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Research article

Multistep collocation technique implementation for a pantograph-type second-kind Volterra integral equation

Shireen Obaid Khaleel, Parviz Darania*, **Saeed Pishbin and Shadi Malek Bagomghaleh**

Department of Mathematics, Faculty of Sciences, Urmia University, P. O. Box 165, Urmia, Iran

* **Correspondence:** Email: p.darania@urmia.ac.ir.

Abstract: In this research, we have elaborated high-rate multistep collocation strategies in order to concern with second-type vanishing delay VIEs. Herein, characteristics of uniqueness, existence, and regularity for both numerical and analytical solutions have been shown. To explore the solvability of the system derived from the numerical method, we have defined particular operators and demonstrated that these operators are both compact and bounded. Solvability is studied by means of the innovative compact operator concepts. The concept of convergence has been examined in greater detail, revealing that the convergence of the method is influenced by the spectral radius of the matrix generated according to the collocation parameters in the difference equation resulting from the method's error. Finally, two numerical examples are given to certify our theoretically gained results. Also, since the proposed numerical method is local in nature, it can be compared to other local methods, such as those used in reference [1]. We will compare our method with [1] in the last section.

Keywords: second-kind Volterra integral equation (VIE); vanishing delay; compact operator; uniform mesh; high-rate multistep collocation techniques
Mathematics Subject Classification: 45G10, 65R20

1. Introduction

This work deals with the numerical and analytical treatment of a second-kind delay Volterra integral equation (DVIE)

$$x(T) = g(T) + \int_0^T B_1(T,\tau)x(\tau)\mathrm{d}\tau + \int_0^{qT} B_2(T,\tau)x(\tau)\mathrm{d}\tau, \quad T \in J := [0, \mathfrak{T}], \tag{1.1}$$

where $g(T)$, $B_1(T,\tau)$, and $B_2(T,\tau)$ are smooth enough on their respective domains in terms of J, $D_1 := \{(T,\tau) : 0 \le \tau \le T \le \mathfrak{T}\}$, and $D_2 := \{(T,\tau) : 0 \le \tau \le qT \le \mathfrak{T}\}$, so that based on [2], Eq (1.1) can be solved for a unique solution $x(T)$, with respect to the variable vanishing delay of qT with

$0 < q < 1$. The reason for calling delay function qT with vanishing delay term more clearly turns back to that, it vanishes at the beginning point of the interval J. Moreover, $\mathbb{X} := C^{M+R}(J := [0, \mathfrak{T}])$ is a Banach space, where $x(T)$, DVIE (1.1) solution lies as long as for the given data function $g(T)$, we have $g \in C^{M+R}(J := [0, \mathfrak{T}])$, too. In more details, M stands for number of used collocation parameters, and R illustrates the first R start solutions obtained via the classical collocation scheme in the multistep collocation strategy as well.

By virtue of Volterra integral equations and also their functional counterparts along with delay terms, whether vanishing or non-vanishing, a wide spectrum of science subjects, namely biology, ecology, physics and chemistry, have been mathematically well-formulated in order to analyse and study underlying phenomena. Particularly, these classes of mathematical modellings can be found in fluid dynamics, viscoelasticity of materials, population growth dynamics, heat conduction, epidemiology, controlled liquidation in obsolete production units, and renovation in economic systems; see [3–7] and references therein.

Taking a look at the works being furnished to solve (1.1) results in constructing fast rate convergent multistep collocation-based strategies. This issue emerges because of the fact that, for example, [8] illustrates a global convergence order of $M + 1$ for under study equation (1.1) on uniform meshes, while when it comes to superconvergence orders, it is notable that the order is at most $2M$, most essentially with $q = 1/2$, even M, as well as the M Gauss-Legendre collocation points. It is worth remarking that Brunner et al. [9] explored geometric meshes for solving (1.1), which yields to reach again a global convergence order of $M + 1$ for iterated collocation methods with the superconvergence order $2M - \epsilon_N$, by regarding hypotheses of $\epsilon_N \to 0$ with $N \to \infty$.

Seemingly, the concept of exploiting non-uniform meshes makes sense to obtain higher superconvergence orders, most specifically for vanishing delay VIEs of the second type like (1.1). Pursuing this goal, Ming et al. [1] utilized quasi-geometric meshes in order to gain $2M$ superconvergence rates for the under consideration second-kind VIEs of pantograph type. As shown, there, quasi-geometric collocation schemes come into $M + 1$ globally convergent numerical approximations, while in this study, we have reached the same or even higher superconvergence orders with a great deal of convenience and pace.

For these reasons, and by following our previous works [10–12], through this paper, we aim to develop multistep collocation techniques for second-type VIEs with vanishing delays (1.1) on uniform meshes, in order to improve global convergence rate up to $M + R$ with R before obtained multistep solutions are considered as starting estimates due to the nature of multistep collocation schemes.

In this regard, the most recent article [7] is state-of-the-art where multistep collocation schemes have been used to solve a second-kind Fredholm integral equation, and most significantly, the strategy of employing graded meshes has been proposed to tackle weakly singular FIEs.

Studies of [13–15] are elaborate evidences for efficiency as well as effectiveness of performing multistep collocation techniques encountering FIEs [7], VIEs [13–15], and DVIEs [9–12], within differential and/or integral equations together with prominent delay arguments. Since these classes of methods provide threefold benefits in terms of easy implementation, high-order convergent solutions, as well as the lowest degrees of freedom and complexity. Conte et al. [14] utilized a multistep collocation strategy to tackle VIEs of second-kind, which may be considered an introduction to multistep collocation methods.

The outline of this study is as follows: The foundation of Section 2 relies on the construction of multistep collocation techniques for numerically solving second-kind VIEs along with vanishing delays on uniform meshes. A comprehensive investigation of collocation-based multistep methods in terms of existence and uniqueness of multistep collocation approximations, error analysis, and more precisely convergence results has been gathered in Section 3. Section 4 presents a numerical illustration of the performance of the proposed numerical scheme. In addition, it will validate the attained analytical findings of Section 3. The final section stands for some concluding remarks that are concerned with VIEs of pantograph type.

2. Multistep collocation strategy with relative preliminaries

In this position, we will first prove analytical solution (exact solution) existence, uniqueness, and regularity, or equivalently solvability of pantograph type second-kind VIE (1.1). Also, the same solvability discussion for the R-step collocation solution will be given further.

Theorem 2.1. *Following regularity assumptions for under consideration vanishing delay VIE* (1.1)*, with some $k \geq 0$ and $J := [0, \mathfrak{T}]$,*

$$B_1 \in C^k(D_1), \quad B_2 \in C^k(D_2), \quad g \in C^k(J),$$

are held. Then, for any given data function g and delay parameter $q \in (0, 1)$, DVIE (1.1) *has a unique solution $x \in C^k(J)$.*

Proof. It completely follows from [2], most precisely proof of Theorem 2.3.11. □

As we know, multistep collocation approximations for vanishing delay VIE (1.1) are given in the following manner

$$u(T_m + \tau h) = \sum_{r=0}^{R-1} \Psi_r(\tau) x_{m-r} + \sum_{s=1}^{M} \rho_s(\tau) X_{m,s}, \quad \tau \in [0, 1], \quad m \geq R - 1, \tag{2.1}$$

which lie in $S^{(-1)}_{M+R-1}(\delta_h)$, a discontinuous piecewise polynomial space being extensively used for collocation-based approaches, which is defined like this

$$S^{(-1)}_{\mu-1}(\delta_h) := \{u \in L^2(J) : u|_{\sigma_m:=(T_m,T_{m+1}]} \in \pi_{\mu-1}, \quad s.t. \quad m = 0, \ldots, N-1\},$$

where $\pi_{\mu-1}$ stands for real polynomials having degrees not greater than $\mu - 1$. In addition, δ_h is considered as the uniform mesh of the introduced interval J and is like the following

$$\delta_h := \{T_m := mh; \quad s.t. \quad m := 0, \ldots, N; \quad h := \mathfrak{T}/N\}.$$

A solution approximation (2.1) is given with the below considerations

$$\Psi_r(\tau) = \prod_{p=1}^{M} \frac{(\tau - c_p)}{(-r - c_p)} \prod_{p=0, p\neq r}^{R-1} \frac{(\tau + p)}{(-r + p)},$$
$$\rho_s(\tau) = \prod_{p=0}^{R-1} \frac{(\tau + p)}{(c_s + p)} \prod_{p=1, p\neq s}^{M} \frac{(\tau - c_p)}{(c_s - c_p)}. \tag{2.2}$$

In addition, we have assumed that $X_{m,s} = u(T_{m,s})$ together with $x_{m-r} = u(T_{m-r})$.

Furthermore, $\mathbb{Z}_h$ stands for collocation points as bellow

$$\mathbb{Z}_h := \{T_{m,p} := T_m + c_p h \quad s.t. \quad 0 < c_1 < \cdots < c_M \leqslant 1; \quad 0 \leqslant m \leqslant N-1\}.$$

Subsequently, the multistep collocation equation of (1.1), is obtained for $p := 1, \cdots, M$ and $m := R, \ldots, N-1$ at the collocation points of $T_{m,p}$ as what comes

$$X_{m,p} = g(T_{m,p}) + \int_0^{T_{m,p}} B_1(T_{m,p}, \tau) u_m(\tau) \mathrm{d}\tau + \int_0^{qT_{m,p}} B_2(T_{m,p}, \tau) u_m(\tau) \mathrm{d}\tau. \tag{2.3}$$

Next, putting the multistep collocation polynomial (2.1) into Eq (2.3), gets us to the following multistep collocation system for $m := R-1, \ldots, N-1$

$$\begin{aligned} X_{m,p} = g(T_{m,p}) &+ h \sum_{i=0}^{R-2} \int_0^1 B_1(T_{m,p}, T_i + \tau h) u(T_i + \tau h) \mathrm{d}\tau \\ &+ h \sum_{i=R-1}^{m-1} \int_0^1 B_1(T_{m,p}, T_i + \tau h) (\sum_{r=0}^{R-1} \Psi_r(\tau) x_{i-r} + \sum_{s=1}^{M} \rho_s(\tau) X_{i,s}) \mathrm{d}\tau \\ &+ h \int_0^{c_p} B_1(T_{m,p}, T_m + \tau h) (\sum_{r=0}^{R-1} \Psi_r(\tau) x_{m-r} + \sum_{s=1}^{M} \rho_s(\tau) X_{m,s}) \mathrm{d}\tau \\ &+ h \sum_{i=0}^{R-2} \int_0^1 B_2(T_{m,p}, T_i + \tau h) u(T_i + \tau h) \mathrm{d}\tau \\ &+ h \sum_{i=R-1}^{q_{m,p}-1} \int_0^1 B_2(T_{m,p}, T_i + \tau h) (\sum_{r=0}^{R-1} \Psi_r(\tau) x_{i-r} + \sum_{s=1}^{M} \rho_s(\tau) X_{i,s}) \mathrm{d}\tau \\ &+ h \int_0^{\eta_{m,p}} B_2(T_{m,p}, T_{q_{m,p}} + \tau h) (\sum_{r=0}^{R-1} \Psi_r(\tau) x_{q_{m,p}-r} + \sum_{s=1}^{M} \rho_s(\tau) X_{q_{m,p},s}) \mathrm{d}\tau. \end{aligned} \tag{2.4}$$

It is worthy to note that, according to the key monograph by Brunner [16], tackling with vanishing delay VIEs is a bit sophisticated in comparison to VIEs without any vanishing delay term or even to a non-vanishing one. In more details, let define $qT_{m,p}$ like what follows:

$$qT_{m,p} := T_{q_{m,p}} + \eta_{m,p} h,$$

in which taking $q_{m,p}$ equal and/or less than m, along with $\eta_{m,p} \in [0, 1)$ yields to have delay term of $qT_{m,p}$ being located in the interval $[T_{q_{m,p}}, T_{q_{m,p}+1}]$.

Thereafter, determination of the location of the variable delay term, i.e., $qT_{m,p}$ in the interval of $[T_{q_{m,p}}, T_{q_{m,p}+1}]$ leads to encounter the threefold phases, which come in the subsequent.

Phase I. $qT_{m,1} > T_m$, equivalently $q_{m,1} = m$.

Phase II. $qT_{m,1} \leqslant T_m < qT_{m,M}$, it means that if there is an integer, namely ii, such that for $0 < ii < M$, then we have $q_{m,ii} = m-1$, together with $q_{m,ii+1} = m$.

Phase III. $qT_{m,M} \leqslant T_m$, which indicates that for integers like q_m and ii, such that $0 < q_m < m-1$ and $0 < ii < M$, accordingly $q_{m,ii} = q_m$ together with $q_{m,ii+1} = q_m + 1$ are satisfied.

Therefore, the phase I multistep collocation equation at $T_{m,p}$ has the following form:

$$\begin{aligned} X_{m,p} &= g(T_{m,p}) + h\sum_{i=0}^{R-2}\int_0^1 B_1(T_{m,p},T_i+h)u(T_i+\tau h)\mathrm{d}\tau \\ &+ h\sum_{i=R-1}^{m-1}\int_0^1 B_1(T_{m,p},T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r}+\sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\ &+ h\int_0^{c_p} B_1(T_{m,p},T_m+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-r}+\sum_{s=1}^{M}\rho_s(\tau)X_{m,s})\mathrm{d}\tau \\ &+ h\sum_{i=0}^{R-2}\int_0^1 B_2(T_{m,p},T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\ &+ h\sum_{i=R-1}^{m-1}\int_0^1 B_2(T_{m,p},T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r}+\sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\ &+ h\int_0^{\eta_{m,p}} B_2(T_{m,p},T_m+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-r}+\sum_{s=1}^{M}\rho_s(\tau)X_{m,s})\mathrm{d}\tau. \end{aligned} \tag{2.5}$$

Then, with respect to the below notations, which are associated with indices of $i := 0,\dots,m;\quad p := 1,\dots,M;\quad r := 0,\dots,R-1$; and $\quad s := 1,\dots,M$,

$$x_i^{(1)} := (x_{i-R+1},x_{i-R+2},\dots,x_i)^{\mathrm{T}};\qquad X_m := (X_{m,1},X_{m,2},\dots,X_{m,M})^{\mathrm{T}},$$
$$G_m := (g(T_{m,1}),g(T_{m,2}),\dots,g(T_{m,M}))^{\mathrm{T}},$$

$$\begin{aligned} \bar{B}_1^{m,i} &:= \Big(\int_0^1 B_1(T_{m,p},T_i+\tau h)\Psi_r(\tau)\mathrm{d}\tau\Big)_{p,r}, & \hat{B}_1^{m,i} &:= \Big(\int_0^1 B_1(T_{m,p},T_i+\tau h)\rho_s(\tau)\mathrm{d}\tau\Big)_{p,s}, \\ \bar{B}_2^{m,i} &:= \Big(\int_0^1 B_2(T_{m,p},T_i+\tau h)\Psi_r(\tau)\mathrm{d}\tau\Big)_{p,r}, & \hat{B}_2^{m,i} &:= \Big(\int_0^1 B_2(T_{m,p},T_i+\tau h)\rho_s(\tau)\mathrm{d}\tau\Big)_{p,s}, \\ \Lambda_1^i &:= \Big(\int_0^1 B_1(T_{m,p},T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau\Big)_{p,1}, & \Lambda_2^i &:= \Big(\int_0^1 B_2(T_{m,p},T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau\Big)_{p,1}, \\ \bar{B}_1^{m} &:= \Big(\int_0^{c_p} B_1(T_{m,p},T_m+\tau h)\Psi_r(\tau)\mathrm{d}\tau\Big)_{p,r}, & \hat{B}_1^{m} &:= \Big(\int_0^{c_p} B_1(T_{m,p},T_m+\tau h)\rho_s(\tau)\mathrm{d}\tau\Big)_{p,s}; \\ \bar{B}_2^{m} &:= \Big(\int_0^{\eta_{m,p}} B_2(T_{m,p},T_m+\tau h)\Psi_r(\tau)\mathrm{d}\tau\Big)_{p,r}, & \hat{B}_2^{m} &:= \Big(\int_0^{\eta_{m,p}} B_2(T_{m,p},T_m+\tau h)\rho_s(\tau)\mathrm{d}\tau\Big)_{p,s}. \end{aligned}$$

Equation (2.5) briefly becomes

$$\begin{aligned} \Big(I-h(\hat{B}_1^m+\hat{B}_2^m)\Big)X_m &= G_m + h\sum_{i=0}^{R-2}(\Lambda_1^i+\Lambda_2^i) + h(\bar{B}_1^m+\bar{B}_2^m)x_m^{(1)} \\ &+ h\sum_{i=R-1}^{m-1}\Big((\bar{B}_1^{m,i}+\bar{B}_2^{m,i})x_i^{(1)}+(\hat{B}_1^{m,i}+\hat{B}_2^{m,i})X_i\Big). \end{aligned} \tag{2.6}$$

Phase II consists of two cases; more directly, for $ii < M$, we have firstly, $q_{m,ii} = m-1$, second $q_{m,ii+1} = m$. Case I includes $i \leqslant ii$ such that (2.4) takes form of below

$$
\begin{aligned}
X_{m,p} = g(T_{m,p}) &+ h\sum_{i=0}^{R-2}\int_0^1 B_1(T_{m,p}, T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\
&+ h\sum_{i=R-1}^{m-1}\int_0^1 B_1(T_{m,p}, T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\
&+ h\int_0^{c_p} B_1(T_{m,p}, T_m+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{m,s})\mathrm{d}\tau \\
&+ h\sum_{i=0}^{R-2}\int_0^1 B_2(T_{m,p}, T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\
&+ h\sum_{i=R-1}^{m-2}\int_0^1 B_2(T_{m,p}, T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\
&+ h\int_0^{\eta_{m,p}} B_2(T_{m,p}, T_{m-1}+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-1-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{m-1,s})\mathrm{d}\tau. \qquad (2.7)
\end{aligned}
$$

In addition, for $p > ii$, the multistep collocation equation at $T_{m,p}$ is

$$
\begin{aligned}
X_{m,p} = g(T_{m,p}) &+ h\sum_{i=0}^{R-2}\int_0^1 B_1(T_{m,p}, T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\
&+ h\sum_{i=R-1}^{m-1}\int_0^1 B_1(T_{m,p}, T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\
&+ h\int_0^{c_p} B_1(T_{m,p}, T_m+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{m,s})\mathrm{d}\tau \\
&+ h\sum_{i=0}^{R-2}\int_0^1 B_2(T_{m,p}, T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\
&+ h\sum_{i=R-1}^{m-1}\int_0^1 B_2(T_{m,p}, T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\
&+ h\int_0^{\eta_{m,p}} B_2(T_{m,p}, T_m+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{m,s})\mathrm{d}\tau. \qquad (2.8)
\end{aligned}
$$

Phase II multistep collocation equation is presented by

$$
\begin{aligned}
\Big(I - h(\hat{B}_1^m + \hat{\hat{B}}_2^m)\Big)X_m = G_m &+ h\sum_{i=0}^{R-2}(\Lambda_1^i + \Lambda_2^i) + h(\bar{B}_1^m + \bar{\bar{B}}_2^m)x_m^{(1)} \\
&+ h\sum_{i=R-1}^{m-1}\Big((\bar{B}_1^{m,i} + \bar{B}_2^{m,i})x_i^{(1)} + (\hat{B}_1^{m,i} + \hat{B}_2^{m,i})X_i\Big), \qquad (2.9)
\end{aligned}
$$

where

$$\bar{\bar{B}}_2^m := \text{diag}(0,\dots,0,1,\dots,1)\bar{B}_2^m; \qquad \hat{\hat{B}}_2^m := \text{diag}(0,\dots,0,1,\dots,1)\hat{B}_2^m,$$

such that in both ii is the number of zeros. We are now concerned with phase III. Herein, we face two cases, such that for integers in terms of q_m and ii with assuming $0 < q_m < m$ together with $0 < ii < M$, we then have $q_{m,ii} = q_m - 1$ and $q_{m,ii+1} = q_m$. Consequently, the phase III multistep collocation equation is as follows:

$$\begin{aligned}
X_{m,p} = g(T_{m,p}) &+ h\sum_{i=0}^{R-2}\int_0^1 B_1(t_{m,p}, T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\
&+ h\sum_{i=R-1}^{m-1}\int_0^1 B_1(T_{m,p}, T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\
&+ h\int_0^{c_p} B_1(T_{m,p}, T_m+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{m-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{m,s})\mathrm{d}\tau \\
&+ h\sum_{i=0}^{R-2}\int_0^1 B_2(T_{m,p}, T_i+\tau h)u(T_i+\tau h)\mathrm{d}\tau \\
&+ h\sum_{i=R-1}^{q_m-1}\int_0^1 B_2(T_{m,p}, T_i+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{i-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{i,s})\mathrm{d}\tau \\
&+ h\int_0^{\eta_{m,p}} B_2(T_{m,p}, T_{q_m}+\tau h)(\sum_{r=0}^{R-1}\Psi_r(\tau)x_{q_m-r} + \sum_{s=1}^{M}\rho_s(\tau)X_{q_m,s})\mathrm{d}\tau, \qquad (2.10)
\end{aligned}$$

which can be written

$$\begin{aligned}
\left(I - h\hat{B}_1^m\right)X_m = G_m &+ h\sum_{i=0}^{R-2}(\Lambda_1^i + \Lambda_2^i) + h\bar{B}_1^m x_m^{(1)} + h\sum_{i=R-1}^{m-1}\left(\bar{B}_1^{m,i}x_i^{(1)} + \hat{B}_1^{m,i}X_i\right) \\
&+ h\sum_{i=R-1}^{q_m-1}\left(\bar{B}_2^{m,i}x_i^{(1)} + \hat{B}_2^{m,i}X_i\right) + h\left(\bar{\bar{B}}_2^{q_m}x_{q_m}^{(1)} + \hat{\hat{B}}_2^{q_m}X_{q_m}\right), \qquad (2.11)
\end{aligned}$$

in which

$$\bar{\bar{B}}_2^{q_m} := \text{diag}(0,\dots,0,1,\dots,1)\bar{B}_2^{m,q_m}, \qquad \hat{\hat{B}}_2^{q_m} := \text{diag}(0,\dots,0,1,\dots,1)\hat{B}_2^{m,q_m},$$

are satisfied such that the number of zeros is ii.

Supposing

$$\|\hat{B}_1^m\|_\infty \leqslant \|B_1\|_\infty\beta, \qquad \|\hat{B}_2^m\|_\infty \leqslant \|B_2\|_\infty\beta, \qquad (2.12)$$

where

$$\beta := \sum_{\substack{s=1\\ \tau\in[0,1]}}^{M} \sup|\rho_s(\tau)|, \qquad (2.13)$$

may lead to helpful underlying bounds, which will be used in solvability proofs. Then, we have what follows:

$$\|\hat{\hat{B}}_2^m\|_\infty \leqslant \|\hat{B}_2^m\|_\infty \leqslant \|B_2\|_\infty \beta. \tag{2.14}$$

In the subsequent discussion, we will investigate the solvability of the system derived from the numerical technique. Hence, by means of the underlying theorem, we have demonstrated the most principle idea of compact operators in the case of second-kind vanishing delay VIEs.

Theorem 2.2. *Recall* $\mathbb{X} := C^{M+R}([0,\mathfrak{T}])$. *Let linear and continuous operators* $\mathfrak{B}_1, \mathfrak{B}_2 : \mathbb{X} \to S^{(-1)}_{M+R-1}(\delta_h)$ *as follows:*

$$(\mathfrak{B}_1 x)(T) := T\int_0^{\mathfrak{T}} B_1(T,\tau(T,\nu))x(\tau(T,\nu))d\nu,$$

$$(\mathfrak{B}_2 x)(T) := (qT)\int_0^{\mathfrak{T}} B_2(T,q\tau(T,\nu))x(q\tau(T,\nu))\mathrm{d}\nu.$$

Then, for operator format of (1.1)*, more precisely the following*

$$\Big(\mathcal{I} - (\mathfrak{B}_1 + \mathfrak{B}_2)\Big)x = g,$$

$\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *stand for compact operators.*

Proof. First, we transform intervals of non-delay and delay integrations, respectively $[0,T]$ and $[0,qT]$ into $[0,\mathfrak{T}]$ with the constant $\mathfrak{T}$, through a transformation like

$$\begin{aligned}&\tau : [0,\mathfrak{T}]\times[0,\mathfrak{T}] \longrightarrow [0,\mathfrak{T}]\\&\tau(T,\nu) := T\nu.\end{aligned} \tag{2.15}$$

In this way, under study equation of (1.1) may be transformed into its operator form as

$$\Big(\mathcal{I} - (\mathfrak{B}_1 + \mathfrak{B}_2)\Big)x = g.$$

Suppose

$$\mathbb{B}_1 := \{\mathfrak{B}_1 x : x \in C^{M+R}([0,\mathfrak{T}])\}, \quad \mathbb{B}_2 := \{\mathfrak{B}_2 x : x \in C^{M+R}([0,\mathfrak{T}])\}.$$

Based on [17], we aim to prove that $\mathbb{B}_1$ and $\mathbb{B}_2$ are equicontinuous and uniformly bounded, and then, immediately, thanks to Arzela-Ascoli Theorem, the compactness of operators $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be concluded. By assuming $L_1 := \|B_1\|_\infty = \sup|B_1(T,\tau(T,\nu))|$ and $L_2 := \|B_2\|_\infty = \sup|B_2(T,\tau(T,\nu))|$ for $T,\tau \in [0,\mathfrak{T}]$, it is straightforward to see

$$\begin{aligned}\|\mathfrak{B}_1 x\|_\infty &= \sup_{T\in[0,\mathfrak{T}]} |\mathfrak{B}_1 x(T)|\\&= \sup_{T\in[0,\mathfrak{T}]} |T\int_0^{\mathfrak{T}} B_1(T,\tau(T,\nu))x(\tau(T,\nu))d\nu| \leq \mathfrak{T}^2 L_1\|x\|_\infty < \infty.\end{aligned}$$

In a similar way,

$$\|\mathfrak{B}_2 x\|_\infty \leq q\mathfrak{T}^2 L_2\|x\|_\infty < \infty.$$

Thereby, the uniformly boundedness property of sets $\mathbb{B}_1$ and $\mathbb{B}_2$ has been derived. When it comes to continuity, we only prove that $\mathfrak{B}_2$ is a continuous operator; the proof for $\mathfrak{B}_1$ is in the same manner. Due to the assumption that kernel B_2 and solution y are sufficiently smooth on their domains, for all $T, \bar{T}$ from $[0, \mathfrak{T}]$

$$\lim_{T\to\bar{T}} |x(\tau(\bar{T}, \nu)) - x(\tau(T, \nu))| = 0,$$

and

$$\lim_{T\to\bar{T}} |B_2(\bar{T}, q\tau(\bar{T}, \nu)) - B_2(T, q\tau(T, \nu))| = 0,$$

are satisfied. Hence, we conduct the following when $T \longrightarrow \bar{T}$.

$$\begin{aligned}
&|\mathfrak{B}_2 x(\bar{T}) - \mathfrak{B}_2 x(T)| \\
&\quad = |\int_0^{\mathfrak{T}} \Big(B_2(\bar{T}, q\tau(\bar{T}, \nu))x(q\tau(\bar{T}, \nu)) - B_2(T, q\tau(T, \nu))x(q\tau(T, \nu))\Big)\mathrm{d}\nu| \\
&\quad = |\int_0^{\mathfrak{T}} \Big(B_2(\bar{T}, q\tau(\bar{T}, \nu))x(q\tau(\bar{T}, \nu)) - B_2(\bar{T}, q\tau(\bar{T}, \nu))x(q\tau(T, \nu)) \\
&\qquad + B_2(\bar{T}, q\tau(\bar{T}, \nu))x(q\tau(T, \nu)) - B_2(T, q\tau(T, \nu))x(q\tau(T, \nu))\Big)\mathrm{d}\nu| \\
&\quad \le \int_0^{\mathfrak{T}} |B_2(\bar{T}, q\tau(\bar{T}, \nu))[x(q\tau(\bar{T}, \nu)) - x(q\tau(T, \nu))]|\mathrm{d}\nu \\
&\qquad + \int_0^{\mathfrak{T}} |[B_2(\bar{T}, q\tau(\bar{T}, \nu)) - B_2(T, q\tau(T, \nu))]x(q\tau(T, \nu))|\mathrm{d}\nu \\
&\quad \le \mathfrak{T}\Big(\|B_2\|_\infty \sup |x(q\tau(\bar{T}, \nu)) - x(q\tau(T, \nu))| \\
&\qquad + \sup |B_2(\bar{T}, q\tau(\bar{T}, \nu)) - B_2(T, q\tau(T, \nu))| \|x\|_\infty\Big) \longrightarrow 0.
\end{aligned}$$

This fulfills the proof. □

The upcoming theorem, i.e., solvability of numerical solutions, has strong ties with the compactness concept of second-kind pantograph delay VIEs operators. Subsequent theorem demonstrates solvability for numerically obtained multistep collocation solution belonging to (2.3).

Theorem 2.3. *Consider $B_1(T, \tau) \in C^{M+R}(D_1)$, $B_2(T, \tau) \in C^{M+R}(D_2)$, and $g \in C^{M+R}(J)$ for vanishing DVIE* (1.1)*. Then, multistep collocation equation* (2.3) *is uniquely solvable for a unique numerical solution $u \in S^{(-1)}_{M+R-1}(\delta_h)$, which has been provided by relation* (2.1) *in a general subinterval $\sigma_m := (T_m, T_{m+1}]$ of the uniform δ_h mesh, such that for a given $\mathfrak{h} > 0$ we have*

$$0 < h < \mathfrak{h} := \frac{1}{M(\|B_1\|_\infty + \|B_2\|_\infty)\beta},$$

where β is defined in (2.13) *and M is the number of collocation parameters.*

Proof. Overall, we aim to show that coefficient matrices in phases I, II, and III, respectively, in terms of

$$\begin{aligned}
&I - h(\hat{B}_1^m + \hat{B}_2^m), \\
&I - h(\hat{B}_1^m + \hat{\hat{B}}_2^m), \\
&I - h\hat{B}_1^m,
\end{aligned} \tag{2.16}$$

with $0 \leq m \leq N-1$ are invertible. For this sake, take into account the obtained multistep collocation equations in three phases, mainly (2.6), (2.9), and (2.11). Recalling achieved bounds of (2.12) and (2.14), there is a well-defined

$$\mathfrak{h} := \frac{1}{M(\|B_1\|_\infty + \|B_2\|_\infty)\beta} > 0,$$

such that in a generic subinterval $\sigma_m := (T_m, T_{m+1}]$ of δ_h with $0 < h < \mathfrak{h}$ for $0 \leq m \leq N-1$, we have the followings

$$\begin{aligned} &\|h(\hat{B}_1^m + \hat{B}_2^m)\|_\infty < 1, \\ &\|h(\hat{B}_1^m + \hat{\hat{B}}_2^m)\|_\infty < 1, \\ &\|h\hat{B}_1^m\|_\infty < 1. \end{aligned} \tag{2.17}$$

Thereafter, thanks to the Neumann theorem, we can display the desired assertion as follows:

$$\begin{aligned} &\|\Big(I - h(\hat{B}_1^m + \hat{B}_2^m)\Big)^{-1}\|_\infty < \infty, \\ &\|\Big(I - h(\hat{B}_1^m + \hat{\hat{B}}_2^m)\Big)^{-1}\|_\infty < \infty, \\ &\|\Big(I - h\hat{B}_1^m\Big)^{-1}\|_\infty < \infty, \end{aligned} \tag{2.18}$$

which asserts that multistep collocation equation (2.3) is uniquely solvable in all three phases for obtaining a unique approximation of (2.1). □

In this position, we are interested in concerns with convergence analysis. In view of the operator form of the under study second-kind pantograph DVIE (1.1), specifically

$$\Big(\mathcal{I} - (\mathfrak{B}_1 + \mathfrak{B}_2)\Big)x = g, \tag{2.19}$$

with respect to compact operators $\mathfrak{B}_1$ and $\mathfrak{B}_2$ introduced in Theorem 2.2, the following properties equip us to establish the convergence theorem in the sequel. In order to be convenient, we name $\mathfrak{B}_1 + \mathfrak{B}_2$ with $\mathscr{B}$, which is a compact operator, as Theorem 2.2 shows. The idea of using a projection interpolation operator for attaining convergence is an immediate outcome of compactness. Regarding this, projection interpolation operator like $\mathscr{P}$ is defined in the following for an arbitrary function of x,

$$\begin{aligned} &\mathscr{P} : C(J := [0, \mathfrak{T}]) \longrightarrow S_{M+R-1}^{(-1)}(\delta_h), \\ &(\mathscr{P}x)(T_{m,p}) = x(T_{m,p}), \qquad T_{m,p} \in \mathbb{Z}_h. \end{aligned} \tag{2.20}$$

Afterwards, in order to obtain operator form of error equation, influence equation (1.1), and multistep collocation equation (2.3) by projection operator of $\mathscr{P}$ as follows

$$\begin{aligned} &\mathscr{P}x = \mathscr{P}\mathscr{B}x + \mathscr{P}g, \\ &\mathscr{P}u = \mathscr{P}\mathscr{B}u + \mathscr{P}g. \end{aligned} \tag{2.21}$$

The operator error equation is obtained straightforwardly in the following way:

$$e = \mathscr{P}e$$

$$
\begin{aligned}
&= \mathscr{P}x - u \\
&= \mathscr{P}\mathscr{B}x - \mathscr{P}\mathscr{B}u \\
&= \mathscr{P}\mathscr{B}(\mathscr{P}e) + \mathscr{P}\mathscr{B}(x - \mathscr{P}x) \\
&= \mathscr{P}\mathscr{B}e + \mathscr{P}\mathscr{B}(\mathcal{I} - \mathscr{P})x \\
&= \mathscr{P}\mathscr{B}e + \mathcal{K},
\end{aligned} \tag{2.22}
$$

in which we have assumed $\mathcal{K} := \mathscr{P}\mathscr{B}(\mathcal{I} - \mathscr{P})x$. Additionally, for the sake of simplicity, take $\mathscr{W} := \mathscr{P}\mathscr{B}$. From (2.22), it can easily be seen that

$$
\|e\|_\infty \leq \|(\mathcal{I} - \mathscr{W})^{-1}\|_\infty \|\mathcal{K}\|_\infty. \tag{2.23}
$$

Lemma 2.1. *By supposing $B_1 \in C^{M+R}(D_1)$, $B_2 \in C^{M+R}(D_2)$, and $g \in C^{M+R}(J)$, then the following key results of*

$$
\lim_{h \to 0} \|\mathscr{W} - \mathscr{B}\|_\infty = 0, \tag{2.24}
$$

$$
\|(\mathcal{I} - \mathscr{W})^{-1}\|_\infty < \infty \tag{2.25}
$$

are held.

Proof. On the one hand $\mathscr{W} := \mathscr{P}\mathscr{B}$, on the other hand, according to Theorem 2.2, $\mathscr{B} \in \mathcal{L}(S^{(-1)}_{M+R-1}(\delta_h))$ is a compact operator, thus it can be immediately concluded that $\mathscr{W}$ is uniformly convergent to $\mathscr{B}$, hence (2.24) is asserted. Again, due to the compactness of $\mathscr{B}$, based on Theorem 2.6 in the study [18], $(\mathcal{I} - \mathscr{B})$ is an invertible operator; however, its inverse is bounded. Most specifically,

$$
\|(\mathcal{I} - \mathscr{B})^{-1}\|_\infty < \infty. \tag{2.26}
$$

Regarding well-established (2.24) and (2.26) yields to

$$
\|\mathscr{W} - \mathscr{B}\|_\infty \|(\mathcal{I} - \mathscr{B})^{-1}\|_\infty \leq 1. \tag{2.27}
$$

(2.27) results in boundedness of operator $\mathscr{W}$, thereby, thanks to the Neumann Theorem, $(\mathcal{I} - \mathscr{W})$ is invertible and bounded simultaneously. In order to attain a bound of $\|(\mathcal{I} - \mathscr{W})^{-1}\|_\infty$ we pursue what follows. Operator multistep collocation equation of (2.21) can be written for suitably assumed $\psi \in S^{(-1)}_{M+R-1}(\delta_h)$ such that

$$
u = \mathscr{W}u + \psi, \tag{2.28}
$$

in which most importantly, $\|\psi\|_\infty = 1$ has been taken. First, subtracting $\mathscr{B}u$ from (2.28), then multiplying by $(\mathcal{I} - \mathscr{B})^{-1}$ derives the following relation:

$$
u = (\mathcal{I} - \mathscr{B})^{-1}(\mathscr{W} - \mathscr{B})u + (\mathcal{I} - \mathscr{B})^{-1}\psi. \tag{2.29}
$$

Taking norm of (2.29), with respect to (2.27) yields

$$
\begin{aligned}
\|u\|_\infty &\leq \frac{1}{1 - \|(\mathcal{I} - \mathscr{B})^{-1}(\mathscr{W} - \mathscr{B})\|_\infty} \|(\mathcal{I} - \mathscr{B})^{-1}\psi\|_\infty \\
&\leq 2\|(\mathcal{I} - \mathscr{B})^{-1}\| < \infty.
\end{aligned} \tag{2.30}
$$

This ends the proof with obtaining the desired result of $\|(\mathcal{I} - \mathscr{W})^{-1}\|_\infty = \|u\|_\infty < \infty$. □

3. Convergence analysis

We will just exploit the spectral radius concept of the matrix based on collocation parameters in the difference equation arising from the method's error in concluding convergence results. We now give the convergence analysis in a theorem frame.

Theorem 3.1. *In pantograph DVIE* (1.1)*, we have supposed that* $g \in C^{M+R}(J)$*,* $B_1 \in C^{M+R}(D_1)$*, and* $B_2 \in C^{M+R}(D_2)$*. For* $\varepsilon := x - u$ *and* $P := M + R$*, suppose that there is a start error of* $\|\varepsilon\|_\infty := \sup_{t\in[0,t_{r-1}]} \mid \varepsilon(t) \mid = O(h^P)$*. Also, with* ρ*, which denotes the spectral radius of the matrix* $\mathbf{S}$*,* $\rho(\mathbf{S}) < 1$ *is satisfied. It should be noted that* $\mathbf{S}$ *will be defined further in the proof stream. Then, the uniquely attained multistep collocation solution* u *uniformly converges to the exact solution* x*, holding the convergence order of* $M + R$

$$\|x - u\|_\infty := \sup_{t\in J:=[0,T]} \mid x(t) - u(t) \mid \leq ch^P.$$

Proof. For the exact solution x with any $\tau \in [0, 1]$, the following representation is satisfied by considering the Peano's theorem on the representation of interpolation error given in the Corollary 1.8.2 of [16].

$$x(T_m + \tau h) = \sum_{r=0}^{R-1} \psi_r(\tau)x(T_{m-r}) + \sum_{p=1}^{M} \rho_p(\tau)x(T_{m,r}) + h^P \tilde{R}_{M,R,m}(\tau), \tag{3.1}$$

where the bases $\psi_r(\tau)$ and $\rho_p(\tau)$ have been previously defined. Also, the remainder function is given as below

$$\tilde{R}_{M,R,m}(\tau) := \int_{-R+1}^{1} \tilde{L}_{M,R}(\tau, v)x^{(p)}(T_m + vh)\mathrm{d}v,$$

in which

$$\tilde{L}_{M,R}(\tau, v) :=$$
$$\frac{1}{(M + R - 1)!}\Big((\tau - v)_+^{M+R-1} - \sum_{r=0}^{R-1} \psi_r(\tau)(-k - v)_+^{M+R-1} - \sum_{p=1}^{M} \rho_p(\tau)(c_p - v)_+^{M+R-1}\Big).$$

Similarly, we have the error representation for $\tau \in [0, 1]$ as follows:

$$\varepsilon(T_m + \tau h) = \sum_{r=0}^{R-1} \psi_r(\tau)\varepsilon_{m-r} + \sum_{p=1}^{M} \rho_p(\tau)\varepsilon_{m,p} + h^P \tilde{R}_{M,R,m}(\tau), \tag{3.2}$$

such that $\varepsilon_{m-r} := x(t_{m-r}) - u(t_{m-r})$, and $\varepsilon_{m,p} := x(t_{m,p}) - u(t_{m,p})$, are held. Recalling the multistep collocation system of (2.4) and considering the fact that the errors satisfy in (2.4), we reach to

$$\begin{aligned}\varepsilon(T_{m,p}) = h &\int_0^{c_p} B_1(T_{m,p}, T_m + \tau h)\varepsilon(T_m + \tau h)\mathrm{d}\tau \\ &+ h \sum_{i=0}^{m-1} \int_0^1 B_1(T_{m,p}, T_i + \tau h)\varepsilon(T_i + \tau h)\mathrm{d}\tau \\ &+ h \int_0^{\eta_{m,p}} B_2(T_{m,p}, T_{q_{m,p}} + \tau h)\varepsilon(T_{q_{m,p}} + \tau h)\mathrm{d}\tau\end{aligned}$$

$$+ h \sum_{i=0}^{q_{m,p}-1} \int_0^1 B_2(T_{m,p}, T_i + \tau h)\varepsilon(T_i + \tau h)\mathrm{d}\tau. \tag{3.3}$$

Subsequently, by taking advantage of (3.2) and assuming that in phase I we have $q_{m,p} = m$, it is straightforward to get to the following, which are indeed the second rows of our error matrices.

$$\begin{aligned}
\varepsilon_{m,p} = {} & h \sum_{r=0}^{R-1} (\int_0^{c_p} B_1(T_{m,p}, T_m + \tau h)\psi_r(\tau)\mathrm{d}\tau)\varepsilon_{m-r} \\
& + h \sum_{p=1}^{M} (\int_0^{c_p} B_1(T_{m,p}, T_m + \tau h)\rho_p(\tau)\mathrm{d}\tau)\varepsilon_{m,p} \\
& + h \sum_{i=0}^{m-1} \sum_{r=0}^{R-1} (\int_0^1 B_1(T_{m,p}, T_i + \tau h)\psi_r(\tau)\mathrm{d}\tau)\varepsilon_{i-r} \\
& + h \sum_{i=0}^{m-1} \sum_{p=1}^{M} (\int_0^1 B_1(T_{m,p}, T_i + \tau h)\rho_p(\tau)\mathrm{d}\tau)\varepsilon_{i,p} \\
& + h \sum_{r=0}^{R-1} (\int_0^{\eta_{m,p}} B_2(T_{m,p}, T_m + \tau h)\psi_r(\tau)\mathrm{d}\tau)\varepsilon_{m-r} \\
& + h \sum_{p=1}^{M} (\int_0^{\eta_{m,p}} B_2(T_{m,p}, T_m + \tau h)\rho_p(\tau)\mathrm{d}\tau)\varepsilon_{m,p} \\
& + h \sum_{i=0}^{m-1} \sum_{r=0}^{R-1} (\int_0^1 B_2(T_{m,p}, T_i + \tau h)\psi_r(\tau)\mathrm{d}\tau)\varepsilon_{i-r} \\
& + h \sum_{i=0}^{m-1} \sum_{p=1}^{M} (\int_0^1 B_2(T_{m,p}, T_i + \tau h)\rho_p(\tau)\mathrm{d}\tau)\varepsilon_{i,p} + O(h^P).
\end{aligned} \tag{3.4}$$

Now, it requires to assume what follows

$$E_m^{(1)} := [\varepsilon_{m-R+1}, \varepsilon_{m-R+2}, \ldots, \varepsilon_m]^T, \qquad E_m^{(2)} := [\varepsilon_{m,1}, \varepsilon_{m,2}, \ldots, \varepsilon_{m,M}]^T.$$

Additionally, by means of Taylor's expansion, it is straightforward to define the following:

$$\begin{aligned}
& \int_0^1 B_1(T_{m,p}, T_i + \tau h)\psi_r(\tau)\mathrm{d}\tau = B_1(T_m, T_m)\sigma_r + O(h), \\
& \int_0^1 B_1(T_{m,p}, T_i + \tau h)\rho_p(\tau)\mathrm{d}\tau = B_1(T_m, T_m)\gamma_p + O(h), \\
& \int_0^{c_p} B_1(T_{m,p}, T_m + \tau h)\psi_r(\tau)\mathrm{d}\tau = B_1(T_m, T_m)l_{pr} + O(h), \\
& \int_0^{c_p} B_1(T_{m,p}, T_m + \tau h)\rho_r(\tau)\mathrm{d}\tau = B_1(T_m, T_m)f_{pr} + O(h), \\
& \int_0^{\eta_{m,p}} B_2(T_{m,p}, T_m + \tau h)\psi_r(\tau)\mathrm{d}\tau = B_2(T_m, T_m)\hat{l}_{pr} + O(h),
\end{aligned}$$

$$\int_0^{\eta_{m,p}} B_2(T_{m,p}, T_m + \tau h)\rho_r(\tau)\mathrm{d}\tau = B_2(T_m, T_m)\hat{f}_{pr} + O(h), \tag{3.5}$$

where

$$\sigma_r := \int_0^1 \psi_r(\tau)\mathrm{d}\tau, \qquad \gamma_p := \int_0^1 \rho_p(\tau)\mathrm{d}\tau.$$
$$\qquad\quad r=0,\cdots,R-1 \qquad\qquad\qquad p=1,\cdots,M$$

In order to achieve error matrices, we use the idea in the article [19]. To do so, assume $\tau = 1$ in (3.2), which leads to

$$E_n^{(1)} = \Omega E_{n-1}^{(1)} + \Upsilon E_{n-1}^{(2)} + h^P \tilde{R}_{M,R,m-1},$$

such that $\tilde{R}_{M,R,m-1} = \begin{bmatrix} \mathbf{0}_{R-1,1} \\ \tilde{R}_{M,R,m-1}(1) \end{bmatrix}$.

Therefore, we have a phase I error system as follows:

$$\begin{bmatrix} \mathbf{I}_{R,R} & \mathbf{0}_{R,M} \\ h(\mathbf{B_1 L} + \mathbf{B_2}\hat{\mathbf{L}}) & \mathbf{I} - h(\mathbf{B_1 F} + \mathbf{B_2}\hat{\mathbf{F}}) \end{bmatrix} \begin{bmatrix} E_m^{(1)} \\ E_m^{(2)} \end{bmatrix}$$
$$= \begin{bmatrix} \Omega & \Upsilon \\ (\mathbf{B_1} + \mathbf{B_2})\mathbf{\Sigma} & (\mathbf{B_1} + \mathbf{B_2})\mathbf{\Gamma} \end{bmatrix}$$
$$\begin{bmatrix} E_{m-1}^{(1)} \\ E_{m-1}^{(2)} \end{bmatrix} + \sum_{i=r-1}^{m-2} \begin{bmatrix} \mathbf{0}_{R,R} & \mathbf{0}_{R,M} \\ (\mathbf{B_1} + \mathbf{B_2})\mathbf{\Sigma} & (\mathbf{B_1} + \mathbf{B_2})\mathbf{\Sigma} \end{bmatrix} \begin{bmatrix} E_i^{(1)} \\ E_i^{(2)} \end{bmatrix}$$
$$+ O(h^P), \tag{3.6}$$

where $\mathbf{B_1}$ and $\mathbf{B_2}$ can be readily derived and

$$\mathbf{L} := \left[l_{pr} := \int_0^{c_p} \psi_r(\tau)\mathrm{d}\tau \right]_{p=1,\ldots,M,\quad r=0,\cdots,R-1}, \qquad \mathbf{F} := \left[f_{pr} := \int_0^{c_p} \rho_r(\tau)\mathrm{d}\tau \right]_{p=1,\ldots,M,\quad r=1,\cdots,M},$$

$$\hat{\mathbf{L}} := \left[\hat{l}_{pr} := \int_0^{\eta_{m,p}} \psi_r(\tau)\mathrm{d}\tau \right]_{p=1,\ldots,M,\quad r=0,\ldots,R-1}, \qquad \hat{\mathbf{F}} := \left[\hat{f}_{pr} := \int_0^{\eta_{m,p}} \rho_r(\tau)\mathrm{d}\tau \right]_{p=1,\ldots,M,\quad r=1,\ldots,M},$$

$$\mathbf{\Omega} := \begin{bmatrix} \mathbf{0}_{R-1,1} & \mathbf{I}_{R-1} \\ \psi_{R-1}(1) & \psi_{R-2}(1), \cdots, \psi_0(1) \end{bmatrix}, \qquad \mathbf{\Upsilon} := \begin{bmatrix} \mathbf{0}_{R-1,M} \\ \rho^T(1) \end{bmatrix},$$

$$\rho(1) := [\rho_1(1), \rho_2(1), \ldots, \rho_M(1)]^T.$$

$$\mathbf{\Sigma} := [\sigma_{R-1}, \ldots, \sigma_0]^T, \qquad \mathbf{\Gamma} := [\gamma_1, \ldots, \gamma_M]^T.$$

Then, after some manipulation, we obtain into

$$\begin{bmatrix} E_m^{(1)} \\ E_m^{(2)} \end{bmatrix} = \mathbf{S} \begin{bmatrix} E_{m-1}^{(1)} \\ E_{m-1}^{(2)} \end{bmatrix} + \sum_{i=r-1}^{m-2} \begin{bmatrix} \mathbf{0}_{R,R} & \mathbf{0}_{R,M} \\ \bar{\mathcal{H}}^{(i)} & \hat{\mathcal{H}}^{(i)} \end{bmatrix} \begin{bmatrix} E_i^{(1)} \\ E_i^{(2)} \end{bmatrix} + O(h^P), \tag{3.7}$$

in which one can derive $\bar{\mathcal{H}}$ and $\hat{\mathcal{H}}$ easily and the matrix of $\mathbf{S}$ is in the following manner

$$\mathbf{S} := \begin{bmatrix} \mathbf{\Omega} & \mathbf{\Upsilon} \\ \Big(\mathbf{I} - h(\mathbf{B_1 F} + \mathbf{B_2 \hat{F}})\Big)^{-1}\Big(- h\mathbf{\Omega}(\mathbf{B_1 L} + \mathbf{B_2 \hat{L}}) + (\mathbf{B_1} + \mathbf{B_2})\mathbf{\Sigma}\Big) & \Big(\mathbf{I} - h(\mathbf{B_1 F} + \mathbf{B_2 \hat{F}})\Big)^{-1}\Big(- h\mathbf{\Upsilon}(\mathbf{B_1 L} + \mathbf{B_2 \hat{L}}) + (\mathbf{B_1} + \mathbf{B_2})\mathbf{\Gamma}\Big) \end{bmatrix}.$$

Thereafter, by means of the Gronwall Lemma together with the assumption of $\rho(\mathbf{S}) < 1$, we can conclude that

$$\|E_m^{(1)}\|_\infty = \mathcal{O}(h^P), \quad \|E_m^{(2)}\|_\infty = \mathcal{O}(h^P).$$

Now, by taking into account (3.1), it is straightforward to see

$$\begin{aligned} \|x - u\|_\infty &= \sup_{T \in J := [0,\mathfrak{T}]} \mid \varepsilon(T) \mid \\ &\leqslant \alpha \max_{0 \leqslant m \leqslant N-1} \Big(\|E_m^{(1)}\|_\infty + \|E_m^{(2)}\|_\infty \Big) + \bar{\beta} h^P \leqslant C h^P, \end{aligned}$$

where C is a general constant along with

$$\alpha := \max\{\|\psi_r\|_\infty; r = 0, \ldots, R-1, \|\rho_p\|_\infty; p = 1, \ldots, M, \},$$

and

$$\bar{\beta} := \max\{\|x^{(p)}\|_\infty, \int_{-R+1}^{0} |\tilde{L}_{M,R}(\tau, v)| \mathrm{d}v\},$$

with $\tau \in [0, 1]$. Pursuing a similar process, one can deduce convergence analysis in both phase II and phase III. This is the end of the proof. □

4. Numerical implementation

In this present section, we concentrate on numerically testing and also verifying the previously attained convergence results of Theorem 3.1. What is more, we try various 2-point and 3-point multistep collocation techniques with $R = 3$ on uniform meshes in order to illustrate optimum L_∞-norm convergence orders. It should be pointed out that 2 and 3 illustrate the number of collocation parameters.

When it comes to comparing with [1], where the same second-type vanishing delay VIE has been analysed and solved, the significant distinct advantages of our proposed technique are twofold. First, the advantage of our method is that it achieves the same error results with significantly fewer spatial grid points specifically, $N = 256$ compared to the $N = 1600$ grid points used in [1]. As we know, N is a commonly used notation for displaying the number of interval divisions. Second, from simplicity and efficiency point of view, in our work a fast rate multistep strategy is implemented on uniform meshes with ease.

Locally achieved absolute multistep collocation errors for $\sigma_m := (T_m, T_{m+1}]$ with $m := 1, \ldots, N-1$ in a defined uniform δ_h mesh are as follows:

$$e_m := \max |u(T_m) - x(T_m)|.$$

Besides, rates of convergence are like the follows:

$$p := \log_2(e_m/e_{2m}).$$

Section 2 of this paper, confirms that the system is solvable for all q values in $(0, 1)$. In addition to that, both one-step [16] and R-step collocation estimate errors together with convergence rates, CPU time based on different delay parameters such as $q = 0.2, 0.5, 0.9$, and various collocation points have been reported in the upcoming two test problems. We have demonstrated that both solvability and convergence of the system are achieved for all values of q in the interval $(0, 1)$. We have chosen a variety of q parameters. It has been compared with the one-step collocation method with the multistep collocation one. The results in both cases have been gathered and shown.

As it can be seen through all tables, with nearly the same computational cost, the multistep collocation method achieves lower errors compared to the one-step collocation method.

For the sake of comparing our results with the ones given in [1], we have provided a selection of the numerical example from [1].

Example 1. [1] In the current test problem, one-step and multistep collocation-based techniques have been applied for the sake of solving the following pantograph DVIE:

$$x(T) = g_1(T) - \int_0^T x(\tau)\mathrm{d}\tau + \frac{1}{2}\int_0^{qT} x(\tau)\mathrm{d}\tau, \quad T \in J := [0, 10], \tag{4.1}$$

with considering the exact solution of $x(T) = \exp(-T)$, and

$$g_1(T) = \frac{1}{2}\Big(1 + \exp(-qT)\Big).$$

Table 1. Multistep and one-step collocation L_∞-norm errors along with their corresponding rates for Eq (4.1) under uniform meshes, delay parameter $q = 0.2$ and collocation parameters $\{c_1 = 7/10, c_2 = 1\}$.

	Multistep CM			One-step CM		
N	L_∞−error	Rate	CPU time(s)	L_∞−error	Rate	CPU time(s)
32	$3.7326E-07$	–	2.250	$2.2872E-03$	–	2.953
64	$1.7579E-08$	4.40	9.063	$6.5226E-04$	1.81	12.657
128	$6.7537E-10$	4.70	38.969	$1.6858E-04$	1.95	56.874
256	$2.1412E-11$	4.97	171.483	$4.3716E-05$	1.94	254.813

Table 2. Multistep and one-step collocation L_∞-norm errors along with their corresponding rates for Eq (4.1) under uniform meshes, delay parameter $q = 0.5$ and collocation parameters $\{c_1 = 7/10, c_2 = 1\}$.

	Multistep CM			One-step CM		
N	$L_\infty-$error	Rate	CPU time(s)	$L_\infty-$error	Rate	CPU time(s)
32	$3.7326E-07$	–	2.718	$2.1145E-03$	–	2.609
64	$1.4659E-08$	4.67	11.455	$5.7019E-04$	1.89	11.827
128	$5.2515E-10$	4.80	48.937	$1.4745E-04$	1.95	48.906
256	$1.7739E-11$	4.88	214.296	$3.7528E-05$	1.97	211.921

Table 3. Multistep and one-step collocation L_∞-norm errors along with their corresponding rates for Eq (4.1) under uniform meshes, delay parameter $q = 0.9$ and collocation parameters $\{c_1 = 7/10, c_2 = 1\}$.

	Multistep CM			One-step CM		
N	$L_\infty-$error	Rate	CPU time(s)	$L_\infty-$error	Rate	CPU time(s)
32	$2.4766E-07$	–	4.296	$1.8352E-03$	–	4.485
64	$1.1967E-08$	4.37	15.876	$5.0747E-04$	1.85	14.656
128	$4.6069E-10$	4.69	64.922	$1.3121E-04$	1.95	62.720
256	$1.5832E-11$	4.86	279.765	$3.4008E-05$	1.94	276.422

As noted, the convergence of one-step methods relies solely on the collocation parameters c_i. In contrast, multistep collocation methods depend not only on these parameters but also on the spectral radius of the coefficient matrix of the difference equation, denoted as $\mathbf{S}$.

The proposed method relies on the parameters c_i, requiring that $\rho(\mathbf{S}) < 1$ for convergence to occur. It is important to note that article [1] reports the highest accuracy at the Gauss points; however, since the condition $\rho(\mathbf{S}) < 1$ is not satisfied at these points, our proposed numerical method does not converge there.

For the sake of comparing with [1], we have employed the collocation parameters of $c_1 = 7/10$ and $c_2 = 1$, with $M = 2$, $R = 3$, and $q = 0.2, 0.5$. Then we have reached the accuracy of 10^{-11} at $N = 256$ grid points, while at [1], it reaches the accuracy of 10^{-11} for $N = 1600$ grid points.

In the reference [1], the authors have exploited the quasi-geometric meshes, which are dependent on delay parameter q. As a result, for $q = 0.9$, they have attained different solutions than the case, $q = 0.2$ and $q = 0.5$. However, here the uniform meshes have been utilized, so we obtain the same results for all values of the parameter q.

For the second test problem, we first take a monotone increasing function of $x(T) = \sqrt{2 + T}$, which is totally different from an exponential function, as an exact solution, and then we have reported the results just for the delay parameter of $q = 0.2$ but for two selections of collocation parameters.

Example 2. Herein, we have considered what follows.

$$x(T) = g_2(T) + \int_0^T (T + \tau)x(\tau)\mathrm{d}\tau + \int_0^{qT} T x(\tau)\mathrm{d}\tau, \quad T \in J := [0, 1], \tag{4.2}$$

For which, we have carried out both classical and multistep collocation schemes by taking $g_2(T)$ such that we have $x(T) = \sqrt{2+T}$.

It is worthwhile to note that in the second test problem the integral interval is $J := [0, 1]$, which differs from the one in the first example.

Table 4. Multistep and one-step collocation L_∞-norm errors along with their corresponding rates for Eq (4.2) under uniform meshes, delay parameter $q = 0.2$ and collocation parameters $\{c_1 = 7/10, c_2 = 1\}$.

	Multistep CM			One-step CM		
N	L_∞−error	Rate	CPU time(s)	L_∞−error	Rate	CPU time(s)
8	$2.0663E-09$	−	0.437	$3.2035E-04$	−	0.374
16	$9.7556E-11$	4.40	0.813	$7.8609E-05$	2.02	0.750
32	$4.0352E-12$	4.59	2.157	$1.9480E-05$	2.01	2.156
64	$1.4210E-13$	4.82	9.359	$4.8319E-06$	2.01	8.220

Table 5. Multistep and one-step collocation L_∞-norm errors along with their corresponding rates for Eq (4.2) under uniform meshes, delay parameter $q = 0.2$ and collocation parameters $\{c_1 = 1/2, c_2 = 7/10, c_3 = 1\}$.

	Multistep CM			One-step CM		
N	L_∞−error	Rate	CPU time(s)	L_∞−error	Rate	CPU time(s)
8	$2.3015E-11$	−	1.095	$2.7042E-06$	−	0.551
16	$6.8789E-13$ 5.06	1.860	$3.3248E-07$	3.02	0.907	
32	$1.2434E-14$	5.78	4.173	$4.1104E-08$	3.01	3.594
64	$2.1102E-16$	5.88	12.76	$5.0948E-09$	3.01	10.23

All presented results in Tables 1–5 are strongly in agreement with the foregoing Theorem 3.1.

5. Conclusions

In this article, we employed a robust high-rate multistep collocation strategy in order to tackle second-kind vanishing delay VIEs. Both numerical and theoretical solution solvability have been proved detailedly. Moreover, a theorem associated with multistep collocation solution error analysis has been presented. In the last section, for the sake of confirming theoretically achieved consequences, two numerical examples were given, which have been investigated for a variety of delay parameters of $q = 0.2, 0.5, 0.9$ on uniform meshes perfectly. Furthermore, it is of great importance to remark that based on our study, high convergence rates for second-kind pantograph delay VIEs (1.1) can be attained for all various delay parameters of q in $(0, 1)$.

Author contributions

Shireen Obaid Khaleel: Writing original draft, Visualization, Formal analysis; Parviz Darania: Writing original draft, Validation, Writing review & editing; Saeed Pishbin: Software, Validation, Writing review & editing; Shadi Malek Bagomghaleh: Writing original draft, Visualization, Formal analysis. All authors discussed the results and revised the draft. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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