Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 30745–30760.
DOI: 10.3934/math.20241485
Received: 10 August 2024
Revised: 10 October 2024
Accepted: 14 October 2024
Published: 30 October 2024

Research article

Global stability of traveling waves in monostable stream-population model

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Abstract: The stability of monotone traveling waves to a stream-population model is established in a particular weighted function space via the method of upper and lower solutions and a squeezing technique. By analyzing the behaviors of the traveling wave for a large time period under a small perturbation, we obtain the results of the local stability. The comparison principle and the squeeze theorem also allows us to prove the global stability of the positive steady-state solutions in the special weighted function space by constructing suitable upper and lower solutions.

Keywords: stream-population model; local stability; global stability; weighted function space; upper and lower solutions
Mathematics Subject Classification: 35B35, 35B40, 35B51, 37C65, 92D25

1. Introduction

In this paper, we mainly focus on the following stream-population model [1]

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} - \alpha\dfrac{\partial u}{\partial x} - \sigma u + \mu v, \\ \dfrac{\partial v}{\partial t} = \epsilon\dfrac{\partial^2 v}{\partial x^2} + \sigma u - \mu v + f(v), \end{cases} \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, \forall x \in \mathbb{R}. \tag{1.2}$$

Here, $u(x,t)$, $v(x,t)$ are the population densities of neuston and benthon respectively; σ represents the per capita drift rate at which the species returns to the benthic population; μ is the per capita rate at which individuals in the benthic population enter the drift; α is the speed of the flow; d, ϵ are the diffusion coefficients of species u, v respectively; the reaction term $f(v)$ is a strictly convex function of second order differentiable satisfying $f(0) = f(1) = 0$, $f'(0) > 0 > f'(1)$, and $f(v) > 0$ for $v \in (0,1)$.

$\sigma, \mu, \alpha, d, \epsilon$ are positive constants and possess the biological meaning [2, 3]. It is worth mentioning that the benthon basically do not move horizontally since $\epsilon \ll 1$. Obviously, $(0, 0)$ and $\left(\frac{\mu}{\sigma}, 1\right)$ are the equilibria and further we can obtain that $(0, 0)$ is unstable and $\left(\frac{\mu}{\sigma}, 1\right)$ is stable for the corresponding spatially homogeneous system of (1.1).

In this paper, we are interested in the nonnegative traveling wave, connecting $(0, 0)$ and $\left(\frac{\mu}{\sigma}, 1\right)$, which possesses the wave profile as

$$u(x, t) = \overline{U}(z), v(x, t) = \overline{V}(z), z = x - ct, \tag{1.3}$$

where the wave speed $c \geq 0$. Substituting (1.3) into (1.1), yields the new wave profile as

$$\begin{cases} -c\overline{U}' = d\overline{U}'' - \alpha\overline{U}' - \sigma\overline{U} + \mu\overline{V}, \\ -c\overline{V}' = \epsilon\overline{V}'' + \sigma\overline{U} - \mu\overline{V} + f(\overline{V}), \end{cases} \tag{1.4}$$

with the boundary value condition

$$(\overline{U}, \overline{V})(-\infty) = \left(\frac{\mu}{\sigma}, 1\right), (\overline{U}, \overline{V})(\infty) = (0, 0). \tag{1.5}$$

It is worth pointing out that the existence of the traveling wave solution of (1.4) was proved, and the determinacy of linear and nonlinear selection of the minimal wave speed was further established by employing the method of upper and lower solutions in [1].

On this basis of the existence, we shall investigate the local and the global stability of the monotone traveling waves. In this progress, we need to conform that the solution of (1.1) is infinitely close to $(\overline{U}, \overline{V})(x - ct)$ when t is large enough for given initial data $u_0(x)$ and $v_0(x)$. To begin with, we set

$$(u, v)(x, t) = (U, V)(z, t), \tag{1.6}$$

and then (1.4) can be transformed into the partial differential model

$$\begin{cases} U_t = dU_{zz} + (c - \alpha)U_z - \sigma U + \mu V, \\ V_t = \epsilon V_{zz} + cV_z + \sigma U - \mu V + f(V), \end{cases} \tag{1.7}$$

subject to

$$U(z, 0) = u_0(z), V(z, 0) = v_0(z), \forall z \in \mathbb{R}. \tag{1.8}$$

From the above system (1.7) we know that $(\overline{U}, \overline{V})(z)$ is also its steady state.

The investigation of the stabilities of traveling wave solutions could be traced back to the original works [4, 5]. Then many works assessing stability are available, but differences among them are frequent. For examples, the stability of the traveling waves with critical speed [6] and noncritical speed [7] in appropriate weighted Banach spaces is proved. The global stability of the forced waves [8, 9] is presented. The time-periodic traveling waves are asymptotically stable [10, 11]. Mei et. al. [12–14] proved the exponential stability of traveling wave fronts, and the exponential stability of stochastic systems is demonstrated [15, 16]. By analyzing the location of the spectrum, the local stability of the traveling waves are investigated [17, 18]. For more related works, we refer to [19–32], and the references cited therein.

Despite the success in the study of the existence and the speed determinacy of traveling waves to the model (1.1), the local and global stability of the traveling waves in the monostable still remains unsolved. In this paper, mainly inspired by the ideas in [18], we devote our attention to finishing the issue of stability for the steady state $(\overline{U}, \overline{V})(z)$. Firstly, the local stability is discussed on the basis of asymptotic behavior of the traveling waves near the equilibrium and the method of spectrum analysis. When analyzing the eigenvalue problem, we need to overcome the great challenge brought by the high-order nonlinear terms. Then by using the result of local stability, we should choose appropriate weighted Banach space. Furthermore, also by constructing upper and lower solutions, we prove that the solution (U, V) in the moving coordinates converges to the steady state. The new results on the global stability is established in the special weighted function space by using the squeeze theorem.

This paper are organized as follows. In section 2, we present some preliminaries and main results. In section 3, we prove the local stability of the traveling waves. The global stability of the steady state $(\overline{U}, \overline{V})(z)$ in a special choice on the weighted function space $L^\infty_\omega(\mathbb{R})$ is investigated in section 4.

2. Preliminaries and main results

First, linearizing the system (1.4) at $(0, 0)$ we obtain

$$\begin{cases} -c\overline{U}' = d\overline{U}'' - \alpha\overline{U}' - \sigma\overline{U} + \mu\overline{V}, \\ -c\overline{V}' = \epsilon\overline{V}'' + \sigma\overline{U} - \mu\overline{V} + f'(0)\overline{V}. \end{cases} \tag{2.1}$$

The transformation

$$(\overline{U}, \overline{V})(z) \sim (\xi_1 e^{-\lambda z}, \xi_2 e^{-\lambda z}) \tag{2.2}$$

with $\lambda > 0$ and $\xi_i (i = 1, 2)$ being positive constants, allows us to transform the discussion of the asymptotic behaviors at infinitely in variable z to the following eigenvalue problem

$$c\lambda\xi = \begin{pmatrix} d\lambda^2 + \alpha\lambda - \sigma & \mu \\ \sigma & \epsilon\lambda^2 - \mu + f'(0) \end{pmatrix}\xi, \tag{2.3}$$

where $\xi = (\xi_1, \xi_2)^T$. Letting

$$A_1(\lambda) = d\lambda^2 + \alpha\lambda - \sigma, A_2(\lambda) = \epsilon\lambda^2 - \mu + f'(0), A(\lambda) = \begin{pmatrix} A_1(\lambda) & \mu \\ \sigma & A_2(\lambda) \end{pmatrix}, \tag{2.4}$$

then the eigenvalue problem (2.3) is equivalent to $r(\lambda)\xi = A(\lambda)\xi$. Further, we can obtain the eigenvalues

$$r_\pm(\lambda) = \frac{A_1(\lambda) + A_2(\lambda) \pm \sqrt{(A_1(\lambda) - A_2(\lambda))^2 + 4\sigma\mu}}{2}. \tag{2.5}$$

As we know, the principal eigenvalue is greater than or equal to the real part of matrix eigenvalues. According to [33], the principal eigenvalue of matrix $A(\lambda)$ is

$$r(\lambda) = r_+(\lambda), \tag{2.6}$$

where r_+ is real number and for any $\lambda \in (0, +\infty)$,

$$\begin{aligned} r_+(\lambda) &= \frac{1}{2}\left[A_1(\lambda) + A_2(\lambda) + \sqrt{(A_1(\lambda) + A_2(\lambda))^2 - 4(A_1(\lambda)A_2(\lambda) - \sigma\mu)}\right] \\ &= \frac{1}{2}\left[(d + \epsilon)\lambda^2 + \alpha\lambda - \sigma - \mu + f'(0) + \sqrt{[(d - \epsilon)\lambda^2 + \alpha\lambda - \sigma + \mu - f'(0)]^2 + 4\sigma\mu}\right] \\ &> 0. \end{aligned} \tag{2.7}$$

Actually, through the classified discussion of the positive and negative of $A_1$and A_2, $r_+(\lambda) > 0$ holds true.

Lemma 2.1. *([1], see Section 2) $r(\lambda)$, defined in (2.7), is a real, continuous, and convex function with respect to $\lambda \in R$. Assume that*

$$c_0 = \inf_{\lambda \in (0,\infty)} \frac{r(\lambda)}{\lambda} \in R^+, \tag{2.8}$$

where c_0 is called the minimal wave speed, then we can obtain that the equation $c\lambda = r(\lambda)$ has

- *no solution, if $c < c_0$;*
- *a unique solution $\lambda(c_0)$, if $c = c_0$;*
- *two solutions $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) < \lambda_2(c)$, if $c > c_0$.*

Based on the above analysis, we could give the asymptotic behaviors as $z \to \infty$ with the following lemma.

Lemma 2.2. *([1], see Section 2) The asymptotic behaviors of the traveling wave solution $(\overline{U}(z), \overline{V}(z))$ (see (2.1)) can be represented as follows:*

$$\begin{pmatrix} \overline{U} \\ \overline{V} \end{pmatrix} = C_1 \begin{pmatrix} \xi_1(\lambda_1(c)) \\ \xi_2(\lambda_1(c)) \end{pmatrix} e^{-\lambda_1(c)z} + C_2 \begin{pmatrix} \xi_1(\lambda_2(c)) \\ \xi_2(\lambda_2(c)) \end{pmatrix} e^{-\lambda_2(c)z} \tag{2.9}$$

with $C_1 > 0$ or $C_1 = 0, C_2 > 0$, where the eigenvectors corresponding to eigenvalues $\lambda_i(c)(i = 1, 2)$ are

$$\begin{pmatrix} \xi_1(\lambda_i(c)) \\ \xi_2(\lambda_i(c)) \end{pmatrix} = \begin{pmatrix} \mu \\ c\lambda_i(c) - A_1(\lambda_i(c)) \end{pmatrix} or \begin{pmatrix} c\lambda_i(c) - A_2(\lambda_i(c)) \\ \sigma \end{pmatrix}. \tag{2.10}$$

The proof of the above lemmas can refer to the proof of Theorem 1 in [18], which will not be repeated here.

Before stating our main results, let us make the following notation.

Notation: Throughout the paper, L^p are function spaces defined by using a natural generalization of the p-norm for finite dimensional vector spaces. The weighted Lebesgue space L^p_ω with $1 \le p < \infty$ can be expressed by

$$L^p_\omega = \left\{ g(z) : \frac{g(z)}{\omega(z)} \in L^p(\mathbb{R}) \right\} \tag{2.11}$$

with the norm

$$\|g(z)\|_{L^p_\omega} = \left(\int_{-\infty}^{\infty} \frac{|g(z)|^p}{|\omega(z)|^p} \, dz \right)^{\frac{1}{p}}, \tag{2.12}$$

where the weighted function

$$\omega(z)=\begin{cases}e^{-\beta(z-z_0)}, & z>z_0,\\ 1, & z\le z_0\end{cases} \tag{2.13}$$

for some positive constants z_0 and β.

With the introduction above, we state our main conclusions for two theorems.

Theorem 1. *(Local stability) For any $c>c_0$, the wavefront $(\overline{U},\overline{V})(z)$ is locally stable in the weighted function space L^p_ω if $\beta\in(\lambda_1,\lambda_2)$ to be chosen.*

Theorem 2. *(Global stability) Assume that $c>c_0$, $\lambda_1<\beta<\lambda_2$ and the initial data $U(z,0)=U_0(z)$ and $V(z,0)=V_0(z)$ satisfy*

$$(0,0)\le(U_0,V_0)(z)\le\left(\tfrac{\mu}{\sigma},1\right),\ \forall z\in\mathbb{R},$$
$$\liminf_{z\to-\infty}(U_0,V_0)(z)>(0,0),$$

and

$$|U_0(z)-\overline{U}(z)|\in L^\infty_\omega(\mathbb{R}),\ |V_0(z)-\overline{V}(z)|\in L^\infty_\omega(\mathbb{R}).$$

Then (1.7) with initial data admits a unique solution $(U,V)(z,t)$ satisfying

$$(0,0)\le(U,V)(z,t)\le\left(\tfrac{\mu}{\sigma},1\right),\ \forall(z,t)\in\mathbb{R}\times\mathbb{R}^+.$$

Moreover, the inequalities

$$\sup_{z\in\mathbb{R}}|U(z,t)-\overline{U}(z)|\le ke^{-\eta t},\ t>0$$

and

$$\sup_{z\in\mathbb{R}}|V(z,t)-\overline{V}(z)|\le ke^{-\eta t},\ t>0$$

hold for some positive constants k and η.

3. Proof of Theorem 1

The purpose in this section is to apply the weighted energy method and spectral analysis to prove the local stability of the traveling waves presented in Theorem 1.

First, we assume that

$$U(z,t)=\overline{U}(z)+\delta\psi_1(z)e^{\chi t},V(z,t)=\overline{V}(z)+\delta\psi_2(z)e^{\chi t}, \tag{3.1}$$

where $\delta\ll1,\chi$ is a parameter, and $\psi_1(z)$ and $\psi_2(z)$ are real functions. Substituting $(U(z,t),V(z,t))$ into system (1.7), and linearizing the new system with respect to $(\overline{U},\overline{V})(z)$, we obtain the following spectral problem:

$$\chi\Psi=\mathfrak{L}\Psi:=D\Psi''+C\Psi'+B\Psi, \tag{3.2}$$

where

$$\Psi=(\psi_1,\psi_2)^T,D=\begin{pmatrix}d&0\\0&\epsilon\end{pmatrix},C=\begin{pmatrix}c-\alpha&0\\0&c\end{pmatrix},B=\begin{pmatrix}-\sigma&\mu\\ \sigma&-\mu+f'(\overline{V})\end{pmatrix}, \tag{3.3}$$

where the sign of the maximal real part to the spectrum of the operator $\mathfrak{L}$ is related to the local stability of the traveling wave solution. To proceed, we set

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \omega\phi_1 \\ \omega\phi_2 \end{pmatrix}, \tag{3.4}$$

where ϕ_1 and ϕ_2 are the functions in the space L^p, ω has been defined in (2.13) with $\lambda_1 < \beta < \lambda_2$. Substitute (3.4) into (3.2) to obtain a new spectral problem as:

$$\chi\Phi = \mathfrak{L}_\omega\Phi := D\Phi'' + H\Phi' + J\Phi, \tag{3.5}$$

where $\Phi = (\phi_1, \phi_2)^T$, D is defined in (3.3) and

$$\begin{aligned} H &= \begin{pmatrix} c - \alpha + 2d\frac{\omega'}{\omega} & 0 \\ 0 & c + 2\epsilon\frac{\omega'}{\omega} \end{pmatrix}, \\ J &= \begin{pmatrix} d\frac{\omega''}{\omega} + (c-\alpha)\frac{\omega'}{\omega} - \sigma & \mu \\ \sigma & \epsilon\frac{\omega''}{\omega} + c\frac{\omega'}{\omega} - \mu + f'(\overline{V}) \end{pmatrix}. \end{aligned} \tag{3.6}$$

Since $H(z)$ and $J(z)$ are bounded real matrix functions and ω, ω', ω'' exist as $z \to \pm\infty$, we define

$$H_\pm = \lim_{z\to\pm\infty} H(z), J_\pm = \lim_{z\to\pm\infty} J(z). \tag{3.7}$$

Further, we analyze the position of the essential spectrum for the operator $\mathfrak{L}_\omega$ and give the following lemma.

Lemma 3.1. *([34], see Theorem A.2) If we define*

$$S_\pm = \{\chi | det(-\theta^2 D + i\theta H_\pm + J_\pm - \chi I) = 0, \ -\infty < \theta < \infty\}, \tag{3.8}$$

then the essential spectrum of $\mathfrak{L}_\omega$ is contained in the union of the regions inside or on the curves S_+ and S_-, which are on the left-half complex plane, when the condition $\lambda_1 < \beta < \lambda_2$ is satisfied.

Proof. We prove the lemma 3.1 for two cases with respect to z.

Case 1. When $z \to +\infty$ in (3.6), we have

$$\begin{aligned} H_+ &= \lim_{z\to+\infty} H(z) = \begin{pmatrix} c - \alpha - 2d\beta & 0 \\ 0 & c - 2\epsilon\beta \end{pmatrix}, \\ J_+ &= \lim_{z\to+\infty} J(z) = \begin{pmatrix} d\beta^2 - (c-\alpha)\beta - \sigma & \mu \\ \sigma & \epsilon\beta^2 - c\beta - \mu + f'(0) \end{pmatrix}. \end{aligned} \tag{3.9}$$

Therefore, the equation $det(-\theta^2 D + i\theta H_+ + J_+ - \chi I) = 0$ has two solutions, i.e.

$$\chi_{1,2} = \frac{1}{2}\left\{(E+G) + i(F+K) \pm \sqrt{[(E-G) + i(F-K)]^2 + 4\sigma\mu}\right\} \tag{3.10}$$

with $E = -d\theta^2 + A_1(\beta) - c\beta$, $F = (c - \alpha - 2d\beta)\theta$, $G = -\epsilon\theta^2 + A_2(\beta) - c\beta$, $K = (c - 2\epsilon\beta)\theta$, where A_1 and A_2 have been noted in (2.4). Assume that

$$S_{+,1} = \{\chi_1 | -\infty < \theta < \infty\}, S_{+,2} = \{\chi_2 | -\infty < \theta < \infty\}. \tag{3.11}$$

By Euler's formula, we obtain the real parts of the eigenvalues $\chi_{1,2}$ as follows:

$$\begin{aligned} Re(\chi_1) &= \frac{1}{2}\Big[(E+G) \\ &+ \sqrt{\frac{1}{2}\left(\sqrt{((E-G)^2+(F-K)^2+4\sigma\mu)^2-16(F-K)^2\sigma\mu}+(E-G)^2-(F-K)^2+4\sigma\mu\right)}\Big], \\ Re(\chi_2) &= \frac{1}{2}\Big[(E+G) \\ &- \sqrt{\frac{1}{2}\left(\sqrt{((E-G)^2+(F-K)^2+4\sigma\mu)^2-16(F-K)^2\sigma\mu}+(E-G)^2-(F-K)^2+4\sigma\mu\right)}\Big]. \end{aligned} \tag{3.12}$$

Obviously, $Re(\chi_1) > Re(\chi_2)$. By a simple calculation, one can obtain that

$$\begin{aligned} Re(\chi_1) &< \frac{1}{2}\left[(E+G)+\sqrt{(E-G)^2+4\sigma\mu}\right] \\ &= \frac{1}{2}\left[-(d+\epsilon)\theta^2+A_1(\beta)+A_2(\beta)+\sqrt{-(d-\epsilon)\theta^2+(A_1(\beta)-A_2(\beta))^2+4\sigma\mu}\right]-c\beta \\ &\le \frac{1}{2}\left[A_1(\beta)+A_2(\beta)+\sqrt{(A_1(\beta)-A_2(\beta))^2+4\sigma\mu}\right]-c\beta \\ &= r_+(\beta)-c\beta \\ &< 0. \end{aligned} \tag{3.13}$$

since the formulas (2.7 and 2.8) and $\lambda_1 < \beta < \lambda_2$ are satisfied for $c > c_0$. That is to say $S_+ = S_{+,1} \cup S_{+,2}$ is on the left-half complex plane.

Case 2. When $z \to -\infty$, it follows that

$$\begin{aligned} H_- &= \lim_{z\to-\infty} H(z) = \begin{pmatrix} c-\alpha & 0 \\ 0 & c \end{pmatrix}, \\ J_- &= \lim_{z\to-\infty} J(z) = \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu+f'(0) \end{pmatrix}. \end{aligned} \tag{3.14}$$

Similar to Case 1, solving the equation $det(-\theta^2 D + i\theta H_- + J_- - \chi I) = 0$, we obtain

$$\chi_{3,4} = \frac{1}{2}\left[(M+P)+i(N+Q) \pm \sqrt{((M-P)+i(N-Q))^2+4\sigma\mu}\right], \tag{3.15}$$

where $M = -d\theta^2 - \sigma$, $N = (c-\alpha)\theta$, $P = -\epsilon\theta^2 - \mu + f'(1)$, $Q = c\theta$. Further, Euler's formulas allow us to obtain the maximal real part of $\chi_{3,4}$ as

$$\begin{aligned} Re(\chi_3) &= \frac{1}{2}\Big[(M+P) \\ &+ \sqrt{\frac{1}{2}\left(\sqrt{((M-P)^2+(N-Q)^2+4\sigma\mu)^2-16(N-Q)^2\sigma\mu}+(M-P)^2-(N-Q)^2+4\sigma\mu\right)}\Big] \\ &\le \frac{1}{2}\left[(M+P)+\sqrt{(M+P)^2-4(MP-\sigma\mu)}\right] \\ &< 0, \end{aligned} \tag{3.16}$$

because of $M+P<0$ and $MP-\sigma\mu>0$. It means that $S_- = \{\chi_3 | -\infty < \theta < \infty\} \cup \{\chi_4 | -\infty < \theta < \infty\}$ is also on the left-half complex plane.

By the above analysis, the essential spectrum of the operator $\mathfrak{L}_\omega$ is on the left-half complex plane. □

If the sign of the real part of the principal eigenvalue in the point spectrum (3.2) is negative, the traveling wave solutions is locally stable. So, we need the following step to check the sign of the principal eigenvalue to ensure the result of local stability.

Finally, we judge the sign of the principal eigenvalue in the point spectrum to prove the local stability. We first discuss the asymptotic behavior of the system (1.4) as $z \to -\infty$. By linearizing the system (1.4) at $(\frac{\mu}{\sigma}, 1)$, we have

$$\begin{cases} -c\overline{U}' = d\overline{U}'' - \alpha\overline{U}' - \sigma\overline{U} + \mu\overline{V}, \\ -c\overline{V}' = \epsilon\overline{V}'' + \sigma\overline{U} - \mu\overline{V} + f'(1)(\overline{V} - 1). \end{cases} \tag{3.17}$$

Let

$$(\overline{U}, \overline{V})(z) = \left(\frac{\mu}{\sigma} - \xi_3 e^{\lambda z}, 1 - \xi_4 e^{\lambda z}\right) \tag{3.18}$$

with $\lambda > 0$ and ξ_3, ξ_4 being positive constants. Substituting (3.18) into (3.17), it follows that

$$\begin{pmatrix} d\lambda^2 + (c-\alpha)\lambda - \sigma & \mu \\ \sigma & \epsilon\lambda^2 + c\lambda - \mu + f'(1) \end{pmatrix} \xi = 0, \tag{3.19}$$

where $\xi = (\xi_3, \xi_4)^T$. Suppose that $\lambda_i (i = 3, 4, 5, 6)$ are the eigenvalues to the left matrix of Eq (3.19). According to the Vieta's theorem, we know these four eigenvalues are two positive numbers and two negative numbers. Without loss of generality, we assume that $\lambda_3 > \lambda_4 > 0 > \lambda_5 > \lambda_6$ for $c > 0$. Then the wave profile has the following asymptotic behaviors:

$$\begin{pmatrix} \overline{U} \\ \overline{V} \end{pmatrix} = \begin{pmatrix} \frac{\mu}{\sigma} \\ 1 \end{pmatrix} - C_3 \begin{pmatrix} \xi_3(\lambda_4) \\ \xi_4(\lambda_4) \end{pmatrix} e^{\lambda_4 z} - C_4 \begin{pmatrix} \xi_3(\lambda_3) \\ \xi_4(\lambda_3) \end{pmatrix} e^{\lambda_3 z}, \ as\ z \to -\infty \tag{3.20}$$

with $C_3 > 0$ or $C_3 = 0, C_4 > 0$.

To associate with (3.2), we consider the following system

$$u_t = Du_{zz} + Cu_z + Bu, \tag{3.21}$$

where $u(z,t) = (u_1(z,t), u_2(z,t))^T$ and D, C, B are defined in (3.3). For a given solution semiflow $Q_t = u(z, t, \psi)$ of (3.21) with any given initial data $\phi \in L^p$, we denote by $e^{\chi t}\Psi$ the solution of (3.21). It is easy to see that Q_t is compact and strongly positive. By the Krein-Rutman theorem (see, e.g., [35]), Q_t has a simple principal eigenvalue χ_{max} with a strongly positive eigenvector, and all other eigenvalues must satisfy $e^{\chi t}\Psi < e^{\chi_{max} t}\Psi$.

Next, we shall discuss the eigenvalue χ for two cases as $\chi = 0$ and $\chi > 0$.

Case 1. When $\chi = 0$, it can be directly calculated that $(-\overline{U}', -\overline{V}')(z)$ is the corresponding positive eigenvector derived form (3.2), where the asymptotic behavior of the solution of (2.1), i.e. $(\overline{U}, \overline{V})(z) \sim (C_1 e^{-\lambda_1 z}, C_1 e^{-\lambda_1 z})$, $C_1 > 0$, as $z \to \infty$. Although it could be check that the positive eigenvector $(-\overline{U}', -\overline{V}')(z)$ is not belong to the weighted space L^p_ω since $\lambda_1 < \beta < \lambda_2$.

Case 2. In this case, we discuss the eigenvalue $\chi > 0$ to produce a contrary for two cases with respect to z, namely, $z \to -\infty$ and $z \to \infty$. Assume that $\chi > 0$ and $\Psi \in L^p_\omega$. Then we know that the minimal positive eigenvalue of the Ψ is larger than β. Since $\overline{\Psi}(z) = (-\overline{U}', -\overline{V}')(z)$ is a positive solution of (3.21), it follows that $\overline{\Psi}(z) > \Psi$ as $z \to \infty$.

When $z \to -\infty$, assume that $\Psi(z)$ has the asymptotic behavior as $ke^{\lambda z}$ for some positive k and λ. Substituting it into the spectral problem (3.5), we obtain the characteristic equation in eigenvalue χ as follows:

$$\begin{vmatrix} d\lambda^2 + (c-\alpha)\lambda - \sigma - \chi & \mu \\ \sigma & \epsilon\lambda^2 + c\lambda - \mu + f'(1) - \chi \end{vmatrix} = 0. \tag{3.22}$$

Next, we shall show the relationship between the four roots of (3.22) denoted by $\hat{\lambda}_i (i = 3, 4, 5, 6)$ and $\lambda_i (i = 3, 4, 5, 6)$, namely, $\hat{\lambda}_3 > \lambda_3$, $\hat{\lambda}_4 > \lambda_4$, $\hat{\lambda}_5 < \lambda_5$, $\hat{\lambda}_6 < \lambda_6$. In fact, when $\chi > 0$, the parabolic

$$\begin{aligned} &p_1 : d\lambda^2 + (c-\alpha)\lambda - \sigma - \chi = 0, \\ &p_2 : d\lambda^2 + (c-\alpha)\lambda - \sigma = 0, \end{aligned} \tag{3.23}$$

have two roots $r^{\pm}_{p_1}$ and $r^{\pm}_{p_2}$ with one positive and one negative, respectively. It is easy to see that $r^{-}_{p_1} < r^{-}_{p_2}$ and $r^{+}_{p_1} < r^{+}_{p_2}$. Similar discussion to

$$\begin{aligned} &p_3 : \epsilon\lambda^2 + c\lambda - \mu + f'(1) - \chi = 0, \\ &p_4 : \epsilon\lambda^2 + c\lambda - \mu + f'(1) = 0, \end{aligned} \tag{3.24}$$

we also obtain that $r^{-}_{p_3} < r^{-}_{p_4}$ and $r^{+}_{p_3} < r^{+}_{p_4}$. Further, we know that $r^{-}_{p_1}$, $r^{+}_{p_1}$, $r^{-}_{p_3}$ and $r^{+}_{p_3}$ are also the roots of

$$p_5 : (d\lambda^2 + (c-\alpha)\lambda - \sigma - \chi)(\epsilon\lambda^2 + c\lambda - \mu + f'(1) - \chi) = 0 \tag{3.25}$$

and $r^{-}_{p_2}$, $r^{+}_{p_2}$, $r^{-}_{p_4}$ and $r^{+}_{p_4}$ are the roots of

$$p_6 : (d\lambda^2 + (c-\alpha)\lambda - \sigma)(\epsilon\lambda^2 + c\lambda - \mu + f'(1)) = 0. \tag{3.26}$$

That is to say that the positive roots of p_5 related to p_6 are moving right. Therefore, when the curves of p_5 and p_6 intersects with the same line $y = \mu\sigma$, the points are denoted by $\hat{\lambda}_i (i = 3, 4, 5, 6)$ and $\lambda_i (i = 3, 4, 5, 6)$, respectively. And the points of intersection satisfy $\hat{\lambda}_3 > \lambda_3$, $\hat{\lambda}_4 > \lambda_4$, $\hat{\lambda}_5 < \lambda_5$, $\hat{\lambda}_6 < \lambda_6$. In other words, the positive roots of λ are increasing with respect to χ. This indicates that $\overline{\Psi}(z) \sim k_1 e^{\lambda_3 z}$ and $\Psi(z) \sim k_2 e^{\hat{\lambda}_3 z}$ as $z \to -\infty$.

Thus, we choose $\bar{k}$ sufficient large such that $\bar{k}\overline{\Psi} \geq |\Psi|$. By the comparison principal for the system (3.21), it must be $\bar{k}\overline{\Psi}(z) \geq |\Psi|e^{\chi t}$. Hence the assumption $\chi > 0$ is incorrect. This implies that the real parts of all eigenvalues χ of (3.5) should not be positive for $\Psi \in L^p_\omega$.

The proof is complete.

4. Proof of Theorem 2

In this section, we will prove Theorem 2. In order to realize it, we need the following conclusion. (**Comparison principle**) Let $(U^+, V^+)(z, t)$ and $(U^-, V^-)(z, t)$ be the solutions of (1.7) with respect to the initial values

$$\begin{aligned} &U_0^+(z) = \max\{U_0(z), \overline{U}(z)\}, \quad V_0^+(z) = \max\{V_0(z), \overline{V}(z)\}, \\ &U_0^-(z) = \min\{U_0(z), \overline{U}(z)\}, \quad V_0^-(z) = \min\{V_0(z), \overline{V}(z)\} \end{aligned} \tag{4.1}$$

respectively; namely

$$\begin{cases} U_t^{\pm} = dU_{zz}^{\pm} + (c-\alpha)U_z^{\pm} - \sigma U^{\pm} + \mu V^{\pm}, \\ V_t^{\pm} = \epsilon V_{zz}^{\pm} + cV_z^{\pm} + \sigma U^{\pm} - \mu V^{\pm} + f(V^{\pm}), \\ (U^{\pm}, V^{\pm})(z,0) = (U_0^{\pm}, V_0^{\pm})(z). \end{cases} \tag{4.2}$$

It holds that

$$\begin{aligned} &(0,0) \le (U_0^-, V_0^-)(z) \le (U_0, V_0)(z) \le (U_0^+, V_0^+)(z) \le \left(\frac{\mu}{\sigma}, 1\right), \\ &(0,0) \le (U_0^-, V_0^-)(z) \le (\overline{U}, \overline{V})(z) \le (U_0^+, V_0^+)(z) \le \left(\frac{\mu}{\sigma}, 1\right). \end{aligned} \tag{4.3}$$

By comparison principle, we have

$$\begin{aligned} &(0,0) \le (U^-, V^-)(z,t) \le (\overline{U}, \overline{V})(z,t) \le (U^+, V^+)(z,t) \le \left(\frac{\mu}{\sigma}, 1\right), \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+, \\ &(0,0) \le (U^-, V^-)(z,t) \le (\overline{U}, \overline{V})(z) \le (U^+, V^+)(z,t) \le \left(\frac{\mu}{\sigma}, 1\right), \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+. \end{aligned} \tag{4.4}$$

If both $(U^+, V^+)(z,t)$ and $(U^-, V^-)(z,t)$ converge to $(\overline{U}, \overline{V})(z)$, then the squeezing theorem ensures the global stability of system (1.7) stated in (2).

Lemma 4.1. *Under the conditions given in Theorem 2, $(U^+, V^+)(z,t)$ converges to $(\overline{U}, \overline{V})(z)$.*

Proof. To begin with, we suppose that

$$R(z,t) = U^+(z,t) - \overline{U}(z), S(z,t) = V^+(z,t) - \overline{V}(z), \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{4.5}$$

which satisfies the initial values

$$R(z,0) = U_0^+(z) - \overline{U}(z), S(z,0) = V_0^+(z) - \overline{V}(z). \tag{4.6}$$

By (4.2) and (4.4), we have

$$(0,0) \le (R,S)(z,t) \le \left(\frac{\mu}{\sigma}, 1\right), \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{4.7}$$

Through (1.4) and (4.3), the following inequality

$$\begin{pmatrix} R \\ S \end{pmatrix}_t \le D \begin{pmatrix} R \\ S \end{pmatrix}_{zz} + C \begin{pmatrix} R \\ S \end{pmatrix}_z + B_0 \begin{pmatrix} R \\ S \end{pmatrix} \tag{4.8}$$

holds for the matrices C and D defined in (3.3), and

$$B_0 = \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu + f'(0) \end{pmatrix}. \tag{4.9}$$

Then we consider the convergence of $(R,S)(z,t)$ for two cases with respect to z.
Case 1. When $z > z_0$, we first define

$$\begin{pmatrix} R \\ S \end{pmatrix}(z,t) = e^{-\beta(z-z_0)} \begin{pmatrix} \overline{R} \\ \overline{S} \end{pmatrix}(z,t), \tag{4.10}$$

so as to investigate the stability in weighted function space L^∞_ω for all $(z,t)\in\mathbb{R}\times\mathbb{R}^+$, where $\overline{R}$ and $\overline{S}$ are the functions in $L^\infty(\mathbb{R})$, the weighted function $\omega(z)$ is defined in (3.3). Substituting (4.10) into (4.8), we obtain

$$\begin{aligned}\begin{pmatrix}\overline{R}\\ \overline{S}\end{pmatrix}_t &\le D\begin{pmatrix}\overline{R}\\ \overline{S}\end{pmatrix}_{zz}+\begin{pmatrix}c-\alpha-2d\beta & 0\\ 0 & c-2\epsilon\beta\end{pmatrix}\begin{pmatrix}\overline{R}\\ \overline{S}\end{pmatrix}_z+\begin{pmatrix}A_1(\beta)-c\beta & \mu\\ \sigma & A_2(\beta)-c\beta\end{pmatrix}\begin{pmatrix}\overline{R}\\ \overline{S}\end{pmatrix}\\ &:=\begin{pmatrix}\mathfrak{L}_1(\overline{R},\overline{S})\\ \mathfrak{L}_2(\overline{R},\overline{S})\end{pmatrix},\end{aligned}\tag{4.11}$$

where $A_1(\beta)$ and $A_2(\beta)$ are defined in (2.4).

Further, we choose

$$\overline{R}_1(z,t)=k_1\zeta_1e^{-\eta_1t},\ \overline{S}_1(z,t)=k_1\zeta_2e^{-\eta_1t},\ \forall(z,t)\in\mathbb{R}\times\mathbb{R}^+,\tag{4.12}$$

for some positive constants k_1 and η_1, where $(\zeta_1,\zeta_2)^T=(\zeta_1(\beta),\zeta_2(\beta))^T$ is the eigenvector of matrix M_1. In fact,

$$M_1=\begin{pmatrix}A_1(\beta)-c\beta & \mu\\ \sigma & A_2(\beta)-c\beta\end{pmatrix}\tag{4.13}$$

which has the eigenvalue

$$\overline{\lambda}_1=\tfrac{1}{2}\left(A_1(\beta)+A_2(\beta)+\sqrt{(A_1(\beta)-A_2(\beta))^2+4\sigma\mu}\right)-c\beta.$$

The associated eigenvectors can be found by direct calculation as follows:

$$\zeta_1(\beta)=\mu,\zeta_2(\beta)=\frac{1}{2}\left(-A_1(\beta)+A_2(\beta)+\sqrt{(A_1(\beta)-A_2(\beta))^2+4\sigma\mu}\right).\tag{4.14}$$

It is not hard to check that, for $\lambda_1<\beta<\lambda_2$, $\zeta_1(\beta)$, $\zeta_2(\beta)$ are positive and $\overline{\lambda}_1$ is negative. Moreover, one can obtain

$$\mathfrak{L}_1(\overline{R}_1,\overline{S}_1)=\overline{\lambda}_1\overline{R}_1<0,\ \ \mathfrak{L}_2(\overline{R}_1,\overline{S}_1)=\overline{\lambda}_1\overline{S}_1<0.\tag{4.15}$$

Thus, we can choose $\eta_1\le-\overline{\lambda}_1$ such that

$$\begin{pmatrix}\overline{R}_1\\ \overline{S}_1\end{pmatrix}_t=-\eta_1k_1\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}e^{-\eta_1t}\ge\begin{pmatrix}\mathfrak{L}_1(\overline{R}_1,\overline{S}_1)\\ \mathfrak{L}_2(\overline{R}_1,\overline{S}_1)\end{pmatrix}.\tag{4.16}$$

Once we choose $k_1\ge\max\limits_{z\in\mathbb{R}}\left\{\frac{\overline{R}(z,0)}{\zeta_1},\frac{\overline{S}(z,0)}{\zeta_2}\right\}$ to make

$$(\overline{R}_1,\overline{S}_1)(z,0)=(k_1\zeta_1,k_2\zeta_2)\ge(\overline{R},\overline{S})(z,0),$$

then by comparison principle on unbounded domain, $\forall(z,t)\in\mathbb{R}\times\mathbb{R}^+$, we obtain

$$(R,S)(z,t)=e^{-\beta(z-z_0)}(\overline{R},\overline{S})(z,t)\le k_1(\zeta_1,\zeta_2)e^{-\beta(z-z_0)-\eta_1t}.\tag{4.17}$$

Therefore, the above inequality holds for any fixed point z_0. That is to say, for $z>z_0$, $(R,S)(z,t)$ converges to $(0,0)$.

Case 2. When $z \le z_0$, the system marked $(R, S)(z,t)$ can be shown as

$$\begin{pmatrix} R \\ S \end{pmatrix}_t = D\begin{pmatrix} R \\ S \end{pmatrix}_{zz} + C\begin{pmatrix} R \\ S \end{pmatrix}_z + \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu \end{pmatrix}\begin{pmatrix} R \\ S \end{pmatrix} + \begin{pmatrix} 0 \\ f(S+\overline{V}) - f(\overline{V}) \end{pmatrix}. \tag{4.18}$$

Here, $f(S+\overline{V}) - f(\overline{V})$ is a second-order differentiable function that can be expressed as $f'(1)S + h(S)$ ($h(S)$ also is second-order differentiable and $h(0) = 0$) as $(\overline{U}, \overline{V})(z) \to \left(\frac{\mu}{\sigma}, 1\right)$. Consequently, we need to select z_0, so that

$$\begin{pmatrix} R \\ S \end{pmatrix}_t \le D\begin{pmatrix} R \\ S \end{pmatrix}_{zz} + C\begin{pmatrix} R \\ S \end{pmatrix}_z + \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu + f'(1) + \varepsilon_1 \end{pmatrix}\begin{pmatrix} R \\ S \end{pmatrix} + \begin{pmatrix} 0 \\ h(S) \end{pmatrix} \tag{4.19}$$

for some given enough small positive ε_1 and $\varepsilon_1 \ll \mu - f'(1)$.

From the following autonomous system

$$\begin{pmatrix} \widehat{R} \\ \widehat{S} \end{pmatrix}_t = \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu + f'(1) + \varepsilon_1 \end{pmatrix}\begin{pmatrix} \widehat{R} \\ \widehat{S} \end{pmatrix} + \begin{pmatrix} 0 \\ h(\widehat{S}) \end{pmatrix} \tag{4.20}$$

with the initial data

$$\widehat{R}(0) \ge \overline{R}(z,0), \widehat{S}(0) \ge \overline{S}(z,0), \forall z \in \mathbb{R}, \tag{4.21}$$

we know that the solution $(\widehat{R}, \widehat{S})(t)$ of (4.20) is an upper solution of system (4.18).

Next, for the convergence of $(R, S)(z,t)$, it is sufficient to prove that $(\widehat{R}, \widehat{S})(t)$ converges to $(0,0)$ as t tends to ∞. A direct calculation shows that the eigenvalues $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ of the Jacobian matrix of system (4.20) at the fixed point $(0,0)$ are less than zero. Therefore, $(0,0)$ is a stable node. From the phase plane analysis, any manifold in $\widehat{RS}$-space for any initial value $(\widehat{R}, \widehat{S})(0)$ in region $[0, \frac{\mu}{\sigma}] \times [0,1]$ converges to the origin $(0,0)$. Then, as $t \to \infty$, we define

$$(\widehat{R}, \widehat{S}) = \widehat{k}_1(\widehat{C}_1, \widehat{C}_2)e^{\widehat{\lambda}_1 t} \tag{4.22}$$

with $\widehat{k}_1 > 0$ and $(\widehat{C}_1, \widehat{C}_2)^T$ is the eigenvector of the Jacobian matrix respect to $\widehat{\lambda}_1$. We can choose $\widehat{k}_1$ large enough and $\widetilde{\lambda}_1 = \min\{\eta_1, -\widehat{\lambda}_1\}$ to be used that we have

$$(R, S)(z_0, t) \le k_1(\zeta_1, \zeta_2)e^{-\eta_1 t} \le \widehat{k}_1(\zeta_1, \zeta_2)e^{-\widetilde{\lambda}_1 t} \tag{4.23}$$

at the boundary $z = z_0$. As a result, by comparison on the domain $(-\infty, z_0] \times [0, \infty)$, see Lemma 3.2 in [36], for all $(z,t) \in (-\infty, z_0] \times \mathbb{R}^+$,

$$(R, S)(z,t) \le \widehat{k}_1(\zeta_1, \zeta_2)e^{-\widetilde{\lambda}_1 t}, \tag{4.24}$$

then $(R, S)(z,t)$ converges to $(0,0)$ for $z \in (-\infty, z_0]$. □

Lemma 4.2. *Under the conditions given in Theorem 2, $(U^-, V^-)(z,t)$ converges to $(\overline{U}, \overline{V})(z)$.*

Proof. Let

$$X(z,t) = \overline{U}(z) - U^-(z,t), Y(z,t) = \overline{V}(z) - V^-(z,t), \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+ \tag{4.25}$$

to satisfy the initial values

$$X(z,0) = \overline{U}(z) - U_0^-(z), Y(z,0) = \overline{V}(z) - V_0^-(z). \tag{4.26}$$

Similar discussion as lemma 4.1, $\forall(z,t) \in \mathbb{R} \times \mathbb{R}^+$, we have

$$(0,0) \le (X,Y)(z,t) \le \left(\frac{\mu}{\sigma}, 1\right), \tag{4.27}$$

and

$$\begin{pmatrix} X \\ Y \end{pmatrix}_t = D\begin{pmatrix} X \\ Y \end{pmatrix}_{zz} + C\begin{pmatrix} X \\ Y \end{pmatrix}_z + B_1\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 0 \\ f(V^- + Y) - f(V^-) \end{pmatrix}, \tag{4.28}$$

where the second-order matrices C and D are consistent with those in (3.3) and

$$B_1 = \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu \end{pmatrix}. \tag{4.29}$$

Now we also consider the convergence for the following two cases with respect to z.

Case 1. $(z > z_0)$. By using the fact $X < \overline{U}$ and $Y < \overline{V}$, we can verify that $\exists \eta_2 > 0$ and $k_2 \ge e^{\beta(z-z_0)} \max\limits_{z\in\mathbb{R}} \left\{\frac{X(z,0)}{\zeta_1}, \frac{Y(z,0)}{\zeta_2}\right\}$ bring

$$(X,Y)(z,t) \le k_2(\zeta_1,\zeta_2)e^{-\eta_2 t} \tag{4.30}$$

for all $(z,t) \in \mathbb{R} \times \mathbb{R}^+$.

Case 2. $(z \le z_0)$. Define $(\widehat{X},\widehat{Y})(z,t)$ as the solution of the following autonomous system

$$\begin{pmatrix} \widehat{X} \\ \widehat{Y} \end{pmatrix}_t = \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu + f'(1) + \varepsilon_1 \end{pmatrix}\begin{pmatrix} \widehat{X} \\ \widehat{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ h(\widehat{Y}) \end{pmatrix} \tag{4.31}$$

with the initial data

$$\widehat{X}(0) \ge \overline{X}(z,0), \widehat{Y}(0) \ge \overline{Y}(z,0), \forall z \in \mathbb{R}. \tag{4.32}$$

Then $(\widehat{X},\widehat{Y})$ is an upper solution to the system

$$\begin{pmatrix} X \\ Y \end{pmatrix}_t = D\begin{pmatrix} X \\ Y \end{pmatrix}_{zz} + C\begin{pmatrix} X \\ Y \end{pmatrix}_z + \begin{pmatrix} -\sigma & \mu \\ \sigma & -\mu \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 0 \\ f(\overline{V}) - f(\overline{V} - Y) \end{pmatrix}. \tag{4.33}$$

Phase plane analysis shows that for any initial values in region $[0, \frac{\mu}{\sigma}]\times[0,1]$, $(\widehat{X},\widehat{Y})(t)$ always converges to $\left(\frac{\mu}{\sigma}, 1\right)$. Similar to the proof of case 2 in Lemma 4.1, for some positive numbers $\widehat{k}_2$ and $\widetilde{\lambda}_2$, we obtain

$$(X,Y)(z,t) \le \widehat{k}_2(\zeta_1,\zeta_2)e^{-\widetilde{\lambda}_2 t}, \quad \forall(z,t) \in (-\infty, z_0] \times \mathbb{R}^+. \tag{4.34}$$

□

Now we will use the squeezing theorem to obtain the conclusion of the Theorem (2).

By the inequalities (4.4), for any $(z,t) \in \mathbb{R} \times \mathbb{R}^+$, it gives

$$\begin{aligned} |X(z,t)| &\le |U(z,t) - \overline{U}(z)| \le |R(z,t)|, \\ |Y(z,t)| &\le |V(z,t) - \overline{V}(z)| \le |S(z,t)|. \end{aligned} \tag{4.35}$$

From Lemmas 4.1 and 4.2 and the squeezing theorem, $\exists k > 0, \eta > 0$ so that $\forall(z,t) \in \mathbb{R} \times \mathbb{R}^+$,

$$\begin{aligned} |U(z,t) - \overline{U}(z)| &\le ke^{-\eta t}, \\ |V(z,t) - \overline{V}(z)| &\le ke^{-\eta t}. \end{aligned} \tag{4.36}$$

To sum up, Theorem 2 is proved completely.

Author contributions

Y. Tang and C. Pan: Methodology; H. Wang and C. Pan: Validation; Y. Tang and C. Pan: Formal analysis; C. Pan: Writing-original draft preparation; Y. Tang and H. Wang: Writing-review and editing; C.Pan and H. Wang: Supervision; C. Pan: Project administration; C. Pan: Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This work is supported by the National Natural Science Foundation of China grant (No.11801263).

Conflict of interest

The authors declare no conflict of interest.

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