) with $j, k = 1, 5, 9$ and parameters $N = 6$ and $h = 3/2, 1$.

(ζ_k, ζ_l)	$\left\|f(\zeta) - \mathcal{G}_{\frac{3}{2},4,4,6}[f](\zeta)\right\|$	$\mathcal{B}_{\frac{3}{2},4,4,6}(\zeta)$	$\left\|f(\zeta) - \mathcal{G}_{1,4,4,6}[f](\zeta)\right\|$	$\mathcal{B}_{1,4,4,6}(\zeta)$
(ζ_1, ζ_1)	7.04519×10^{-10}	5.1421×10^{-9}	1.94289×10^{-13}	1.1191×10^{-12}
(ζ_5, ζ_1)	7.89457×10^{-10}	5.1421×10^{-9}	1.43774×10^{-13}	1.1191×10^{-12}
(ζ_9, ζ_1)	1.07908×10^{-9}	5.1421×10^{-9}	1.37446×10^{-13}	1.1191×10^{-12}
(ζ_1, ζ_5)	6.58279×10^{-10}	5.1421×10^{-9}	2.67564×10^{-13}	1.1191×10^{-12}
(ζ_5, ζ_5)	5.88276×10^{-11}	5.1421×10^{-9}	5.58442×10^{-14}	1.1191×10^{-12}
(ζ_9, ζ_5)	4.67437×10^{-11}	5.1421×10^{-9}	5.04041×10^{-14}	1.1191×10^{-12}
(ζ_1, ζ_9)	1.12024×10^{-9}	5.1421×10^{-9}	3.13416×10^{-13}	1.1191×10^{-12}
(ζ_5, ζ_9)	3.98303×10^{-11}	5.1421×10^{-9}	6.12843×10^{-14}	1.1191×10^{-12}
(ζ_9, ζ_9)	5.04961×10^{-11}	5.1421×10^{-9}	2.90878×10^{-14}	1.1191×10^{-12}



(a) **(b)**

Figure 2. Illustrations related to Example 6.2. The figure in (a) shows the error $f(\zeta) - \mathcal{G}_{3/2,4,4,6}[f](\zeta)$ with $\zeta \in [0, 10]^2$, while the figure in (b) depicts the error $f(\zeta) - \mathcal{G}_{1,4,4,6}[f](\zeta)$, also with $\zeta \in [0, 10]^2$. Both figures use the same parameters $J_1 = J_2 = 4$, and $N = 6$, except that $h = 3/2$ in (a) and $h = 1$ in (b).



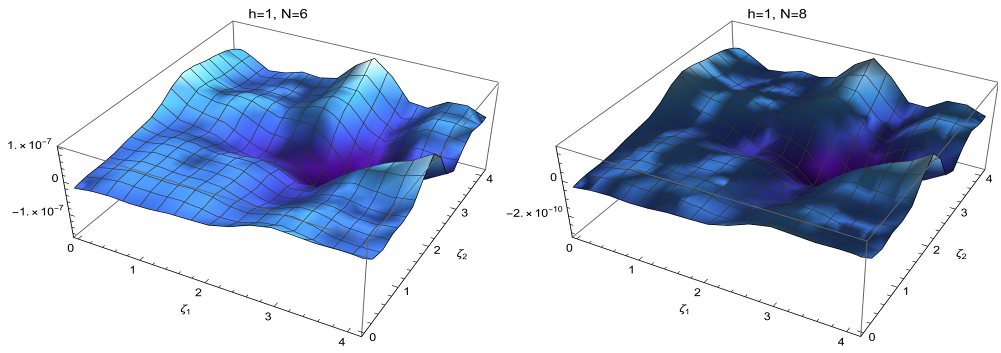
(a) **(b)**

Figure 3. The figure in (a) illustrates the bound $\mathcal{B}_{h,4,4,6}(\zeta)$ in Example 6.2, where $\zeta \in [0,4]^2$, with parameters $J_1 = J_2 = 4$, $N = 6$, and $h = 3/2$, while the figure in (b) presents the same bound with identical values for J_1, J_2, and N, but with $h = 1$.

Example 6.3. *Consider the function* $f(z_1, z_2) = (1 + z_1^2)(1 + z_2^2)\cos(z_1 + z_2)$*, where* $(z_1, z_2) \in \mathbb{C}^2$*. This function exhibits polynomial growth along the axes of* $\mathbb{R}^2$ *and fulfills the conditions specified in Corollary* 4.4 *with* $\Omega' = 1$*,* $\nu_1 = \nu_1 = 1$*, and* $C = 1$*. In this example, we will denote the real-valued bound of Case II, expressed in Eq (4.14), as:*

$$\begin{aligned}\mathcal{B}_{h,J_1,J_2,N}[f](x) &= \sum_{l=1}^{2} 2^{J_l-1}\omega_{h,J_l}(x_l)\mathcal{A}_{N,J_l}(0)\frac{N^{\nu_l-1/2}\, e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}} \\ &+ \prod_{l=1}^{2} 2^{J_l-1}\omega_{h,J_l}(x_l)\mathcal{A}_{N,J_l}(0)\frac{N^{\nu_l-1/2}\, e^{-\alpha J_l N}}{\sqrt{\pi\alpha J_l N}},\end{aligned}$$

where $x \in \mathbb{R}^2$ *and the functions* $\mathcal{A}_{N,J_l}$*, and* $\omega_{h,J_l}(z_l)$ *are provided in (4.15) and (4.2), respectively. Since the function* f *is an increasing function on the axes of* $\mathbb{R}_+^2$*, we utilize the relative error and the bound* $\mathcal{R}_{h,J_1,J_2,N}[f](x) := \mathcal{B}_{h,J_1,J_2,N}[f](x)/f[x]$*, where* $x \in \mathbb{R}^2$ *instead of the absolute error and the bound* $\mathcal{B}_{h,J_1,J_2,N}[f]$ *to describe the approximation results. In this example, we will use the sampling points* $\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$ *with* $x_{[J_1]} = (0.4, 1.6, 1.7)$ *and* $x_{[J_2]} = (0.3, 0.9, 1.4)$*,* $h = 1$*,* $\Omega = 1.1$*, and* $N = 6, 9$*. The numerical results are presented in Table 3 at the points* (ζ_k, ζ_l)*,* $k, l = 1, 3, 5$*, and are illustrated in Figures 4(a) and 4(b).*



(a) **(b)**

Figure 4. The figure in (a) illustrates the error $f(\zeta) - \mathcal{G}_{1,3,3,6}[f](\zeta)$ in Example 6.3, where $\zeta \in [0,4]^2$, with parameters $J_1 = J_2 = 3$, $N = 6$, and $h = 1$, while the figure in (b) shows the same relative error with identical values for J_1, J_2, and h, but with $N = 8$.

Table 3. Relative errors associated with approximating f in Example 6.3 and their relative bounds at the points (ζ_j, ζ_k) with $j, k = 1, 3, 5$ and parameters $h = 1$ and $N = 6, 8$.

(ζ_k, ζ_l)	$\frac{f(\zeta)-\mathcal{G}_{1,3,3,6}[f](\zeta)}{f(\zeta)}$	$\mathcal{R}_{1,3,3,6}(\zeta)$	$\frac{f(\zeta)-\mathcal{G}_{1,3,3,8}[f](\zeta)}{f(\zeta)}$	$\mathcal{R}_{1,3,3,8}(\zeta)$
(ζ_1, ζ_1)	6.64752×10^{-10}	5.63904×10^{-9}	1.15453×10^{-12}	1.06100×10^{-11}
(ζ_3, ζ_1)	7.04764×10^{-10}	1.92038×10^{-9}	1.30941×10^{-12}	3.55420×10^{-12}
(ζ_5, ζ_1)	9.82559×10^{-10}	1.43331×10^{-9}	1.83733×10^{-12}	2.61379×10^{-12}
(ζ_1, ζ_3)	1.50094×10^{-10}	2.09183×10^{-9}	1.29913×10^{-13}	3.85316×10^{-12}
(ζ_3, ζ_3)	2.36410×10^{-10}	7.7962×10^{-10}	3.75806×10^{-13}	1.41688×10^{-12}
(ζ_5, ζ_3)	3.79282×10^{-10}	6.6780×10^{-10}	6.41416×10^{-13}	1.19889×10^{-12}
(ζ_1, ζ_5)	2.41005×10^{-10}	1.67440×10^{-9}	6.36655×10^{-13}	3.03417×10^{-12}
(ζ_3, ζ_5)	5.91103×10^{-11}	7.1697×10^{-10}	1.07359×10^{-13}	1.28464×10^{-12}
(ζ_5, ζ_5)	2.50180×10^{-10}	8.2529×10^{-10}	3.33395×10^{-13}	1.46346×10^{-12}

Example 6.4. *Consider the function* $f(z_1, z_2) = \cosh(z_1 + z_2)$, *where* $(z_1, z_2) \in \mathbb{C}^2$. *It is evident that* $|\cosh(z)| \le e^{|\Re z_1| + |\Re z_2|}$ *for all* $z = (z_1, z_2) \in \mathbb{C}^2$. *This function belongs to the space* $E_0^2(\varphi)$ *and satisfies the exponential growth condition* (4.16) *on* $\mathbb{R}^2$. *Therefore, we apply the Corollary* 4.5 *with* $C = 1$, $\kappa = 1$, *and* $\Omega = 0$. *In this example, we will denote the real-valued bound of Case III, expressed in Eq (4.17), as:*

$$\begin{aligned} \mathcal{B}_{h,J_1,J_2,N}[f](x) \;&:=\; \prod_{k=1}^{2} e^{\kappa|x_k|} \sum_{l=1}^{2} 2^{J_l-1} \omega_{h,J_l}(x_l) \chi_{N,J_l}(0) \frac{e^{-(\alpha-\kappa h)J_l N}}{\sqrt{\pi \alpha J_l N}} \\ &+ \prod_{l=1}^{2} 2^{J_l-1} e^{\kappa|x_l|} \omega_{h,J_l}(z_l) \chi_{N,J_l}(0) \frac{e^{-(\alpha-\kappa h)J_l N}}{\sqrt{\pi \alpha J_l N}}, \end{aligned}$$

where the functions $\chi_{J_l,N}$ *and* ω_{h,J_l} *are defined in (1.8) and (4.2), respectively. In this case, since the function* f *is increasing on the positive axes of* $\mathbb{R}^2$, *we will use the relative error and the bound*

$\mathcal{R}_{h,J_1,J_2,N}[f](x) := \mathcal{B}_{h,J_1,J_2,N}[f](x)/f[x]$, where $x \in \mathbb{R}^2$, to describe the approximation results. For this example, we will utilize the sampling points $\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$ with $x_{[J_1]} = (0.4, 1.2)$ and $x_{[J_2]} = (0.4, 0.9)$ and $h = 1/2$. The numerical results are presented in Table 4 at the points (x_k, x_l), $k, l = 3, 5, 7$, and are illustrated in Figures 5(a) and 5(b).

Table 4. Relative errors associated with approximating f in Example 6.4 and their relative bounds at the points (ζ_j, ζ_k) with $j, k = 3, 5, 7$ and parameters $h = 1/2$, $J_1 = J_2 = 2$, and $N = 5, 8$.

(ζ_k, ζ_l)	$\left\lvert\frac{f(\zeta)-\mathcal{G}_{\frac{1}{2},2,2,5}[f](\zeta)}{f(\zeta)}\right\rvert$	$\mathcal{R}_{\frac{1}{2},2,2,5}(\zeta)$	$\left\lvert\frac{f(\zeta)-\mathcal{G}_{\frac{1}{2},2,2,8}[f](\zeta)}{f(\zeta)}\right\rvert$	$\mathcal{R}_{\frac{1}{2},2,2,8}(\zeta)$
(ζ_3, ζ_3)	5.31600×10^{-8}	1.63621×10^{-5}	3.10988×10^{-12}	1.99919×10^{-8}
(ζ_5, ζ_3)	5.21026×10^{-8}	1.63613×10^{-5}	4.24068×10^{-12}	1.99928×10^{-8}
(ζ_7, ζ_3)	5.20833×10^{-8}	1.63613×10^{-5}	4.25930×10^{-12}	1.99928×10^{-8}
(ζ_3, ζ_5)	5.21026×10^{-8}	1.63613×10^{-5}	4.24068×10^{-12}	1.99928×10^{-8}
(ζ_5, ζ_5)	5.20833×10^{-8}	1.63613×10^{-5}	4.26155×10^{-12}	1.99928×10^{-8}
(ζ_7, ζ_5)	5.20829×10^{-8}	1.63613×10^{-5}	4.26076×10^{-12}	1.99928×10^{-8}
(ζ_3, ζ_7)	5.20833×10^{-8}	1.63613×10^{-5}	4.26133×10^{-12}	1.99928×10^{-8}
(ζ_5, ζ_7)	5.20829×10^{-8}	1.63613×10^{-5}	4.26149×10^{-12}	1.99928×10^{-8}
(ζ_7, ζ_7)	5.20829×10^{-8}	1.63613×10^{-5}	4.26071×10^{-12}	1.99928×10^{-8}

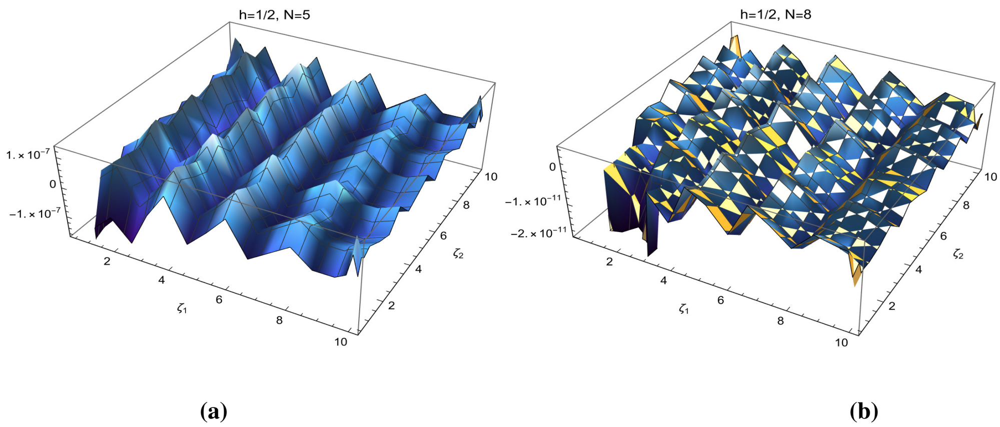


Figure 5. The figure in (a) illustrates the relative error $\left(f(\zeta) - \mathcal{G}_{\frac{1}{2},2,2,N}[f](\zeta)\right)/f(\zeta)$ in Example 6.4 with parameters $J_1 = J_2 = 2$, $N = 5$, and $h = 1/2$, while the figure in (b) shows the same relative error with identical values for J_1, J_2, and h, but with $N = 8$.

Example 6.5. *The function $f(z) = \frac{4}{(z_1^2+4)(z_2^2+4)}$, where $z = (z_1, z_2) \in \mathbb{C}^2$, is an analytic function defined on the 2-dimensional horizontal strip $\mathcal{S}_2^2$ as specified in Eq (3.10). Therefore, f belongs to the class $A_2^2(\varphi)$, allowing us to utilize Theorem* Theorem 5.1 *with the parameters $d = 2$ and $N = 6, 8$, and the sampling points $\left(\tau_{j_1,n_1,h}, \tau_{j_2,n_2,h}\right)$ with $x_{[J_1]} = (0.1, 0.2)$ and $x_{[J_2]} = (0.1, 0.3)$. In this example, we will denote the real-valued bound of Theorem 5.1, expressed in Eq (4.17), with $\varphi = 4$ as:*

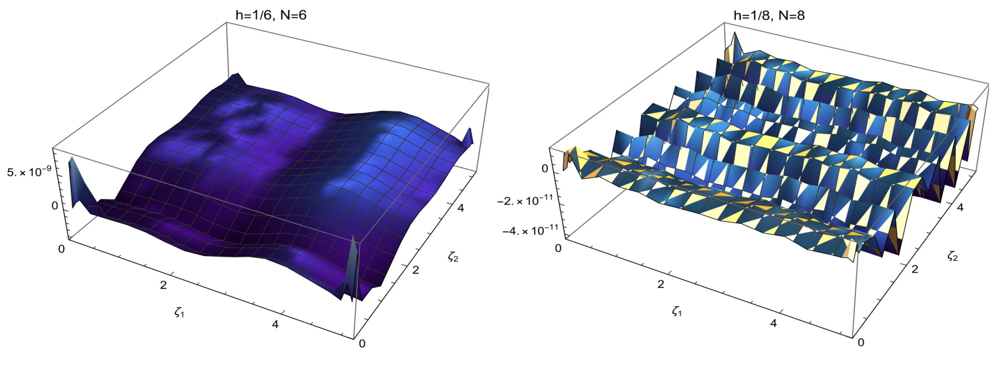
$$\mathcal{B}_{\frac{d}{N},J_1,J_2,N}[f](x) := 4\sum_{l=1}^{2} 2^{J_l+1/2}\omega_{\frac{d}{N},J_1}(x_l)\gamma_{N,l}(0)\,\frac{e^{-\frac{\pi J_l N}{2}}}{\pi\sqrt{N}}$$

$$+ \quad 4\prod_{l=1}^{2} 2^{J_l+1/2}\omega_{\frac{d}{N},J_1}(x_l)\gamma_{N,l}(0)\,\frac{e^{-\frac{\pi J_l N}{2}}}{\pi\sqrt{N}},$$

where the functions γ_{N,J_l} and $\omega_{\frac{d}{N},J_1}$ are defined as (1.11) and (4.2), respectively. In this case, since the function f is decreasing on the positive axes of $\mathbb{R}^2$, we will use the relative error and the bound the bound $\mathcal{R}_{\frac{d}{N},J_1,J_2,N}[f](x) := \mathcal{B}_{h,J_1,J_2,N}[f](x)/f[x]$, where $x \in \mathbb{R}^2$, to describe the approximation results. In Table 5, we present a summary of the approximations of the function f at intermediate points (ζ_k, ζ_l), where $k, l = 1, 2, 3$. Additionally, visual representations of the results are provided in Figures 6(a) and 6(b).

Table 5. Relative errors associated with approximating f in Example 6.5 and their relative bounds at the points (ζ_j, ζ_k) with $j, k = 1, 2, 3$ and parameters $d = 2$, $J_l = J_2 = 2$, and $N = 6, 8$.

(ζ_k, ζ_l)	$\left\lvert\frac{f(\zeta)-\mathcal{G}_{\frac{1}{6},2,2,6}[f](\zeta)}{f(\zeta)}\right\rvert$	$\mathcal{R}_{\frac{1}{6},2,2,6}(\zeta)$	$\left\lvert\frac{f(\zeta)-\mathcal{G}_{\frac{1}{8},2,2,8}[f](\zeta)}{f(\zeta)}\right\rvert$	$\mathcal{R}_{\frac{1}{8},2,2,8}(\zeta)$
(ζ_1, ζ_1)	5.32426×10^{-9}	3.55458×10^{-8}	9.69024×10^{-12}	7.64733×10^{-11}
(ζ_2, ζ_1)	3.94306×10^{-9}	3.75070×10^{-8}	1.37515×10^{-11}	7.88538×10^{-11}
(ζ_3, ζ_1)	3.10719×10^{-9}	4.14293×10^{-8}	1.49181×10^{-11}	8.36148×10^{-11}
(ζ_1, ζ_2)	8.08044×10^{-9}	3.75070×10^{-8}	1.18241×10^{-11}	7.88538×10^{-11}
(ζ_2, ζ_2)	1.46201×10^{-9}	3.95764×10^{-8}	1.58860×10^{-11}	8.13084×10^{-11}
(ζ_3, ζ_2)	2.29787×10^{-9}	4.37151×10^{-8}	1.70515×10^{-11}	8.62175×10^{-11}
(ζ_1, ζ_3)	1.12048×10^{-9}	4.14293×10^{-8}	1.19993×10^{-11}	8.36148×10^{-11}
(ζ_2, ζ_3)	2.50168×10^{-9}	4.37151×10^{-8}	1.60607×10^{-11}	8.62175×10^{-11}
(ζ_3, ζ_3)	3.33754×10^{-9}	4.82866×10^{-8}	1.72268×10^{-11}	9.14231×10^{-11}



(a) **(b)**

Figure 6. The figure in (a) illustrates the relative error $\left(f(\zeta) - \mathcal{G}_{\frac{1}{N},2,2,N}[f](\zeta)\right)/f(\zeta)$ in Example 6.5 with parameters $J_1 = J_2 = 2$, $N = 6$, and $d = 2$, while the figure in (b) shows the same relative error with identical values for J_1, J_2, and d, but with $N = 8$.

7. Conclusions

In recent times, there has been significant exploration into enhancing the convergence rate of the one-dimensional periodic nonuniform sampling series through the incorporation of a Gaussian multiplier. Notable contributions in this field have been made by Wang et al. (2019) and Rasdad (2022). Building upon these advancements, this paper takes it a step further and focuses on accelerating the convergence of the two-dimensional periodic nonuniform sampling series by introducing a bivariate Gaussian multiplier. The two-dimensional periodic nonuniform sampling series goes back to Butzer and Hinsen (1989). The convergence rate of the Butzer-Hinsen expansion is relatively slow, on the order of $O(N^{-p})$ with $p \geq 1$. By applying this acceleration technique, the convergence rate significantly improves to an exponential order, specifically $e^{-\alpha N}$, where $\alpha > 0$. The approach employed in this paper utilizes complex-analytic techniques and is applicable to a broad range of functions. Specifically, it applies to two classes of functions. The first class includes bivariate entire functions of exponential type that satisfy a decay condition. The second class comprises bivariate analytic functions defined on a bivariate horizontal strip. To support the theoretical analysis, numerical experiments are conducted to validate the effectiveness of the approach. Moreover, this technique holds the potential for future research in expediting the convergence rate of multidimensional classical and Hermite periodic nonuniform sampling series, opening up promising possibilities for further enhancing the efficiency of these sampling methods.

Author contributions

The authors contributed equally and they both read and approved the final manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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